

**MATH 221
FIRST SEMESTER
CALCULUS**

fall 2009

Typeset: June 8, 2010

MATH 221 – 1st SEMESTER CALCULUS
LECTURE NOTES VERSION 2.0 (fall 2009)

This is a self contained set of lecture notes for Math 221. The notes were written by Sigurd Angenent, starting from an extensive collection of notes and problems compiled by Joel Robbin. The $\text{\LaTeX}$ and `Python` files which were used to produce these notes are available at the following web site

<http://www.math.wisc.edu/~angenent/Free-Lecture-Notes>

They are meant to be freely available in the sense that “free software” is free. More precisely:

Copyright (c) 2006 Sigurd B. Angenent. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled “GNU Free Documentation License”.

Contents

Chapter 1. Numbers and Functions	5	3. Exercises	64
1. What is a number?	5	4. Finding sign changes of a function	65
2. Exercises	7	5. Increasing and decreasing functions	66
3. Functions	8	6. Examples	67
4. Inverse functions and Implicit functions	10	7. Maxima and Minima	69
5. Exercises	13	8. Must there always be a maximum?	71
Chapter 2. Derivatives (1)	15	9. Examples – functions with and without maxima or minima	71
1. The tangent to a curve	15	10. General method for sketching the graph of a function	72
2. An example – tangent to a parabola	16	11. Convexity, Concavity and the Second Derivative	74
3. Instantaneous velocity	17	12. Proofs of some of the theorems	75
4. Rates of change	17	13. Exercises	76
5. Examples of rates of change	18	14. Optimization Problems	77
6. Exercises	18	15. Exercises	78
Chapter 3. Limits and Continuous Functions	21	Chapter 6. Exponentials and Logarithms (naturally)	81
1. Informal definition of limits	21	1. Exponents	81
2. The formal, authoritative, definition of limit	22	2. Logarithms	82
3. Exercises	25	3. Properties of logarithms	83
4. Variations on the limit theme	25	4. Graphs of exponential functions and logarithms	83
5. Properties of the Limit	27	5. The derivative of a^x and the definition of e	84
6. Examples of limit computations	27	6. Derivatives of Logarithms	85
7. When limits fail to exist	29	7. Limits involving exponentials and logarithms	86
8. What's in a name?	32	8. Exponential growth and decay	86
9. Limits and Inequalities	33	9. Exercises	87
10. Continuity	34	Chapter 7. The Integral	91
11. Substitution in Limits	35	1. Area under a Graph	91
12. Exercises	36	2. When f changes its sign	92
13. Two Limits in Trigonometry	36	3. The Fundamental Theorem of Calculus	93
14. Exercises	38	4. Exercises	94
Chapter 4. Derivatives (2)	41	5. The indefinite integral	95
1. Derivatives Defined	41	6. Properties of the Integral	97
2. Direct computation of derivatives	42	7. The definite integral as a function of its integration bounds	98
3. Differentiable implies Continuous	43	8. Method of substitution	99
4. Some non-differentiable functions	43	9. Exercises	100
5. Exercises	44	Chapter 8. Applications of the integral	105
6. The Differentiation Rules	45	1. Areas between graphs	105
7. Differentiating powers of functions	48	2. Exercises	106
8. Exercises	49	3. Cavalieri's principle and volumes of solids	106
9. Higher Derivatives	50	4. Examples of volumes of solids of revolution	109
10. Exercises	51	5. Volumes by cylindrical shells	111
11. Differentiating Trigonometric functions	51	6. Exercises	113
12. Exercises	52	7. Distance from velocity, velocity from acceleration	113
13. The Chain Rule	52	8. The length of a curve	116
14. Exercises	57	9. Examples of length computations	117
15. Implicit differentiation	58	10. Exercises	118
16. Exercises	60	11. Work done by a force	118
Chapter 5. Graph Sketching and Max-Min Problems	63	12. Work done by an electric current	119
1. Tangent and Normal lines to a graph	63	Chapter 9. Answers and Hints	121
2. The Intermediate Value Theorem	63	GNU Free Documentation License	125

1. APPLICABILITY AND DEFINITIONS	125
2. VERBATIM COPYING	125
3. COPYING IN QUANTITY	125
4. MODIFICATIONS	125
5. COMBINING DOCUMENTS	126
6. COLLECTIONS OF DOCUMENTS	126
7. AGGREGATION WITH INDEPENDENT WORKS	126
8. TRANSLATION	126
9. TERMINATION	126
10. FUTURE REVISIONS OF THIS LICENSE	126
11. RELICENSING	126

CHAPTER 1

Numbers and Functions

The subject of this course is “functions of one real variable” so we begin by wondering what a real number “really” is, and then, in the next section, what a function is.

1. What is a number?

1.1. Different kinds of numbers. The simplest numbers are the *positive integers*

$$1, 2, 3, 4, \dots$$

the number *zero*

$$0,$$

and the *negative integers*

$$\dots, -4, -3, -2, -1.$$

Together these form the integers or “whole numbers.”

Next, there are the numbers you get by dividing one whole number by another (nonzero) whole number. These are the so called fractions or *rational numbers* such as

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \dots$$

or

$$-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{2}{4}, -\frac{3}{4}, -\frac{4}{3}, \dots$$

By definition, any whole number is a rational number (in particular zero is a rational number.)

You can add, subtract, multiply and divide any pair of rational numbers and the result will again be a rational number (provided you don’t try to divide by zero).

One day in middle school you were told that there are other numbers besides the rational numbers, and the first example of such a number is the square root of two. It has been known ever since the time of the greeks that no rational number exists whose square is exactly 2, i.e. you can’t find a fraction $\frac{m}{n}$ such that

$$\left(\frac{m}{n}\right)^2 = 2, \text{ i.e. } m^2 = 2n^2.$$

Nevertheless, if you compute x^2 for some values of x between 1 and 2, and check if you get more or less than 2, then it looks like there should be some number x between 1.4 and 1.5 whose square is exactly 2. So, we *assume* that there is such a number, and we call it the square root of 2, written as $\sqrt{2}$. This raises several questions. How do we know there really is a number between 1.4 and 1.5 for which $x^2 = 2$? How many other such numbers are we going to assume into existence? Do these new numbers obey the same algebra rules (like $a + b = b + a$) as the rational numbers? If we knew precisely what these numbers (like $\sqrt{2}$) were then we could perhaps answer such questions. It turns out to be rather difficult to give a precise description of what a number is, and in this course we won’t try to get anywhere near the bottom of this issue. Instead we will think of numbers as “infinite decimal expansions” as follows.

One can represent certain fractions as decimal fractions, e.g.

$$\frac{279}{25} = \frac{1116}{100} = 11.16.$$

x	x^2
1.2	1.44
1.3	1.69
1.4	$1.96 < 2$
1.5	$2.25 > 2$
1.6	2.56

Not all fractions can be represented as decimal fractions. For instance, expanding $\frac{1}{3}$ into a decimal fraction leads to an unending decimal fraction

$$\frac{1}{3} = 0.333\ 333\ 333\ 333\ 333\ \dots$$

It is impossible to write the complete decimal expansion of $\frac{1}{3}$ because it contains infinitely many digits. But we can describe the expansion: each digit is a three. An electronic calculator, which always represents numbers as **finite** decimal numbers, can never hold the number $\frac{1}{3}$ exactly.

Every fraction can be written as a decimal fraction which may or may not be finite. If the decimal expansion doesn't end, then it must repeat. For instance,

$$\frac{1}{7} = 0.142857\ 142857\ 142857\ 142857\ \dots$$

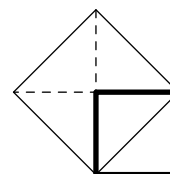
Conversely, any infinite repeating decimal expansion represents a rational number.

A **real number** is specified by a possibly unending decimal expansion. For instance,

$$\sqrt{2} = 1.414\ 213\ 562\ 373\ 095\ 048\ 801\ 688\ 724\ 209\ 698\ 078\ 569\ 671\ 875\ 376\ 9\ \dots$$

Of course you can never write *all* the digits in the decimal expansion, so you only write the first few digits and hide the others behind dots. To give a precise description of a real number (such as $\sqrt{2}$) you have to explain how you could *in principle* compute as many digits in the expansion as you would like. During the next three semesters of calculus we will not go into the details of how this should be done.

1.2. A reason to believe in $\sqrt{2}$. The Pythagorean theorem says that the hypotenuse of a right triangle with sides 1 and 1 must be a line segment of length $\sqrt{2}$. In middle or high school you learned something similar to the following geometric construction of a line segment whose length is $\sqrt{2}$. Take a square with side of length 1, and construct a new square one of whose sides is the diagonal of the first square. The figure you get consists of 5 triangles of equal area and by counting triangles you see that the larger square has exactly twice the area of the smaller square. Therefore the diagonal of the smaller square, being the side of the larger square, is $\sqrt{2}$ as long as the side of the smaller square.



Why are real numbers called real? All the numbers we will use in this first semester of calculus are “real numbers.” At some point (in 2nd semester calculus) it becomes useful to assume that there is a number whose square is -1 . No real number has this property since the square of any real number is positive, so it was decided to call this new imagined number “imaginary” and to refer to the numbers we already have (rationals, $\sqrt{2}$ -like things) as “real.”

1.3. The real number line and intervals. It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the **origin**. We also decide which direction we call “left” and hence which we call “right.” Some draw the number line vertically and use the words “up” and “down.”

To plot any real number x one marks off a distance x from the origin, to the right (up) if $x > 0$, to the left (down) if $x < 0$.

The **distance along the number line** between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number.

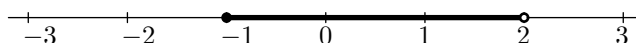


Figure 1. To draw the half open interval $[-1, 2)$ use a filled dot to mark the endpoint which is included and an open dot for an excluded endpoint.

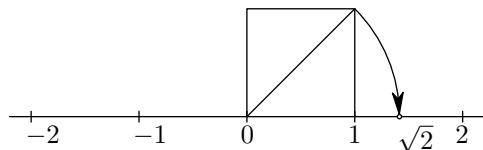


Figure 2. To find $\sqrt{2}$ on the real line you draw a square of sides 1 and drop the diagonal onto the real line.

Almost every equation involving variables x , y , etc. we write down in this course will be true for some values of x but not for others. In modern abstract mathematics a collection of real numbers (or any other kind of mathematical objects) is called a **set**. Below are some examples of sets of real numbers. We will use the notation from these examples throughout this course.

The collection of all real numbers between two given real numbers form an interval. The following notation is used

- (a, b) is the set of all real numbers x which satisfy $a < x < b$.
- $[a, b)$ is the set of all real numbers x which satisfy $a \leq x < b$.
- $(a, b]$ is the set of all real numbers x which satisfy $a < x \leq b$.
- $[a, b]$ is the set of all real numbers x which satisfy $a \leq x \leq b$.

If the endpoint is not included then it may be ∞ or $-\infty$. E.g. $(-\infty, 2]$ is the interval of all real numbers (both positive and negative) which are ≤ 2 .

1.4. Set notation. A common way of describing a set is to say it is the collection of all real numbers which satisfy a certain condition. One uses this notation

$$\mathcal{A} = \{x \mid x \text{ satisfies this or that condition}\}$$

Most of the time we will use upper case letters in a calligraphic font to denote sets. $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \dots)$

For instance, the interval (a, b) can be described as

$$(a, b) = \{x \mid a < x < b\}$$

The set

$$\mathcal{B} = \{x \mid x^2 - 1 > 0\}$$

consists of all real numbers x for which $x^2 - 1 > 0$, i.e. it consists of all real numbers x for which either $x > 1$ or $x < -1$ holds. This set consists of two parts: the interval $(-\infty, -1)$ and the interval $(1, \infty)$.

You can try to draw a set of real numbers by drawing the number line and coloring the points belonging to that set red, or by marking them in some other way.

Some sets can be very difficult to draw. For instance,

$$\mathcal{C} = \{x \mid x \text{ is a rational number}\}$$

can't be accurately drawn. In this course we will try to avoid such sets.

Sets can also contain just a few numbers, like

$$\mathcal{D} = \{1, 2, 3\}$$

which is the set containing the numbers one, two and three. Or the set

$$\mathcal{E} = \{x \mid x^3 - 4x^2 + 1 = 0\}$$

which consists of the solutions of the equation $x^3 - 4x^2 + 1 = 0$. (There are three of them, but it is not easy to give a formula for the solutions.)

If $\mathcal{A}$ and $\mathcal{B}$ are two sets then **the union of $\mathcal{A}$ and $\mathcal{B}$** is the set which contains all numbers that belong either to $\mathcal{A}$ or to $\mathcal{B}$. The following notation is used

$$\mathcal{A} \cup \mathcal{B} = \{x \mid x \text{ belongs to } \mathcal{A} \text{ or to } \mathcal{B} \text{ or both.}\}$$

Similarly, the *intersection of two sets $\mathcal{A}$ and $\mathcal{B}$* is the set of numbers which belong to both sets. This notation is used:

$$\mathcal{A} \cap \mathcal{B} = \{x \mid x \text{ belongs to both } \mathcal{A} \text{ and } \mathcal{B}\}$$

2. Exercises

1. What is the 2007th digit after the period in the expansion of $\frac{1}{7}$?

2. Which of the following fractions have finite decimal expansions?

$$a = \frac{2}{3}, \quad b = \frac{3}{25}, \quad c = \frac{276937}{15625}.$$

3. Draw the following sets of real numbers. Each of these sets is the union of one or more intervals. Find those intervals. Which of these sets are finite?

$$\mathcal{A} = \{x \mid x^2 - 3x + 2 \leq 0\}$$

$$\mathcal{B} = \{x \mid x^2 - 3x + 2 \geq 0\}$$

$$\mathcal{C} = \{x \mid x^2 - 3x > 3\}$$

$$\mathcal{D} = \{x \mid x^2 - 5 > 2x\}$$

$$\mathcal{E} = \{t \mid t^2 - 3t + 2 \leq 0\}$$

$$\mathcal{F} = \{\alpha \mid \alpha^2 - 3\alpha + 2 \geq 0\}$$

$$\mathcal{G} = (0, 1) \cup (5, 7]$$

$$\mathcal{H} = (\{1\} \cup \{2, 3\}) \cap (0, 2\sqrt{2})$$

$$\mathcal{Q} = \{\theta \mid \sin \theta = \frac{1}{2}\}$$

$$\mathcal{R} = \{\varphi \mid \cos \varphi > 0\}$$

4. Suppose $\mathcal{A}$ and $\mathcal{B}$ are intervals. Is it always true that $\mathcal{A} \cap \mathcal{B}$ is an interval? How about $\mathcal{A} \cup \mathcal{B}$?

5. Consider the sets

$$\mathcal{M} = \{x \mid x > 0\} \text{ and } \mathcal{N} = \{y \mid y > 0\}.$$

Are these sets the same?

6. **Group Problem.**

Write the numbers

$$x = 0.3131313131\dots, \quad y = 0.273273273273\dots$$

$$\text{and } z = 0.21541541541541541\dots$$

as fractions (i.e. write them as $\frac{m}{n}$, specifying m and n .)

(Hint: show that $100x = x + 31$. A similar trick works for y , but z is a little harder.)

7. **Group Problem.**

Is the number whose decimal expansion after the period consists only of nines, i.e.

$$x = 0.9999999999999999\dots$$

an integer?

3. Functions

Wherein we meet the main characters of this semester

3.1. Definition. To specify a *function* f you must

- (1) give a *rule* which tells you how to compute the value $f(x)$ of the function for a given real number x , and:
- (2) say for which real numbers x the rule may be applied.

The set of numbers for which a function is defined is called its *domain*. The set of all possible numbers $f(x)$ as x runs over the domain is called the *range* of the function. The rule must be *unambiguous*: the same x must always lead to the same $f(x)$.

For instance, one can define a function f by putting $f(x) = \sqrt{x}$ for all $x \geq 0$. Here the rule defining f is “take the square root of whatever number you’re given”, and the function f will accept all nonnegative real numbers.

The rule which specifies a function can come in many different forms. Most often it is a formula, as in the square root example of the previous paragraph. Sometimes you need a few formulas, as in

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases} \quad \text{domain of } g = \text{all real numbers.}$$

Functions which are defined by different formulas on different intervals are sometimes called *piecewise defined functions*.

3.2. Graphing a function. You get the *graph of a function* f by drawing all points whose coordinates are (x, y) where x must be in the domain of f and $y = f(x)$.

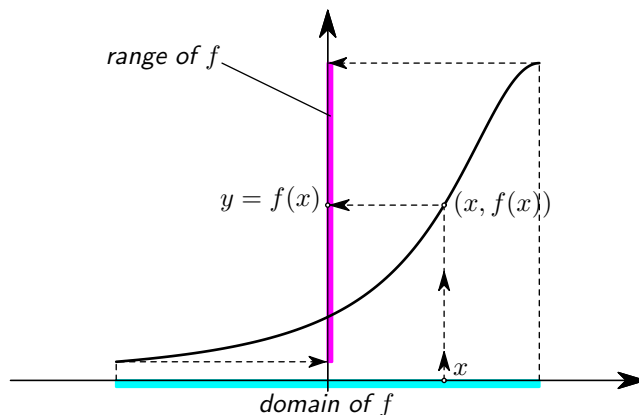


Figure 3. The graph of a function f . The domain of f consists of all x values at which the function is defined, and the range consists of all possible values f can have.

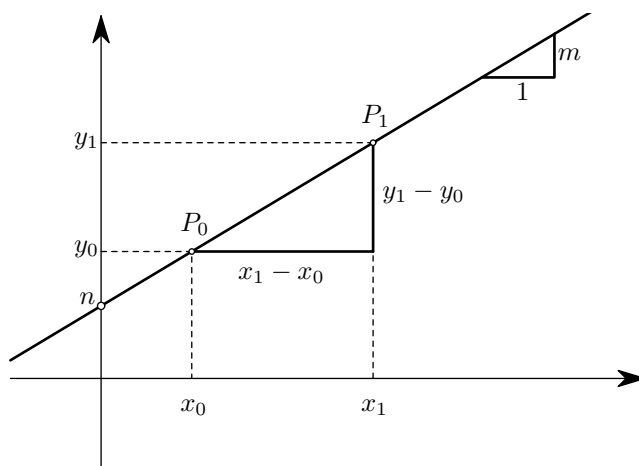


Figure 4. A straight line and its slope. The line is the graph of $f(x) = mx + n$. It intersects the y -axis at height n , and the ratio between the amounts by which y and x increase as you move from one point to another on the line is $\frac{y_1 - y_0}{x_1 - x_0} = m$.

3.3. Linear functions. A function which is given by the formula

$$f(x) = mx + n$$

where m and n are constants is called a **linear function**. Its graph is a straight line. The constants m and n are the **slope** and **y -intercept** of the line. Conversely, any straight line which is not vertical (i.e. not parallel to the y -axis) is the graph of a linear function. If you know two points (x_0, y_0) and (x_1, y_1) on the line, then then one can compute the slope m from the “rise-over-run” formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

This formula actually contains a theorem from Euclidean geometry, namely it says that the ratio $(y_1 - y_0) : (x_1 - x_0)$ is the same for every pair of points (x_0, y_0) and (x_1, y_1) that you could pick on the line.

3.4. Domain and “biggest possible domain.” In this course we will usually not be careful about specifying the domain of the function. When this happens the domain is understood to be the set of all x for which the rule which tells you how to compute $f(x)$ is meaningful. For instance, if we say that h is the function

$$h(x) = \sqrt{x}$$

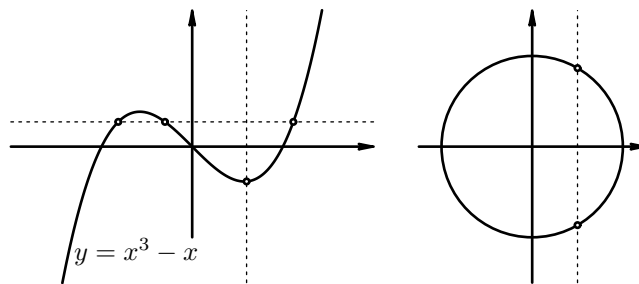


Figure 5. The graph of $y = x^3 - x$ fails the “horizontal line test,” but it passes the “vertical line test.” The circle fails both tests.

then the domain of h is understood to be the set of all nonnegative real numbers

$$\text{domain of } h = [0, \infty)$$

since $\sqrt{x}$ is well-defined for all $x \geq 0$ and undefined for $x < 0$.

A systematic way of finding the domain and range of a function for which you are only given a formula is as follows:

- The domain of f consists of all x for which $f(x)$ is well-defined (“makes sense”)
- The range of f consists of all y for which you can solve the equation $f(x) = y$.

3.5. Example – find the domain and range of $f(x) = 1/x^2$. The expression $1/x^2$ can be computed for all real numbers x except $x = 0$ since this leads to division by zero. Hence the domain of the function $f(x) = 1/x^2$ is

$$\text{“all real numbers except 0”} = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty).$$

To find the range we ask “for which y can we solve the equation $y = f(x)$ for x ,” i.e. we for which y can you solve $y = 1/x^2$ for x ?

If $y = 1/x^2$ then we must have $x^2 = 1/y$, so first of all, since we have to divide by y , y can’t be zero. Furthermore, $1/y = x^2$ says that y must be positive. On the other hand, if $y > 0$ then $y = 1/x^2$ has a solution (in fact two solutions), namely $x = \pm 1/\sqrt{y}$. This shows that the range of f is

$$\text{“all positive real numbers”} = \{x \mid x > 0\} = (0, \infty).$$

3.6. Functions in “real life.” One can describe the motion of an object using a function. If some object is moving along a straight line, then you can define the following function: Let $x(t)$ be the distance from the object to a fixed marker on the line, at the time t . Here the domain of the function is the set of all times t for which we know the position of the object, and the rule is

Given t , measure the distance between the object and the marker at time t .

There are many examples of this kind. For instance, a biologist could describe the growth of a cell by defining $m(t)$ to be the mass of the cell at time t (measured since the birth of the cell). Here the domain is the interval $[0, T]$, where T is the life time of the cell, and the rule that describes the function is

Given t , weigh the cell at time t .

3.7. The Vertical Line Property. Generally speaking graphs of functions are curves in the plane but they distinguish themselves from arbitrary curves by the way they intersect vertical lines: **The graph of a function cannot intersect a vertical line “ $x = \text{constant}$ ” in more than one point.** The reason why this is true is very simple: if two points lie on a vertical line, then they have the same x coordinate, so if they also lie on the graph of a function f , then their y -coordinates must also be equal, namely $f(x)$.

3.8. Examples. The graph of $f(x) = x^3 - x$ “goes up and down,” and, even though it intersects several horizontal lines in more than one point, it intersects *every* vertical line in exactly one point.

The collection of points determined by the equation $x^2 + y^2 = 1$ is a circle. It is not the graph of a function since the vertical line $x = 0$ (the y -axis) intersects the graph in two points $P_1(0, 1)$ and $P_2(0, -1)$. See Figure 6.

4. Inverse functions and Implicit functions

For many functions the rule which tells you how to compute it is not an explicit formula, but instead an equation which you still must solve. A function which is defined in this way is called an “implicit function.”

4.1. Example. One can define a function f by saying that for each x the value of $f(x)$ is the solution y of the equation

$$x^2 + 2y - 3 = 0.$$

In this example you can solve the equation for y ,

$$y = \frac{3 - x^2}{2}.$$

Thus we see that the function we have defined is $f(x) = (3 - x^2)/2$.

Here we have two definitions of the same function, namely

- (i) “ $y = f(x)$ is defined by $x^2 + 2y - 3 = 0$,” and
- (ii) “ f is defined by $f(x) = (3 - x^2)/2$.”

The first definition is the implicit definition, the second is explicit. You see that with an “implicit function” it isn’t the function itself, but rather the way it was defined that’s implicit.

4.2. Another example: domain of an implicitly defined function. Define g by saying that for any x the value $y = g(x)$ is the solution of

$$x^2 + xy - 3 = 0.$$

Just as in the previous example one can then solve for y , and one finds that

$$g(x) = y = \frac{3 - x^2}{x}.$$

Unlike the previous example this formula does not make sense when $x = 0$, and indeed, for $x = 0$ our rule for g says that $g(0) = y$ is the solution of

$$0^2 + 0 \cdot y - 3 = 0, \text{ i.e. } y \text{ is the solution of } 3 = 0.$$

That equation has no solution and hence $x = 0$ does not belong to the domain of our function g .

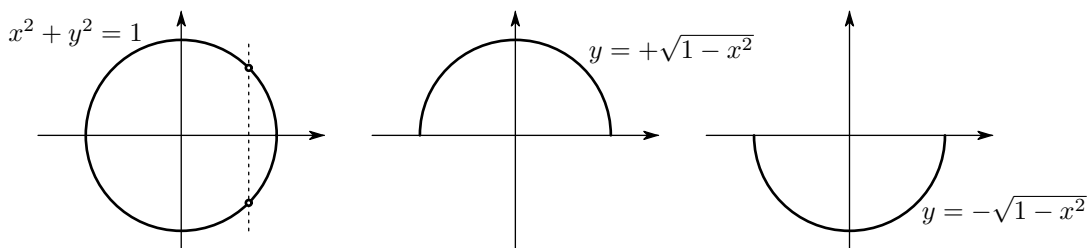


Figure 6. The circle determined by $x^2 + y^2 = 1$ is not the graph of a function, but it contains the graphs of the two functions $h_1(x) = \sqrt{1 - x^2}$ and $h_2(x) = -\sqrt{1 - x^2}$.

4.3. Example: the equation alone does not determine the function. Define $y = h(x)$ to be the solution of

$$x^2 + y^2 = 1.$$

If $x > 1$ or $x < -1$ then $x^2 > 1$ and there is no solution, so $h(x)$ is at most defined when $-1 \leq x \leq 1$. But when $-1 < x < 1$ there is another problem: not only does the equation have a solution, but it even has two solutions:

$$x^2 + y^2 = 1 \iff y = \sqrt{1 - x^2} \text{ or } y = -\sqrt{1 - x^2}.$$

The rule which defines a function must be unambiguous, and since we have not specified which of these two solutions is $h(x)$ the function is not defined for $-1 < x < 1$.

One can fix this by making a choice, but there are many possible choices. Here are three possibilities:

$$h_1(x) = \text{the nonnegative solution } y \text{ of } x^2 + y^2 = 1$$

$$h_2(x) = \text{the nonpositive solution } y \text{ of } x^2 + y^2 = 1$$

$$h_3(x) = \begin{cases} h_1(x) & \text{when } x < 0 \\ h_2(x) & \text{when } x \geq 0 \end{cases}$$

4.4. Why use implicit functions? In all the examples we have done so far we could replace the implicit description of the function with an explicit formula. This is not always possible or if it is possible the implicit description is much simpler than the explicit formula. For instance, you can define a function f by saying that $y = f(x)$ if and only if

$$(1) \quad y^3 + 3y + 2x = 0.$$

This means that the recipe for computing $f(x)$ for any given x is “solve the equation $y^3 + 3y + 2x = 0$.” E.g. to compute $f(0)$ you set $x = 0$ and solve $y^3 + 3y = 0$. The only solution is $y = 0$, so $f(0) = 0$. To compute $f(1)$ you have to solve $y^3 + 3y + 2 \cdot 1 = 0$, and if you’re lucky you see that $y = -1$ is the solution, and $f(1) = -1$.

In general, no matter what x is, the equation (1) turns out to have exactly one solution y (which depends on x , this is how you get the function f). Solving (1) is not easy. In the early 1500s Cardano and Tartaglia discovered a formula¹ for the solution. Here it is:

$$y = f(x) = \sqrt[3]{-x + \sqrt{1 + x^2}} - \sqrt[3]{x + \sqrt{1 + x^2}}.$$

The implicit description looks a lot simpler, and when we try to differentiate this function later on, it will be much easier to use “implicit differentiation” than to use the Cardano-Tartaglia formula directly.

4.5. Inverse functions. If you have a function f , then you can try to define a new function f^{-1} , the so-called **inverse function of f** , by the following prescription:

$$(2) \quad \text{For any given } x \text{ we say that } y = f^{-1}(x) \text{ if } y \text{ is the solution to the equation } f(y) = x.$$

So to find $y = f^{-1}(x)$ you solve the equation $x = f(y)$. If this is to define a function then the prescription (2) must be unambiguous and the equation $f(y) = x$ has to have a solution and cannot have more than one solution.

¹To see the solution and its history visit

http://www.gap-system.org/~history/HistTopics/Quadratic_etc_equations.html

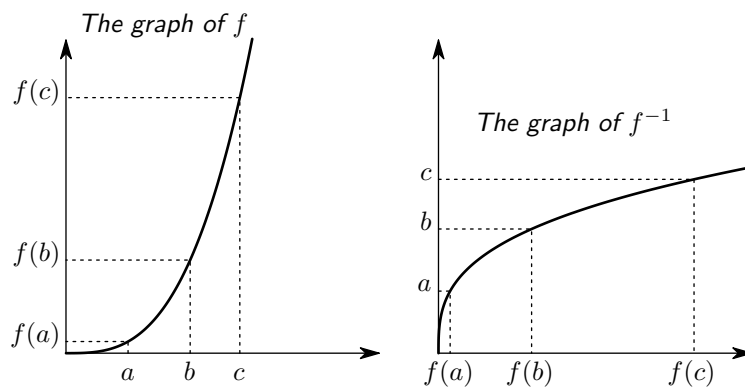


Figure 7. The graph of a function and its inverse are mirror images of each other.

4.6. Examples. Consider the function f with $f(x) = 2x + 3$. Then the equation $f(y) = x$ works out to be

$$2y + 3 = x$$

and this has the solution

$$y = \frac{x - 3}{2}.$$

So $f^{-1}(x)$ is defined for all x , and it is given by $f^{-1}(x) = (x - 3)/2$.

Next we consider the function $g(x) = x^2$ with domain all positive real numbers. To see for which x the inverse $g^{-1}(x)$ is defined we try to solve the equation $g(y) = x$, i.e. we try to solve $y^2 = x$. If $x < 0$ then this equation has no solutions since $y \geq 0$ for all y . But if $x \geq 0$ then $y = \sqrt{x}$ does have a solution, namely $y = \sqrt{x}$.

So we see that $g^{-1}(x)$ is defined for all nonnegative real numbers x , and that it is given by $g^{-1}(x) = \sqrt{x}$.

4.7. Inverse trigonometric functions. The familiar trigonometric functions Sine, Cosine and Tangent have inverses which are called arcsine, arccosine and arctangent.

$y = f(x)$		$x = f^{-1}(y)$	
$y = \sin x$	$(-\pi/2 \leq x \leq \pi/2)$	$x = \arcsin(y)$	$(-1 \leq y \leq 1)$
$y = \cos x$	$(0 \leq x \leq \pi)$	$x = \arccos(y)$	$(-1 \leq y \leq 1)$
$y = \tan x$	$(-\pi/2 < x < \pi/2)$	$x = \arctan(y)$	

The notations $\arcsin y = \sin^{-1} y$, $\arccos x = \cos^{-1} x$, and $\arctan u = \tan^{-1} u$ are also commonly used for the inverse trigonometric functions. We will avoid the $\sin^{-1} y$ notation because it is ambiguous. Namely, everybody writes the square of $\sin y$ as

$$(\sin y)^2 = \sin^2 y.$$

Replacing the 2's by -1 's would lead to

$$\arcsin y = \sin^{-1} y \stackrel{???}{=} (\sin y)^{-1} = \frac{1}{\sin y}, \quad \text{which is not true!}$$

5. Exercises

- 8.** The functions f and g are defined by

$$f(x) = x^2 \text{ and } g(s) = s^2.$$

Are f and g the same functions or are they different?

- 9.** Find a formula for the function f which is defined by

$$y = f(x) \iff x^2 y + y = 7.$$

What is the domain of f ?

- 10.** Find a formula for the function f which is defined by

$$y = f(x) \iff x^2 y - y = 6.$$

What is the domain of f ?

- 11.** Let f be the function defined by $y = f(x) \iff y$ is the largest solution of

$$y^2 = 3x^2 - 2xy.$$

Find a formula for f . What are the domain and range of f ?

12. Find a formula for the function f which is defined by $y = f(x) \iff 2x + 2xy + y^2 = 5$ and $y > -x$.

Find the domain of f .

13. Use a calculator to compute $f(1.2)$ in three decimals where f is the implicitly defined function from §4.4. (There are (at least) two different ways of finding $f(1.2)$)

14. Group Problem.

(a) True or false:

for all x one has $\sin(\arcsin x) = x$?

(b) True or false:

for all x one has $\arcsin(\sin x) = x$?

15. On a graphing calculator plot the graphs of the following functions, and explain the results. (Hint: first do the previous exercise.)

$$f(x) = \arcsin(\sin x), \quad -2\pi \leq x \leq 2\pi$$

$$g(x) = \arcsin(x) + \arccos(x), \quad 0 \leq x \leq 1$$

$$h(x) = \arctan \frac{\sin x}{\cos x}, \quad |x| < \pi/2$$

$$k(x) = \arctan \frac{\cos x}{\sin x}, \quad |x| < \pi/2$$

$$l(x) = \arcsin(\cos x), \quad -\pi \leq x \leq \pi$$

$$m(x) = \cos(\arcsin x), \quad -1 \leq x \leq 1$$

16. Find the inverse of the function f which is given by $f(x) = \sin x$ and whose domain is $\pi \leq x \leq 2\pi$. Sketch the graphs of both f and f^{-1} .

17. Find a number a such that the function $f(x) = \sin(x + \pi/4)$ with domain $a \leq x \leq a + \pi$ has an inverse. Give a formula for $f^{-1}(x)$ using the arcsine function.

18. Draw the graph of the function h_3 from §4.3.

19. A function f is given which satisfies

$$f(2x + 3) = x^2$$

for all real numbers x .

Compute

$$(a) f(0) \quad (b) f(3) \quad (c) f(x)$$

$$(d) f(y) \quad (e) f(f(2))$$

where x and y are arbitrary real numbers.

What are the range and domain of f ?

20. A function f is given which satisfies

$$f\left(\frac{1}{x+1}\right) = 2x - 12.$$

for all real numbers x .

Compute

$$(a) f(1) \quad (b) f(0) \quad (c) f(x)$$

$$(d) f(t) \quad (e) f(f(2))$$

where x and t are arbitrary real numbers.

What are the range and domain of f ?

21. Does there exist a function f which satisfies

$$f(x^2) = x + 1$$

for all real numbers x ?

* * *

The following exercises review precalculus material involving quadratic expressions $ax^2 + bx + c$ in one way or another.

22. Explain how you "complete the square" in a quadratic expression like $ax^2 + bx$.

23. Find the range of the following functions:

$$f(x) = 2x^2 + 3$$

$$g(x) = -2x^2 + 4x$$

$$h(x) = 4x + x^2$$

$$k(x) = 4 \sin x + \sin^2 x$$

$$\ell(x) = 1/(1 + x^2)$$

$$m(x) = 1/(3 + 2x + x^2).$$

24. Group Problem.

For each real number a we define a line ℓ_a with equation $y = ax + a^2$.

(a) Draw the lines corresponding to $a = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.

(b) Does the point with coordinates $(3, 2)$ lie on one or more of the lines ℓ_a (where a can be any number, not just the five values from part (a))? If so, for which values of a does $(3, 2)$ lie on ℓ_a ?

(c) Which points in the plane lie on at least one of the lines ℓ_a ?

25. For which values of m and n does the graph of $f(x) = mx + n$ intersect the graph of $g(x) = 1/x$ in exactly one point and also contain the point $(-1, 1)$?

26. For which values of m and n does the graph of $f(x) = mx + n$ not intersect the graph of $g(x) = 1/x$?

Derivatives (1)

To work with derivatives you have to know what a limit is, but to motivate why we are going to study limits let's first look at the two classical problems that gave rise to the notion of a derivative: the tangent to a curve, and the instantaneous velocity of a moving object.

1. The tangent to a curve

Suppose you have a function $y = f(x)$ and you draw its graph. If you want to find the tangent to the graph of f at some given point on the graph of f , how would you do that?

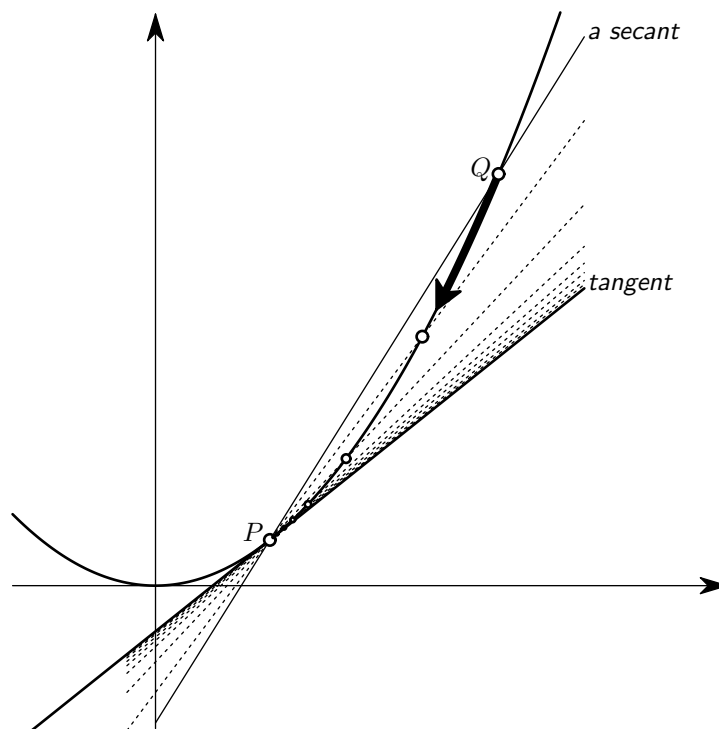


Figure 1. Constructing the tangent by letting $Q \rightarrow P$

Let P be the point on the graph at which you want to draw the tangent. If you are making a real paper and ink drawing you would take a ruler, make sure it goes through P and then turn it until it doesn't cross the graph anywhere else.

If you are using equations to describe the curve and lines, then you could pick a point Q on the graph and construct the line through P and Q ("construct" means "find an equation for"). This line is called a "secant," and it is of course not the tangent that you're looking for. But if you choose Q to be very close to P then the secant will be close to the tangent.

So this is our recipe for constructing the tangent through P : pick another point Q on the graph, find the line through P and Q , and see what happens to this line as you take Q closer and closer to P . The resulting secants will then get closer and closer to some line, and that line is the tangent.

We'll write this in formulas in a moment, but first let's worry about how close Q should be to P . We can't set Q equal to P , because then P and Q don't determine a line (you need *two* points to determine a line). If you choose Q different from P then you don't get the tangent, but at best something that is "close" to it. Some people have suggested that one should take Q "infinitely close" to P , but it isn't clear what that would mean. The concept of a limit is meant to solve this confusing problem.

2. An example – tangent to a parabola

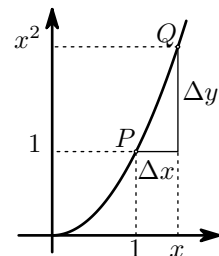
To make things more concrete, suppose that the function we had was $f(x) = x^2$, and that the point was $(1, 1)$. The graph of f is of course a parabola.

Any line through the point $P(1, 1)$ has equation

$$y - 1 = m(x - 1)$$

where m is the slope of the line. So instead of finding the equation of the secant and tangent lines we will find their slopes.

Let Q be the other point on the parabola, with coordinates (x, x^2) . We can "move Q around on the graph" by changing x . Whatever x we choose, it must be different from 1, for otherwise P and Q would be the same point. What we want to find out is how the line through P and Q changes if x is changed (and in particular, if x is chosen very close to a). Now, as one changes x one thing stays the same, namely, the secant still goes through P . So to describe the secant we only need to know its slope. By the "rise over run" formula, the slope of the secant line joining P and Q is



$$m_{PQ} = \frac{\Delta y}{\Delta x} \quad \text{where} \quad \Delta y = x^2 - 1 \quad \text{and} \quad \Delta x = x - 1.$$

By factoring $x^2 - 1$ we can rewrite the formula for the slope as follows

$$(3) \quad m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1.$$

As x gets closer to 1, the slope m_{PQ} , being $x + 1$, gets closer to the value $1 + 1 = 2$. We say that

the limit of the slope m_{PQ} as Q approaches P is 2.

In symbols,

$$\lim_{Q \rightarrow P} m_{PQ} = 2,$$

or, since Q approaching P is the same as x approaching 1,

$$(4) \quad \lim_{x \rightarrow 1} m_{PQ} = 2.$$

So we find that the tangent line to the parabola $y = x^2$ at the point $(1, 1)$ has equation

$$y - 1 = 2(x - 1), \text{ i.e. } y = 2x - 1.$$

A warning: you cannot substitute $x = 1$ in equation (3) to get (4) even though it looks like that's what we did. The reason why you can't do that is that when $x = 1$ the point Q coincides with the point P so "the line through P and Q " is not defined; also, if $x = 1$ then $\Delta x = \Delta y = 0$ so that the rise-over-run formula for the slope gives

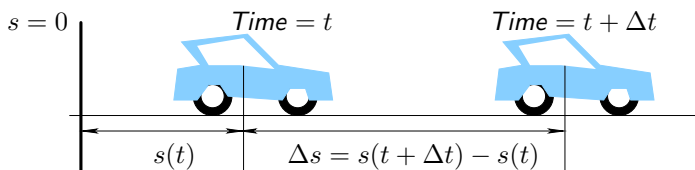
$$m_{PQ} = \frac{\Delta x}{\Delta y} = \frac{0}{0} = \text{undefined}.$$

It is only after the algebra trick in (3) that setting $x = 1$ gives something that is well defined. But if the intermediate steps leading to $m_{PQ} = x + 1$ aren't valid for $x = 1$ why should the final result mean anything for $x = 1$?

Something more complicated has happened. We did a calculation which is valid for all $x \neq 1$, and later looked at what happens if x gets “very close to 1.” This is the concept of a limit and we’ll study it in more detail later in this section, but first another example.

3. Instantaneous velocity

If you try to define “instantaneous velocity” you will again end up trying to divide zero by zero. Here is how it goes: When you are driving in your car the speedometer tells you how fast you are going, i.e. what your velocity is. What is this velocity? What does it mean if the speedometer says “50mph”?



We all know what **average velocity** is. Namely, if it takes you two hours to cover 100 miles, then your average velocity was

$$\frac{\text{distance traveled}}{\text{time it took}} = 50 \text{ miles per hour.}$$

This is not the number the speedometer provides you – it doesn’t wait two hours, measure how far you went and compute distance/time. If the speedometer in your car tells you that you are driving 50mph, then that should be your velocity **at the moment** that you look at your speedometer, i.e. “distance traveled over time it took” at the moment you look at the speedometer. But during the moment you look at your speedometer no time goes by (because a moment has no length) and you didn’t cover any distance, so your velocity at that moment is $\frac{0}{0}$, i.e. undefined. Your velocity at **any** moment is undefined. But then what is the speedometer telling you?

To put all this into formulas we need to introduce some notation. Let t be the time (in hours) that has passed since we got onto the road, and let $s(t)$ be the distance we have covered since then.

Instead of trying to find the velocity exactly at time t , we find a formula for the average velocity during some (short) time interval beginning at time t . We’ll write Δt for the length of the time interval.

At time t we have traveled $s(t)$ miles. A little later, at time $t + \Delta t$ we have traveled $s(t + \Delta t)$. Therefore during the time interval from t to $t + \Delta t$ we have moved $s(t + \Delta t) - s(t)$ miles. Our average velocity in that time interval is therefore

$$\frac{s(t + \Delta t) - s(t)}{\Delta t} \text{ miles per hour.}$$

The shorter you make the time interval, i.e. the smaller you choose Δt , the closer this number should be to the instantaneous velocity at time t .

So we have the following formula (definition, really) for the velocity at time t

$$(5) \quad v(t) = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}.$$

4. Rates of change

The two previous examples have much in common. If we ignore all the details about geometry, graphs, highways and motion, the following happened in both examples:

We had a function $y = f(x)$, and we wanted to know how much $f(x)$ changes if x changes. If you change x to $x + \Delta x$, then y will change from $f(x)$ to $f(x + \Delta x)$. The change in y is therefore

$$\Delta y = f(x + \Delta x) - f(x),$$

and the average rate of change is

$$(6) \quad \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

This is the average rate of change of f over the interval from x to $x + \Delta x$. To define **the rate of change of the function f at x** we let the length Δx of the interval become smaller and smaller, in the hope that the average rate of change over the shorter and shorter time intervals will get closer and closer to some number. If that happens then that “limiting number” is called the rate of change of f at x , or, the **derivative** of f at x . It is written as

$$(7) \quad f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Derivatives and what you can do with them are what the first half of this semester is about. The description we just went through shows that to understand what a derivative is you need to know what a limit is. In the next chapter we’ll study limits so that we get a less vague understanding of formulas like (7).

5. Examples of rates of change

5.1. Acceleration as the rate at which velocity changes. As you are driving in your car your velocity does not stay constant, it changes with time. Suppose $v(t)$ is your velocity at time t (measured in miles per hour). You could try to figure out how fast your velocity is changing by measuring it at one moment in time (you get $v(t)$), then measuring it a little later (you get $v(t + \Delta t)$). You conclude that your velocity increased by $\Delta v = v(t + \Delta t) - v(t)$ during a time interval of length Δt , and hence

$$\left\{ \begin{array}{l} \text{average rate at which} \\ \text{your velocity changed} \end{array} \right\} = \frac{\Delta v}{\Delta t} = \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$

This rate of change is called your *average acceleration* (over the time interval from t to $t + \Delta t$). Your *instantaneous acceleration* at time t is the limit of your average acceleration as you make the time interval shorter and shorter:

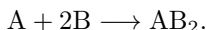
$$\{\text{acceleration at time } t\} = a = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$

th the average and instantaneous accelerations are measured in “miles per hour per hour,” i.e. in

$$(\text{mi/h})/\text{h} = \text{mi/h}^2.$$

Or, if you had measured distances in meters and time in seconds then velocities would be measured in meters per second, and acceleration in meters per second per second, which is the same as meters per second², i.e. “meters per squared second.”

5.2. Reaction rates. Think of a chemical reaction in which two substances A and B react to form AB₂ according to the reaction



If the reaction is taking place in a closed reactor, then the “amounts” of A and B will be decreasing, while the amount of AB₂ will increase. Chemists write $[A]$ for the amount of “A” in the chemical reactor (measured in moles). Clearly $[A]$ changes with time so it defines a *function*. We’re mathematicians so we will write “ $[A](t)$ ” for the number of moles of A present at time t .

To describe how fast the amount of A is changing we consider the derivative of $[A]$ with respect to time, i.e.

$$[A]'(t) = \lim_{\Delta t \rightarrow 0} \frac{[A](t + \Delta t) - [A](t)}{\Delta t}.$$

This quantity is the rate of change of $[A]$. The notation “ $[A]'(t)$ ” is really only used by calculus professors. If you open a paper on chemistry you will find that the derivative is written in LEIBNIZ notation:

$$\frac{d[A]}{dt}$$

More on this in §1.2

How fast does the reaction take place? If you add more A or more B to the reactor then you would expect that the reaction would go faster, i.e. that more AB₂ is being produced per second. The law of *mass-action*

kinetics from chemistry states this more precisely. For our particular reaction it would say that the rate at which A is consumed is given by

$$\frac{d[A]}{dt} = k[A][B]^2,$$

in which the constant k is called the *reaction constant*. It's a constant that you could try to measure by timing how fast the reaction goes.

6. Exercises

27. Repeat the reasoning in §2 to find the slope at the point $(\frac{1}{2}, \frac{1}{4})$, or more generally at any point (a, a^2) on the parabola with equation $y = x^2$.

28. Repeat the reasoning in §2 to find the slope at the point $(\frac{1}{2}, \frac{1}{8})$, or more generally at any point (a, a^3) on the curve with equation $y = x^3$.

29. Group Problem.

Should you trust your calculator?

Find the slope of the tangent to the parabola $y = x^2$ at the point $(\frac{1}{3}, \frac{1}{9})$ (You have already done this: see exercise 27).

Instead of doing the algebra you could try to compute the slope by using a calculator. This exercise is about how you do that and what happens if you try (too hard).

Compute $\frac{\Delta y}{\Delta x}$ for various values of Δx :

$$\Delta x = 0.1, 0.01, 0.001, 10^{-6}, 10^{-12}.$$

As you choose Δx smaller your computed $\frac{\Delta y}{\Delta x}$ ought to get closer to the actual slope. Use at least 10 decimals and organize your results in a table like this:

Δx	$f(a)$	$f(a + \Delta x)$	Δy	$\Delta y / \Delta x$
0.1	...	...	...	...
0.01	...	...	...	...
0.001	...	...	...	...
10^{-6}	...	...	...	...
10^{-12}	...	...	...	...

Look carefully at the ratios $\Delta y / \Delta x$. Do they look like they are converging to some number? Compare the values of $\frac{\Delta y}{\Delta x}$ with the true value you got in the beginning of this problem.

30. Simplify the algebraic expressions you get when you compute Δy and $\Delta y / \Delta x$ for the following functions

(a) $y = x^2 - 2x + 1$

(b) $y = \frac{1}{x}$

(c) $y = 2^x$

31. Look ahead at Figure 3 in the next chapter. What is the derivative of $f(x) = x \cos \frac{\pi}{x}$ at the points A and B on the graph?

32. Suppose that some quantity y is a function of some other quantity x , and suppose that y is a mass, i.e. y is measured in pounds, and x is a length, measured in feet. What units do the increments Δy and Δx , and the derivative dy/dx have?

33. A tank is filling with water. The volume (in gallons) of water in the tank at time t (seconds) is $V(t)$. What units does the derivative $V'(t)$ have?

34. Group Problem.

Let $A(x)$ be the area of an equilateral triangle whose sides measure x inches.

(a) Show that $\frac{dA}{dx}$ has the units of a length.

(b) Which length does $\frac{dA}{dx}$ represent geometrically? [Hint: draw two equilateral triangles, one with side x and another with side $x + \Delta x$. Arrange the triangles so that they both have the origin as their lower left hand corner, and so their base is on the x-axis.]

35. Group Problem.

Let $A(x)$ be the area of a square with side x , and let $L(x)$ be the perimeter of the square (sum of the lengths of all its sides). Using the familiar formulas for $A(x)$ and $L(x)$ show that $A'(x) = \frac{1}{2}L(x)$.

Give a geometric interpretation that explains why $\Delta A \approx \frac{1}{2}L(x)\Delta x$ for small Δx .

36. Let $A(r)$ be the area enclosed by a circle of radius r , and let $L(r)$ be the length of the circle. Show that $A'(r) = L(r)$. (Use the familiar formulas from geometry for the area and perimeter of a circle.)

37. Let $V(r)$ be the volume enclosed by a sphere of radius r , and let $S(r)$ be its surface area. Show that $V'(r) = S(r)$. (Use the formulas $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$.)

Limits and Continuous Functions

1. Informal definition of limits

While it is easy to define precisely in a few words what a square root is ($\sqrt{a}$ is the positive number whose square is a) the definition of the limit of a function runs over several terse lines, and most people don't find it very enlightening when they first see it. (See §2.) So we postpone this for a while and fine tune our intuition for another page.

1.1. Definition of limit (1st attempt). If f is some function then

$$\lim_{x \rightarrow a} f(x) = L$$

is read “the limit of $f(x)$ as x approaches a is L .” It means that if you choose values of x which are close **but not equal** to a , then $f(x)$ will be close to the value L ; moreover, $f(x)$ gets closer and closer to L as x gets closer and closer to a .

The following alternative notation is sometimes used

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow a;$$

(read “ $f(x)$ approaches L as x approaches a ” or “ $f(x)$ goes to L as x goes to a ”.)

1.2. Example. If $f(x) = x + 3$ then

$$\lim_{x \rightarrow 4} f(x) = 7,$$

is true, because if you substitute numbers x close to 4 in $f(x) = x + 3$ the result will be close to 7.

1.3. Example: substituting numbers to guess a limit. What (if anything) is

$$\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x^2 - 4}?$$

Here $f(x) = (x^2 - 2x)/(x^2 - 4)$ and $a = 2$.

We first try to substitute $x = 2$, but this leads to

$$f(2) = \frac{2^2 - 2 \cdot 2}{2^2 - 4} = \frac{0}{0}$$

which does not exist. Next we try to substitute values of x close but not equal to 2. Table 1 suggests that $f(x)$ approaches 0.5.

x	$f(x)$	x	$g(x)$
3.000000	0.600000	1.000000	1.009990
2.500000	0.555556	0.500000	1.009980
2.100000	0.512195	0.100000	1.009899
2.010000	0.501247	0.010000	1.008991
2.001000	0.500125	0.001000	1.000000

Table 1. Finding limits by substituting values of x “close to a .” (Values of $f(x)$ and $g(x)$ rounded to six decimals.)

1.4. Example: Substituting numbers can suggest the wrong answer. The previous example shows that our first definition of “limit” is not very precise, because it says “ x close to a ,” but how close is close enough? Suppose we had taken the function

$$g(x) = \frac{101\,000x}{100\,000x + 1}$$

and we had asked for the limit $\lim_{x \rightarrow 0} g(x)$.

Then substitution of some “small values of x ” could lead us to believe that the limit is $1.000\dots$. Only when you substitute even smaller values do you find that the limit is 0 (zero)!

See also problem [29](#).

2. The formal, authoritative, definition of limit

The informal description of the limit uses phrases like “closer and closer” and “really very small.” In the end we don’t really know what they mean, although they are suggestive. “Fortunately” there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x \rightarrow a} f(x)$ equals some number L or not. Here is the definition. It takes a while to digest, so read it once, look at the examples, do a few exercises, read the definition again. Go on to the next sections. Throughout the semester come back to this section and read it again.

2.1. Definition of $\lim_{x \rightarrow a} f(x) = L$. We say that L is the limit of $f(x)$ as $x \rightarrow a$, if

- (1) $f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a .
- (2) for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all x in the domain of f one has

$$(8) \quad |x - a| < \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Why the absolute values? The quantity $|x - y|$ is the distance between the points x and y on the number line, and one can measure how close x is to y by calculating $|x - y|$. The inequality $|x - y| < \delta$ says that “the distance between x and y is less than δ ,” or that “ x and y are closer than δ .”

What are ε and δ ? The quantity ε is how close you would like $f(x)$ to be to its limit L ; the quantity δ is how close you have to choose x to a to achieve this. To prove that $\lim_{x \rightarrow a} f(x) = L$ you must assume that someone has given you an unknown $\varepsilon > 0$, and then find a positive δ for which (8) holds. The δ you find will depend on ε .

2.2. Show that $\lim_{x \rightarrow 5} 2x + 1 = 11$. We have $f(x) = 2x + 1$, $a = 5$ and $L = 11$, and the question we must answer is “how close should x be to 5 if want to be sure that $f(x) = 2x + 1$ differs less than ε from $L = 11$?”

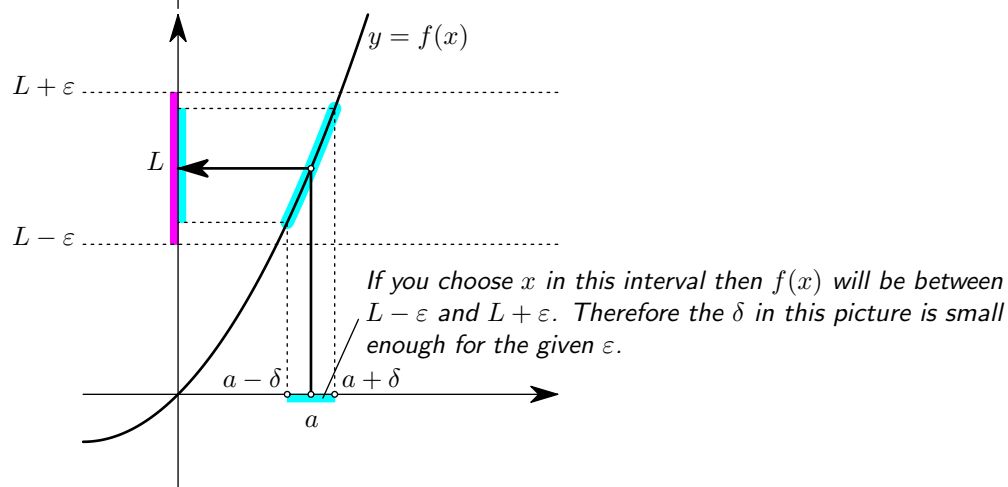
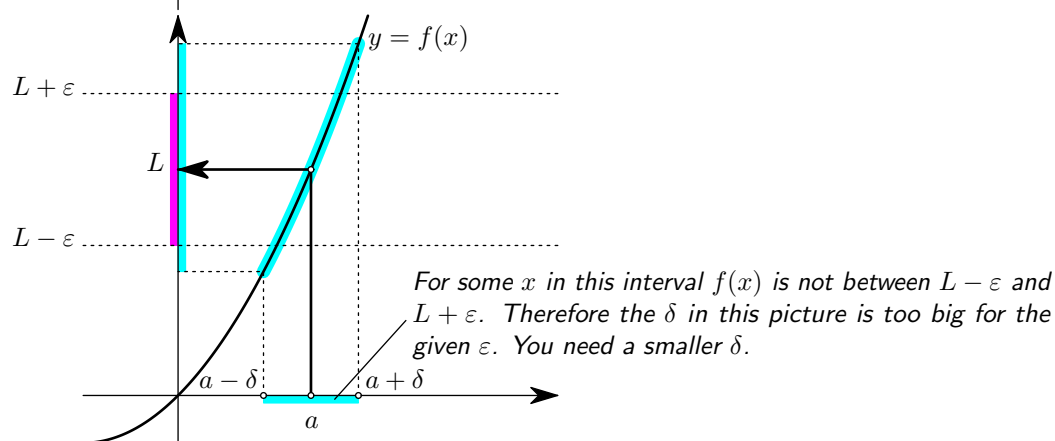
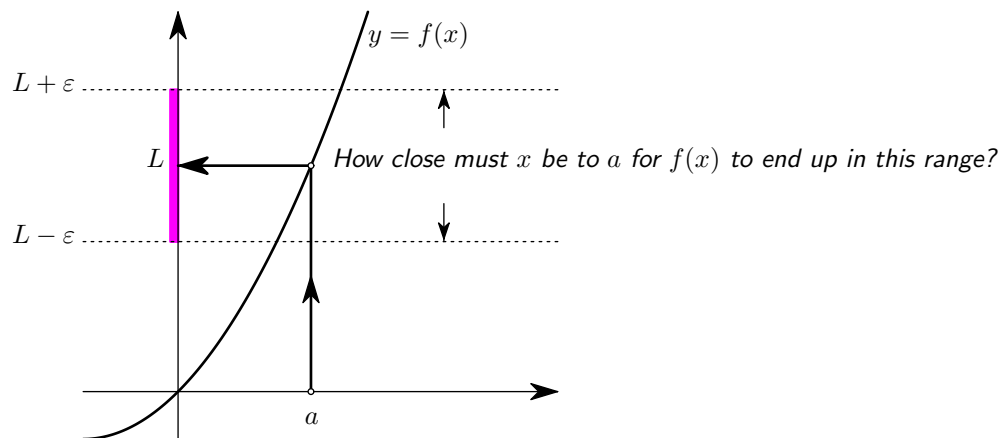
To figure this out we try to get an idea of how big $|f(x) - L|$ is:

$$|f(x) - L| = |(2x + 1) - 11| = |2x - 10| = 2 \cdot |x - 5| = 2 \cdot |x - a|.$$

So, if $2|x - a| < \varepsilon$ then we have $|f(x) - L| < \varepsilon$, i.e.

$$\text{if } |x - a| < \frac{1}{2}\varepsilon \text{ then } |f(x) - L| < \varepsilon.$$

We can therefore choose $\delta = \frac{1}{2}\varepsilon$. No matter what $\varepsilon > 0$ we are given our δ will also be positive, and if $|x - 5| < \delta$ then we can guarantee $|(2x + 1) - 11| < \varepsilon$. That shows that $\lim_{x \rightarrow 5} 2x + 1 = 11$.



2.3. The limit $\lim_{x \rightarrow 1} x^2 = 1$ and the “don’t choose $\delta > 1$ ” trick. We have $f(x) = x^2$, $a = 1$, $L = 1$, and again the question is, “how small should $|x - 1|$ be to guarantee $|x^2 - 1| < \varepsilon$?”

We begin by estimating the difference $|x^2 - 1|$

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x + 1| \cdot |x - 1|.$$

Propagation of errors – another interpretation of ε and δ

According to the limit definition “ $\lim_{x \rightarrow R} \pi x^2 = A$ ” is true if *for every $\varepsilon > 0$ you can find a $\delta > 0$ such that $|x - R| < \delta$ implies $|\pi x^2 - A| < \varepsilon$* . Here's a more concrete situation in which ε and δ appear in exactly the same roles:

Suppose you are given a circle drawn on a piece of paper, and you want to know its area. You decide to measure its radius, R , and then compute the area of the circle by calculating

$$\text{Area} = \pi R^2.$$

The area is a function of the radius, and we'll call that function f :

$$f(x) = \pi x^2.$$

When you measure the radius R you will make an error, simply because you can never measure anything with infinite precision. Suppose that R is the real value of the radius, and that x is the number you measured. Then the size of the error you made is

$$\text{error in radius measurement} = |x - R|.$$

When you compute the area you also won't get the exact value: you would get $f(x) = \pi x^2$ instead of $A = f(R) = \pi R^2$. The error in your computed value of the area is

$$\text{error in area} = |f(x) - f(R)| = |f(x) - A|.$$

Now you can ask the following question:

Suppose you want to know the area with an error of at most ε , then what is the largest error that you can afford to make when you measure the radius?

The answer will be something like this: if you want the computed area to have an error of at most $|f(x) - A| < \varepsilon$, then the error in your radius measurement should satisfy $|x - R| < \delta$. You have to do the algebra with inequalities to compute δ when you know ε , as in the examples in this section.

You would expect that if your measured radius x is close enough to the real value R , then your computed area $f(x) = \pi x^2$ will be close to the real area A .

In terms of ε and δ this means that you would expect that no matter how accurately you want to know the area (i.e. how small you make ε) you can always achieve that precision by making the error in your radius measurement small enough (i.e. by making δ sufficiently small).

As x approaches 1 the factor $|x - 1|$ becomes small, and if the other factor $|x + 1|$ were a constant (e.g. 2 as in the previous example) then we could find δ as before, by dividing ε by that constant.

Here is a trick that allows you to replace the factor $|x + 1|$ with a constant. We hereby agree *that we always choose our δ so that $\delta \leq 1$* . If we do that, then we will always have

$$|x - 1| < \delta \leq 1, \text{ i.e. } |x - 1| < 1,$$

and x will always be between 0 and 2. Therefore

$$|x^2 - 1| = |x + 1| \cdot |x - 1| < 3|x - 1|.$$

If we now want to be sure that $|x^2 - 1| < \varepsilon$, then this calculation shows that we should require $3|x - 1| < \varepsilon$, i.e. $|x - 1| < \frac{1}{3}\varepsilon$. So we should choose $\delta \leq \frac{1}{3}\varepsilon$. We must also live up to our promise never to choose $\delta > 1$, so if we are handed an ε for which $\frac{1}{3}\varepsilon > 1$, then we choose $\delta = 1$ instead of $\delta = \frac{1}{3}\varepsilon$. To summarize, we are going to choose

$$\delta = \text{the smaller of } 1 \text{ and } \frac{1}{3}\varepsilon.$$

We have shown that if you choose δ this way, then $|x - 1| < \delta$ implies $|x^2 - 1| < \varepsilon$, no matter what $\varepsilon > 0$ is.

The expression “the smaller of a and b ” shows up often, and is abbreviated to $\min(a, b)$. We could therefore say that in this problem we will choose δ to be

$$\delta = \min(1, \frac{1}{3}\varepsilon).$$

2.4. Show that $\lim_{x \rightarrow 4} 1/x = 1/4$. Solution: We apply the definition with $a = 4$, $L = 1/4$ and $f(x) = 1/x$. Thus, for any $\varepsilon > 0$ we try to show that if $|x - 4|$ is small enough then one has $|f(x) - 1/4| < \varepsilon$.

We begin by estimating $|f(x) - 1/4|$ in terms of $|x - 4|$:

$$|f(x) - 1/4| = \left| \frac{1}{x} - \frac{1}{4} \right| = \left| \frac{4 - x}{4x} \right| = \frac{|x - 4|}{|4x|} = \frac{1}{|4x|} |x - 4|.$$

As before, things would be easier if $1/|4x|$ were a constant. To achieve that we again agree not to take $\delta > 1$. If we always have $\delta \leq 1$, then we will always have $|x - 4| < 1$, and hence $3 < x < 5$. How large can $1/|4x|$ be in this situation? Answer: the quantity $1/|4x|$ increases as you decrease x , so if $3 < x < 5$ then it will never be larger than $1/(4 \cdot 3) = \frac{1}{12}$.

We see that if we never choose $\delta > 1$, we will always have

$$|f(x) - \frac{1}{4}| \leq \frac{1}{12} |x - 4| \quad \text{for } |x - 4| < \delta.$$

To guarantee that $|f(x) - \frac{1}{4}| < \varepsilon$ we could therefore require

$$\frac{1}{12} |x - 4| < \varepsilon, \quad \text{i.e. } |x - 4| < 12\varepsilon.$$

Hence if we choose $\delta = 12\varepsilon$ or any smaller number, then $|x - 4| < \delta$ implies $|f(x) - 1/4| < \varepsilon$. Of course we have to honor our agreement never to choose $\delta > 1$, so our choice of δ is

$$\delta = \text{the smaller of } 1 \text{ and } 12\varepsilon = \min(1, 12\varepsilon).$$

3. Exercises

38. Group Problem.

Joe offers to make square sheets of paper for Bruce. Given $x > 0$ Joe plans to mark off a length x and cut out a square of side x . Bruce asks Joe for a square with area 4 square foot. Joe tells Bruce that he can't measure **exactly** 2 foot and the area of the square he produces will only be approximately 4 square foot. Bruce doesn't mind as long as the area of the square doesn't differ more than 0.01 square foot from what he really asked for (namely, 4 square foot).

(a) What is the biggest error Joe can afford to make when he marks off the length x ?

(b) Jen also wants square sheets, with area 4 square feet. However, she needs the error in the area to be less than 0.00001 square foot. (She's paying).

How accurate must Joe measure the side of the squares he's going to cut for Jen?

Use the ε - δ definition to prove the following limits

39. $\lim_{x \rightarrow 1} 2x - 4 = 6$

40. $\lim_{x \rightarrow 2} x^2 = 4$.

41. $\lim_{x \rightarrow 2} x^2 - 7x + 3 = -7$

42. $\lim_{x \rightarrow 3} x^3 = 27$

43. $\lim_{x \rightarrow 2} x^3 + 6x^2 = 32$.

44. $\lim_{x \rightarrow 4} \sqrt{x} = 2$.

45. $\lim_{x \rightarrow 3} \sqrt{x+6} = 9$.

46. $\lim_{x \rightarrow 2} \frac{1+x}{4+x} = \frac{1}{2}$.

47. $\lim_{x \rightarrow 1} \frac{2-x}{4-x} = \frac{1}{3}$.

48. $\lim_{x \rightarrow 3} \frac{x}{6-x} = 1$.

49. $\lim_{x \rightarrow 0} \sqrt{|x|} = 0$

50. Group Problem.

(Joe goes cubic.) Joe is offering to build cubes of side x . Airline regulations allow you take a cube on board provided its volume and surface area add up to less than 33 (everything measured in feet). For instance, a cube with 2 foot sides has volume+area equal to $2^3 + 6 \times 2^2 = 32$.

If you ask Joe to build a cube whose volume plus total surface area is 32 cubic feet with an error of at most ε , then what error can he afford to make when he measures the side of the cube he's making?

51. Our definition of a derivative in (7) contains a limit. What is the function " f " there, and what is the variable?

4. Variations on the limit theme

Not all limits are "for $x \rightarrow a$." here we describe some possible variations on the concept of limit.

4.1. Left and right limits. When we let “ x approach a ” we allow x to be both larger or smaller than a , as long as x gets close to a . If we explicitly want to study the behaviour of $f(x)$ as x approaches a through values larger than a , then we write

$$\lim_{x \searrow a} f(x) \text{ or } \lim_{x \rightarrow a+} f(x) \text{ or } \lim_{x \rightarrow a+0} f(x) \text{ or } \lim_{x \rightarrow a, x > a} f(x).$$

All four notations are in use. Similarly, to designate the value which $f(x)$ approaches as x approaches a through values below a one writes

$$\lim_{x \nearrow a} f(x) \text{ or } \lim_{x \rightarrow a-} f(x) \text{ or } \lim_{x \rightarrow a-0} f(x) \text{ or } \lim_{x \rightarrow a, x < a} f(x).$$

The precise definition of right limits goes like this:

4.2. Definition of right-limits. *Let f be a function. Then*

$$(9) \quad \lim_{x \searrow a} f(x) = L.$$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \varepsilon$$

holds for all x in the domain of f .

The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined in a similar way. The following theorem tells you how to use one-sided limits to decide if a function $f(x)$ has a limit at $x = a$.

4.3. Theorem. *If both one-sided limits*

$$\lim_{x \searrow a} f(x) = L_+, \text{ and } \lim_{x \nearrow a} f(x) = L_-$$

exist, then

$$\lim_{x \rightarrow a} f(x) \text{ exists } \iff L_+ = L_-.$$

In other words, if a function has both left- and right-limits at some $x = a$, then that function has a limit at $x = a$ if the left- and right-limits are equal.

4.4. Limits at infinity. Instead of letting x approach some finite number, one can let x become “larger and larger” and ask what happens to $f(x)$. If there is a number L such that $f(x)$ gets arbitrarily close to L if one chooses x sufficiently large, then we write

$$\lim_{x \rightarrow \infty} f(x) = L, \text{ or } \lim_{x \uparrow \infty} f(x) = L, \text{ or } \lim_{x \nearrow \infty} f(x) = L.$$

(“The limit for x going to infinity is L .”)

4.5. Example – Limit of $1/x$. The larger you choose x , the smaller its reciprocal $1/x$ becomes. Therefore, it seems reasonable to say

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Here is the precise definition:

4.6. Definition of limit at ∞ . *Let f be some function which is defined on some interval $x_0 < x < \infty$. If there is a number L such that for every $\varepsilon > 0$ one can find an A such that*

$$x > A \implies |f(x) - L| < \varepsilon$$

for all x , then we say that the limit of $f(x)$ for $x \rightarrow \infty$ is L .

The definition is very similar to the original definition of the limit. Instead of δ which specifies how close x should be to a , we now have a number A which says how large x should be, which is a way of saying “how close x should be to infinity.”

4.7. Example – Limit of $1/x$ (again) . To *prove* that $\lim_{x \rightarrow \infty} 1/x = 0$ we apply the definition to $f(x) = 1/x$, $L = 0$.

For given $\varepsilon > 0$ we need to show that

$$(10) \quad \left| \frac{1}{x} - L \right| < \varepsilon \text{ for all } x > A$$

provided we choose the right A .

How do we choose A ? A is not allowed to depend on x , but it may depend on ε .

If we assume for now that we will only consider positive values of x , then (10) simplifies to

$$\frac{1}{x} < \varepsilon$$

which is equivalent to

$$x > \frac{1}{\varepsilon}.$$

This tells us how to choose A . Given any positive ε , we will simply choose

$$A = \frac{1}{\varepsilon}$$

Then one has $|\frac{1}{x} - 0| = \frac{1}{x} < \varepsilon$ for all $x > A$. Hence we have proved that $\lim_{x \rightarrow \infty} 1/x = 0$.

5. Properties of the Limit

The precise definition of the limit is not easy to use, and fortunately we won't use it very often in this class. Instead, there are a number of properties that limits have which allow you to compute them without having to resort to "epsilon-ness."

The following properties also apply to the variations on the limit from 4. I.e. the following statements remain true if one replaces each limit by a one-sided limit, or a limit for $x \rightarrow \infty$.

Limits of constants and of x . If a and c are constants, then

$$(P_1) \quad \lim_{x \rightarrow a} c = c$$

and

$$(P_2) \quad \lim_{x \rightarrow a} x = a.$$

Limits of sums, products and quotients. Let F_1 and F_2 be two given functions whose limits for $x \rightarrow a$ we know,

$$\lim_{x \rightarrow a} F_1(x) = L_1, \quad \lim_{x \rightarrow a} F_2(x) = L_2.$$

Then

$$(P_3) \quad \lim_{x \rightarrow a} (F_1(x) + F_2(x)) = L_1 + L_2,$$

$$(P_4) \quad \lim_{x \rightarrow a} (F_1(x) - F_2(x)) = L_1 - L_2,$$

$$(P_5) \quad \lim_{x \rightarrow a} (F_1(x) \cdot F_2(x)) = L_1 \cdot L_2$$

Finally, if $\lim_{x \rightarrow a} F_2(x) \neq 0$,

$$(P_6) \quad \lim_{x \rightarrow a} \frac{F_1(x)}{F_2(x)} = \frac{L_1}{L_2}.$$

In other words the limit of the sum is the sum of the limits, etc. One can prove these laws using the definition of limit in §2 but we will not do this here. However, I hope these laws seem like common sense: if, for x close to a , the quantity $F_1(x)$ is close to L_1 and $F_2(x)$ is close to L_2 , then certainly $F_1(x) + F_2(x)$ should be close to $L_1 + L_2$.

There are two more properties of limits which we will add to this list later on. They are the “Sandwich Theorem” (§9) and the substitution theorem (§10).

6. Examples of limit computations

6.1. Find $\lim_{x \rightarrow 2} x^2$. One has

$$\begin{aligned}\lim_{x \rightarrow 2} x^2 &= \lim_{x \rightarrow 2} x \cdot x \\ &= \left(\lim_{x \rightarrow 2} x\right) \cdot \left(\lim_{x \rightarrow 2} x\right) && \text{by } (P_5) \\ &= 2 \cdot 2 = 4.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{x \rightarrow 2} x^3 &= \lim_{x \rightarrow 2} x \cdot x^2 \\ &= \left(\lim_{x \rightarrow 2} x\right) \cdot \left(\lim_{x \rightarrow 2} x^2\right) && (P_5) \text{ again} \\ &= 2 \cdot 4 = 8,\end{aligned}$$

and, by (P_4)

$$\lim_{x \rightarrow 2} x^2 - 1 = \lim_{x \rightarrow 2} x^2 - \lim_{x \rightarrow 2} 1 = 4 - 1 = 3,$$

and, by (P_4) again,

$$\lim_{x \rightarrow 2} x^3 - 1 = \lim_{x \rightarrow 2} x^3 - \lim_{x \rightarrow 2} 1 = 8 - 1 = 7,$$

Putting all this together, one gets

$$\lim_{x \rightarrow 2} \frac{x^3 - 1}{x^2 - 1} = \frac{2^3 - 1}{2^2 - 1} = \frac{8 - 1}{4 - 1} = \frac{7}{3}$$

because of (P_6) . To apply (P_6) we must check that the denominator (“ L_2 ”) is not zero. Since the denominator is 3 everything is OK, and we were allowed to use (P_6) .

6.2. Try the examples 1.3 and 1.4 using the limit properties. To compute $\lim_{x \rightarrow 2} (x^2 - 2x)/(x^2 - 4)$ we first use the limit properties to find

$$\lim_{x \rightarrow 2} x^2 - 2x = 0 \text{ and } \lim_{x \rightarrow 2} x^2 - 4 = 0.$$

to complete the computation we would like to apply the last property (P_6) about quotients, but this would give us

$$\lim_{x \rightarrow 2} f(x) = \frac{0}{0}.$$

The denominator is zero, so we were not allowed to use (P_6) (and the result doesn’t mean anything anyway). We have to do something else.

The function we are dealing with is a **rational function**, which means that it is the quotient of two polynomials. For such functions there is an algebra trick which always allows you to compute the limit even if you first get $\frac{0}{0}$. The thing to do is to divide numerator and denominator by $x - 2$. In our case we have

$$x^2 - 2x = (x - 2) \cdot x, \quad x^2 - 4 = (x - 2) \cdot (x + 2)$$

so that

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{(x - 2) \cdot x}{(x - 2) \cdot (x + 2)} = \lim_{x \rightarrow 2} \frac{x}{x + 2}.$$

After this simplification we **can** use the properties $(P_{\dots})$ to compute

$$\lim_{x \rightarrow 2} f(x) = \frac{2}{2 + 2} = \frac{1}{2}.$$

6.3. Example – Find $\lim_{x \rightarrow 2} \sqrt{x}$. Of course, you would think that $\lim_{x \rightarrow 2} \sqrt{x} = \sqrt{2}$ and you can indeed prove this using δ & ε (See problem 44.) But is there an easier way? There is nothing in the limit properties which tells us how to deal with a square root, and using them we can't even prove that there is a limit. However, if you *assume* that the limit exists then the limit properties allow us to find this limit.

The argument goes like this: suppose that there is a number L with

$$\lim_{x \rightarrow 2} \sqrt{x} = L.$$

Then property (P_5) implies that

$$L^2 = \left(\lim_{x \rightarrow 2} \sqrt{x}\right) \cdot \left(\lim_{x \rightarrow 2} \sqrt{x}\right) = \lim_{x \rightarrow 2} \sqrt{x} \cdot \sqrt{x} = \lim_{x \rightarrow 2} x = 2.$$

In other words, $L^2 = 2$, and hence L must be either $\sqrt{2}$ or $-\sqrt{2}$. We can reject the latter because whatever x does, its squareroot is always a positive number, and hence it can never “get close to” a negative number like $-\sqrt{2}$.

Our conclusion: if the limit exists, then

$$\lim_{x \rightarrow 2} \sqrt{x} = \sqrt{2}.$$

The result is not surprising: if x gets close to 2 then $\sqrt{x}$ gets close to $\sqrt{2}$.

6.4. Example – The derivative of $\sqrt{x}$ at $x = 2$. Find

$$\lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$$

assuming the result from the previous example.

Solution: The function is a fraction whose numerator and denominator vanish when $x = 2$, i.e. the limit is of the form $\frac{0}{0}$. We use the same algebra trick as before, namely we factor numerator and denominator:

$$\frac{\sqrt{x} - \sqrt{2}}{x - 2} = \frac{\sqrt{x} - \sqrt{2}}{(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})} = \frac{1}{\sqrt{x} + \sqrt{2}}.$$

Now one can use the limit properties to compute

$$\lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} = \lim_{x \rightarrow 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

6.5. Limit as $x \rightarrow \infty$ of rational functions. A rational function is the quotient of two polynomials, so

$$(11) \quad R(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}.$$

We have seen that

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

We even proved this in example 4.7. Using this you can find the limit at ∞ for any rational function $R(x)$ as in (11). One could turn the outcome of the calculation of $\lim_{x \rightarrow \infty} R(x)$ into a recipe/formula involving the degrees n and m of the numerator and denominator, and also their coefficients a_i , b_j , which students would then memorize, but it is better to remember “the trick.”

To find $\lim_{x \rightarrow \infty} R(x)$ divide numerator and denominator by x^m (the highest power of x occurring in the denominator).

For example, let's compute

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 3}{5x^2 + 7x - 39}.$$

Remember the trick and divide top and bottom by x^2 , and you get

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3x^2 + 3}{5x^2 + 7x - 39} &= \lim_{x \rightarrow \infty} \frac{3 + 3/x^2}{5 + 7/x - 39/x^2} \\ &= \frac{\lim_{x \rightarrow \infty} 3 + 3/x^2}{\lim_{x \rightarrow \infty} 5 + 7/x - 39/x^2} \\ &= \frac{3}{5}\end{aligned}$$

Here we have used the limit properties (P_*) to break the limit down into little pieces like $\lim_{x \rightarrow \infty} 39/x^2$ which we can compute as follows

$$\lim_{x \rightarrow \infty} 39/x^2 = \lim_{x \rightarrow \infty} 39 \cdot \left(\frac{1}{x}\right)^2 = \left(\lim_{x \rightarrow \infty} 39\right) \cdot \left(\lim_{x \rightarrow \infty} \frac{1}{x}\right)^2 = 39 \cdot 0^2 = 0.$$

6.6. Another example with a rational function . Compute

$$\lim_{x \rightarrow \infty} \frac{x}{x^3 + 5}.$$

We apply “the trick” again and divide numerator and denominator by x^3 . This leads to

$$\lim_{x \rightarrow \infty} \frac{x}{x^3 + 5} = \lim_{x \rightarrow \infty} \frac{1/x^2}{1 + 5/x^3} = \frac{\lim_{x \rightarrow \infty} 1/x^2}{\lim_{x \rightarrow \infty} 1 + 5/x^3} = \frac{0}{1} = 0.$$

To show all possible ways a limit of a rational function can turn out we should do yet another example, but that one belongs in the next section (see example 7.6.)

7. When limits fail to exist

In the last couple of examples we worried about the possibility that a limit $\lim_{x \rightarrow a} g(x)$ actually might not exist. This can actually happen, and in this section we’ll see a few examples of what failed limits look like. First let’s agree on what we will call a “failed limit.”

7.1. Definition. *If there is no number L such that $\lim_{x \rightarrow a} f(x) = L$, then we say that the limit $\lim_{x \rightarrow a} f(x)$ does not exist.*

7.2. The sign function near $x = 0$. The “sign function¹” is defined by

$$\text{sign}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

Note that “the sign of zero” is defined to be zero. But does the sign function have a limit at $x = 0$, i.e. does $\lim_{x \rightarrow 0} \text{sign}(x)$ exist? And is it also zero? The answers are **no** and **no**, and here is why: suppose that for some number L one had

$$\lim_{x \rightarrow 0} \text{sign}(x) = L,$$

then since for arbitrary small positive values of x one has $\text{sign}(x) = +1$ one would think that $L = +1$. But for arbitrarily small negative values of x one has $\text{sign}(x) = -1$, so one would conclude that $L = -1$. But one number L can’t be both $+1$ and -1 at the same time, so there is no such L , i.e. there is no limit.

$$\lim_{x \rightarrow 0} \text{sign}(x) \text{ does not exist.}$$

¹Some people don’t like the notation $\text{sign}(x)$, and prefer to write

$$g(x) = \frac{x}{|x|}$$

instead of $g(x) = \text{sign}(x)$. If you think about this formula for a moment you’ll see that $\text{sign}(x) = x/|x|$ for all $x \neq 0$. When $x = 0$ the quotient $x/|x|$ is of course not defined.

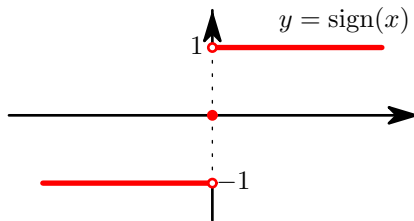


Figure 1. The sign function.

In this example the one-sided limits do exist, namely,

$$\lim_{x \searrow 0} \text{sign}(x) = 1 \text{ and } \lim_{x \nearrow 0} \text{sign}(x) = -1.$$

All this says is that when x approaches 0 through positive values, its sign approaches +1, while if x goes to 0 through negative values, then its sign approaches -1.

7.3. The example of the backward sine. Contemplate the limit as $x \rightarrow 0$ of the “backward sine,” i.e.

$$\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{x}\right).$$

When $x = 0$ the function $f(x) = \sin(\pi/x)$ is not defined, because its definition involves division by x . What happens to $f(x)$ as $x \rightarrow 0$? First, π/x becomes larger and larger (“goes to infinity”) as $x \rightarrow 0$. Then, taking the sine, we see that $\sin(\pi/x)$ oscillates between +1 and -1 infinitely often as $x \rightarrow 0$. This means that $f(x)$ gets close to any number between -1 and +1 as $x \rightarrow 0$, but that the function $f(x)$ *never stays close* to any particular value because it keeps oscillating up and down.

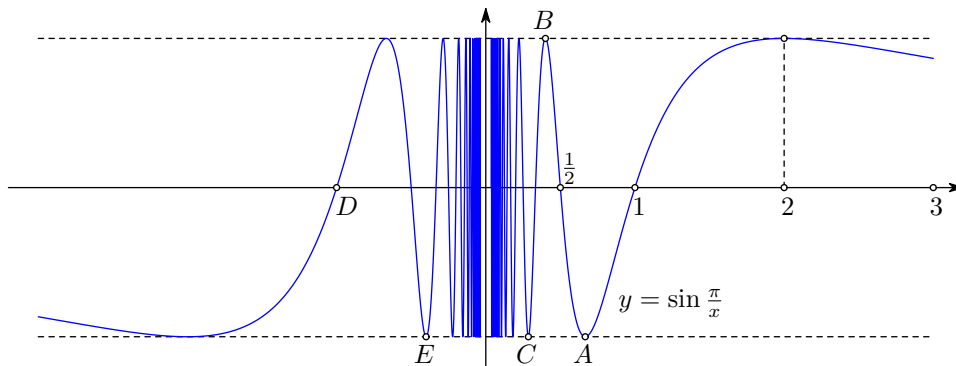


Figure 2. Graph of $y = \sin \frac{\pi}{x}$ for $-3 < x < 3$, $x \neq 0$.

Here again, the limit $\lim_{x \rightarrow 0} f(x)$ does not exist. We have arrived at this conclusion by only considering what $f(x)$ does for small positive values of x . So the limit fails to exist in a stronger way than in the example of the sign-function. There, even though the limit didn’t exist, the one-sided limits existed. In the present example we see that even the one-sided limit

$$\lim_{x \searrow 0} \sin \frac{\pi}{x}$$

does not exist.

7.4. Trying to divide by zero using a limit. The expression $1/0$ is not defined, but what about

$$\lim_{x \rightarrow 0} \frac{1}{x}?$$

This limit also does not exist. Here are two reasons:

It is common wisdom that if you divide by a small number you get a large number, so as $x \searrow 0$ the quotient $1/x$ will not be able to stay close to any particular finite number, and the limit can't exist.

“Common wisdom” is not always a reliable tool in mathematical proofs, so here is a better argument. The limit can't exist, because that would contradict the limit properties $(P_1) \cdots (P_6)$. Namely, suppose that there were an number L such that

$$\lim_{x \rightarrow 0} \frac{1}{x} = L.$$

Then the limit property (P_5) would imply that

$$\lim_{x \rightarrow 0} \left(\frac{1}{x} \cdot x \right) = \left(\lim_{x \rightarrow 0} \frac{1}{x} \right) \cdot \left(\lim_{x \rightarrow 0} x \right) = L \cdot 0 = 0.$$

On the other hand $\frac{1}{x} \cdot x = 1$ so the above limit should be 1! A number can't be both 0 and 1 at the same time, so we have a contradiction. The assumption that $\lim_{x \rightarrow 0} 1/x$ exists is to blame, so it must go.

7.5. Using limit properties to show a limit does *not* exist. The limit properties tell us how to prove that certain limits exist (and how to compute them). Although it is perhaps not so obvious at first sight, they also allow you to prove that certain limits do not exist. The previous example shows one instance of such use. Here is another.

Property (P_3) says that if both $\lim_{x \rightarrow a} g(x)$ and $\lim_{x \rightarrow a} h(x)$ exist then $\lim_{x \rightarrow a} g(x) + h(x)$ also must exist. You can turn this around and say that if $\lim_{x \rightarrow a} g(x) + h(x)$ does not exist then either $\lim_{x \rightarrow a} g(x)$ or $\lim_{x \rightarrow a} h(x)$ does not exist (or both limits fail to exist).

For instance, the limit

$$\lim_{x \rightarrow 0} \frac{1}{x} - x$$

can't exist, for if it did, then the limit

$$\lim_{x \rightarrow 0} \frac{1}{x} = \lim_{x \rightarrow 0} \left(\frac{1}{x} - x + x \right) = \lim_{x \rightarrow 0} \left(\frac{1}{x} - x \right) + \lim_{x \rightarrow 0} x$$

would also have to exist, and we know $\lim_{x \rightarrow 0} \frac{1}{x}$ doesn't exist.

7.6. Limits at ∞ which don't exist. If you let x go to ∞ , then x will not get “closer and closer” to any particular number L , so it seems reasonable to guess that

$$\lim_{x \rightarrow \infty} x \text{ does not exist.}$$

One can prove this from the limit definition (and see exercise 72).

Let's consider

$$L = \lim_{x \rightarrow \infty} \frac{x^2 + 2x - 1}{x + 2}.$$

Once again we divide numerator and denominator by the highest power in the denominator (i.e. x)

$$L = \lim_{x \rightarrow \infty} \frac{x + 2 - \frac{1}{x}}{1 + 2/x}$$

Here the denominator has a limit ('tis 1), but the numerator does not, for if $\lim_{x \rightarrow \infty} x + 2 - \frac{1}{x}$ existed then, since $\lim_{x \rightarrow \infty} (2 - 1/x) = 2$ exists,

$$\lim_{x \rightarrow \infty} x = \lim_{x \rightarrow \infty} \left[\left(x + 2 - \frac{1}{x} \right) - \left(2 - \frac{1}{x} \right) \right]$$

would also have to exist, and $\lim_{x \rightarrow \infty} x$ doesn't exist.

So we see that L is the limit of a fraction in which the denominator has a limit, but the numerator does not. In this situation the limit L itself can never exist. If it did, then

$$\lim_{x \rightarrow \infty} \left(x + 2 - \frac{1}{x} \right) = \lim_{x \rightarrow \infty} \frac{x + 2 - \frac{1}{x}}{1 + 2/x} \cdot (1 + 2/x)$$

would also have to have a limit.

8. What's in a name?

There is a big difference between the variables x and a in the formula

$$\lim_{x \rightarrow a} 2x + 1,$$

namely a is a **free variable**, while x is a **dummy variable** (or “placeholder” or a “bound variable.”)

The difference between these two kinds of variables is this:

- if you replace a dummy variable in some formula consistently by some other variable then the value of the formula does not change. On the other hand, it never makes sense to substitute a number for a dummy variable.
- the value of the formula may depend on the value of the free variable.

To understand what this means consider the example $\lim_{x \rightarrow a} 2x + 1$ again. The limit is easy to compute:

$$\lim_{x \rightarrow a} 2x + 1 = 2a + 1.$$

If we replace x by, say u (systematically) then we get

$$\lim_{u \rightarrow a} 2u + 1$$

which is again equal to $2a + 1$. This computation says that *if some number gets close to a then two times that number plus one gets close to $2a + 1$* . This is a very wordy way of expressing the formula, and you can shorten things by giving a name (like x or u) to the number which approaches a . But the result of our computation shouldn't depend on the name we choose, i.e. it doesn't matter if we call it x or u .

Since the name of the variable x doesn't matter it is called a dummy variable. Some prefer to call x a bound variable, meaning that in

$$\lim_{x \rightarrow a} 2x + 1$$

the x in the expression $2x + 1$ is bound to the x written underneath the limit – you can't change one without changing the other.

Substituting a number for a dummy variable usually leads to complete nonsense. For instance, let's try setting $x = 3$ in our limit, i.e. what is

$$\lim_{3 \rightarrow a} 2 \cdot 3 + 1?$$

Of course $2 \cdot 3 + 1 = 7$, but what does 7 do when 3 gets closer and closer to the number a ? That's a silly question, because 3 is a constant and it doesn't “get closer” to some other number like a ! If you ever see 3 get closer to another number then it's time to take a vacation.

On the other hand the variable a is free: you can assign it particular values, and its value will affect the value of the limit. For instance, if we set $a = 3$ (but leave x alone) then we get

$$\lim_{x \rightarrow 3} 2x + 1$$

and there's nothing strange about that (the limit is $2 \cdot 3 + 1 = 7$, no problem.) You could substitute other values of a and you would get a different answer. In general you get $2a + 1$.

9. Limits and Inequalities

This section has two theorems which let you compare limits of different functions. The properties in these theorems are not formulas that allow you to compute limits like the properties $(P_1) \dots (P_6)$ from §5. Instead, they allow you to **reason** about limits, i.e. they let you say that this or that limit is positive, or that it must be the same as some other limit which you find easier to think about.

The first theorem should not surprise you – all it says is that bigger functions have bigger limits.

9.1. Theorem. *Let f and g be functions whose limits for $x \rightarrow a$ exist, and assume that $f(x) \leq g(x)$ holds for all x . Then*

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

A useful special case arises when you set $f(x) = 0$. The theorem then says that if a function g never has negative values, then its limit will also never be negative.

The statement may seem obvious, but it still needs a proof, starting from the ε - δ definition of limit. This will be done in lecture.

Here is the second theorem about limits and inequalities.

9.2. The Sandwich Theorem. *Suppose that*

$$f(x) \leq g(x) \leq h(x)$$

(for all x) and that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x).$$

Then

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x).$$

The theorem is useful when you want to know the limit of g , and when you can **sandwich** it between two functions f and h whose limits are easier to compute. The Sandwich Theorem looks like the first theorem of this section, but there is an important difference: in the Sandwich Theorem you don't have to assume that the limit of g exists. The inequalities $f \leq g \leq h$ combined with the circumstance that f and h have the same limit are enough to guarantee that the limit of g exists.

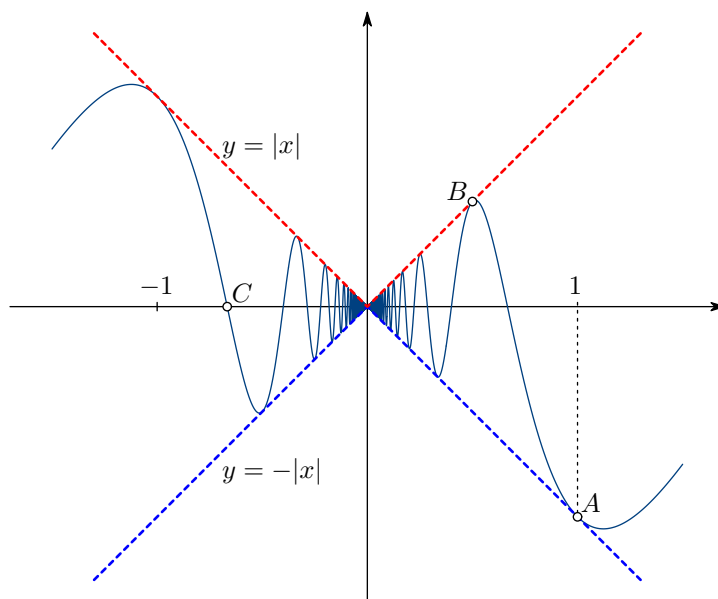


Figure 3. Graphs of $|x|$, $-|x|$ and $x \cos \frac{\pi}{x}$ for $-1.2 < x < 1.2$

9.3. Example: a Backward Cosine Sandwich. The Sandwich Theorem says that if the function $g(x)$ is sandwiched between two functions $f(x)$ and $h(x)$ and the limits of the outside functions f and h exist and are equal, then the limit of the inside function g exists and equals this common value. For example

$$-|x| \leq x \cos \frac{1}{x} \leq |x|$$

since the cosine is always between -1 and 1 . Since

$$\lim_{x \rightarrow 0} -|x| = \lim_{x \rightarrow 0} |x| = 0$$

the sandwich theorem tells us that

$$\lim_{x \rightarrow 0} x \cos \frac{1}{x} = 0.$$

Note that the limit $\lim_{x \rightarrow 0} \cos(1/x)$ does **not** exist, for the same reason that the “backward sine” did not have a limit for $x \rightarrow 0$ (see example 7.3). Multiplying with x changed that.

10. Continuity

10.1. Definition. A function g is **continuous** at a if

$$(12) \quad \lim_{x \rightarrow a} g(x) = g(a)$$

A function is continuous if it is continuous at every a in its domain.

Note that when we say that a function is continuous on some interval it is understood that the domain of the function includes that interval. For example, the function $f(x) = 1/x^2$ is continuous on the interval $1 < x < 5$ but is **not** continuous on the interval $-1 < x < 1$.

10.2. Polynomials are continuous. For instance, let us show that $P(x) = x^2 + 3x$ is continuous at $x = 2$. To show that you have to prove that

$$\lim_{x \rightarrow 2} P(x) = P(2),$$

i.e.

$$\lim_{x \rightarrow 2} x^2 + 3x = 2^2 + 3 \cdot 2.$$

You can do this two ways: using the definition with ε and δ (i.e. the hard way), or using the limit properties $(P_1) \dots (P_6)$ from §5 (just as good, and easier, even though it still takes a few lines to write it out – do both!)

10.3. Rational functions are continuous. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, and let a be any number in the domain of R , i.e. any number for which $Q(a) \neq 0$. Then one has

$$\begin{aligned} \lim_{x \rightarrow a} R(x) &= \lim_{x \rightarrow a} \frac{P(x)}{Q(x)} \\ &= \frac{\lim_{x \rightarrow a} P(x)}{\lim_{x \rightarrow a} Q(x)} && \text{property } (P_6) \\ &= \frac{P(a)}{Q(a)} && P \text{ and } Q \text{ are continuous} \\ &= R(a). \end{aligned}$$

This shows that R is indeed continuous at a .

10.4. Some discontinuous functions. If $\lim_{x \rightarrow a} g(x)$ does not exist, then it certainly cannot be equal to $g(a)$, and therefore any failed limit provides an example of a discontinuous function.

For instance, the sign function $g(x) = \text{sign}(x)$ from example ?? is not continuous at $x = 0$.

Is the backward sine function $g(x) = \sin(1/x)$ from example ?? also discontinuous at $x = 0$? No, it is not, for two reasons: first, the limit $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist, and second, we haven’t even defined the function $g(x)$ at $x = 0$, so even if the limit existed, we would have no value $g(0)$ to compare it with.

10.5. How to make functions discontinuous. Here is a discontinuous function:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 3, \\ 47 & \text{if } x = 3. \end{cases}$$

In other words, we take a continuous function like $g(x) = x^2$, and change its value somewhere, e.g. at $x = 3$. Then

$$\lim_{x \rightarrow 3} f(x) = 9 \neq 47 = f(3).$$

The reason that the limit is 9 is that our new function $f(x)$ coincides with our old continuous function $g(x)$ for all x except $x = 3$. Therefore the limit of $f(x)$ as $x \rightarrow 3$ is the same as the limit of $g(x)$ as $x \rightarrow 3$, and since g is continuous this is $g(3) = 9$.

10.6. Sandwich in a bow tie. We return to the function from example ???. Consider

$$f(x) = \begin{cases} x \cos\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

Then f is continuous at $x = 0$ by the Sandwich Theorem (see Example ??).

If we change the definition of f by picking a different value at $x = 0$ the new function will not be continuous, since changing f at $x = 0$ does not change the limit $\lim_{x \rightarrow 0} f(x)$. Since this limit is zero, $f(0) = 0$ is the only possible choice of $f(0)$ which makes f continuous at $x = 0$.

11. Substitution in Limits

Given two functions f and g one can consider their composition $h(x) = f(g(x))$. To compute the limit

$$\lim_{x \rightarrow a} f(g(x))$$

we write $u = g(x)$, so that we want to know

$$\lim_{x \rightarrow a} f(u) \text{ where } u = g(x).$$

Suppose that you can find the limits

$$L = \lim_{x \rightarrow a} g(x) \text{ and } \lim_{u \rightarrow L} f(u) = M.$$

Then it seems reasonable that as x approaches a , $u = g(x)$ will approach L , and $f(g(x))$ approaches M .

This is in fact a theorem:

11.1. Theorem. *If $\lim_{x \rightarrow a} g(x) = L$, and if the function f is continuous at $u = L$, then*

$$\lim_{x \rightarrow a} f(g(x)) = \lim_{u \rightarrow L} f(u) = f(L).$$

Another way to write this is

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right).$$

11.2. Example: compute $\lim_{x \rightarrow 3} \sqrt{x^3 - 3x^2 + 2}$. The given function is the composition of two functions, namely

$$\sqrt{x^3 - 3x^2 + 2} = \sqrt{u}, \text{ with } u = x^3 - 3x^2 + 2,$$

or, in function notation, we want to find $\lim_{x \rightarrow 3} h(x)$ where

$$h(x) = f(g(x)), \text{ with } g(x) = x^3 - 3x^2 + 2 \text{ and } f(u) = \sqrt{u}.$$

Either way, we have

$$\lim_{x \rightarrow 3} x^3 - 3x^2 + 2 = 2 \quad \text{and} \quad \lim_{u \rightarrow 2} \sqrt{u} = \sqrt{2}.$$

You get the first limit from the limit properties $(P_1) \dots (P_5)$. The second limit says that taking the square root is a continuous function, which it is. We have not proved that (yet), but this particular limit is the one from example 6.3. Putting these two limits together we conclude that the limit is $\sqrt{2}$.

Normally, you write this whole argument as follows:

$$\lim_{x \rightarrow 3} \sqrt{x^3 - 3x^2 + 2} = \sqrt{\lim_{x \rightarrow 3} x^3 - 3x^2 + 2} = \sqrt{2},$$

where you must point out that $f(x) = \sqrt{x}$ is a continuous function to justify the first step.

Another possible way of writing this is

$$\lim_{x \rightarrow 3} \sqrt{x^3 - 3x^2 + 2} = \lim_{u \rightarrow 2} \sqrt{u} = \sqrt{2},$$

where you must say that you have substituted $u = x^3 - 3x^2 + 2$.

12. Exercises

Find the following limits.

52. $\lim_{x \rightarrow -7} (2x + 5)$

53. $\lim_{x \rightarrow 7^-} (2x + 5)$

54. $\lim_{x \rightarrow -\infty} (2x + 5)$

55. $\lim_{x \rightarrow -4} (x + 3)^{2006}$

56. $\lim_{x \rightarrow -4} (x + 3)^{2007}$

57. $\lim_{x \rightarrow -\infty} (x + 3)^{2007}$

58. $\lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1}$

59. $\lim_{t \nearrow 1} \frac{t^2 + t - 2}{t^2 - 1}$

60. $\lim_{t \rightarrow -1} \frac{t^2 + t - 2}{t^2 - 1}$

61. $\lim_{x \rightarrow \infty} \frac{x^2 + 3}{x^2 + 4}$

62. $\lim_{x \rightarrow \infty} \frac{x^5 + 3}{x^2 + 4}$

63. $\lim_{x \rightarrow \infty} \frac{x^2 + 1}{x^5 + 2}$

64. $\lim_{x \rightarrow \infty} \frac{(2x + 1)^4}{(3x^2 + 1)^2}$

65. $\lim_{u \rightarrow \infty} \frac{(2u + 1)^4}{(3u^2 + 1)^2}$

66. $\lim_{t \rightarrow 0} \frac{(2t + 1)^4}{(3t^2 + 1)^2}$

67. What are the coordinates of the points labeled $A, \dots, E$ in Figure 2 (the graph of $y = \sin \pi/x$).

68. If $\lim_{x \rightarrow a} f(x)$ exists then f is continuous at $x = a$. *True or false?*

69. Give two examples of functions for which $\lim_{x \searrow 0} f(x)$ does not exist.

70. Group Problem.

If $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ both do not exist, then $\lim_{x \rightarrow 0} (f(x) + g(x))$ also does not exist. *True or false?*

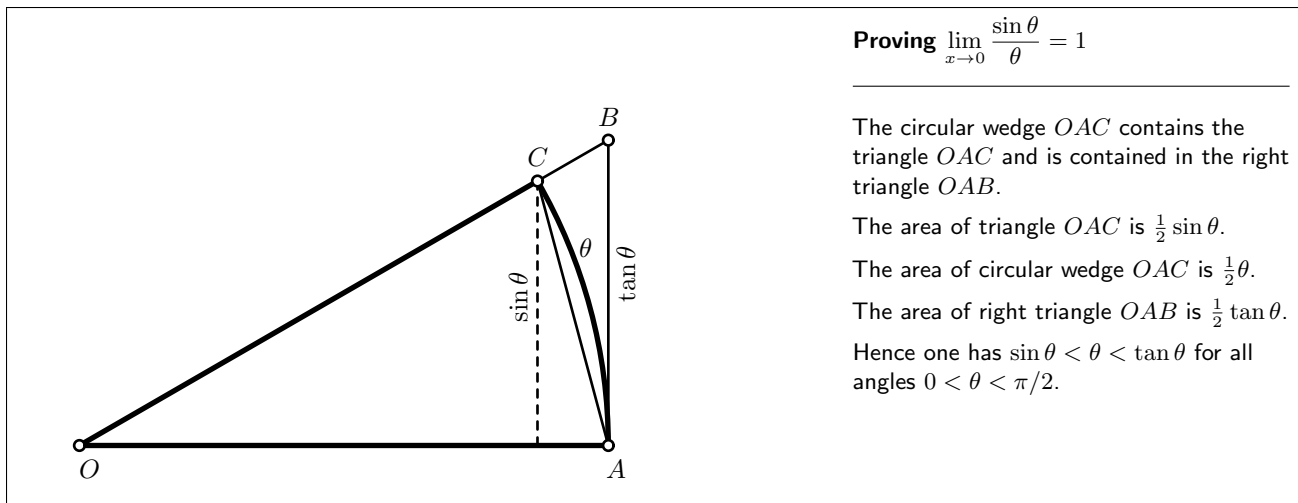
71. Group Problem.

If $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ both do not exist, then $\lim_{x \rightarrow 0} (f(x)/g(x))$ also does not exist. *True or false?*

72. Group Problem.

In the text we proved that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$. Show that this implies that $\lim_{x \rightarrow \infty} x$ does not exist. Hint: Suppose $\lim_{x \rightarrow \infty} x = L$ for some number L . Apply the limit properties to $\lim_{x \rightarrow \infty} x \cdot \frac{1}{x}$.

73. Evaluate $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$. Hint: Multiply top and bottom by $\sqrt{x} + 3$.



74. Evaluate $\lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$.

75. Evaluate $\lim_{x \rightarrow 2} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}}}{x - 2}$.

76. A function f is defined by

$$f(x) = \begin{cases} x^3 & \text{for } x < -1 \\ ax + b & \text{for } -1 \leq x < 1 \\ x^2 + 2 & \text{for } x \geq 1. \end{cases}$$

where a and b are constants. The function f is continuous. What are a and b ?

77. Find a constant k such that the function

$$f(x) = \begin{cases} 3x + 2 & \text{for } x < 2 \\ x^2 + k & \text{for } x \geq 2. \end{cases}$$

is continuous. Hint: Compute the one-sided limits.

78. Find constants a and c such that the function

$$f(x) = \begin{cases} x^3 + c & \text{for } x < 0 \\ ax + c^2 & \text{for } 0 \leq x < 1 \\ \arctan x & \text{for } x \geq 1. \end{cases}$$

is continuous for all x .

13. Two Limits in Trigonometry

In this section we'll derive a few limits involving the trigonometric functions. You can think of them as saying that for small angles θ one has

$$\sin \theta \approx \theta \quad \text{and} \quad \cos \theta \approx 1 - \frac{1}{2}\theta^2.$$

We will use these limits when we compute the derivatives of Sine, Cosine and Tangent.

13.1. Theorem. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.

PROOF. The proof requires a few sandwiches and some geometry.

We begin by only considering positive angles, and in fact we will only consider angles $0 < \theta < \pi/2$.

Since the wedge OAC contains the triangle OAC its area must be larger. The area of the wedge is $\frac{1}{2}\theta$ and the area of the triangle is $\frac{1}{2}\sin \theta$, so we find that

$$(13) \quad 0 < \sin \theta < \theta \text{ for } 0 < \theta < \frac{\pi}{2}.$$

The Sandwich Theorem implies that

$$(14) \quad \lim_{\theta \searrow 0} \sin \theta = 0.$$

Moreover, we also have

$$(15) \quad \lim_{\theta \searrow 0} \cos \theta = \lim_{\theta \searrow 0} \sqrt{1 - \sin^2 \theta} = 1.$$

Next we compare the areas of the wedge OAC and the larger triangle OAB . Since OAB has area $\frac{1}{2} \tan \theta$ we find that

$$\theta < \tan \theta$$

for $0 < \theta < \frac{\pi}{2}$. Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$ we can multiply with $\cos \theta$ and divide by θ to get

$$\cos \theta < \frac{\sin \theta}{\theta} \text{ for } 0 < \theta < \frac{\pi}{2}$$

If we go back to (15) and divide by θ , then we get

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

The Sandwich Theorem can be used once again, and now it gives

$$\lim_{\theta \searrow 0} \frac{\sin \theta}{\theta} = 1.$$

This is a one-sided limit. To get the limit in which $\theta \nearrow 0$, you use that $\sin \theta$ is an odd function. □

13.2. An example. We will show that

$$(16) \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2}.$$

This follows from $\sin^2 \theta + \cos^2 \theta = 1$. Namely,

$$\begin{aligned} \frac{1 - \cos \theta}{\theta^2} &= \frac{1}{1 + \cos \theta} \frac{1 - \cos^2 \theta}{\theta^2} \\ &= \frac{1}{1 + \cos \theta} \frac{\sin^2 \theta}{\theta^2} \\ &= \frac{1}{1 + \cos \theta} \left\{ \frac{\sin \theta}{\theta} \right\}^2. \end{aligned}$$

We have just shown that $\cos \theta \rightarrow 1$ and $\frac{\sin \theta}{\theta} \rightarrow 1$ as $\theta \rightarrow 0$, so (16) follows.

14. Exercises

Find each of the following limits or show that it does not exist. Distinguish between limits which are infinite and limits which do not exist.

79. $\lim_{\alpha \rightarrow 0} \frac{\sin 2\alpha}{\sin \alpha}$ (two ways: with and without the double angle formula!)

80. $\lim_{x \rightarrow 0} \frac{\sin 3x}{2x}$.

81. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta}$.

82. $\lim_{\alpha \rightarrow 0} \frac{\tan 4\alpha}{\sin 2\alpha}$.

83. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x}$.

84. $\lim_{\theta \rightarrow \pi/2} \frac{1 - \sin \theta}{\theta - \pi/2}$

85. $\lim_{x \rightarrow \infty} \frac{2x^3 + 3x^2 \cos x}{(x+2)^3}$.

86. $\lim_{x \rightarrow 0} \frac{\sin^2 x}{1 - \cos x}$.

87. $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2}$.

88. $\lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{\tan^3 x}$.

89. $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{1 - \cos x}$.

90. $\lim_{x \rightarrow \pi/2} \frac{x - \frac{\pi}{2}}{\cos x}$.

91. $\lim_{x \rightarrow \pi/2} (x - \frac{\pi}{2}) \tan x$.

92. $\lim_{x \rightarrow 0} \frac{\cos x}{x^2 + 9}$.

93. $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$.

94. $\lim_{x \rightarrow 0} \frac{\sin x}{x + \sin x}$.

95. $A = \lim_{x \rightarrow \infty} \frac{\sin x}{x}$. **(!!)**

96. $B = \lim_{x \rightarrow \infty} \frac{\cos x}{x}$. **(!! again)**

97. Is there a constant k such that the function

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ k & \text{for } x = 0. \end{cases}$$

is continuous? If so, find it; if not, say why.

98. Find a constant A so that the function

$$f(x) = \begin{cases} \frac{\sin x}{2x} & \text{for } x \neq 0 \\ A & \text{when } x = 0 \end{cases}$$

99. Compute $\lim_{x \rightarrow \infty} x \sin \frac{\pi}{x}$ and $\lim_{x \rightarrow \infty} x \tan \frac{\pi}{x}$. (Hint: substitute something).

100. Group Problem.

(*Geometry & Trig review*) Let A_n be the area of the regular n -gon inscribed in the unit circle, and let B_n be the area of the regular n -gon whose inscribed circle has radius 1.

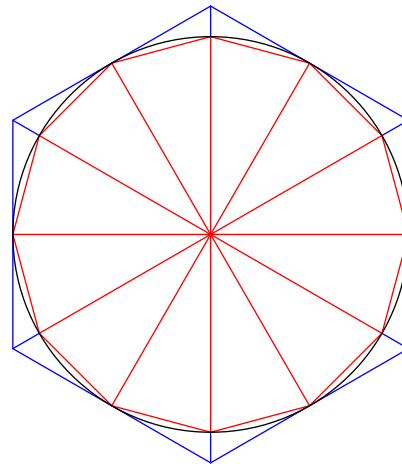
(a) Show that $A_n < \pi < B_n$.

(b) Show that

$$A_n = \frac{n}{2} \sin \frac{2\pi}{n} \text{ and } B_n = n \tan \frac{\pi}{n}$$

(c) Compute $\lim_{n \rightarrow \infty} A_n$ and $\lim_{n \rightarrow \infty} B_n$.

Here is a picture of A_{12} , B_6 and π :



On a historical note: Archimedes managed to compute A_{96} and B_{96} and by doing this got the most accurate approximation for π that was known in his time. See also:

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pi_through_the_ages.html

CHAPTER 4

Derivatives (2)

“Leibniz never thought of the derivative as a limit”

<http://www.gap-system.org/~history/Biographies/Leibniz.html>

In chapter 2 we saw two mathematical problems which led to expressions of the form $\frac{0}{0}$. Now that we know how to handle limits, we can state the definition of the derivative of a function. After computing a few derivatives using the definition we will spend most of this section developing the *differential calculus*, which is a collection of rules that allow you to compute derivatives without always having to use basic definition.

1. Derivatives Defined

1.1. Definition. Let f be a function which is defined on some interval (c, d) and let a be some number in this interval.

The **derivative of the function f at a** is the value of the limit

$$(17) \quad f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

f is said to be **differentiable at a** if this limit exists.

f is called **differentiable on the interval (c, d)** if it is differentiable at every point a in (c, d) .

1.2. Other notations. One can substitute $x = a + h$ in the limit (17) and let $h \rightarrow 0$ instead of $x \rightarrow a$. This gives the formula

$$(18) \quad f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h},$$

Often you will find this equation written with x instead of a and Δx instead of h , which makes it look like this:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The interpretation is the same as in equation (6) from §4. The numerator $f(x + \Delta x) - f(x)$ represents the amount by which the function value of f changes if one increases its argument x by a (small) amount Δx . If you write $y = f(x)$ then we can call the increase in f

$$\Delta y = f(x + \Delta x) - f(x),$$

so that the derivative $f'(x)$ is

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

GOTTFRIED WILHELM VON LEIBNIZ, one of the inventors of calculus, came up with the idea that one should write this limit as

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x},$$

the idea being that after letting Δx go to zero it didn't vanish, but instead became an infinitely small quantity which Leibniz called “ dx .” The result of increasing x by this infinitely small quantity dx is that $y = f(x)$ increased by another infinitely small quantity dy . The ratio of these two infinitely small quantities is what we call the derivative of $y = f(x)$.

There are no “infinitely small real numbers,” and this makes Leibniz’ notation difficult to justify. In the 20th century mathematicians have managed to create a consistent theory of “infinitesimals” which allows you to compute with “ dx and dy ” as Leibniz and his contemporaries would have done. This theory is called “non standard analysis.” We won’t mention it any further¹. Nonetheless, even though we won’t use infinitely small numbers, Leibniz’ notation is very useful and we will use it.

2. Direct computation of derivatives

2.1. Example – The derivative of $f(x) = x^2$ is $f'(x) = 2x$. We have done this computation before in §2. The result was

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} (2x+h) = 2x.$$

Leibniz would have written

$$\frac{dx^2}{dx} = 2x.$$

2.2. The derivative of $g(x) = x$ is $g'(x) = 1$. Indeed, one has

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

In Leibniz’ notation:

$$\frac{dx}{dx} = 1.$$

This is an example where Leibniz’ notation is most misleading, because if you divide dx by dx then you should of course get 1. Nonetheless, this is not what is going on. The expression $\frac{dx}{dx}$ is not really a fraction since there are no two “infinitely small” quantities dx which we are dividing.

2.3. The derivative of any constant function is zero . Let $k(x) = c$ be a constant function. Then we have

$$k'(x) = \lim_{h \rightarrow 0} \frac{k(x+h) - k(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0.$$

Leibniz would have said that if c is a constant, then

$$\frac{dc}{dx} = 0.$$

2.4. Derivative of x^n for $n = 1, 2, 3, \dots$. To differentiate $f(x) = x^n$ one proceeds as follows:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}.$$

We need to simplify the fraction $(x^n - a^n)/(x - a)$. For $n = 2$ we have

$$\frac{x^2 - a^2}{x - a} = x + a.$$

For $n = 1, 2, 3, \dots$ the geometric sum formula tells us that

$$(19) \quad \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + xa^{n-2} + a^{n-1}.$$

¹But if you want to read more on this you should see Keisler’s calculus text at

<http://www.math.wisc.edu/~keisler/calc.html>

I would not recommend using Keisler’s text and this text at the same time, but if you like math you should remember that it exists, and look at it (later, say, after you pass 221.)

If you don't remember the geometric sum formula, then you could also just verify (19) by carefully multiplying both sides with $x - a$. For instance, when $n = 3$ you would get

$$\begin{array}{rcl} x \times (x^2 + xa + a^2) & = & x^3 + ax^2 + a^2x \\ -a \times (x^2 + xa + a^2) & = & -ax^2 - a^2x - a^3 \\ \hline (x - a) \times (x^2 + xa + a^2) & = & x^3 - a^3 \end{array}$$

With formula (19) in hand we can now easily find the derivative of x^n :

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \rightarrow a} \{x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}\} \\ &= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots + a a^{n-2} + a^{n-1}. \end{aligned}$$

Here there are n terms, and they all are equal to a^{n-1} , so the final result is

$$f'(a) = na^{n-1}.$$

One could also write this as $f'(x) = nx^{n-1}$, or, in Leibniz' notation

$$\frac{dx^n}{dx} = nx^{n-1}.$$

This formula turns out to be true in general, but here we have only proved it for the case in which n is a positive integer.

3. Differentiable implies Continuous

3.1. Theorem. *If a function f is differentiable at some a in its domain, then f is also continuous at a .*

PROOF. We are given that

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists, and we must show that

$$\lim_{x \rightarrow a} f(x) = f(a).$$

This follows from the following computation

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (f(x) - f(a) + f(a)) && \text{(algebra)} \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) && \text{(more algebra)} \\ &= \left\{ \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \right\} \cdot \lim_{x \rightarrow a} (x - a) + \lim_{x \rightarrow a} f(a) && \text{(Limit Properties)} \\ &= f'(a) \cdot 0 + f(a) && (f'(a) \text{ exists}) \\ &= f(a). \end{aligned}$$

□

4. Some non-differentiable functions

4.1. A graph with a corner. Consider the function

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

This function is continuous at all x , but it is not differentiable at $x = 0$.

To see this try to compute the derivative at 0,

$$f'(0) = \lim_{x \rightarrow 0} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x} = \lim_{x \rightarrow 0} \text{sign}(x).$$

We know this limit does not exist (see §7.2)

If you look at the graph of $f(x) = |x|$ then you see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin.

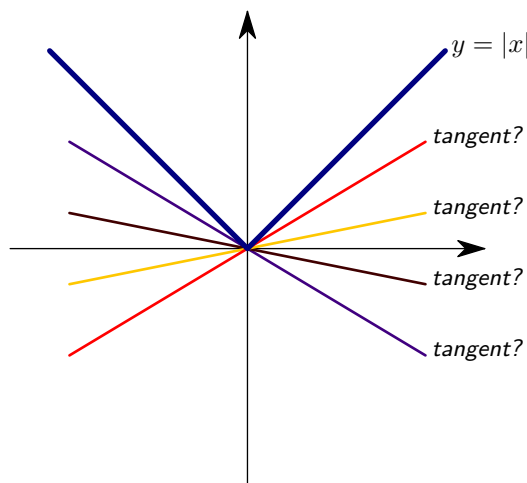


Figure 1. The graph of $y = |x|$ has no tangent at the origin.

4.2. A graph with a cusp. Another example of a function without a derivative at $x = 0$ is

$$f(x) = \sqrt{|x|}.$$

When you try to compute the derivative you get this limit

$$f'(0) = \lim_{x \rightarrow 0} \frac{\sqrt{|x|}}{x} = ?$$

The limit from the right is

$$\lim_{x \searrow 0} \frac{\sqrt{|x|}}{x} = \lim_{x \searrow 0} \frac{1}{\sqrt{x}},$$

which does not exist (it is $+\infty$). Likewise, the limit from the left also does not exist ('tis $-\infty$). Nonetheless, a drawing for the graph of f suggests an obvious tangent to the graph at $x = 0$, namely, the y -axis. That observation does not give us a derivative, because the y -axis is vertical and hence has no slope.

4.3. A graph with absolutely no tangents, *anywhere*. The previous two examples were about functions which did not have a derivative at $x = 0$. In both examples the point $x = 0$ was the only point where the function failed to have a derivative. It is easy to give examples of functions which are not differentiable at more than one value of x , but here I would like to show you a function f which doesn't have a derivative *anywhere in its domain*.

To keep things short I won't write a formula for the function, and merely show you a graph. In this graph you see a typical path of a Brownian motion, i.e. t is time, and $x(t)$ is the position of a particle which undergoes a Brownian motion – come to lecture for further explanation (see also the article on wikipedia). To see a similar graph check the Dow Jones or Nasdaq in the upper left hand corner of the web page at <http://finance.yahoo.com> in the afternoon on any weekday.

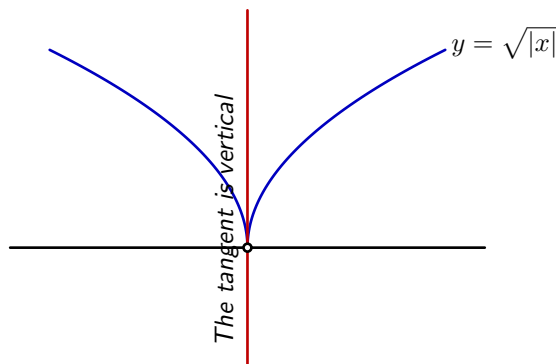


Figure 2. Tangent to the graph of $y = |x|^{1/2}$ at the origin

5. Exercises

101. Compute the derivative of the following functions

$$f(x) = x^2 - 2x$$

$$g(x) = \frac{1}{x}$$

$$k(x) = x^3 - 17x$$

$$u(x) = \frac{2}{1+x}$$

$$v(x) = \sqrt{x}$$

$$w(x) = \frac{1}{\sqrt{x}}$$

using either (17) or (18).

102. Which of the following functions is differentiable at $x = 0$?

$$f(x) = x|x|, \quad g(x) = x\sqrt{|x|},$$

$$h(x) = x + |x|, \quad k(x) = x^2 \sin \frac{\pi}{x},$$

$$\ell(x) = x \sin \frac{\pi}{x}.$$

These formulas do not define k and ℓ at $x = 0$. We define $k(0) = \ell(0) = 0$.

103. For which value(s) is the function defined by

$$f(x) = \begin{cases} ax + b & \text{for } x < 0 \\ x - x^2 & \text{for } x \geq 0 \end{cases}$$

differentiable at $x = 0$? Sketch the graph of the function f for the values a and b you found.

104. For which value(s) is the function defined by

$$f(x) = \begin{cases} ax^2 + b & \text{for } x < 1 \\ x - x^2 & \text{for } x \geq 1 \end{cases}$$

differentiable at $x = 0$? Sketch the graph of the function f for the values a and b you found.

105. For which value(s) is the function defined by

$$f(x) = \begin{cases} ax^2 & \text{for } x < 2 \\ x + b & \text{for } x \geq 2 \end{cases}$$

differentiable at $x = 0$? Sketch the graph of the function f for the values a and b you found.

106. Group Problem.

True or false: If a function f is continuous at some $x = a$ then it must also be differentiable at $x = a$?

107. Group Problem.

True or false: If a function f is differentiable at some $x = a$ then it must also be continuous at $x = a$?

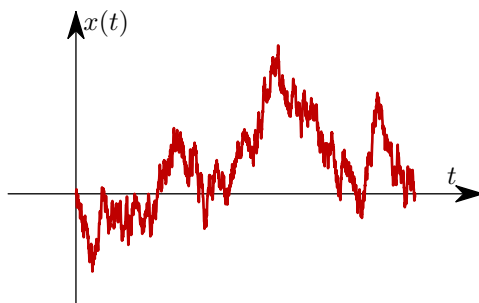


Figure 3. A Brownian motion. Note how the graph doesn't seem to have a tangent anywhere at all.

6. The Differentiation Rules

You could go on and compute more derivatives from the definition. Each time you would have to compute a new limit, and hope that there is some trick that allows you to find that limit. This is fortunately not necessary. It turns out that if you know a few basic derivatives (such as $dx^n/dx = nx^{n-1}$) the you can find derivatives of arbitrarily complicated functions by breaking them into smaller pieces. In this section we'll look at rules which tell you how to differentiate a function which is either the sum, difference, product or quotient of two other functions.

<i>Constant rule:</i>	$c' = 0$	$\frac{dc}{dx} = 0$
<i>Sum rule:</i>	$(u \pm v)' = u' \pm v'$	$\frac{du \pm v}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
<i>Product rule:</i>	$(u \cdot v)' = u' \cdot v + u \cdot v'$	$\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$
<i>Quotient rule:</i>	$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$	$\frac{d\frac{u}{v}}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Table 1. The differentiation rules

The situation is analogous to that of the “limit-properties” $(P_1) \dots (P_6)$ from the previous chapter which allowed us to compute limits without always having to go back to the epsilon-delta definition.

6.1. Sum, product and quotient rules. In the following c and n are constants, u and v are functions of x , and $'$ denotes differentiation. The Differentiation Rules in function notation, and Leibniz notation, are listed in figure 1.

Note that we already proved the Constant Rule in example 2.2. We will now prove the sum, product and quotient rules.

6.2. Proof of the Sum Rule. Suppose that $f(x) = u(x) + v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
 f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} && \text{(definition of } f') \\
 &= \lim_{x \rightarrow a} \frac{(u(x) + v(x)) - (u(a) + v(a))}{x - a} && \text{(use } f = u + v) \\
 &= \lim_{x \rightarrow a} \left(\frac{u(x) - u(a)}{x - a} + \frac{v(x) - v(a)}{x - a} \right) && \text{(algebra)} \\
 &= \lim_{x \rightarrow a} \frac{u(x) - u(a)}{x - a} + \lim_{x \rightarrow a} \frac{v(x) - v(a)}{x - a} && \text{(limit property)} \\
 &= u'(a) + v'(a) && \text{(definition of } u', v')
 \end{aligned}$$

6.3. Proof of the Product Rule. Let $f(x) = u(x)v(x)$. To find the derivative we must express the change of f in terms of the changes of u and v

$$\begin{aligned}
 f(x) - f(a) &= u(x)v(x) - u(a)v(a) \\
 &= u(x)v(x) - u(x)v(a) + u(x)v(a) - u(a)v(a) \\
 &= u(x)(v(x) - v(a)) + (u(x) - u(a))v(a)
 \end{aligned}$$

Now divide by $x - a$ and let $x \rightarrow a$:

$$\begin{aligned}\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \rightarrow a} u(x) \frac{v(x) - v(a)}{x - a} + \frac{u(x) - u(a)}{x - a} v(a) \\ &\quad \text{(use the limit properties)} \\ &= \left(\lim_{x \rightarrow a} u(x) \right) \left(\lim_{x \rightarrow a} \frac{v(x) - v(a)}{x - a} \right) + \left(\lim_{x \rightarrow a} \frac{u(x) - u(a)}{x - a} \right) v(a) \\ &= u(a)v'(a) + u'(a)v(a),\end{aligned}$$

as claimed. In this last step we have used that

$$\lim_{x \rightarrow a} \frac{u(x) - u(a)}{x - a} = u'(a) \quad \text{and} \quad \lim_{x \rightarrow a} \frac{v(x) - v(a)}{x - a} = v'(a)$$

and also that

$$\lim_{x \rightarrow a} u(x) = u(a)$$

This last limit follows from the fact that u is continuous, which in turn follows from the fact that u is differentiable.

6.4. Proof of the Quotient Rule . We can break the proof into two parts. First we do the special case where $f(x) = 1/v(x)$, and then we use the product rule to differentiate

$$f(x) = \frac{u(x)}{v(x)} = u(x) \cdot \frac{1}{v(x)}.$$

So let $f(x) = 1/v(x)$. We can express the change in f in terms of the change in v

$$f(x) - f(a) = \frac{1}{v(x)} - \frac{1}{v(a)} = \frac{v(x) - v(a)}{v(x)v(a)}.$$

Dividing by $x - a$ we get

$$\frac{f(x) - f(a)}{x - a} = \frac{1}{v(x)v(a)} \frac{v(x) - v(a)}{x - a}.$$

Now we want to take the limit $x \rightarrow a$. We are given the v is differentiable, so it must also be continuous and hence

$$\lim_{x \rightarrow a} v(x) = v(a).$$

Therefore we find

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{1}{v(x)v(a)} \lim_{x \rightarrow a} \frac{v(x) - v(a)}{x - a} = \frac{v'(a)}{v(a)^2}.$$

That completes the first step of the proof. In the second step we use the product rule to differentiate $f = u/v$

$$f' = \left(\frac{u}{v} \right)' = \left(u \cdot \frac{1}{v} \right)' = u' \cdot \frac{1}{v} + u \cdot \left(\frac{1}{v} \right)' = \frac{u'}{v} - u \frac{v'}{v^2} = \frac{u'v - uv'}{v^2}.$$

6.5. A shorter, but not quite perfect derivation of the Quotient Rule . The Quotient Rule can be derived from the Product Rule as follows: if $w = u/v$ then

$$(20) \quad w \cdot v = u$$

By the product rule we have

$$w' \cdot v + w \cdot v' = u',$$

so that

$$w' = \frac{u' - w \cdot v'}{v} = \frac{u' - (u/v) \cdot v'}{v} = \frac{u' \cdot v - u \cdot v'}{v^2}.$$

Unlike the proof in §6.4 above, this argument does not prove that w is differentiable if u and v are. It only says that *if the derivative exists* then it must be what the Quotient Rule says it is.

The trick which is used here, is a special case of a method called “implicit differentiation.” We have an equation (20) which the quotient w satisfies, and from by differentiating this equation we find w' .

6.6. Differentiating a constant multiple of a function . Note that the rule

$$(cu)' = cu'$$

follows from the Constant Rule and the Product Rule.

6.7. Picture of the Product Rule. If u and v are quantities which depend on x , and if increasing x by Δx causes u and v to change by Δu and Δv , then the product of u and v will change by

$$(21) \quad \Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u\Delta v.$$

If u and v are differentiable functions of x , then the changes Δu and Δv will be of the same order of magnitude as Δx , and thus one expects $\Delta u\Delta v$ to be much smaller. One therefore ignores the last term in (21), and thus arrives at

$$\Delta(uv) = u\Delta v + v\Delta u.$$

Leibniz would now divide by Δx and replace Δ 's by d 's to get the product rule:

$$\frac{\Delta(uv)}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x}.$$

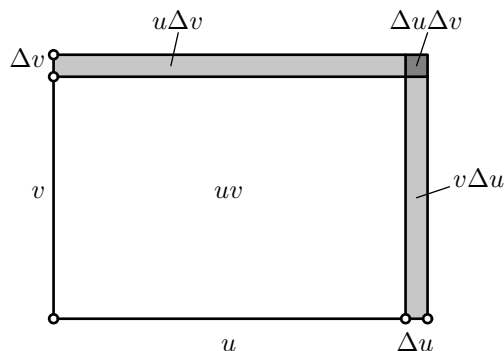


Figure 4. The Product Rule. *How much does the area of a rectangle change if its sides u and v are increased by Δu and Δv ? Most of the increase is accounted for by the two thin rectangles whose areas are $u\Delta v$ and $v\Delta u$. So the increase in area is approximately $u\Delta v + v\Delta u$, which explains why the product rule says $(uv)' = uv' + vu'$.*

7. Differentiating powers of functions

7.1. Product rule with more than one factor. If a function is given as the product of n functions, i.e.

$$f(x) = u_1(x) \times u_2(x) \times \cdots \times u_n(x),$$

then you can differentiate it by applying the product rule $n - 1$ times (there are n factors, so there are $n - 1$ multiplications.)

After the first step you would get

$$f' = u_1'(u_2 \cdots u_n) + u_1(u_2 \cdots u_n)'.$$

In the second step you apply the product rule to $(u_2 u_3 \cdots u_n)'$. This yields

$$\begin{aligned} f' &= u_1' u_2 \cdots u_n + u_1 [u_2' u_3 \cdots u_n + u_2 (u_3 \cdots u_n)'] \\ &= u_1' u_2 \cdots u_n + u_1 u_2' u_3 \cdots u_n + u_1 u_2 (u_3 \cdots u_n)'. \end{aligned}$$

Continuing this way one finds after $n - 1$ applications of the product rule that

$$(22) \quad (u_1 \cdots u_n)' = u_1' u_2 \cdots u_n + u_1 u_2' u_3 \cdots u_n + \cdots + u_1 u_2 u_3 \cdots u_n'.$$

7.2. The Power rule . If all n factors in the previous paragraph are the same, so that the function f is the n^{th} power of some other function,

$$f(x) = (u(x))^n,$$

then all terms in the right hand side of (22) are the same, and, since there are n of them, one gets

$$f'(x) = nu^{n-1}(x)u'(x),$$

or, in Leibniz' notation,

$$(23) \quad \frac{du^n}{dx} = nu^{n-1} \frac{du}{dx}.$$

7.3. The Power Rule for Negative Integer Exponents . We have just proved the power rule (23) assuming n is a positive integer. The rule actually holds for all real exponents n , but the proof is harder.

Here we prove the Power Rule for negative exponents using the Quotient Rule. Suppose $n = -m$ where m is a positive integer. Then the Quotient Rule tells us that

$$(u^n)' = (u^{-m})' = \left(\frac{1}{u^m} \right)' \stackrel{\text{Q.R.}}{=} -\frac{(u^m)'}{(u^m)^2}.$$

Since m is a positive integer, we can use (23), so $(u^m)' = mu^{m-1}$, and hence

$$(u^n)' = -\frac{mu^{m-1} \cdot u'}{u^{2m}} = -mu^{-m-1} \cdot u' = nu^{n-1}u'.$$

7.4. The Power Rule for Rational Exponents . So far we have proved that the power law holds if the exponent n is an integer.

We will now see how you can show that the power law holds even if the exponent n is any fraction, $n = p/q$. The following derivation contains the trick called *implicit differentiation* which we will study in more detail in Section 15.

So let $n = p/q$ where p and q are integers and consider the function

$$w(x) = u(x)^{p/q}.$$

Assuming that both u and w are differentiable functions, we will show that

$$(24) \quad w'(x) = \frac{p}{q} u(x)^{\frac{p}{q}-1} u'(x)$$

Raising both sides to the q th power gives

$$w(x)^q = u(x)^p.$$

Here the exponents p and q are integers, so we may apply the Power Rule to both sides. We get

$$qw^{q-1} \cdot w' = pu^{p-1} \cdot u'.$$

Dividing both sides by qw^{q-1} and substituting $u^{p/q}$ for w gives

$$w' = \frac{pu^{p-1} \cdot u'}{qw^{q-1}} = \frac{pu^{p-1} \cdot u'}{qu^{p(q-1)/q}} = \frac{pu^{p-1} \cdot u'}{qu^{p-(p/q)}} = \frac{p}{q} \cdot u^{(p/q)-1} \cdot u'$$

which is the Power Rule for $n = p/q$.

This proof is flawed because we did not show that $w(x) = u(x)^{p/q}$ is differentiable: we only showed what the derivative should be, *if it exists*.

7.5. Derivative of x^n for integer n . If you choose the function $u(x)$ in the Power Rule to be $u(x) = x$, then $u'(x) = 1$, and hence the derivative of $f(x) = u(x)^n = x^n$ is

$$f'(x) = nu(x)^{n-1}u'(x) = nx^{n-1} \cdot 1 = nx^{n-1}.$$

We already knew this of course.

7.6. Example – differentiate a polynomial. Using the Differentiation Rules you can easily differentiate any polynomial and hence any rational function. For example, using the Sum Rule, the Power Rule with $u(x) = x$, the rule $(cu)' = cu'$, the derivative of the polynomial

$$f(x) = 2x^4 - x^3 + 7$$

is

$$f'(x) = 8x^3 - 3x^2.$$

7.7. Example – differentiate a rational function. By the Quotient Rule the derivative of the function

$$g(x) = \frac{2x^4 - x^3 + 7}{1 + x^2}$$

is

$$\begin{aligned} g'(x) &= \frac{(8x^3 - 3x^2)(1 + x^2) - (2x^4 - x^3 + 7)2x}{(1 + x^2)^2} \\ &= \frac{6x^5 - x^4 + 8x^3 - 3x^2 - 14x}{(1 + x^2)^2}. \end{aligned}$$

If you compare this example with the previous then you see that polynomials simplify when you differentiate them while rational functions become more complicated.

7.8. Derivative of the square root. The derivative of $f(x) = \sqrt{x} = x^{1/2}$ is

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

where we used the power rule with $n = 1/2$ and $u(x) = x$.

8. Exercises

108. Let $f(x) = (x^2 + 1)(x^3 + 3)$. Find $f'(x)$ in two ways:

(a) by multiplying and then differentiating,

(b) by using the product rule.

Are your answers the same?

109. Let $f(x) = (1 + x^2)^4$. Find $f'(x)$ in two ways, first by expanding to get an expression for $f(x)$ as a polynomial in x and then differentiating, and then by using the power rule. Are the answers the same?

110. Prove the statement in §6.6, i.e. show that $(cu)' = c(u')$ follows from the product rule.

Compute the derivatives of the following functions.

(try to simplify your answers)

111. $f(x) = x + 1 + (x + 1)^2$

112. $f(x) = \frac{x - 2}{x^4 + 1}$

113. $f(x) = \left(\frac{1}{1 + x}\right)^{-1}$

114. $f(x) = \sqrt{1 - x^2}$

115. $f(x) = \frac{ax + b}{cx + d}$

116. $f(x) = \frac{1}{(1 + x^2)^2}$

117. $f(x) = \frac{x}{1 + \sqrt{x}}$

118. $f(x) = \sqrt{\frac{1 - x}{1 + x}}$

119. $f(x) = \sqrt[3]{x + \sqrt{x}}$

120. $\varphi(t) = \frac{t}{1 + \sqrt{t}}$

121. $g(s) = \sqrt{\frac{1 - s}{1 + s}}$

122. $h(\rho) = \sqrt[3]{\rho + \sqrt{\rho}}$

123. Group Problem.

Using derivatives to approximate numbers.

(a) Find the derivative of $f(x) = x^{4/3}$.

(b) Use (a) to estimate the number

$$\frac{127^{4/3} - 125^{4/3}}{2}$$

approximately without a calculator. Your answer should have the form p/q where p and q are integers. [Hint: Note that $5^3 = 125$ and take a good look at equation (17).]

(c) Approximate in the same way the numbers $\sqrt{143}$ and $\sqrt{145}$ (Hint: $12 \times 12 = 144$).

124. Group Problem.

(Making the product and quotient rules look nicer.)

Instead of looking at the derivative of a function you can look at the ratio of its derivative to the function itself, i.e. you can compute f'/f . This quantity is called the **logarithmic derivative of the function** f for reasons that will become clear later this semester.

(a) Compute the logarithmic derivative of these functions (i.e. find $f'(x)/f(x)$)

$$\begin{aligned} F(x) &= x, & g(x) &= 3x, & h(x) &= x^2 \\ k(x) &= -x^2, & \ell(x) &= 2007x^2, & m(x) &= x^{2007} \end{aligned}$$

(b) Show that for any pair of functions u and v one has

$$\begin{aligned} \frac{(uv)'}{uv} &= \frac{u'}{u} + \frac{v'}{v} \\ \frac{(u/v)'}{u/v} &= \frac{u'}{u} - \frac{v'}{v} \\ \frac{(u^n)'}{u^n} &= n \frac{u'}{u} \end{aligned}$$

125. (a) Find $f'(x)$ and $g'(x)$ if

$$f(x) = \frac{1+x^2}{2x^4+7}, \quad g(x) = \frac{2x^4+7}{1+x^2}.$$

Note that $f(x) = 1/g(x)$.

(b) Is it true that $f'(x) = 1/g'(x)$?

(c) Is it true that $f(x) = g^{-1}(x)$?

(d) Is it true that $f(x) = g(x)^{-1}$?

126. Group Problem. (a) Let $x(t) = (1-t^2)/(1+t^2)$, $y(t) = 2t/(1+t^2)$ and $u(t) = y(t)/x(t)$. Find dx/dt , dy/dt .

(b) Now that you've done (a) there are two different ways of finding du/dt . What are they, and use one of both to find du/dt .

9. Higher Derivatives

9.1. The derivative is a function. If the derivative $f'(a)$ of some function f exists for all a in the domain of f , then we have a new function: namely, for each number in the domain of f we compute the derivative of f at that number. This function is called the **derivative function** of f , and it is denoted by f' . Now that we have agreed that the derivative of a function is a function, we can repeat the process and try to differentiate the derivative. The result, if it exists, is called the **second derivative of f** . It is denoted f'' . The derivative of the second derivative is called the third derivative, written f''' , and so on.

The n th derivative of f is denoted $f^{(n)}$. Thus

$$f^{(0)} = f, \quad f^{(1)} = f', \quad f^{(2)} = f'', \quad f^{(3)} = f''', \dots$$

Leibniz' notation for the n th derivative of $y = f(x)$ is

$$\frac{d^n y}{dx^n} = f^{(n)}(x).$$

9.2. Example. If $f(x) = x^2 - 2x + 3$ then

$$f(x) = x^2 - 2x + 3$$

$$f'(x) = 2x - 2$$

$$f''(x) = 2$$

$$f^{(3)}(x) = 0$$

$$f^{(4)}(x) = 0$$

$\vdots$

All further derivatives of f are zero.

9.3. Operator notation. A common variation on Leibniz' notation for derivatives is the so-called *operator notation*, as in

$$\frac{d(x^3 - x)}{dx} = \frac{d}{dx}(x^3 - x) = 3x^2 - 1.$$

For higher derivatives one can write

$$\frac{d^2 y}{dx^2} = \left(\frac{d}{dx} \right)^2 y$$

Be careful to distinguish the second derivative from the square of the first derivative. Usually

$$\frac{d^2 y}{dx^2} \neq \left(\frac{dy}{dx} \right)^2 \text{ !!!!}$$

10. Exercises

127. The equation

$$(\dagger) \quad \frac{2x}{x^2 - 1} = \frac{1}{x + 1} + \frac{1}{x - 1}$$

holds for all values of x (except $x = \pm 1$), so you should get the same answer if you differentiate both sides. Check this.

Compute the third derivative of $f(x) = 2x/(x^2 - 1)$ by using either the left or right hand side (your choice) of $(\dagger)$.

128. Compute the first, second and third derivatives of the following functions

$$f(x) = (x + 1)^4$$

$$g(x) = (x^2 + 1)^4$$

$$h(x) = \sqrt{x - 2}$$

$$k(x) = \sqrt[3]{x - \frac{1}{x}}$$

129. Find the derivatives of 10th order of the functions

$$f(x) = x^{12} + x^8$$

$$g(x) = \frac{1}{x}$$

$$h(x) = \frac{12}{1 - x}$$

$$k(x) = \frac{x^2}{1 - x}$$

130. Find $f'(x)$, $f''(x)$ and $f^{(3)}(x)$ if

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$

131. Group Problem.

(a) Find the 12th derivative of the function $f(x) = \frac{1}{x + 2}$.

(b) Find the n^{th} order derivative of $f(x) = \frac{1}{x + 2}$ (i.e. find a formula for $f^{(n)}(x)$ which is valid for all $n = 0, 1, 2, 3, \dots$).

(c) Find the n^{th} order derivative of $g(x) = \frac{x}{x + 2}$.

132. (About notation.)

(a) Find dy/dx and d^2y/dx^2 if $y = x/(x + 2)$.

(b) Find du/dt and d^2u/dt^2 if $u = t/(t + 2)$. Hint: See previous problem.

(c) Find $\frac{d}{dx} \left(\frac{x}{x + 2} \right)$ and $\frac{d^2}{dx^2} \left(\frac{x}{x + 2} \right)$. Hint: See previous problem.

(d) Find $\frac{d}{dx} \left(\frac{x}{x + 2} \right) \Big|_{x=1}$ and $\frac{d}{dx} \left(\frac{1}{1 + 2} \right)$.

133. Find d^2y/dx^2 and $(dy/dx)^2$ if $y = x^3$.

11. Differentiating Trigonometric functions

The trigonometric functions Sine, Cosine and Tangent are differentiable, and their derivatives are given by the following formulas

$$(25) \quad \frac{d \sin x}{dx} = \cos x, \quad \frac{d \cos x}{dx} = -\sin x, \quad \frac{d \tan x}{dx} = \frac{1}{\cos^2 x}.$$

Note the minus sign in the derivative of the cosine!

PROOF. By definition one has

$$\sin'(x) = \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin(x)}{h}.$$

To simplify the numerator we use the trigonometric addition formula

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

with $\alpha = x$ and $\beta = h$, which results in

$$\begin{aligned}\frac{\sin(x+h) - \sin(x)}{h} &= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \cos(x)\frac{\sin(h)}{h} + \sin(x)\frac{\cos(h) - 1}{h}\end{aligned}$$

Hence by the formulas

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0$$

from Section 13 we have

$$\begin{aligned}\sin'(x) &= \lim_{h \rightarrow 0} \cos(x)\frac{\sin(h)}{h} + \sin(x)\frac{\cos(h) - 1}{h} \\ &= \cos(x) \cdot 1 + \sin(x) \cdot 0 \\ &= \cos(x).\end{aligned}$$

A similar computation leads to the stated derivative of $\cos x$.

To find the derivative of $\tan x$ we apply the quotient rule to

$$\tan x = \frac{\sin x}{\cos x} = \frac{f(x)}{g(x)}.$$

We get

$$\tan'(x) = \frac{\cos(x)\sin'(x) - \sin(x)\cos'(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

as claimed. □

12. Exercises

Find the derivatives of the following functions (try to simplify your answers)

134. $f(x) = \sin(x) + \cos(x)$

135. $f(x) = 2\sin(x) - 3\cos(x)$

136. $f(x) = 3\sin(x) + 2\cos(x)$

137. $f(x) = x\sin(x) + \cos(x)$

138. $f(x) = x\cos(x) - \sin x$

139. $f(x) = \frac{\sin x}{x}$

140. $f(x) = \cos^2(x)$

141. $f(x) = \sqrt{1 - \sin^2 x}$

142. $f(x) = \sqrt{\frac{1 - \sin x}{1 + \sin x}}$

143. $\cot(x) = \frac{\cos x}{\sin x}$.

144. Can you find a and b so that the function

$$f(x) = \begin{cases} \cos x & \text{for } x \leq \frac{\pi}{4} \\ a + bx & \text{for } x > \frac{\pi}{4} \end{cases}$$

is differentiable at $x = \pi/4$?

145. Can you find a and b so that the function

$$f(x) = \begin{cases} \tan x & \text{for } x < \frac{\pi}{6} \\ a + bx & \text{for } x \geq \frac{\pi}{6} \end{cases}$$

is differentiable at $x = \pi/4$?

146. If f is a given function, and you have another function g which satisfies $g(x) = f(x) + 12$ for all x , then f and g have the same derivatives. Prove this. [*Hint: it's a short proof – use the differentiation rules.*]

147. **Group Problem.**

Show that the functions

$$f(x) = \sin^2 x \text{ and } g(x) = -\cos^2 x$$

have the same derivative by computing $f'(x)$ and $g'(x)$.

With hindsight this was to be expected – why?

148. Find the first and second derivatives of the functions

$$f(x) = \tan^2 x \text{ and } g(x) = \frac{1}{\cos^2 x}.$$

Hint: remember your trig to reduce work!

A depends on B depends on C depends on . . .

Someone is pumping water into a balloon. Assuming that the balloon is spherical you can say how large it is by specifying its radius R . For a growing balloon this radius will change with time t .

The volume of the balloon is a function of its radius, since the volume of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^3.$$

We now have two functions, the first f turns tells you the radius r of the balloon at time t ,

$$r = f(t)$$

and the second tells you the volume of the balloon given its radius

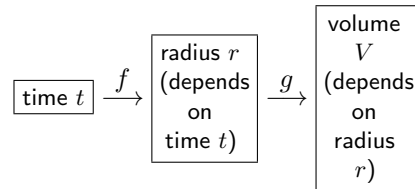
$$V = g(r).$$

The volume of the balloon at time t is then given by

$$V = g(f(t)) = g \circ f(t),$$

i.e. the function which tells you the volume of the balloon at time t is the composition of first f and then g .

Schematically we can summarize this chain of cause-and-effect relations as follows: you could either say that V depends on r , and r depends on t ,



or you could say that V depends directly on t :

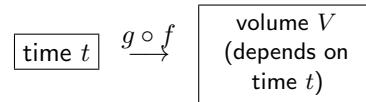


Figure 5. A “real world example” of a composition of functions.

13. The Chain Rule

13.1. Composition of functions. Given two functions f and g , one can define a new function called the **composition of f and g** . The notation for the composition is $f \circ g$, and it is defined by the formula

$$f \circ g(x) = f(g(x)).$$

The domain of the composition is the set of all numbers x for which this formula gives you something well-defined.

For instance, if $f(x) = x^2 + x$ and $g(x) = 2x + 1$ then

$$f \circ g(x) = f(2x + 1) = (2x + 1)^2 + (2x + 1)$$

$$\text{and } g \circ f(x) = g(x^2 + x) = 2(x^2 + x) + 1$$

Note that $f \circ g$ and $g \circ f$ are not the same function in this example (they hardly ever are the same).

If you think of functions as expressing dependence of one quantity on another, then the composition of functions arises as follows. If a quantity z is a function of another quantity y , and if y itself depends on x , then z depends on x via y .

To get $f \circ g$ from the previous example, we could say $z = f(y)$ and $y = g(x)$, so that

$$z = f(y) = y^2 + y \text{ and } y = 2x + 1.$$

Given x one can compute y , and from y one can then compute z . The result will be

$$z = y^2 + y = (2x + 1)^2 + (2x + 1),$$

in other notation,

$$z = f(y) = f(g(x)) = f \circ g(x).$$

One says that **the composition of f and g is the result of substituting g in f** .

13.2. Theorem (Chain Rule). *If f and g are differentiable, so is the composition $f \circ g$.*

The derivative of $f \circ g$ is given by

$$(f \circ g)'(x) = f'(g(x)) g'(x).$$

The chain rule tells you how to find the derivative of the composition $f \circ g$ of two functions f and g provided you now how to differentiate the two functions f and g .

When written in Leibniz' notation the chain rule looks particularly easy. Suppose that $y = g(x)$ and $z = f(y)$, then $z = f \circ g(x)$, and the derivative of z with respect to x is the derivative of the function $f \circ g$. The derivative of z with respect to y is the derivative of the function f , and the derivative of y with respect to x is the derivative of the function g . In short,

$$\frac{dz}{dx} = (f \circ g)'(x), \quad \frac{dz}{dy} = f'(y) \quad \text{and} \quad \frac{dy}{dx} = g'(x)$$

so that the chain rule says

$$(26) \quad \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}.$$

FIRST PROOF OF THE CHAIN RULE (USING LEIBNIZ' NOTATION). We first consider difference quotients instead of derivatives, i.e. using the same notation as above, we consider the effect of an increase of x by an amount Δx on the quantity z .

If x increases by Δx , then $y = g(x)$ will increase by

$$\Delta y = g(x + \Delta x) - g(x),$$

and $z = f(y)$ will increase by

$$\Delta z = f(y + \Delta y) - f(y).$$

The ratio of the increase in $z = f(g(x))$ to the increase in x is

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}.$$

In contrast to dx , dy and dz in equation (26), the Δx , etc. here are finite quantities, so this equation is just algebra: you can cancel the two Δy s. If you let the increase Δx go to zero, then the increase Δy will also go to zero, and the difference quotients converge to the derivatives,

$$\frac{\Delta z}{\Delta x} \longrightarrow \frac{dz}{dx}, \quad \frac{\Delta z}{\Delta y} \longrightarrow \frac{dz}{dy}, \quad \frac{\Delta y}{\Delta x} \longrightarrow \frac{dy}{dx}$$

which immediately leads to Leibniz' form of the quotient rule. □

PROOF OF THE CHAIN RULE. We verify the formula in Theorem 13.2 at some arbitrary value $x = a$, i.e. we will show that

$$(f \circ g)'(a) = f'(g(a)) g'(a).$$

By definition the left hand side is

$$(f \circ g)'(a) = \lim_{x \rightarrow a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

The two derivatives on the right hand side are given by

$$g'(a) = \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}$$

and

$$f'(g(a)) = \lim_{y \rightarrow g(a)} \frac{f(y) - f(g(a))}{y - g(a)}.$$

Since g is a differentiable function it must also be a continuous function, and hence $\lim_{x \rightarrow a} g(x) = g(a)$. So we can substitute $y = g(x)$ in the limit defining $f'(g(a))$

$$(27) \quad f'(g(a)) = \lim_{y \rightarrow a} \frac{f(y) - f(g(a))}{y - g(a)} = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}.$$

Put all this together and you get

$$\begin{aligned} (f \circ g)'(a) &= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a} \\ &= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} \\ &= f'(g(a)) \cdot g'(a) \end{aligned}$$

which is what we were supposed to prove – the proof seems complete.

There is one flaw in this proof, namely, we have divided by $g(x) - g(a)$, which is not allowed when $g(x) - g(a) = 0$. This flaw can be fixed but we will not go into the details here.² $\square$

13.3. First example. We go back to the functions

$$z = f(y) = y^2 + y \text{ and } y = g(x) = 2x + 1$$

from the beginning of this section. The composition of these two functions is

$$z = f(g(x)) = (2x + 1)^2 + (2x + 1) = 4x^2 + 6x + 2.$$

We can compute the derivative of this composed function, i.e. the derivative of z with respect to x in two ways. First, you simply differentiate the last formula we have:

$$(28) \quad \frac{dz}{dx} = \frac{d(4x^2 + 6x + 2)}{dx} = 8x + 6.$$

The other approach is to use the chain rule:

$$\frac{dz}{dy} = \frac{d(y^2 + y)}{dy} = 2y + 1,$$

and

$$\frac{dy}{dx} = \frac{d(2x + 1)}{dx} = 2.$$

Hence, by the chain rule one has

$$(29) \quad \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = (2y + 1) \cdot 2 = 4y + 2.$$

The two answers (28) and (29) should be the same. Once you remember that $y = 2x + 1$ you see that this is indeed true:

$$y = 2x + 1 \implies 4y + 2 = 4(2x + 1) + 2 = 8x + 6.$$

The two computations of dz/dx therefore lead to the same answer. In this example there was no clear advantage in using the chain rule. The chain rule becomes useful when the functions f and g become more complicated.

² Briefly, you have to show that the function

$$h(y) = \begin{cases} \{f(y) - f(g(a))\}/(y - g(a)) & y \neq a \\ f'(g(a)) & y = a \end{cases}$$

is continuous.

13.4. Example where you really need the Chain Rule. We know what the derivative of $\sin x$ with respect to x is, but none of the rules we have found so far tell us how to differentiate $f(x) = \sin(2x)$.

The function $f(x) = \sin 2x$ is the composition of two simpler functions, namely

$$f(x) = g(h(x)) \text{ where } g(u) = \sin u \text{ and } h(x) = 2x.$$

We know how to differentiate each of the two functions g and h :

$$g'(u) = \cos u, \quad h'(x) = 2.$$

Therefore the chain rule implies that

$$f'(x) = g'(h(x))h'(x) = \cos(2x) \cdot 2 = 2 \cos 2x.$$

Leibniz would have decomposed the relation $y = \sin 2x$ between y and x as

$$y = \sin u, \quad u = 2x$$

and then computed the derivative of $\sin 2x$ with respect to x as follows

$$\frac{d \sin 2x}{dx} \stackrel{u=2x}{=} \frac{d \sin u}{du} = \frac{d \sin u}{du} \cdot \frac{du}{dx} = \cos u \cdot 2 = 2 \cos 2x.$$

13.5. The Power Rule and the Chain Rule. The Power Rule, which says that for any function f and any rational number n one has

$$\frac{d}{dx}(f(x)^n) = n f(x)^{n-1} f'(x),$$

is a special case of the Chain Rule, for one can regard $y = f(x)^n$ as the composition of two functions

$$y = g(u), \quad u = f(x)$$

where $g(u) = u^n$. Since $g'(u) = nu^{n-1}$ the Chain Rule implies that

$$\frac{du^n}{dx} = \frac{du^n}{du} \cdot \frac{du}{dx} = nu^{n-1} \frac{du}{dx}.$$

Setting $u = f(x)$ and $\frac{du}{dx} = f'(x)$ then gives you the Power Rule.

13.6. The volume of an inflating balloon. Consider the “real world example” from page 53 again. There we considered a growing water balloon of radius

$$r = f(t).$$

The volume of this balloon is

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi f(t)^3.$$

We can regard this as the composition of two functions, $V = g(r) = \frac{4}{3}\pi r^3$ and $r = f(t)$.

According to the chain rule the rate of change of the volume with time is now

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$$

i.e. it is the product of the rate of change of the volume with the radius of the balloon and the rate of change of the balloon’s radius with time. From

$$\frac{dV}{dr} = \frac{d\frac{4}{3}\pi r^3}{dr} = 4\pi r^2$$

we see that

$$\frac{dV}{dr} = 4\pi r^2 \frac{dr}{dt}.$$

For instance, if the radius of the balloon is growing at 0.5inch/sec, and if its radius is $r = 3.0$ inch, then the volume is growing at a rate of

$$\frac{dV}{dt} = 4\pi(3.0\text{inch})^2 \times 0.5\text{inch/sec} \approx 57\text{inch}^3/\text{sec}.$$

13.7. A more complicated example. Suppose you needed to find the derivative of

$$y = h(x) = \frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2}$$

We can write this function as a composition of two simpler functions, namely,

$$y = f(u), \quad u = g(x),$$

with

$$f(u) = \frac{u}{(u+1)^2} \text{ and } g(x) = \sqrt{x+1}.$$

The derivatives of f and g are

$$f'(u) = \frac{1 \cdot (u+1)^2 - u \cdot 2(u+1)}{(u+1)^4} = \frac{u+1-2}{(u+1)^3} = \frac{u-1}{(u+1)^3},$$

and

$$g'(x) = \frac{1}{2\sqrt{x+1}}.$$

Hence the derivative of the composition is

$$h'(x) = \frac{d}{dx} \left\{ \frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2} \right\} = f'(u)g'(x) = \frac{u-1}{(u+1)^3} \cdot \frac{1}{2\sqrt{x+1}}.$$

The result should be a function of x , and we achieve this by replacing all u 's with $u = \sqrt{x+1}$:

$$\frac{d}{dx} \left\{ \frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2} \right\} = \frac{\sqrt{x+1}-1}{(\sqrt{x+1}+1)^3} \cdot \frac{1}{2\sqrt{x+1}}.$$

The last step (where you replace u by its definition in terms of x) is important because the problem was presented to you with only x and y as variables while u was a variable you introduced yourself to do the problem.

Sometimes it is possible to apply the Chain Rule without introducing new letters, and you will simply think “the derivative is the derivative of the outside with respect to the inside times the derivative of the inside.” For instance, to compute

$$\frac{d}{dx} \frac{4 + \sqrt{7+x^3}}{dx}$$

you could set $u = 7 + x^3$, and compute

$$\frac{d}{dx} \frac{4 + \sqrt{7+x^3}}{dx} = \frac{d}{du} \frac{4 + \sqrt{u}}{du} \cdot \frac{du}{dx}.$$

Instead of writing all this explicitly, you could think of $u = 7 + x^3$ as the function “inside the square root,” and think of $4 + \sqrt{u}$ as “the outside function.” You would then immediately write

$$\frac{d}{dx} (4 + \sqrt{7+x^3}) = \frac{1}{2\sqrt{7+x^3}} \cdot 3x^2.$$

13.8. The Chain Rule and composing more than two functions. Often we have to apply the Chain Rule more than once to compute a derivative. Thus if $y = f(u)$, $u = g(v)$, and $v = h(x)$ we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

In functional notation this is

$$(f \circ g \circ h)'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

Note that each of the three derivatives on the right is evaluated at a different point. Thus if $b = h(a)$ and $c = g(b)$ the Chain Rule is

$$\left. \frac{dy}{dx} \right|_{x=a} = \left. \frac{dy}{du} \right|_{u=c} \cdot \left. \frac{du}{dv} \right|_{v=b} \cdot \left. \frac{dv}{dx} \right|_{x=a}.$$

For example, if $y = \frac{1}{1 + \sqrt{9 + x^2}}$, then $y = 1/(1 + u)$ where $u = 1 + \sqrt{v}$ and $v = 9 + x^2$ so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = -\frac{1}{(1+u)^2} \cdot \frac{1}{2\sqrt{v}} \cdot 2x.$$

so

$$\left. \frac{dy}{dx} \right|_{x=4} = \left. \frac{dy}{du} \right|_{u=6} \cdot \left. \frac{du}{dv} \right|_{v=25} \cdot \left. \frac{dv}{dx} \right|_{x=4} = -\frac{1}{7} \cdot \frac{1}{10} \cdot 8.$$

14. Exercises

- 149.** Let $y = \sqrt{1 + x^3}$ and find dy/dx using the Chain Rule. Say what plays the role of $y = f(u)$ and $u = g(x)$.

- 150.** Repeat the previous exercise with
 $y = (1 + \sqrt{1 + x})^3$.

- 151.** Alice and Bob differentiated $y = \sqrt{1 + x^3}$ with respect to x differently. Alice wrote $y = \sqrt{u}$ and $u = 1 + x^3$ while Bob wrote $y = \sqrt{1 + v}$ and $v = x^3$. Assuming neither one made a mistake, did they get the same answer?

- 152.** Let $y = u^3 + 1$ and $u = 3x + 7$. Find $\frac{dy}{dx}$ and $\frac{dy}{du}$. Express the former in terms of x and the latter in terms of u .

- 153.** Suppose that $f(x) = \sqrt{x}$, $g(x) = 1 + x^2$, $v(x) = f \circ g(x)$, $w(x) = g \circ f(x)$. Find formulas for $v(x)$, $w(x)$, $v'(x)$, and $w'(x)$.

Compute the following derivatives

- 154.** $f(x) = \sin 2x - \cos 3x$

- 155.** $f(x) = \sin \frac{\pi}{x}$

- 156.** $f(x) = \sin(\cos 3x)$

- 157.** $f(x) = \frac{\sin x^2}{x^2}$

- 158.** $f(x) = \tan \sqrt{1 + x^2}$

- 159.** $f(x) = \cos^2 x - \cos x^2$

160. Group Problem.

Moe is pouring water into a glass. At time t (seconds) the height of the water in the glass is $h(t)$ (inch). The ACME glass company, which made the glass, says that the volume in the glass to height h is $V = 1.2 h^2$ (fluid ounces).

(a) The water height in the glass is rising at 2 inch per second at the moment that the height is 2 inch. How fast is Moe pouring water into the glass?

(b) If Moe pours water at a rate of 1 ounce per second, then how fast is the water level in the glass going up when it is 3 inches?

(c) Moe pours water at 1 ounce per second, and at some moment the water level is going up at 0.5 inch per second. What is the water level at that moment?

- 161.** Find the derivative of $f(x) = x \cos \frac{\pi}{x}$ at the point C in Figure 3.

- 162.** Suppose that $f(x) = x^2 + 1$, $g(x) = x + 5$, and
 $v = f \circ g$, $w = g \circ f$, $p = f \cdot g$, $q = g \cdot f$.
Find $v(x)$, $w(x)$, $p(x)$, and $q(x)$.

163. Group Problem.

Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	-4	-1/3	-8/3

Define

$$\begin{aligned} v(x) &= f(g(x)), & w(x) &= g(f(x)), \\ p(x) &= f(x)g(x), & q(x) &= g(x)f(x). \end{aligned}$$

Evaluate $v(0)$, $w(0)$, $p(0)$, $q(0)$, $v'(0)$ and $w'(0)$, $p'(0)$, $q'(0)$. If there is insufficient information to answer the question, so indicate.

- 164.** A differentiable function f satisfies $f(3) = 5$, $f(9) = 7$, $f'(3) = 11$ and $f'(9) = 13$. Find an equation for the tangent line to the curve $y = f(x^2)$ at the point $(x, y) = (3, 7)$.

- 165.** There is a function f whose second derivative satisfies
 $(\dagger) \quad f''(x) = -64f(x)$.

(a) One such function is $f(x) = \sin ax$, provided you choose the right constant a : Which value should a have?

(b) For which choices of the constants A , a and b does the function $f(x) = A \sin(ax + b)$ satisfy $(\dagger)$?

166. Group Problem.

A cubical sponge, hereafter referred to as 'Bob', is absorbing water, which causes him to expand. His side at time t is $S(t)$. His volume is $V(t)$.

(a) What is the relation between $S(t)$ and $V(t)$, i.e. can you find a function f so that $V(t) = f(S(t))$?

(b) Describe the meaning of the derivatives $S'(t)$ and $V'(t)$ in one plain english sentence each. If we measure lengths in inches and time in minutes, then what units do t , $S(t)$, $V(t)$, $S'(t)$ and $V'(t)$ have?

(c) What is the relation between $S'(t)$ and $V'(t)$?

(d) At the moment that Bob's volume is 8 cubic inches, he is absorbing water at a rate of 2 cubic inch per minute. How fast is his side $S(t)$ growing?

15. Implicit differentiation

15.1. The recipe. Recall that an implicitly defined function is a function $y = f(x)$ which is defined by an equation of the form

$$F(x, y) = 0.$$

We call this equation the **defining equation** for the function $y = f(x)$. To find $y = f(x)$ for a given value of x you must solve the defining equation $F(x, y) = 0$ for y .

Here is a recipe for computing the derivative of an implicitly defined function.

- (1) Differentiate the equation $F(x, y) = 0$; you may need the chain rule to deal with the occurrences of y in $F(x, y)$;
- (2) You can rearrange the terms in the result of step 1 so as to get an equation of the form

$$(30) \quad G(x, y) \frac{dy}{dx} + H(x, y) = 0,$$

where G and H are expressions containing x and y but not the derivative.

- (3) Solve the equation in step 2 for $\frac{dy}{dx}$:

$$(31) \quad \frac{dy}{dx} = -\frac{H(x, y)}{G(x, y)}$$

- (4) If you also have an explicit description of the function (i.e. a formula expressing $y = f(x)$ in terms of x) then you can substitute $y = f(x)$ in the expression (31) to get a formula for dy/dx in terms of x only.

Often no explicit formula for y is available and you can't take this last step. In that case (31) is as far as you can go.

Observe that by following this procedure you will get a formula for the derivative $\frac{dy}{dx}$ which contains both x and y .

15.2. Dealing with equations of the form $F_1(x, y) = F_2(x, y)$. If the implicit definition of the function is not of the form $F(x, y) = 0$ but rather of the form $F_1(x, y) = F_2(x, y)$ then you move all terms to the left hand side, and proceed as above. E.g. to deal with a function $y = f(x)$ which satisfies

$$y^2 + x = xy$$

you rewrite this equation as

$$y^2 + x - xy = 0$$

and set $F(x, y) = y^2 + x - xy$.

15.3. Example – Derivative of $\sqrt[4]{1-x^4}$. Consider the function

$$f(x) = \sqrt[4]{1-x^4}, \quad -1 \leq x \leq 1.$$

We will compute its derivative in two ways: first the direct method, and then using the method of implicit differentiation (i.e. the recipe above).

The direct approach goes like this:

$$\begin{aligned} f'(x) &= \frac{d(1-x^4)^{1/4}}{dx} \\ &= \frac{1}{4}(1-x^4)^{-3/4} \frac{d(1-x^4)}{dx} \\ &= \frac{1}{4}(1-x^4)^{-3/4} (-4x^3) \\ &= -\frac{x^3}{(1-x^4)^{3/4}} \end{aligned}$$

To find the derivative using implicit differentiation we must first find a nice implicit description of the function. For instance, we could decide to get rid of all roots or fractional exponents in the function and point out that $y = \sqrt[4]{1-x^4}$ satisfies the equation $y^4 = 1-x^4$. So our implicit description of the function $y = f(x) = \sqrt[4]{1-x^4}$ is

$$x^4 + y^4 - 1 = 0; \quad \text{The defining function is therefore } F(x, y) = x^4 + y^4 - 1$$

Differentiate both sides with respect to x (and remember that $y = f(x)$, so y here is a function of x), and you get

$$\frac{dx^4}{dx} + \frac{dy^4}{dx} - \frac{d1}{dx} = 0 \implies 4x^3 + 4y^3 \frac{dy}{dx} = 0.$$

The expressions G and H from equation (30) in the recipe are $G(x, y) = 4y^3$ and $H(x, y) = 4x^3$.

This last equation can be solved for dy/dx :

$$\frac{dy}{dx} = -\frac{x^3}{y^3}.$$

This is a nice and short form of the derivative, but it contains y as well as x . To express dy/dx in terms of x only, and remove the y dependency we use $y = \sqrt[4]{1-x^4}$. The result is

$$f'(x) = \frac{dy}{dx} = -\frac{x^3}{y^3} = -\frac{x^3}{(1-x^4)^{3/4}}.$$

15.4. Another example. Let f be a function defined by

$$y = f(x) \iff 2y + \sin y = x, \text{ i.e. } 2y + \sin y - x = 0.$$

For instance, if $x = 2\pi$ then $y = \pi$, i.e. $f(2\pi) = \pi$.

To find the derivative dy/dx we differentiate the defining equation

$$\frac{d(2y + \sin y - x)}{dx} = \frac{d0}{dx} \implies 2\frac{dy}{dx} + \cos y \frac{dy}{dx} - \frac{dx}{dx} = 0 \implies (2 + \cos y) \frac{dy}{dx} - 1 = 0.$$

Solve for $\frac{dy}{dx}$ and you get

$$f'(x) = \frac{1}{2 + \cos y} = \frac{1}{2 + \cos f(x)}.$$

If we were asked to find $f'(2\pi)$ then, since we know $f(2\pi) = \pi$, we could answer

$$f'(2\pi) = \frac{1}{2 + \cos \pi} = \frac{1}{2 - 1} = 1.$$

If we were asked $f'(\pi/2)$, then all we would be able to say is

$$f'(\pi/2) = \frac{1}{2 + \cos f(\pi/2)}.$$

To say more we would first have to find $y = f(\pi/2)$, which one does by solving

$$2y + \sin y = \frac{\pi}{2}.$$

15.5. Derivatives of Arc Sine and Arc Tangent. Recall that

$$y = \arcsin x \iff x = \sin y \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2},$$

and

$$y = \arctan x \iff x = \tan y \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

15.6. Theorem.

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$$

PROOF. If $y = \arcsin x$ then $x = \sin y$. Differentiate this relation

$$\frac{dx}{dx} = \frac{d \sin y}{dx}$$

and apply the chain rule. You get

$$1 = (\cos y) \frac{dy}{dx},$$

and hence

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

How do we get rid of the y on the right hand side? We know $x = \sin y$, and also $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Therefore

$$\sin^2 y + \cos^2 y = 1 \implies \cos y = \pm \sqrt{1 - \sin^2 y} = \pm \sqrt{1 - x^2}.$$

Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ we know that $\cos y \geq 0$, so we must choose the positive square root. This leaves us with $\cos y = \sqrt{1 - x^2}$, and hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The derivative of $\arctan x$ is found in the same way, and you should really do this yourself. $\square$

16. Exercises

For each of the following problems find the derivative $f'(x)$ if $y = f(x)$ satisfies the given equation. State what the expressions $F(x, y)$, $G(x, y)$ and $H(x, y)$ from the recipe in the beginning of this section are.

If you can find an explicit description of the function $y = f(x)$, say what it is.

167. $xy = \frac{\pi}{6}$

168. $\sin(xy) = \frac{1}{2}$

169. $\frac{xy}{x+y} = 1$

170. $x + y = xy$

171. $(y-1)^2 + x = 0$

172. $(y+1)^2 + y - x = 0$

173. $(y-x)^2 + x = 0$

174. $(y+x)^2 + 2y - x = 0$

175. $(y^2 - 1)^2 + x = 0$

176. $(y^2 + 1)^2 - x = 0$

177. $x^3 + xy + y^3 = 3$

178. $\sin x + \sin y = 1$

179. $\sin x + xy + y^5 = \pi$

180. $\tan x + \tan y = 1$

For each of the following explicitly defined functions find an implicit definition which does not involve taking roots. Then use this description to find the derivative dy/dx .

181. $y = f(x) = \sqrt{1-x}$

182. $y = f(x) = \sqrt[4]{x+x^2}$

183. $y = f(x) = \sqrt{1-\sqrt{x}}$

184. $y = f(x) = \sqrt[4]{x-\sqrt{x}}$

185. $y = f(x) = \sqrt[3]{\sqrt{2x+1} - x^2}$

186. $y = f(x) = \sqrt[4]{x+x^2}$

187. $y = f(x) = \sqrt[3]{x - \sqrt{2x + 1}}$

188. $y = f(x) = \sqrt[4]{\sqrt[3]{x}}$

189. Group Problem.

(Inverse trig review) Simplify the following expressions, and indicate for which values of x (or θ , or ...) your simplification is valid. In case of doubt, try plotting the function on a graphing calculator.

(a) $\sin \arcsin x$	(e) $\tan \arctan z$
(b) $\cos \arcsin x$	(f) $\tan \arcsin \theta$
(c) $\arctan(\tan \theta)$	(g) $\arcsin(\sin \theta)$
(d) $\cot \arctan x$	(h) $\cot \arcsin x$

Now that you know the derivatives of $\arcsin$ and $\arctan$, you can find the derivatives of the following functions. What are they?

190. $f(x) = \arcsin(2x)$

191. $f(x) = \arcsin \sqrt{x}$

192. $f(x) = \arctan(\sin x)$

193. $f(x) = \sin \arctan x$

194. $f(x) = (\arcsin x)^2$

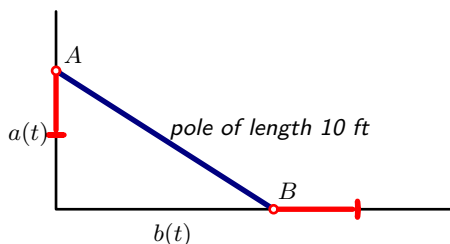
195. $f(x) = \frac{1}{1 + (\arctan x)^2}$

196. $f(x) = \sqrt{1 - (\arcsin x)^2}$

197. $f(x) = \frac{\arctan x}{\arcsin x}$

**PROBLEMS ON
RELATED RATES**

198. A 10 foot long pole has one end (B) on the floor and another (A) against a wall. If the bottom of the pole is 8 feet away from the wall, and if it is sliding away from the wall at 7 feet per second, then with what speed is the top (A) going down?



199. A pole 10 feet long rests against a vertical wall. If the bottom of the pole slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the pole and the wall changing when the angle is $\pi/4$ radians?

200. A pole 13 meters long is leaning against a wall. The bottom of the pole is pulled along the ground away from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the pole is 5 m away from the wall?

201. Group Problem.

A television camera is positioned 4000 ft from the base of a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) How fast is the camera's angle of elevation changing at that same moment? (Assume that the television camera points toward the rocket.)

202. Group Problem.

A 2-foot tall dog is walking away from a streetlight which is on a 10-foot pole. At a certain moment, the tip of the dog's shadow is moving away from the streetlight at 5 feet per second. How fast is the dog walking at that moment?

203. An isosceles triangle is changing its shape: the lengths of the two equal sides remain fixed at 2 inch, but the angle $\theta(t)$ between them changes.

Let $A(t)$ be the area of the triangle at time t . If the area increases at a constant rate of $0.5 \text{ inch}^2/\text{sec}$, then how fast is the angle increasing or decreasing when $\theta = 60^\circ$?

204. A point P is moving in the first quadrant of the plane. Its motion is parallel to the x -axis; its distance to the x -axis is always 10 (feet). Its velocity is 3 feet per second to the left. We write θ for the angle between the positive x -axis and the line segment from the origin to P .

(a) Make a drawing of the point P .

(b) Where is the point when $\theta = \pi/3$?

(c) Compute the rate of change of the angle θ at the moment that $\theta = \frac{\pi}{3}$.

205. The point Q is moving on the line $y = x$ with velocity 3 m/sec. Find the rate of change of the following quantities at the moment in which Q is at the point $(1, 1)$:

(a) the distance from Q to the origin,

(b) the distance from Q to the point $R(2, 0)$,

(c) the angle $\angle ORQ$ where R is again the point $R(2, 0)$.

206. A point P is sliding on the parabola with equation $y = x^2$. Its x -coordinate is increasing at a constant rate of 2 feet/minute.

Find the rate of change of the following quantities at the moment that P is at $(3, 9)$:

(a) the distance from P to the origin,

(b) the area of the rectangle whose lower left corner is the origin and whose upper right corner is P ,

- (c) the slope of the tangent to the parabola at P ,
- (d) the angle $\angle OPQ$ where Q is the point $(0, 3)$.

207. Group Problem.

A certain amount of gas is trapped in a cylinder with a piston. The **ideal gas law** from thermodynamics says that if the cylinder is not heated, and if the piston moves slowly, then one has

$$pV = CT$$

where p is the pressure in the gas, V is its volume, T its temperature (in degrees Kelvin) and C is a constant depending on the amount of gas trapped in the cylinder.

(a) If the pressure is 10psi (pounds per square inch), if the volume is 25inch^3 , and if the piston is moving so

that the gas volume is expanding at a rate of 2inch^3 per minute, then what is the rate of change of the pressure?

(b) The ideal gas law turns out to be only approximately true. A more accurate description of gases is given by **van der Waals' equation of state**, which says that

$$\left(p + \frac{a}{V^2}\right)(V - b) = C$$

where a, b, C are constants depending on the temperature and the amount and type of gas in the cylinder.

Suppose that the cylinder contains fictitious gas for which one has $a = 12$ and $b = 3$. Suppose that at some moment the volume of gas is 12in^3 , the pressure is 25psi and suppose the gas is expanding at 2inch^3 per minute. Then how fast is the pressure changing?

Graph Sketching and Max-Min Problems

The signs of the first and second derivatives of a function tell us something about the shape of its graph. In this chapter we learn how to find that information.

1. Tangent and Normal lines to a graph

The slope of the tangent the tangent to the graph of f at the point $(a, f(a))$ is

$$(32) \quad m = f'(a)$$

and hence the equation for the tangent is

$$(33) \quad y = f(a) + f'(a)(x - a).$$

The slope of the normal line to the graph is $-1/m$ and thus one could write the equation for the normal as

$$(34) \quad y = f(a) - \frac{x - a}{f'(a)}.$$

When $f'(a) = 0$ the tangent is horizontal, and hence the normal is vertical. In this case the equation for the normal cannot be written as in (34), but instead one gets the simpler equation

$$y = f(a).$$

Both cases are covered by this form of the equation for the normal

$$(35) \quad x = a + f'(a)(f(a) - y).$$

Both (35) and (34) are formulas that you shouldn't try to remember. It is easier to remember that if the slope of the tangent is $m = f'(a)$, then the slope of the normal is $-1/m$.

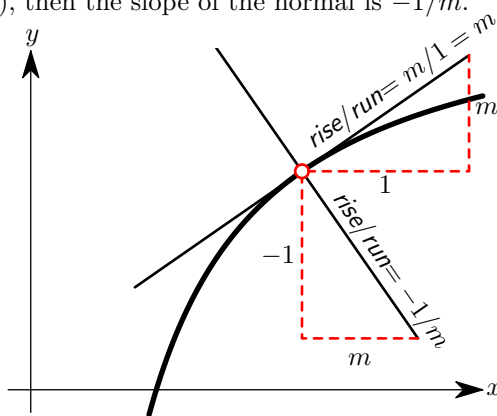


Figure 1. Why “slope of normal = $\frac{-1}{\text{slope of tangent}}$.”

2. The Intermediate Value Theorem

It is said that a function is continuous if you can draw its graph without taking your pencil off the paper. A more precise version of this statement is the *Intermediate Value Theorem*:

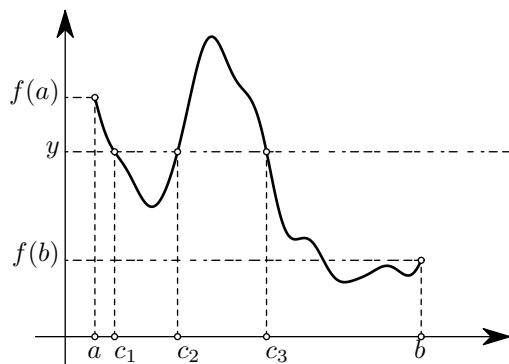


Figure 2. The Intermediate Value Theorem says that a continuous function must attain any given value y between $f(a)$ and $f(b)$ at least once. In this example there are three values of c for which $f(c) = y$ holds.

2.1. Intermediate Value Theorem. If f is a continuous function on an interval $a \leq x \leq b$, and if y is some number between $f(a)$ and $f(b)$, then there is a number c with $a \leq c \leq b$ such that $f(c) = y$.

Here “ y between $f(a)$ and $f(b)$ ” means that $f(a) \leq y \leq f(b)$ if $f(a) \leq f(b)$, and $f(b) \leq y \leq f(a)$ if $f(b) \leq f(a)$.

2.2. Example – Square root of 2. Consider the function $f(x) = x^2$. Since $f(1) < 2$ and $f(2) = 4 > 2$ the intermediate value theorem with $a = 1$, $b = 2$, $y = 2$ tells us that there is a number c between 1 and 2 such that $f(c) = 2$, i.e. for which $c^2 = 2$. So the theorem tells us that the square root of 2 exists.

2.3. Example – The equation $\theta + \sin \theta = \frac{\pi}{2}$. Consider the function $f(x) = x + \sin x$. It is a continuous function at all x , so from $f(0) = 0$ and $f(\pi) = \pi$ it follows that there is a number θ between 0 and π such that $f(\theta) = \pi/2$. In other words, the equation

$$(36) \quad \theta + \sin \theta = \frac{\pi}{2}$$

has a solution θ with $0 \leq \theta \leq \pi/2$. Unlike the previous example, where we knew the solution was $\sqrt{2}$, there is no simple formula for the solution to (36).

2.4. Example – Solving $1/x = 0$. If we apply the intermediate value theorem to the function $f(x) = 1/x$ on the interval $[a, b] = [-1, 1]$, then we see that for any y between $f(a) = f(-1) = -1$ and $f(b) = f(1) = 1$ there is a number c in the interval $[-1, 1]$ such that $1/c = y$. For instance, we could choose $y = 0$ (that’s between -1 and $+1$), and conclude that there is some c with $-1 \leq c \leq 1$ and $1/c = 0$.

But there is no such c , because $1/c$ is never zero! So we have done something wrong, and the mistake we made is that we overlooked that our function $f(x) = 1/x$ is not defined on the **whole** interval $-1 \leq x \leq 1$ because it is not defined at $x = 0$. *The moral: always check the hypotheses of a theorem before you use it!*

3. Exercises

208. Where does the normal to the graph of $y = x^2$ at the point $(1, 1)$ intersect the x -axis?

209. Where does the tangent to the graph of $y = x^2$ at the point (a, a^2) intersect the x -axis?

210. Where does the normal to the graph of $y = x^2$ at the point (a, a^2) intersect the x -axis?

211. Where does the normal to the graph of $y = \sqrt{x}$ at the point $(a, \sqrt{a})$ intersect the x -axis?

212. Does the graph of $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Does the graph of the same function have any vertical tangents?

Does it have vertical normals?

Does it have horizontal normals?

213. At some point $(a, f(a))$ on the graph of $f(x) = -1 + 2x - x^2$ the tangent to this graph goes through the origin. Which point is it?

214. Find equations for the tangent and normal lines

to the curve ... at the point...

(a) $y = 4x/(1+x^2)$	(1, 2)
(b) $y = 8/(4+x^2)$	(2, 1)
(c) $y^2 = 2x + x^2$	(2, 2)
(d) $xy = 3$	(1, 3)

satisfies $f(-1) = -2$ and $f(+1) = +2$, so, by the Intermediate Value Theorem, there should be some value c between -1 and $+1$ such that $f(c) = 0$. **True or False?**

215. Group Problem.

The function

$$f(x) = \frac{x^2 + |x|}{x}$$

216. Find the equation for the tangents to the graph of the Backward Sine at the points $x = 1$, $x = \frac{1}{2}$ and at D (see Figure 2 in §7.3.)

217. Find the equation for the tangent to the graph of the Backward Cosine in a Bow Tie at the point C (see Figure 3 in §9.3)

4. Finding sign changes of a function

The intermediate value theorem implies the following very useful fact.

4.1. Theorem. *If f is continuous function on some interval $a < x < b$, and if $f(x) \neq 0$ for all x in this interval, then $f(x)$ is either positive for all $a < x < b$ or else it is negative for all $a < x < b$.*

PROOF. The theorem says that there can't be two numbers $a < x_1 < x_2 < b$ such that $f(x_1)$ and $f(x_2)$ have opposite signs. If there were two such numbers then the intermediate value theorem would imply that somewhere between x_1 and x_2 there was a c with $f(c) = 0$. But we are assuming that $f(c) \neq 0$ whenever $a < c < b$. $\square$

4.2. Example. Consider

$$f(x) = (x-3)(x-1)^2(2x+1)^3.$$

The zeros of f (i.e. the solutions of $f(x) = 0$) are $-\frac{1}{2}, 1, 3$. These numbers split the real line into four intervals

$$(-\infty, -\frac{1}{2}), \quad (-\frac{1}{2}, 1), \quad (1, 3), \quad (3, \infty).$$

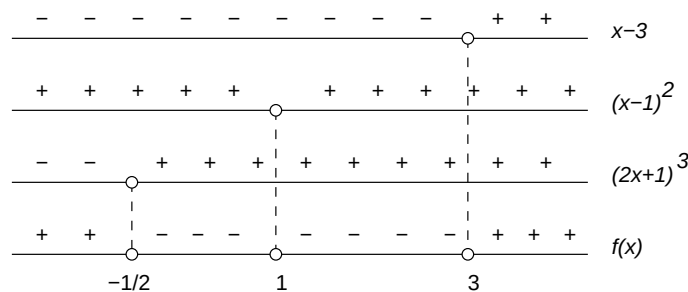
Theorem 4.1 tells us that $f(x)$ cannot change its sign in any of these intervals. For instance, $f(x)$ has the same sign for all x in the first interval $(-\infty, -\frac{1}{2})$. Now we choose a number we like from this interval (e.g. -1) and find the sign of $f(-1)$: $f(-1) = (-4)(-2)^2(-3)^3$ is positive. Therefore $f(x) > 0$ for all x in the interval $(-\infty, -\frac{1}{2})$. In the same we find

$$\begin{aligned} f(-1) &= (-4)(-2)^2(-3)^3 > 0 &\implies f(x) > 0 \text{ for } x < -\frac{1}{2} \\ f(0) &= (-3)(-1)^2(1)^3 < 0 &\implies f(x) < 0 \text{ for } -\frac{1}{2} < x < 1 \\ f(2) &= (-1)(1)^2(5)^3 < 0 &\implies f(x) < 0 \text{ for } 1 < x < 3 \\ f(4) &= (1)(3)^2(9)^3 > 0 &\implies f(x) > 0 \text{ for } x > 3. \end{aligned}$$

If you know all the zeroes of a continuous function, then this method allows you to decide where the function is positive or negative. However, when the given function is factored into easy functions, as in this example, there is a different way of finding the signs of f . For each of the factors $x-3$, $(x-1)^2$ and $(2x+1)^3$ it is easy to determine the sign, for any given x . These signs can only change at a zero of the factor. Thus we have

- $x-3$ is positive for $x > 3$ and negative for $x < 3$;
- $(x-1)^2$ is always positive (except at $x = 1$);
- $(2x+1)^3$ is positive for $x > -\frac{1}{2}$ and negative for $x < -\frac{1}{2}$.

Multiplying these signs we get the same conclusions as above. We can summarize this computation in the following diagram:



5. Increasing and decreasing functions

Here are four very similar definitions – look closely to see how they differ.

- A function is called **increasing** if $a < b$ implies $f(a) < f(b)$ for all numbers a and b in the domain of f .
- A function is called **decreasing** if $a < b$ implies $f(a) > f(b)$ for all numbers a and b in the domain of f .
- The function f is called **non-decreasing** if $a < b$ implies $f(a) \leq f(b)$ for all numbers a and b in the domain of f .
- The function f is called **non-increasing** if $a < b$ implies $f(a) \geq f(b)$ for all numbers a and b in the domain of f .

You can summarize these definitions as follows:

f is ...	if for all a and b one has...
Increasing:	$a < b \implies f(a) < f(b)$
Decreasing:	$a < b \implies f(a) > f(b)$
Non-increasing:	$a < b \implies f(a) \geq f(b)$
Non-decreasing:	$a < b \implies f(a) \leq f(b)$

The sign of the derivative of f tells you if f is increasing or not. More precisely:

5.1. Theorem. *If a function is non-decreasing on an interval $a < x < b$ then $f'(x) \geq 0$ for all x in that interval.*

If a function is non-increasing on an interval $a < x < b$ then $f'(x) \leq 0$ for all x in that interval.

For instance, if f is non-decreasing, then for any given x and any positive Δx one has $f(x + \Delta x) \geq f(x)$ and hence

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0.$$

Now let $\Delta x \searrow 0$ and you find that

$$f'(x) = \lim_{\Delta x \searrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0.$$

What about the converse, i.e. if you know the sign of f' then what can you say about f ? For this we have the following

5.2. Theorem. *Suppose f is a differentiable function on an interval (a, b) .*

If $f'(x) > 0$ for all $a < x < b$, then f is increasing.

If $f'(x) < 0$ for all $a < x < b$, then f is decreasing.

The proof is based on the Mean Value theorem which also finds use in many other situations:

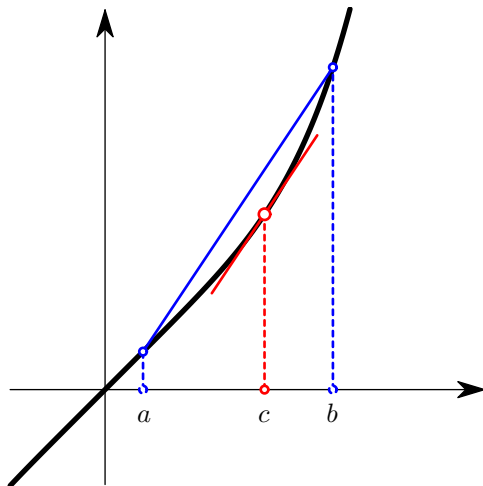


Figure 3. According to the Mean Value Theorem there always is some number c between a and b such that the tangent to the graph of f is parallel to the line segment connecting the two points $(a, f(a))$ and $(b, f(b))$. This is true for any choice of a and b ; c depends on a and b of course.

5.3. The Mean Value Theorem. *If f is a differentiable function on the interval $a \leq x \leq b$, then there is some number c , with $a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

PROOF OF THEOREM 5.2. We show that $f'(x) > 0$ for all x implies that f is increasing. Let $x_1 < x_2$ be two numbers between a and b . Then the Mean Value Theorem implies that there is some c between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

or

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since we know that $f'(c) > 0$ and $x_2 - x_1 > 0$ it follows that $f(x_2) - f(x_1) > 0$, i.e. $f(x_2) > f(x_1)$. □

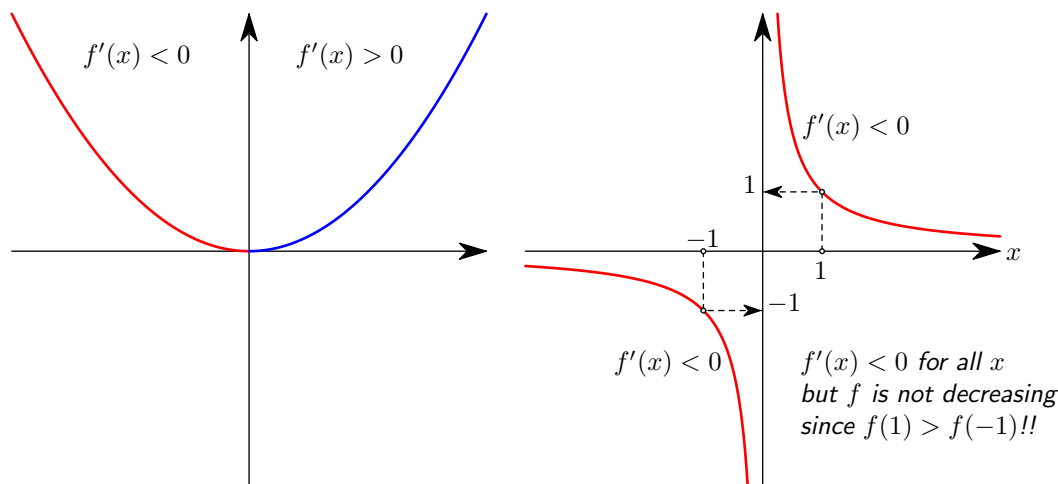
6. Examples

Armed with these theorems we can now split the graph of any function into increasing and decreasing parts simply by computing the derivative $f'(x)$ and finding out where $f'(x) > 0$ and where $f'(x) < 0$ – i.e. we apply the method from the previous section to f' rather than f .

6.1. Example: the parabola $y = x^2$. The familiar graph of $f(x) = x^2$ consists of two parts, one decreasing and one increasing. You can see this from the derivative which is

$$f'(x) = 2x \begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } x < 0. \end{cases}$$

Therefore the function $f(x) = x^2$ is **decreasing for $x < 0$** and **increasing for $x > 0$** .



6.2. Example: the hyperbola $y = 1/x$. The derivative of the function $f(x) = 1/x = x^{-1}$ is

$$f'(x) = -\frac{1}{x^2}$$

which is always negative. You would therefore think that this function is decreasing, or at least non-increasing: if $a < b$ then $1/a \geq 1/b$. But this isn't true if you take $a = -1$ and $b = 1$:

$$a = -1 < 1 = b, \text{ but } \frac{1}{a} = -1 < 1 = \frac{1}{b} !!$$

The problem is that we used theorem 5.2, but if you carefully read that theorem then you see that it applies to functions **that are defined on an interval**. The function in this example, $f(x) = 1/x$, is not defined on the interval $-1 < x < 1$ because it isn't defined at $x = 0$. That's why you can't conclude that the $f(x) = 1/x$ is increasing from $x = -1$ to $x = +1$.

On the other hand, the function is defined and differentiable on the interval $0 < x < \infty$, so theorem 5.2 tells us that $f(x) = 1/x$ is decreasing for $x > 0$. This means, that as long as x is positive, increasing x will decrease $1/x$.

6.3. Graph of a cubic function. Consider the function

$$y = f(x) = x^3 - x.$$

Its derivative is

$$f'(x) = 3x^2 - 1.$$

We try to find out where f' is positive, and where it is negative by factoring $f'(x)$

$$f'(x) = 3\left(x^2 - \frac{1}{3}\right) = 3\left(x + \frac{1}{3}\sqrt{3}\right)\left(x - \frac{1}{3}\sqrt{3}\right)$$

from which you see that

$$f'(x) > 0 \text{ for } x < -\frac{1}{3}\sqrt{3}$$

$$f'(x) < 0 \text{ for } -\frac{1}{3}\sqrt{3} < x < \frac{1}{3}\sqrt{3}$$

$$f'(x) > 0 \text{ for } x > \frac{1}{3}\sqrt{3}$$

Therefore the function f is

$$\text{increasing on } \left(-\infty, -\frac{1}{3}\sqrt{3}\right), \quad \text{decreasing on } \left(-\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right), \quad \text{increasing on } \left(\frac{1}{3}\sqrt{3}, \infty\right).$$

At the two points $x = \pm\frac{1}{3}\sqrt{3}$ one has $f'(x) = 0$ so there the tangent will be horizontal. This leads us to the following picture of the graph of f :

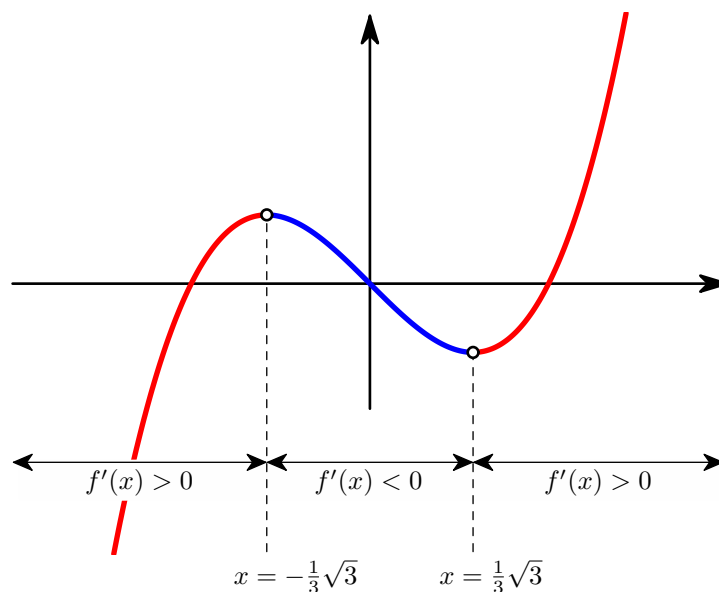


Figure 4. The graph of $f(x) = x^3 - x$.

6.4. A function whose tangent turns up and down infinitely often near the origin. We end with a weird example. Somewhere in the mathematician's zoo of curious functions the following will be on exhibit. Consider the function

$$f(x) = \frac{x}{2} + x^2 \sin \frac{\pi}{x}.$$

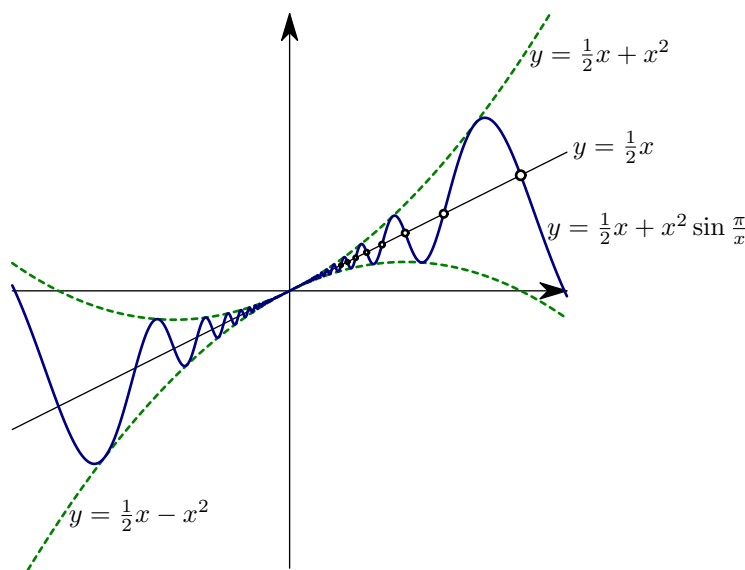


Figure 5. Positive derivative at a point ($x = 0$) does not mean that the function is “increasing near that point.” The slopes at the intersection points alternate between $\frac{1}{2} - \pi$ and $\frac{1}{2} + \pi$.

For $x = 0$ this formula is undefined, and we are free to define $f(0) = 0$. This makes the function continuous at $x = 0$. In fact, this function is differentiable at $x = 0$, with derivative given by

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{1}{2} + x \sin \frac{\pi}{x} = \frac{1}{2}.$$

(To find the limit apply the sandwich theorem to $-|x| \leq x \sin \frac{\pi}{x} \leq |x|$.)

So the slope of the tangent to the graph at the origin is positive ($\frac{1}{2}$), and one would *think* that the function should be increasing near $x = 0$ (i.e. bigger x gives bigger $f(x)$.) The point of this example is that this turns out not to be true.

To explain why not, we must compute the derivative of this function for $x \neq 0$. It is given by

$$f'(x) = \frac{1}{2} - \pi \cos \frac{\pi}{x} + 2x \sin \frac{\pi}{x}.$$

Now consider the sequence of intersection points $P_1, P_2, \dots$ of the graph with the line $y = x/2$. They are

$$P_k(x_k, y_k), \quad x_k = \frac{1}{k}, \quad y_k = f(x_k).$$

For larger and larger k the points P_k tend to the origin (the x coordinate is $\frac{1}{k}$ which goes to 0 as $k \rightarrow \infty$). The slope of the tangent at P_k is given by

$$\begin{aligned} f'(x_k) &= \frac{1}{2} - \pi \cos \frac{\pi}{1/k} + 2 \frac{1}{k} \sin \frac{\pi}{1/k} \\ &= \frac{1}{2} - \pi \underbrace{\cos k\pi}_{=(-1)^k} + \frac{2}{k} \underbrace{\sin k\pi}_{=0} \\ &= \begin{cases} -\frac{1}{2} - \pi \approx -2.64159265358979 \dots & \text{for } k \text{ even} \\ \frac{1}{2} + \pi \approx +3.64159265358979 \dots & \text{for } k \text{ odd} \end{cases} \end{aligned}$$

In other words, along the sequence of points P_k the slope of the tangent flip-flops between $\frac{1}{2} - \pi$ and $\frac{1}{2} + \pi$, i.e. between a positive and a negative number.

In particular, the slope of the tangent at the odd intersection points is negative, and so you would expect the function to be decreasing there. In other words we see that ***even though the derivative at $x = 0$ of this function is positive, there are points on the graph arbitrarily close to the origin where the tangent has negative slope.***

7. Maxima and Minima

A function has a **global maximum** at some a in its domain if $f(x) \leq f(a)$ for all other x in the domain of f . Global maxima are sometimes also called “absolute maxima.”

A function has a **local maximum** at some a in its domain if there is a small $\delta > 0$ such that $f(x) \leq f(a)$ for all x with $a - \delta < x < a + \delta$ which lie in the domain of f .

Every global maximum is a local maximum, but a local maximum doesn't have to be a global maximum.

7.1. Where to find local maxima and minima. Any x value for which $f'(x) = 0$ is called a **stationary point** for the function f .

7.2. Theorem. Suppose f is a differentiable function on some interval $[a, b]$.

Every local maximum or minimum of f is either one of the end points of the interval $[a, b]$, or else it is a stationary point for the function f .

PROOF. Suppose that f has a local maximum at x and suppose that x is not a or b . By assumption the left and right hand limits

$$f'(x) = \lim_{\Delta x \nearrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{and} \quad f'(x) = \lim_{\Delta x \searrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

both exist and they are equal.

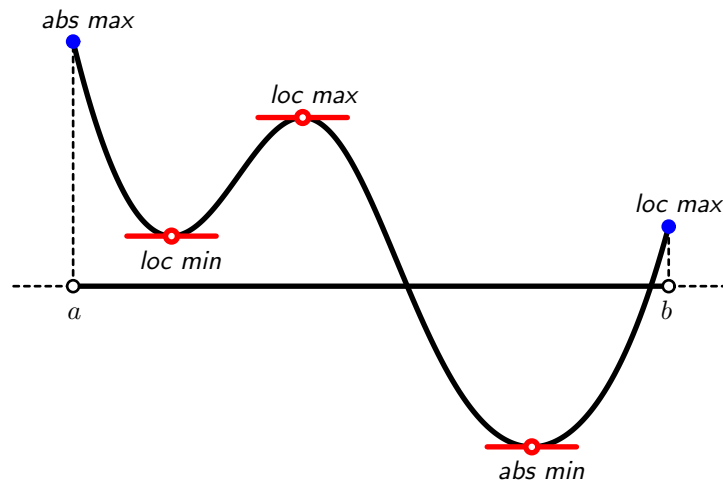


Figure 6. A function defined on an interval $[a, b]$ with one interior absolute minimum, another interior local minimum, an interior local maximum, and two local maxima on the boundary, one of which is in fact an absolute maximum.

Since f has a local maximum at x we have $f(x + \Delta x) - f(x) \leq 0$ if $-\delta < \Delta x < \delta$. In the first limit we also have $\Delta x < 0$, so that

$$\lim_{\Delta x \nearrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \leq 0$$

Hence $f'(x) \leq 0$.

In the second limit we have $\Delta x > 0$, so

$$\lim_{\Delta x \searrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0$$

which implies $f'(x) \geq 0$.

Thus we have shown that $f'(x) \leq 0$ and $f'(x) \geq 0$ at the same time. This can only be true if $f'(x) = 0$. $\square$

7.3. How to tell if a stationary point is a maximum, a minimum, or neither. If $f'(c) = 0$ then c is a stationary point (by definition), and it might be local maximum or a local minimum. You can tell what kind of stationary point c is by looking at the signs of $f'(x)$ for x near c .

7.4. Theorem. *If in some small interval $(c - \delta, c + \delta)$ you have $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ then f has a local minimum at $x = c$.*

If in some small interval $(c - \delta, c + \delta)$ you have $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ then f has a local maximum at $x = c$. The reason is simple: if f increases to the left of c and decreases to the right of c then it has a maximum at c . More precisely:

if $f'(x) > 0$ for x between $c - \delta$ and c , then f is increasing for $c - \delta < x < c$ and therefore $f(x) < f(c)$ for x between $c - \delta$ and c .

If in addition $f'(x) < 0$ for $x > c$ then f is decreasing for x between c and $c + \delta$, so that $f(x) < f(c)$ for those x .

Combine these two facts and you get $f(x) \leq f(c)$ for $c - \delta < x < c + \delta$.

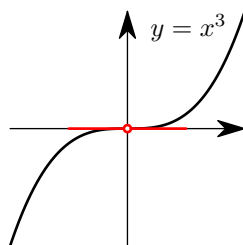
7.5. Example – local maxima and minima of $f(x) = x^3 - x$. In §6.3 we had found that the function $f(x) = x^3 - x$ is decreasing when $-\infty < x < -\frac{1}{3}\sqrt{3}$, and also when $\frac{1}{3}\sqrt{3} < x < \infty$, while it is increasing when $-\frac{1}{3}\sqrt{3} < x < \frac{1}{3}\sqrt{3}$. It follows that the function has a local minimum at $x = -\frac{1}{3}\sqrt{3}$, and a local maximum at $x = \frac{1}{3}\sqrt{3}$.

Neither the local maximum nor the local minimum are global max or min since

$$\lim_{x \rightarrow -\infty} f(x) = +\infty \text{ and } \lim_{x \rightarrow \infty} f(x) = -\infty.$$

7.6. A stationary point that is neither a maximum nor a minimum. If you look for stationary points of the function $f(x) = x^3$ you find that there's only one, namely $x = 0$. The derivative $f'(x) = 3x^2$ does not change sign at $x = 0$, so the test in Theorem 7.4 does not tell us anything.

And in fact, $x = 0$ is neither a local maximum nor a local minimum since $f(x) < f(0)$ for $x < 0$ and $f(x) > 0$ for $x > 0$.



8. Must there always be a maximum?

Theorem 7.2 is very useful since it tells you how to find (local) maxima and minima. The following theorem is also useful, but in a different way. It doesn't say how to find maxima or minima, but it tells you that they do exist, and hence that you are not wasting your time trying to find a maximum or minimum.

8.1. Theorem. *Let f be continuous function defined on the **closed** interval $a \leq x \leq b$. Then f attains its maximum and also its minimum somewhere in this interval. In other words there exist real numbers c and d such that*

$$f(c) \leq f(x) \leq f(d)$$

whenever $a \leq x \leq b$. The proof of this theorem requires a more careful definition of the real numbers than we have given in Chapter 1, and we will take the theorem for granted.

9. Examples – functions with and without maxima or minima

In the following three example we explore what can happen if some of the hypotheses in Theorem 8.1 are not met.

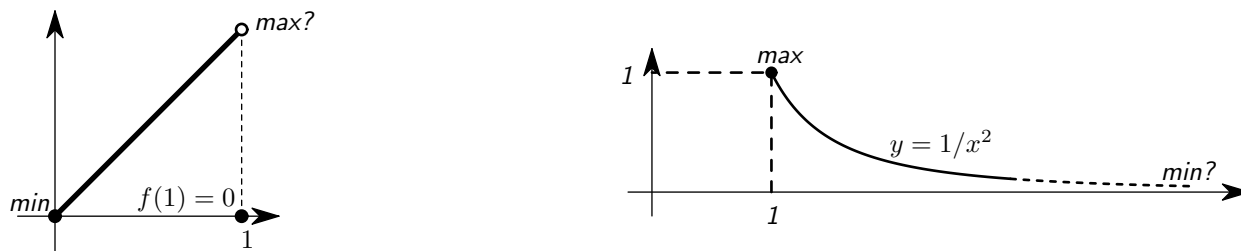


Figure 7. The function on the left has no maximum, and the one on the right has no minimum.

9.1. Question: Does the function

$$f(x) = \begin{cases} x & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x = 1. \end{cases}$$

have a maximum on the interval $0 \leq x \leq 1$?

Answer: No. What would the maximal value be? Since

$$\lim_{x \nearrow 1} f(x) = \lim_{x \nearrow 1} x = 1$$

The maximal value cannot be less than 1. On the other hand the function is never larger than 1. So if there were a number a in the interval $[0, 1]$ such that $f(a)$ was the maximal value of f , then we would have $f(a) = 1$. If you now search the interval for numbers a with $f(a) = 1$, then you notice that such an a does not exist. Conclusion: this function does **not** attain its maximum on the interval $[0, 1]$.

What about Theorem 8.1? That theorem only applies to continuous functions, and the function f in this example is not continuous at $x = 1$. For at $x = 1$ one has

$$f(1) = 0 \neq 1 = \lim_{x \nearrow 1} f(x).$$

So all it takes for the Theorem to fail is that the function f be discontinuous at just one point.

9.2. Question: Does the function

$$f(x) = \frac{1}{x^2}, \quad 1 \leq x < \infty$$

have a maximum or minimum?

Answer: The function has a maximum at $x = 1$, but it has no minimum.

Concerning the maximum: if $x \geq 1$ then $f(x) = 1/x^2 \leq 1$, while $f(1) = 1$. Hence $f(x) \leq f(1)$ for all x in the interval $[1, \infty)$ and that is why f attains its maximum at $x = 1$.

If we look for a minimal value of f then we note that $f(x) \geq 0$ for all x in the interval $[1, \infty)$, and also that

$$\lim_{x \rightarrow \infty} f(x) = 0,$$

so that **if** f attains a minimum at some a with $1 \leq a < \infty$, then the minimal value $f(a)$ must be zero. However, the equation $f(a) = 0$ has no solution – f does not attain its minimum.

Why does Theorem 8.1 not apply? In this example the function f is continuous on the whole interval $[1, \infty)$, but this interval is not a closed interval, i.e. it is not of the form $[a, b]$ (it does not include its endpoints).

10. General method for sketching the graph of a function

Given a differentiable function f defined on some interval $a \leq x \leq b$, you can find the increasing and decreasing parts of the graph, as well as all the local maxima and minima by following this procedure:

- (1) find all solutions of $f'(x) = 0$ in the interval $[a, b]$: these are called the *critical* or *stationary* points for f .
- (2) find the sign of $f'(x)$ at all other points
- (3) each stationary point at which $f'(x)$ actually changes sign is a local maximum or local minimum. Compute the function value $f(x)$ at each stationary point.
- (4) compute the function values at the endpoints of the interval, i.e. compute $f(a)$ and $f(b)$.
- (5) the absolute maximum is attained at the stationary point or the boundary point with the highest function value; the absolute minimum occurs at the boundary or stationary point with the smallest function value.

If the interval is unbounded, i.e. if the function is defined for $-\infty < x < \infty$ then you can't compute the values $f(a)$ and $f(b)$, but instead you should compute $\lim_{x \rightarrow \pm\infty} f(x)$.

10.1. Example – the graph of a rational function. Let's “sketch the graph” of the function

$$f(x) = \frac{x(1-x)}{1+x^2}.$$

By looking at the signs of numerator and denominator we see that

$$\begin{aligned} f(x) &> 0 \text{ for } 0 < x < 1 \\ f(x) &< 0 \text{ for } x < 0 \text{ and also for } x > 1. \end{aligned}$$

We compute the derivative of f

$$f'(x) = \frac{1-2x-x^2}{(1+x^2)^2}.$$

Hence $f'(x) = 0$ holds if and only if

$$1 - 2x - x^2 = 0$$

and the solutions to this quadratic equation are $-1 \pm \sqrt{2}$. These two roots will appear several times and it will shorten our formulas if we abbreviate

$$A = -1 - \sqrt{2} \text{ and } B = -1 + \sqrt{2}.$$

To see if the derivative changes sign we factor the numerator and denominator. The denominator is always positive, and the numerator is

$$-x^2 - 2x + 1 = -(x^2 + 2x - 1) = -(x - A)(x - B).$$

Therefore

$$f'(x) \begin{cases} < 0 & \text{for } x < A \\ > 0 & \text{for } A < x < B \\ < 0 & \text{for } x > B \end{cases}$$

It follows that f is decreasing on the interval $(-\infty, A)$, increasing on the interval (A, B) and decreasing again on the interval (B, ∞) . Therefore

A is a local minimum, and B is a local maximum.

Are these global maxima and minima?

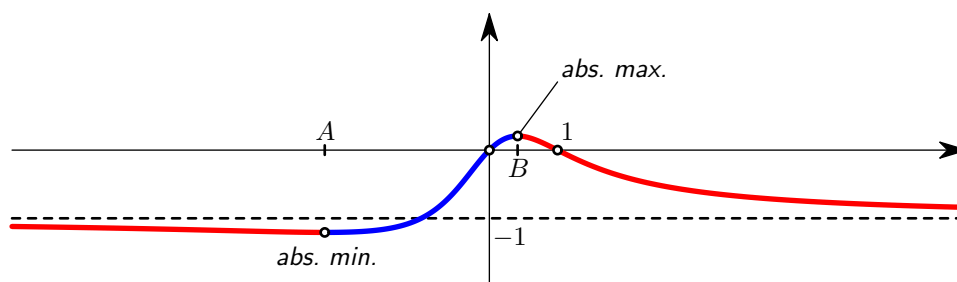


Figure 8. The graph of $f(x) = (x - x^2)/(1 + x^2)$

Since we are dealing with an unbounded interval we must compute the limits of $f(x)$ as $x \rightarrow \pm\infty$. You find

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = -1.$$

Since f is decreasing between $-\infty$ and A , it follows that

$$f(A) \leq f(x) < -1 \text{ for } -\infty < x \leq A.$$

Similarly, f is decreasing from B to $+\infty$, so

$$-1 < f(x) \leq f(-1 + \sqrt{2}) \text{ for } B < x < \infty.$$



Figure 9. If a graph is convex then all chords lie above the graph. If it is not convex then some chords will cross the graph or lie below it.

Between the two stationary points the function is increasing, so

$$f(-1 - \sqrt{2}) \leq f(x) \leq f(B) \text{ for } A \leq x \leq B.$$

From this it follows that $f(x)$ is the smallest it can be when $x = A = -1 - \sqrt{2}$ and at its largest when $x = B = -1 + \sqrt{2}$: the local maximum and minimum which we found are in fact a global maximum and minimum.

11. Convexity, Concavity and the Second Derivative

By definition, a function f is **convex** on some interval $a < x < b$ if the line segment connecting any pair of points on the graph lies *above* the piece of the graph between those two points.

The function is called **concave** if the line segment connecting any pair of points on the graph lies *below* the piece of the graph between those two points.

A point on the graph of f where $f''(x)$ changes sign is called an **inflection point**.

Instead of “convex” and “concave” one often says “curved upwards” or “curved downwards.”

You can use the second derivative to tell if a function is concave or convex.

11.1. Theorem. A function f is convex on some interval $a < x < b$ if and only if $f''(x) \geq 0$ for all x on that interval.

11.2. Theorem. A function f is concave on some interval $a < x < b$ if and only if the derivative $f'(x)$ is a nondecreasing function on that interval.

A proof using the Mean Value Theorem will be given in class.

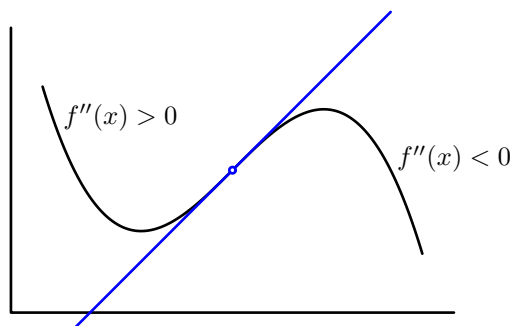


Figure 10. At an inflection point the tangent crosses the graph.

11.3. Example – the cubic function $f(x) = x^3 - x$. The second derivative of the function $f(x) = x^3 - x$ is

$$f''(x) = 6x$$

which is positive for $x > 0$ and negative for $x < 0$. Hence, in the graph in §6.3, the origin is an inflection point, and the piece of the graph where $x > 0$ is convex, while the piece where $x < 0$ is concave.

11.4. The second derivative test. In §7.3 we saw how you can tell if a stationary point is a local maximum or minimum by looking at the sign changes of $f'(x)$. There is another way of distinguishing between local maxima and minima which involves computing the second derivative.

11.5. Theorem. *If c is a stationary point for a function f , and if $f''(c) < 0$ then f has a local maximum at $x = c$.*

If $f''(c) > 0$ then f has a local minimum at c . The theorem doesn't say what happens when $f''(c) = 0$. In that case you must go back to checking the signs of the first derivative near the stationary point.

The basic reason why this theorem is true is that if c is a stationary point with $f''(c) > 0$ then " $f'(x)$ is increasing near $x = c$ " and hence $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$. So the function f is decreasing for $x < c$ and increasing for $x > c$, and therefore it reaches a local minimum at $x = c$.

11.6. Example – that cubic function again. Consider the function $f(x) = x^3 - x$ from §6.3 and §11.3. We had found that this function has two stationary points, namely at $x = \pm\frac{1}{3}\sqrt{3}$. By looking at the sign of $f'(x) = 3x^2 - 1$ we concluded that $-\frac{1}{3}\sqrt{3}$ is a local maximum while $+\frac{1}{3}\sqrt{3}$ is a local minimum. Instead of looking at $f'(x)$ we could also have computed $f''(x)$ at $x = \pm\frac{1}{3}\sqrt{3}$ and applied the second derivative test. Here is how it goes:

Since $f''(x) = 6x$ we have

$$f''\left(-\frac{1}{3}\sqrt{3}\right) = -2\sqrt{3} < 0 \text{ and } f''\left(\frac{1}{3}\sqrt{3}\right) = 2\sqrt{3} > 0.$$

Therefore f has a local maximum at $-\frac{1}{3}\sqrt{3}$ and a local minimum at $\frac{1}{3}\sqrt{3}$.

11.7. When the second derivative test doesn't work. Usually the second derivative test will work, but sometimes a stationary point c has $f''(c) = 0$. In this case the second derivative test gives no information at all. The figure below shows you the graphs of three functions, all three of which have a stationary point at $x = 0$. In all three cases the second derivative vanishes at $x = 0$ so the second derivative test says nothing. As you can see, the stationary point can be a local maximum, a local minimum, or neither.

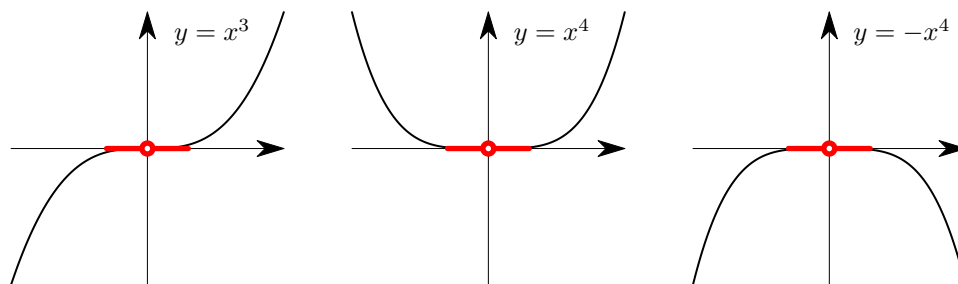


Figure 11. Three functions for which the second derivative test doesn't work.

12. Proofs of some of the theorems

12.1. Proof of the Mean Value Theorem. Let m be the slope of the chord connecting the points $(a, f(a))$ and $(b, f(b))$, i.e.

$$m = \frac{f(b) - f(a)}{b - a},$$

and consider the function

$$g(x) = f(x) - f(a) - m(x - a).$$

This function is continuous (since f is continuous), and g attains its maximum and minimum at two numbers $c_{\min}$ and $c_{\max}$.

There are now two possibilities: either at least one of $c_{\min}$ or $c_{\max}$ is an interior point, or else both $c_{\min}$ and $c_{\max}$ are endpoints of the interval $a \leq x \leq b$.

Consider the first case: one of these two numbers is an interior point, i.e. if $a < c_{\min} < b$ or $a < c_{\max} < b$, then the derivative of g must vanish at $c_{\min}$ or $c_{\max}$. If one has $g'(c_{\min}) = 0$, then one has

$$0 = g'(c_{\min}) = f'(c_{\min}) - m, \text{ i.e. } m = f'(c_{\min}).$$

The definition of m implies that one gets

$$f'(c_{\min}) = \frac{f(b) - f(a)}{b - a}.$$

If $g'(c_{\max}) = 0$ then one gets $m = f'(c_{\max})$ and hence

$$f'(c_{\max}) = \frac{f(b) - f(a)}{b - a}.$$

We are left with the remaining case, in which both $c_{\min}$ and $c_{\max}$ are end points. To deal with this case note that at the endpoints one has

$$g(a) = 0 \text{ and } g(b) = 0.$$

Thus the maximal and minimal values of g are both zero! This means that $g(x) = 0$ for all x , and thus that $g'(x) = 0$ for all x . Therefore we get $f'(x) = m$ for **all** x , and not just for some c .

12.2. Proof of Theorem 5.1. If f is a non-increasing function and if it is differentiable at some interior point a , then we must show that $f'(a) \geq 0$.

Since f is non-decreasing, one has $f(x) \geq f(a)$ for all $x > a$. Hence one also has

$$\frac{f(x) - f(a)}{x - a} \geq 0$$

for all $x > a$. Let $x \searrow a$, and you get

$$f'(a) = \lim_{x \searrow a} \frac{f(x) - f(a)}{x - a} \geq 0.$$

12.3. Proof of Theorem 5.2. Suppose f is a differentiable function on an interval $a < x < b$, and suppose that $f'(x) \geq 0$ on that interval. We must show that f is non-decreasing on that interval, i.e. we have to show that if $x_1 < x_2$ are two numbers in the interval (a, b) , then $f(x_1) \leq f(x_2)$. To prove this we use the Mean Value Theorem: given x_1 and x_2 the Mean Value Theorem hands us a number c with $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

We don't know where c is exactly, but it doesn't matter because we do know that wherever c is we have $f'(c) \geq 0$. Hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0.$$

Multiply with $x_2 - x_1$ (which we are allowed to do since $x_2 > x_1$ so $x_2 - x_1 > 0$) and you get

$$f(x_2) - f(x_1) \geq 0,$$

as claimed.

13. Exercises

218. What does the Intermediate Value Theorem say?

219. What does the Mean Value Theorem say?

220. Group Problem.

If $f(a) = 0$ and $f(b) = 0$ then there is a c between a and b such that $f'(c) = 0$. Show that this follows from the Mean Value Theorem. (Help! A proof! Relax: this one is not difficult. Make a drawing of the situation, then read the Mean Value Theorem again.)

221. What is a stationary point?

222. Group Problem.

How can you tell if a local maximum is a global maximum?

223. Group Problem.

If $f''(a) = 0$ then the graph of f has an inflection point at $x = a$. **True or False?**

224. What is an inflection point?

225. Give an example of a function for which $f'(0) = 0$ even though the function f has neither a local maximum or a local minimum at $x = 0$.

226. Group Problem.

Draw four graphs of functions, one for each of the following four combinations

$$\begin{array}{ll} f' > 0 \text{ and } f'' > 0 & f' > 0 \text{ and } f'' < 0 \\ f' < 0 \text{ and } f'' > 0 & f' < 0 \text{ and } f'' < 0 \end{array}$$

227. Group Problem.

Which of the following combinations are possible:

$$\begin{array}{l} f'(x) > 0 \text{ and } f''(x) = 0 \text{ for all } x \\ f'(x) = 0 \text{ and } f''(x) > 0 \text{ for all } x \end{array}$$

Sketch the graph of the following functions. You should

- (1) find where f , f' and f'' are positive or negative
- (2) find all stationary points
- (3) decide which stationary points are local maxima or minima
- (4) decide which local max/minima are in fact global max/minima
- (5) find all inflection points
- (6) find "horizontal asymptotes," i.e. compute the limits $\lim_{x \rightarrow \pm\infty} f(x)$ when appropriate.

228. $y = x^3 + 2x^2$

229. $y = x^3 - 4x^2$

230. $y = x^4 + 27x$

231. $y = x^4 - 27x$

232. $y = x^4 + 2x^2 - 3$

233. $y = x^4 - 5x^2 + 4$

234. $y = x^5 + 16x$

235. $y = x^5 - 16x$

236. $y = \frac{x}{x+1}$

237. $y = \frac{x}{1+x^2}$

238. $y = \frac{x^2}{1+x^2}$

239. $y = \frac{1+x^2}{1+x}$

240. $y = x + \frac{1}{x}$

241. $y = x - \frac{1}{x}$

242. $y = x^3 + 2x^2 + x$

243. $y = x^3 + 2x^2 - x$

244. $y = x^4 - x^3 - x$

245. $y = x^4 - 2x^3 + 2x$

246. $y = \sqrt{1+x^2}$

247. $y = \sqrt{1-x^2}$

248. $y = \sqrt[4]{1+x^2}$

249. $y = \frac{1}{1+x^4}$

The following functions are periodic, i.e. they satisfy $f(x+L) = f(x)$ for all x , where the constant L is called the period of the function. The graph of a periodic function repeats itself indefinitely to the left and to the right. It therefore has infinitely many (local) minima and maxima, and infinitely many inflections points. Sketch the graphs of the following functions as in the previous problem, but only list those "interesting points" that lie in the interval $0 \leq x < 2\pi$.

250. $y = \sin x$

251. $y = \sin x + \cos x$

252. $y = \sin x + \sin^2 x$

253. $y = 2 \sin x + \sin^2 x$

254. $y = 4 \sin x + \sin^2 x$

255. $y = 2 \cos x + \cos^2 x$

256. $y = \frac{4}{2 + \sin x}$

257. $y = (2 + \sin x)^2$

Find the domain and sketch the graphs of each of the following functions

258. $y = \arcsin x$

259. $y = \arctan x$

260. $y = 2 \arctan x - x$

261. $y = \arctan(x^2)$

262. $y = 3 \arcsin(x) - 5x$

263. $y = 6 \arcsin(x) - 10x^2$

264. In the following two problems it is not possible to solve the equation $f'(x) = 0$, but you can tell something from the second derivative.

(a) Show that the function $f(x) = x \arctan x$ is convex. Then sketch the graph of f .

(b) Show that the function $g(x) = x \arcsin x$ is convex. Then sketch the graph of g .

For each of the following functions use the derivative to decide if they are increasing, decreasing or neither on the indicated intervals

265. $f(x) = \frac{x}{1+x^2} \quad 10 < x < \infty$

266. $f(x) = \frac{2+x^2}{x^3-x} \quad 1 < x < \infty$

267. $f(x) = \frac{2+x^2}{x^3-x} \quad 0 < x < 1$

268. $f(x) = \frac{2+x^2}{x^3-x} \quad 0 < x < \infty$

14. Optimization Problems

Often a problem can be phrased as

For which value of x in the interval $a \leq x \leq b$ is $f(x)$ the largest?

In other words you are given a function f on an interval $[a, b]$ and you must find all global maxima of f on this interval.

If the function is continuous then according to theorem 8.1 there always is at least one x in the interval $[a, b]$ which maximizes $f(x)$.

If f is differentiable then we know what to do: any local maximum is either a stationary point or one of the end points a and b . Therefore you can find the global maxima by following this recipe:

- (1) Find all stationary points of f ;
- (2) Compute $f(x)$ at each stationary point you found in step (1);
- (3) Compute $f(a)$ and $f(b)$;
- (4) The global maxima are those stationary- or endpoints from steps (2) and (3) which have the largest function value.

Usually there is only one global maximum, but sometimes there can be more.

If you have to *minimize* rather than *maximize* a function, then you must look for global minima. The same recipe works (of course you should look for the smallest function value instead of the largest in step 4.)

The difficulty in optimization problems frequently lies not with the calculus part, but rather with setting up the problem. Choosing which quantity to call x and finding the function f is half the job.

14.1. Example – The rectangle with largest area and given perimeter. Which rectangle has the largest area, among all those rectangles for which the total length of the sides is 1?

Solution: If the sides of the rectangle have lengths x and y , then the total length of the sides is

$$L = x + x + y + y = 2(x + y)$$

and the area of the rectangle is

$$A = xy.$$

So are asked to find the largest possible value of $A = xy$ provided $2(x + y) = 1$. The lengths of the sides can also not be negative, so x and y must satisfy $x \geq 0$, $y \geq 0$.

We now want to turn this problem into a question of the form “maximize a function over some interval.” The quantity which we are asked to maximize is A , but it depends on two variables x and y instead of just one variable. However, the variables x and y are not independent since we are only allowed to consider rectangles with $L = 1$. From this equation we get

$$L = 1 \implies y = \frac{1}{2} - x.$$

Hence we must find the maximum of the quantity

$$A = xy = x\left(\frac{1}{2} - x\right)$$

The values of x which we are allowed to consider are only limited by the requirements $x \geq 0$ and $y \geq 0$, i.e. $x \leq \frac{1}{2}$. So we end up with this problem:

Find the maximum of the function $f(x) = x(\frac{1}{2} - x)$ on the interval $0 \leq x \leq \frac{1}{2}$.

Before we start computing anything we note that the function f is a polynomial so that it is differentiable, and hence continuous, and also that the interval $0 \leq x \leq \frac{1}{2}$ is closed. Therefore the theory guarantees that there is a maximum and our recipe will show us where it is.

The derivative is given by

$$f'(x) = \frac{1}{2} - 2x,$$

and hence the only stationary point is $x = \frac{1}{4}$. The function value at this point is

$$f\left(\frac{1}{4}\right) = \frac{1}{4}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{16}.$$

At the endpoints one has $x = 0$ or $x = \frac{1}{2}$, which corresponds to a rectangle one of whose sides has length zero. The area of such rectangles is zero, and so this is not the maximal value we are looking for.

We conclude that the largest area is attained by the rectangle whose sides have lengths

$$x = \frac{1}{4}, \text{ and } y = \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

i.e. by a square with sides $\frac{1}{4}$.

15. Exercises

269. By definition, the perimeter of a rectangle is the sum of the lengths of its four sides. Which rectangle, of all those whose perimeter is 1, has the smallest area? Which one has the largest area?

270. Which rectangle of area 100in^2 minimizes its height plus two times its length?

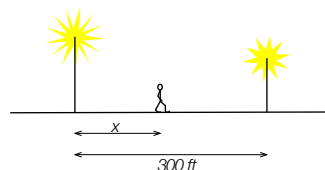
271. You have 1 yard of string from which you make a circular wedge with radius R and opening angle θ . Which choice of θ and R will give you the wedge with the largest area? Which choice leads to the smallest area?

[A circular wedge is the figure consisting of two radii of a circle and the arc connecting them. So the yard of string is used to form the two radii and the arc.]

272. Group Problem.

(The lamp post problem)

In a street two lamp posts are 300 feet apart. The light intensity at a distance d from the first lamp post is $1000/d^2$, the light intensity at distance d from the second (weaker) lamp post is $125/d^2$ (in both cases the light intensity is inversely proportional to the square of the distance to the light source).



The *combined light intensity* is the sum of the two light intensities coming from both lamp posts.

(a) If you are in between the lamp posts, at distance x feet from the stronger light, then give a formula for the combined light intensity coming from both lamp posts as a function of x .

(b) What is the darkest spot between the two lights, i.e. where is the combined light intensity the smallest?

273. (a) You have a sheet of metal with area 100 in^2 from which you are to make a cylindrical soup can. If r is the radius of the can and h its height, then which h and r will give you the can with the largest volume?

(b) If instead of making a plain cylinder you replaced the flat top and bottom of the cylinder with two spherical

caps, then (using the same 100in^2 of sheet metal), then which choice of radius and height of the cylinder give you the container with the largest volume?

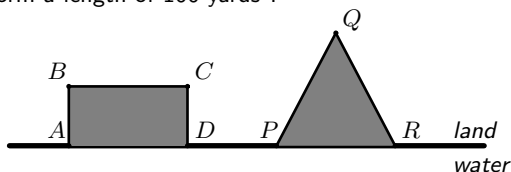
(c) Suppose you only replace the top of the cylinder with a spherical cap, and leave the bottom flat, then which choice of height and radius of the cylinder result in the largest volume?

274. A triangle has one vertex at the origin $O(0,0)$, another at the point $A(2a,0)$ and the third at $(a, a/(1+a^3))$. What are the largest and smallest areas this triangle can have if $0 \leq a < \infty$?

275. Group Problem.

Queen Dido's problem

According to tradition Dido was the founder and first Queen of Carthage. When she arrived on the north coast of Africa (~800BC) the locals allowed her to take as much land as could be enclosed with the hide of one ox. She cut the hide into thin strips and put these together to form a length of 100 yards¹.



(a) If Dido wanted a rectangular region, then how wide should she choose it to enclose as much area as possible (the coastal edge of the boundary doesn't count, so in this problem the length $AB + BC + CD$ is 100 yards.)

(b) If Dido chose a region in the shape of an isosceles triangle PQR , then how wide should she make it to maximize its area (again, don't include the coast in the perimeter: $PQ + QR$ is 100 yards long, and $PQ = QR$.)

276. The product of two numbers x, y is 16. We know $x \geq 1$ and $y \geq 1$. What is the greatest possible sum of the two numbers?

277. What are the smallest and largest values that $(\sin x)(\sin y)$ can have if $x + y = \pi$ and if x and y are both nonnegative?

278. What are the smallest and largest values that $(\cos x)(\cos y)$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

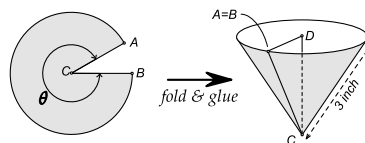
279. (a) What are the smallest and largest values that $\tan x + \tan y$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

(b) What are the smallest and largest values that $\tan x + 2 \tan y$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

280. The cost per hour of fuel to run a locomotive is $v^2/25$ dollars, where v is speed (in miles per hour), and other costs are \$100 per hour regardless of speed. What is the speed that minimizes cost per mile?

281. Group Problem.

Josh is in need of coffee. He has a circular filter with 3 inch radius. He cuts out a wedge and glues the two edges AC and BC together to make a conical filter to hold the ground coffee. The volume V of the coffee cone depends the angle θ of the piece of filter paper Josh made.



(a) Find the volume in terms of the angle θ . (Hint: how long is the circular arc AB on the left? How long is the circular top of the cone on the right? If you know that you can find the radius $AD = BD$ of the top of the cone, and also the height CD of the cone.)

(b) Which angle θ maximizes the volume V ?

¹I made that number up. For the rest start at <http://en.wikipedia.org/wiki/Dido>

Exponentials and Logarithms (naturally)

In this chapter we first recall some facts about exponentials (x^y with $x > 0$ and y arbitrary): they should be familiar from algebra, or “precalculus.” What is new is perhaps the definition of x^y when y is not a fraction: e.g., $2^{3/4}$ is the 4th root of the third power of 2 ($\sqrt[4]{2^3}$), but what is $2^{\sqrt{2}}$?

Then we ask “what is the derivative of $f(x) = a^x$?” The answer leads us to the famous number $e \approx 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,352\,662\,497\,757\,247\,093\,699\,95 \dots$.

Finally, we compute the derivative of $f(x) = \log_a x$, and we look at things that “grow exponentially.”

1. Exponents

Here we go over the definition of x^y when x and y are arbitrary real numbers, with $x > 0$.

For any real number x and any positive integer $n = 1, 2, 3, \dots$ one defines

$$x^n = \overbrace{x \cdot x \cdot \dots \cdot x}^{n \text{ times}}$$

and, if $x \neq 0$,

$$x^{-n} = \frac{1}{x^n}.$$

One defines $x^0 = 1$ for any $x \neq 0$.

To define $x^{p/q}$ for a general fraction $\frac{p}{q}$ one must assume that the number x is positive. One then defines

$$(37) \quad x^{p/q} = \sqrt[q]{x^p}.$$

This does not tell us how to define x^a if the exponent a is not a fraction. One can define x^a for irrational numbers a by taking limits. For example, to define $2^{\sqrt{2}}$, we look at the sequence of numbers you get by truncating the decimal expansion of $\sqrt{2}$, i.e.

$$a_1 = 1, \quad a_2 = 1.4 = \frac{14}{10}, \quad a_3 = 1.41 = \frac{141}{100}, \quad a_4 = 1.414 = \frac{1414}{1000}, \quad \dots$$

Each a_n is a fraction, so that we know what 2^{a_n} is, e.g. $2^{a_4} = \sqrt[1000]{2^{1414}}$. Our definition of $2^{\sqrt{2}}$ then is

$$2^{\sqrt{2}} = \lim_{n \rightarrow \infty} 2^{a_n},$$

i.e. we define $2^{\sqrt{2}}$ as the limit of the sequence of numbers

$$2, \sqrt[10]{2^{14}}, \sqrt[100]{2^{141}}, \sqrt[1000]{2^{1414}}, \dots$$

(See table 1.)

Here one ought to prove that this limit exists, and that its value does not depend on the particular choice of numbers a_n tending to a . We will not go into these details in this course.

It is shown in precalculus texts that the exponential functions satisfy the following properties:

$$(38) \quad \boxed{x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad (x^a)^b = x^{ab}}$$

provided a and b are fractions. One can show that these properties still hold if a and b are real numbers (not necessarily fractions.) Again, we won’t go through the proofs here.

x	2^x
1.0000000000	2.000000000000
1.4000000000	2.639015821546
1.4100000000	2.657371628193
1.4140000000	2.664749650184
1.4142000000	2.665119088532
1.4142100000	2.665137561794
1.4142130000	2.665143103798
1.4142135000	2.665144027466
$\vdots$	$\vdots$

Table 1. Approximating $2^{\sqrt{2}}$. Note that as x gets closer to $\sqrt{2}$ the quantity 2^x appears to converge to some number. This limit is our definition of $2^{\sqrt{2}}$.

Now instead of considering x^a as a function of x we can pick a positive number a and consider the function $f(x) = a^x$. This function is defined for all real numbers x (as long as the base a is positive.).

1.1. The trouble with powers of negative numbers. The cube root of a negative number is well defined. For instance $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$. In view of the definition (37) of $x^{p/q}$ we can write this as

$$(-8)^{1/3} = \sqrt[3]{(-8)^1} = \sqrt[3]{-8} = -2.$$

But there is a problem: since $\frac{2}{6} = \frac{1}{3}$ you would think that $(-8)^{2/6} = (-8)^{1/3}$. However our definition (37) tells us that

$$(-8)^{2/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{+64} = +2.$$

Another example:

$$(-4)^{1/2} = \sqrt{-4} \text{ is not defined}$$

but, even though $\frac{1}{2} = \frac{2}{4}$,

$$(-4)^{2/4} = \sqrt[4]{(-4)^2} = \sqrt[4]{+16} = 2 \text{ is defined.}$$

There are two ways out of this mess:

- (1) avoid taking fractional powers of negative numbers
- (2) when you compute $x^{p/q}$ first simplify the fraction by removing common divisors of p and q .

The safest is just not to take fractional powers of negative numbers.

Given that fractional powers of negative numbers cause all these headaches it is not surprising that we didn't try to define x^a for negative x if a is irrational. For example, $(-8)^\pi$ is not defined¹.

2. Logarithms

Briefly, $y = \log_a x$ is the inverse function to $y = a^x$. This means that, by definition,

$$y = \log_a x \iff x = a^y.$$

In other words, $\log_a x$ is the answer to the question "for which number y does one have $x = a^y$?" The number $\log_a x$ is called **the logarithm with base a of x** . In this definition both a and x must be positive.

For instance,

$$2^3 = 8, \quad 2^{1/2} = \sqrt{2}, \quad 2^{-1} = \frac{1}{2}$$

so

$$\log_2 8 = 3, \quad \log_2(\sqrt{2}) = \frac{1}{2}, \quad \log_2 \frac{1}{2} = -1.$$

¹There is a definition of $(-8)^\pi$ which uses complex numbers. You will see this next semester if you take math 222.

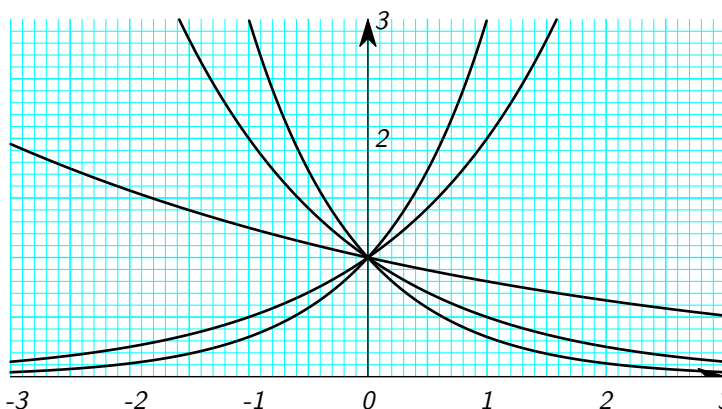


Figure 1. The graphs of $y = 2^x, 3^x, (1/2)^x, (0.1)^x$ and $y = (4/5)^x$. The graphs are purposely not labeled: can you figure out which is which?

Also:

$\log_2(-3)$ doesn't exist

because there is no number y for which $2^y = -3$ (2^y is always positive) and

$\log_{-3} 2$ doesn't exist either

because $y = \log_{-3} 2$ would have to be some real number which satisfies $(-3)^y = 2$, and we don't take non-integer powers of negative numbers.

3. Properties of logarithms

In general one has

$$\log_a a^x = x, \text{ and } a^{\log_a x} = x.$$

There is a subtle difference between these formulas: the first one holds for all real numbers x , but the second only holds for $x > 0$, since $\log_a x$ doesn't make sense for $x \leq 0$.

Again, one finds the following formulas in precalculus texts:

$$(39) \quad \begin{array}{l} \log_a xy = \log_a x + \log_a y \\ \log_a \frac{x}{y} = \log_a x - \log_a y \\ \log_a x^y = y \log_a x \\ \log_a x = \frac{\log_b x}{\log_b a} \end{array}$$

They follow from (38).

4. Graphs of exponential functions and logarithms

Figure 1 shows the graphs of some exponential functions $y = a^x$ with different values of a , and figure 2 shows the graphs of $y = \log_2 x$, $y = \log_3 x$, $\log_{1/2} x$, $\log_{1/3}(x)$ and $y = \log_{10} x$. Can you tell which is which? (Yes, you can.)

From algebra/precalc recall:

If $a > 1$ then $f(x) = a^x$ is an increasing function.

and

If $0 < a < 1$ then $f(x) = a^x$ is a decreasing function.

In other words, for $a > 1$ it follows from $x_1 < x_2$ that $a^{x_1} < a^{x_2}$; if $0 < a < 1$, then $x_1 < x_2$ implies $a^{x_1} > a^{x_2}$.

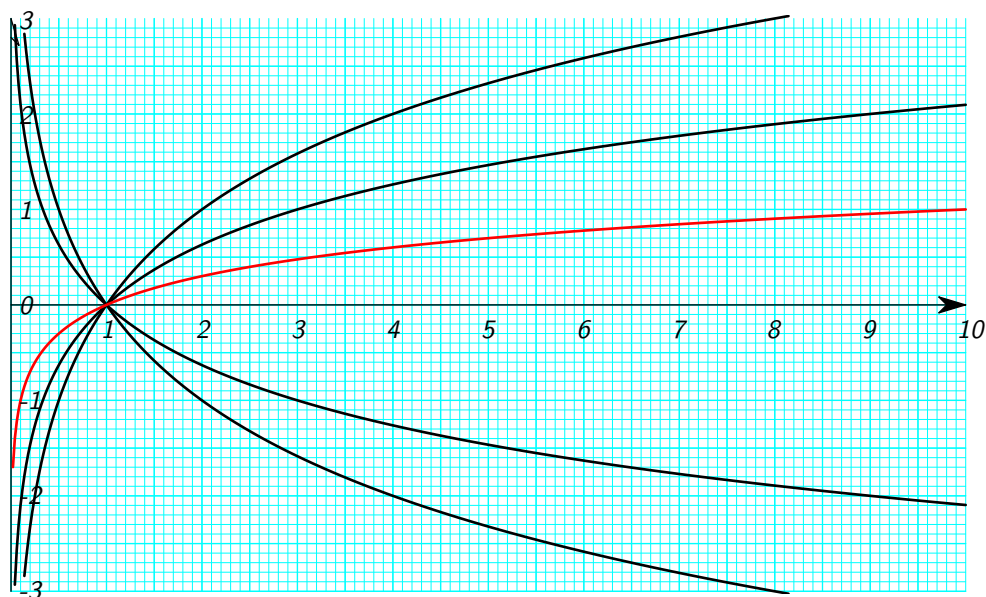


Figure 2. Graphs of some logarithms. Each curve is the graph of a function $y = \log_a x$ for various values of $a > 0$. Can you tell what a is for each graph?

5. The derivative of a^x and the definition of e

To begin, we try to differentiate the function $y = 2^x$:

$$\begin{aligned} \frac{d2^x}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{2^{x+\Delta x} - 2^x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2^x 2^{\Delta x} - 2^x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 2^x \frac{2^{\Delta x} - 1}{\Delta x} \\ &= 2^x \lim_{\Delta x \rightarrow 0} \frac{2^{\Delta x} - 1}{\Delta x}. \end{aligned}$$

So if we assume that the limit

$$\lim_{\Delta x \rightarrow 0} \frac{2^{\Delta x} - 1}{\Delta x} = C$$

exists then we have

$$(40) \quad \frac{d2^x}{dx} = C2^x.$$

On your calculator you can compute $\frac{2^{\Delta x} - 1}{\Delta x}$ for smaller and smaller values of Δx , which leads you to suspect that the limit actually exists, and that $C \approx 0.693147 \dots$. One can in fact prove that the limit exists, but we will not do this here.

Once we know (40) we can compute the derivative of a^x for any other positive number a . To do this we write $a = 2^{\log_2 a}$, and hence

$$a^x = (2^{\log_2 a})^x = 2^{x \cdot \log_2 a}.$$

By the chain rule we therefore get

$$\begin{aligned}\frac{da^x}{dx} &= \frac{d2^{x \cdot \log_2 a}}{dx} \\ &= C 2^{x \cdot \log_2 a} \frac{dx \cdot \log_2 a}{dx} \\ &= (C \log_2 a) 2^{x \cdot \log_2 a} \\ &= (C \log_2 a) a^x.\end{aligned}$$

So the derivative of a^x is just some constant times a^x , the constant being $C \log_2 a$. This is essentially our formula for the derivative of a^x , but one can make the formula look nicer by introducing a special number, namely, we define

$$e = 2^{1/C} \text{ where } C = \lim_{\Delta x \rightarrow 0} \frac{2^{\Delta x} - 1}{\Delta x}.$$

One has

$$e \approx 2.718\,281\,818\,459 \dots$$

This number is special because if you set $a = e$, then

$$C \log_2 a = C \log_2 e = C \log_2 2^{1/C} = C \cdot \frac{1}{C} = 1,$$

and therefore the derivative of the function $y = e^x$ is

$$(41) \quad \boxed{\frac{de^x}{dx} = e^x}.$$

Read that again: the function e^x is its own derivative!

The logarithm with base e is called the **Natural Logarithm**, and is written

$$\ln x = \log_e x.$$

Thus we have

$$(42) \quad \boxed{e^{\ln x} = x \quad \ln e^x = x}$$

where the second formula holds for all real numbers x but the first one only makes sense for $x > 0$.

For any positive number a we have $a = e^{\ln a}$, and also

$$a^x = e^{x \ln a}.$$

By the chain rule you then get

$$(43) \quad \boxed{\frac{da^x}{dx} = a^x \ln a}.$$

6. Derivatives of Logarithms

Since the natural logarithm is the inverse function of $f(x) = e^x$ we can find its derivative by implicit differentiation. Here is the computation (which you should do yourself)

The function $f(x) = \log_a x$ satisfies

$$a^{f(x)} = x$$

Differentiate both sides, and use the chain rule on the left,

$$(\ln a) a^{f(x)} f'(x) = 1.$$

Then solve for $f'(x)$ to get

$$f'(x) = \frac{1}{(\ln a) a^{f(x)}}.$$

Finally we remember that $a^{f(x)} = x$ which gives us the derivative of a^x

$$\frac{da^x}{dx} = \frac{1}{x \ln a}.$$

In particular, the natural logarithm has a very simple derivative, namely, since $\ln e = 1$ we have

$$(44) \quad \boxed{\frac{d \ln x}{dx} = \frac{1}{x}}.$$

7. Limits involving exponentials and logarithms

7.1. Theorem. *Let r be any real number. Then, if $a > 1$,*

$$\lim_{x \rightarrow \infty} x^r a^{-x} = 0,$$

i.e.

$$\lim_{x \rightarrow \infty} \frac{x^r}{a^x} = 0.$$

This theorem says that *any exponential will beat any power of x as $x \rightarrow \infty$* . For instance, as $x \rightarrow \infty$ both x^{1000} and $(1.001)^x$ go to infinity, but

$$\lim_{x \rightarrow \infty} \frac{x^{1000}}{(1.001)^x} = 0,$$

so, in the long run, for very large x , 1.001^x will be much larger than 1000^x .

PROOF WHEN $a = e$. We want to show $\lim_{x \rightarrow \infty} x^r e^{-x} = 0$. To do this consider the function $f(x) = x^{r+1} e^{-x}$. Its derivative is

$$f'(x) = \frac{dx^{r+1}e^{-x}}{dx} = ((r+1)x^r - x^{r+1})e^{-x} = (r+1-x)x^r e^{-x}.$$

Therefore $f'(x) < 0$ for $x > r+1$, i.e. $f(x)$ is decreasing for $x > r+1$. It follows that $f(x) < f(r+1)$ for all $x > r+1$, i.e.

$$x^{r+1} e^{-x} < (r+1)^{r+1} e^{-(r+1)} \text{ for } x > r+1.$$

Divide by x , abbreviate $A = (r+1)^{r+1} e^{-(r+1)}$, and we get

$$0 < x^r e^{-x} < \frac{A}{x} \text{ for all } x > r+1.$$

The Sandwich Theorem implies that $\lim_{x \rightarrow \infty} x^r e^{-x} = 0$, which is what we had promised to show. □

Here are some related limits:

$$a > 1 \implies \lim_{x \rightarrow \infty} \frac{a^x}{x^r} = \infty \quad (D.N.E.)$$

$$m > 0 \implies \lim_{x \rightarrow \infty} \frac{\ln x}{x^m} = 0$$

$$m > 0 \implies \lim_{x \rightarrow 0} x^m \ln x = 0$$

The second limit says that even though $\ln x$ becomes infinitely large as $x \rightarrow \infty$, it is always much less than any power x^m with $m > 0$ real. To prove it you set $x = e^t$ and then $t = s/m$, which leads to

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x^m} \stackrel{x=e^t}{=} \lim_{t \rightarrow \infty} \frac{t}{e^{mt}} \stackrel{t=s/m}{=} \frac{1}{m} \lim_{t \rightarrow \infty} \frac{s}{e^s} = 0.$$

The third limit follows from the second by substituting $x = 1/y$ and using $\ln \frac{1}{x} = -\ln x$.

8. Exponential growth and decay

A quantity X which depends on time t is said to grow or decay exponentially if it is given by

$$(45) \quad X(t) = X_0 e^{kt}.$$

The constant X_0 is the value of $X(t)$ at time $t = 0$ (sometimes called “the initial value of X ”).

The derivative of an exponentially growing quantity, i.e. its rate of change with time, is given by $X'(t) = X_0 k e^{kt}$ so that

$$(46) \quad \frac{dX(t)}{dt} = kX(t).$$

In words, *for an exponentially growing quantity the rate of change is always proportional to the quantity itself*. The proportionality constant is k and is sometimes called “the relative growth rate.”

This property of exponential functions completely describes them, by which I mean that any function which satisfies (46) automatically satisfies (45). To see that this is true, suppose you have a function $X(t)$ for which $X'(t) = kX(t)$ holds at all times t . Then

$$\begin{aligned} \frac{dX(t)e^{-kt}}{dt} &= X(t) \frac{de^{-kt}}{dt} + \frac{dX(t)}{dt} e^{-kt} \\ &= -kX(t)e^{-kt} + X'(t)e^{-kt} \\ &= (X'(t) - kX(t))e^{-kt} \\ &= 0. \end{aligned}$$

It follows that $X(t)e^{-kt}$ does not depend on t . At $t = 0$ one has

$$X(t)e^{-kt} = X(0)e^0 = X_0$$

and therefore we have

$$X(t)e^{-kt} = X_0 \text{ for all } t.$$

Multiply with e^{kt} and we end up with

$$X(t) = X_0 e^{kt}.$$

8.1. Half time and doubling time. If $X(t) = X_0 e^{kt}$ then one has

$$X(t+T) = X_0 e^{kt+ktT} = X_0 e^{kt} e^{kT} = e^{kT} X(t).$$

In words, after time T goes by an exponentially growing (decaying) quantity changes by a factor e^{kT} . If $k > 0$, so that the quantity is actually growing, then one calls

$$T = \frac{\ln 2}{k}$$

the **doubling time** for X because $X(t)$ changes by a factor $e^{kT} = e^{\ln 2} = 2$ every T time units: $X(t)$ doubles every T time units.

If $k < 0$ then $X(t)$ is decaying and one calls

$$T = \frac{\ln 2}{-k}$$

the **half life** because $X(t)$ is reduced by a factor $e^{kT} = e^{-\ln 2} = \frac{1}{2}$ every T time units.

8.2. Determining X_0 and k . The general exponential growth/decay function (45) contains only two constants, X_0 and k , and if you know the values of $X(t)$ at two different times then you can compute these constants.

Suppose that you know

$$X_1 = X(t_1) \text{ and } X_2 = X(t_2).$$

Then we have

$$X_0 e^{kt_1} = X_1 \text{ and } X_2 = X_0 e^{kt_2}$$

in which t_1, t_2, X_1, X_2 are given and k and X_0 are unknown. One first finds k from

$$\frac{X_1}{X_2} = \frac{X_0 e^{kt_1}}{X_0 e^{kt_2}} = e^{k(t_1 - t_2)} \implies \ln \frac{X_1}{X_2} = k(t_1 - t_2)$$

which implies

$$k = \frac{\ln X_1 - \ln X_2}{t_1 - t_2}.$$

Once you have computed k you can find X_0 from

$$X_0 = \frac{X_1}{e^{kt_1}} = \frac{X_2}{e^{kt_2}}.$$

(both expressions should give the same result.)

9. Exercises

Sketch the graphs of the following functions.

(Hint for some of these: if you have to solve something like $e^{4x} - 3e^{3x} + e^x = 0$, then call $w = e^x$, and you get a polynomial equation for w , namely $w^4 - 3w^3 + w = 0$.)

282. $y = e^x$

283. $y = e^{-x}$

284. $y = e^x + e^{-2x}$

285. $y = e^{3x} - 4e^x$

286. $y = \frac{e^x}{1 + e^x}$

287. $y = \frac{2e^x}{1 + e^{2x}}$

288. $y = xe^{-x}$

289. $y = \sqrt{x}e^{-x/4}$

290. $y = x^2 e^{x+2}$

291. $y = e^{x/2} - x$

292. $y = \ln \sqrt{x}$

293. $y = \ln \frac{1}{x}$

294. $y = x \ln x$

295. $y = \frac{-1}{\ln x} \quad (0 < x < \infty, x \neq 1)$

296. $y = (\ln x)^2 \quad (x > 0)$

297. $y = \frac{\ln x}{x} \quad (x > 0)$

298. $y = \ln \sqrt{\frac{1+x}{1-x}} \quad (|x| < 1)$

299. $y = \ln(1 + x^2)$

300. $y = \ln(x^2 - 3x + 2) \quad (x > 2)$

301. $y = \ln \cos x \quad (|x| < \frac{\pi}{2})$

302. The function $f(x) = e^{-x^2}$ plays a central role in statistics and its graph is called **the bell curve** (because of its shape). Sketch the graph of f .

303. Sketch the part of the graph of the function

$$f(x) = e^{-\frac{1}{x}}$$

with $x > 0$.

Find the limits

$$\lim_{x \searrow 0} \frac{f(x)}{x^n} \text{ and } \lim_{x \rightarrow \infty} f(x)$$

where n can be any positive integer (hint: substitute $x = \dots$?)

304. A **damped oscillation** is a function of the form

$$f(x) = e^{-ax} \cos bx \text{ or } f(x) = e^{-ax} \sin bx$$

where a and b are constants.

Sketch the graph of $f(x) = e^{-x} \sin 10x$ (i.e. find zeroes, local max and mins, inflection points) and draw (with pencil on paper) the piece of the graph with $0 \leq x \leq 2\pi$.

This function has many local maxima and minima. What is the ratio between the function values at two consecutive local maxima? (Hint: the answer does not depend on which pair of consecutive local maxima you consider.)

305. Find the inflection points on the graph of $f(x) = (1+x)\ln x$ ($x > 0$).

306. (a) If x is large, which is bigger: 2^x or x^2 ?
 (b) The graphs of $f(x) = x^2$ and $g(x) = 2^x$ intersect at $x = 2$ (since $2^2 = 2^2$). How many more intersections do these graphs have (with $-\infty < x < \infty$)?

Find the following limits.

307. $\lim_{x \rightarrow \infty} \frac{e^x - 1}{e^x + 1}$

308. $\lim_{x \rightarrow \infty} \frac{e^x - x^2}{e^x + x}$

309. $\lim_{x \rightarrow \infty} \frac{2^x}{3^x - 2^x}$

310. $\lim_{x \rightarrow \infty} \frac{e^x - x^2}{e^{2x} + e^{-x}}$

311. $\lim_{x \rightarrow \infty} \frac{e^{-x} - e^{-x/2}}{\sqrt{e^x + 1}}$

312. $\lim_{x \rightarrow \infty} \frac{\sqrt{x + e^{4x}}}{e^{2x} + x}$

313. $\lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{\sqrt{e^x + 1}}$

314. $\lim_{x \rightarrow \infty} \ln(1+x) - \ln x$

315. $\lim_{x \rightarrow \infty} \frac{\ln x}{\ln x^2}$

316. $\lim_{x \rightarrow 0} x \ln x$

317. $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x} + \ln x}$

318. $\lim_{x \rightarrow 0} \frac{\ln x}{\sqrt{x} + \ln x}$

319. Find the tenth derivative of xe^x .

320. For which real number x is $2^x - 3^x$ the largest?

321. Find $\frac{dx^x}{dx}$, $\frac{dx^{x^x}}{dx}$, and $\frac{d(x^x)^x}{dx}$.

Hint: $x^x = e^{x \ln x}$.

322. Group Problem.

About logarithmic differentiation:

(a) Let $y = (x+1)^2(x+3)^4(x+5)^6$ and $u = \ln y$. Find du/dx . Hint: Use the fact that $\ln$ converts multiplication to addition before you differentiate. It will simplify the calculation.

(b) Check that the derivative of $\ln u(x)$ is the logarithmic derivative of the function u (as defined in the exercises following §25, chapter 4.)

323. After 3 days a sample of radon-222 decayed to 58% of its original amount.

(a) What is the half life of radon-222?

(b) How long would it take the sample to decay to 10% of its original amount?

324. Polonium-210 has a half life of 140 days.

(a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days.

(b) Find the mass after 100 days.

(c) When will the mass be reduced to 10 mg?

(d) Sketch the graph of the mass as a function of time.

325. Current agricultural experts believe that the world's farms can feed about 10 billion people. The 1950 world population was 2.517 billion and the 1992 world population was 5.4 billion. When can we expect to run out of food?

326. Group Problem.

The ACME company runs two ads on Sunday mornings. One says that "when this baby is old enough to vote, the world will have one billion new mouths to feed" and the other says "in thirty six years, the world will have to set eight billion places at the table." What does ACME think the population of the world is at present? How fast does ACME think the population is increasing? Use units of billions of people so you can write 8 instead of 8,000,000,000. (Hint: $36 = 2 \times 18$.)

327. The population of California grows exponentially at an instantaneous rate of 2% per year. The population of California on January 1, 2000 was 20,000,000.

(a) Write a formula for the population $N(t)$ of California t years after January 1, 2000.

(b) Each Californian consumes pizzas at the rate of 70 pizzas per year. At what rate is California consuming pizzas t years after 1990?

(c) How many pizzas were consumed in California from January 1, 2005 to January 1, 2009?

328. The population of the country of Farfarawaï grows exponentially.

(a) If its population in the year 1980 was 1,980,000 and its population in the year 1990 was 1,990,000, what is its population in the year 2000?

(b) How long will it take the population to double? (Your answer may be expressed in terms of exponentials and natural logarithms.)

329. The hyperbolic functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

(a) Prove the following identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh 2x = 2 \sinh x \cosh x.$$

(b) Show that

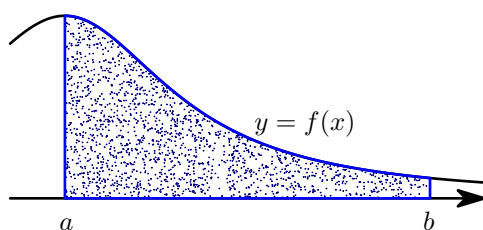
$$\begin{aligned}\frac{d \sinh x}{dx} &= \cosh x, \\ \frac{d \cosh x}{dx} &= \sinh x, \\ \frac{d \tanh x}{dx} &= \frac{1}{\cosh^2 x}.\end{aligned}$$

(c) Sketch the graphs of the three hyperbolic functions.

The Integral

In this chapter we define the integral of a function on some interval $[a, b]$. The most common interpretation of the integral is in terms of the area under the graph of the given function, so that is where we begin.

1. Area under a Graph



Let f be a function which is defined on some interval $a \leq x \leq b$ and assume it is positive, i.e. assume that its graph lies above the x axis. *How large is the area of the region caught between the x axis, the graph of $y = f(x)$ and the vertical lines $y = a$ and $y = b$?*

You can try to compute this area by approximating the region with many thin rectangles. Look at figure 1 before you read on. To make the approximating region you choose a *partition* of the interval $[a, b]$, i.e. you pick numbers $x_1 < \dots < x_n$ with

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

These numbers split the interval $[a, b]$ into n sub-intervals

$$[x_0, x_1], \quad [x_1, x_2], \quad \dots, \quad [x_{n-1}, x_n]$$

whose lengths are

$$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1, \quad \dots, \quad \Delta x_n = x_n - x_{n-1}.$$

In each interval we choose a point c_k , i.e. in the first interval we choose $x_0 \leq c_1 \leq x_1$, in the second interval we choose $x_1 \leq c_2 \leq x_2$, $\dots$, and in the last interval we choose some number $x_{n-1} \leq c_n \leq x_n$. See figure 1.

We then define n rectangles: the base of the k^{th} rectangle is the interval $[x_{k-1}, x_k]$ on the x -axis, while its height is $f(c_k)$ (here k can be any integer from 1 to n .)

The area of the k^{th} rectangle is of course the product of its height and width, i.e. its area is $f(c_k)\Delta x_k$. Adding these we see that the total area of the rectangles is

$$(47) \quad R = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \dots + f(c_n)\Delta x_n.$$

This kind of sum is called a **Riemann sum**.

If the partition is sufficiently fine then one would expect this sum, i.e. the total area of all rectangles to be a good approximation of the area of the region under the graph. Replacing the partition by a finer partition, with more division points, should improve the approximation. So you would expect the area to be the limit of Riemann-sums like R “as the partition becomes finer and finer.” A precise formulation of the definition goes like this:

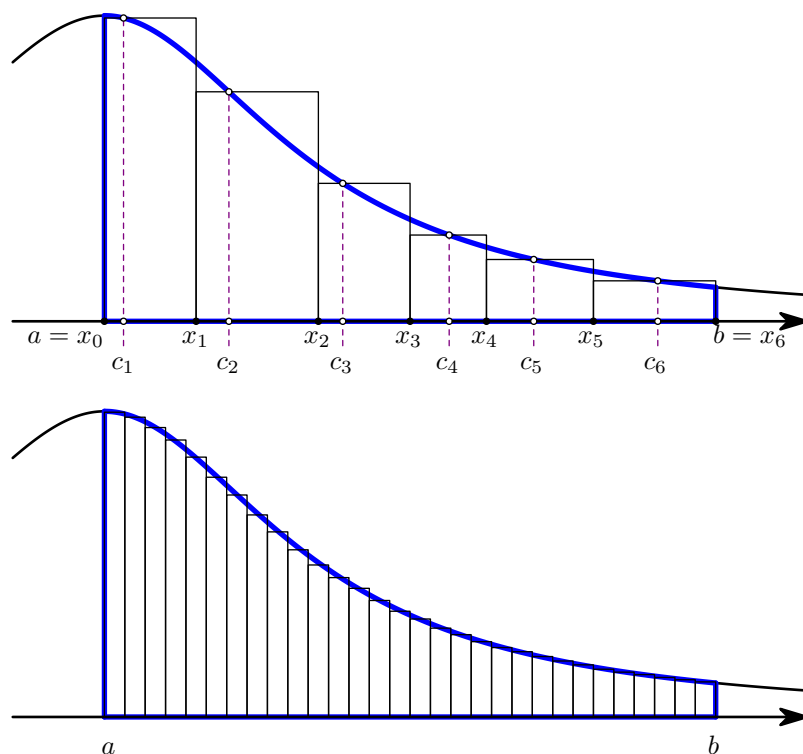


Figure 1. TOP. A Riemann sum in which the interval $a < x < b$ has been cut up into six smaller intervals. In each of those intervals a point c_i has been chosen at random, and the resulting rectangles with heights $f(c_1), \dots, f(c_6)$ were drawn. The total area under the graph of the function is roughly equal to the total area of the rectangles.

BOTTOM. Refining the partition. After adding more partition points the combined area of the rectangles will be a better approximation of the area under the graph of the function f .

1.1. Definition. If f is a function defined on an interval $[a, b]$, then we say that

$$\int_a^b f(x)dx = I,$$

i.e. the integral of “ $f(x)$ from $x = a$ to b ” equals I , if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$\left| f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \dots + f(c_n)\Delta x_n - I \right| < \varepsilon$$

holds for every partition all of whose intervals have length $\Delta x_k < \delta$.

2. When f changes its sign

If the function f is not necessarily positive everywhere in the interval $a \leq x \leq b$, then we still define the integral in exactly the same way: as a limit of Riemann sums whose mesh size becomes smaller and smaller. However the interpretation of the integral as “the area of the region between the graph and the x -axis” has a twist to it.

Let f be some function on an interval $a \leq x \leq b$, and form the Riemann sum

$$R = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \dots + f(c_n)\Delta x_n$$

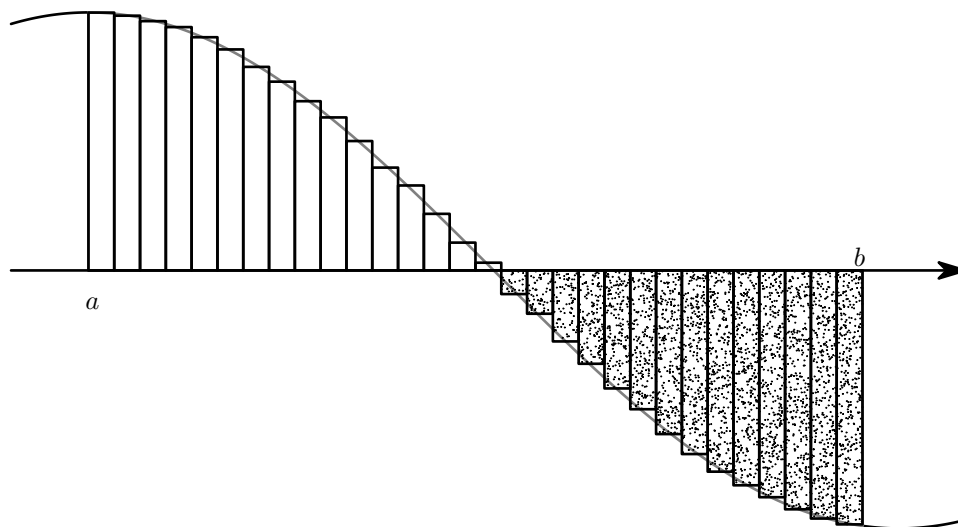


Figure 2. Illustrating a Riemann sum for a function whose sign changes. Always remember this: **AREAS ARE ALWAYS POSITIVE NUMBERS**. The Riemann-sum corresponding to this picture is the total area of the rectangles above the x -axis *minus* the total area of the rectangles below the x -axis.

that goes with some partition, and some choice of c_k .

When f can be positive or negative, then the terms in the Riemann sum can also be positive or negative. If $f(c_k) > 0$ then the quantity $f(c_k)\Delta x_k$ is the area of the corresponding rectangle, but if $f(c_k) < 0$ then $f(c_k)\Delta x_k$ is a negative number, namely *minus* the area of the corresponding rectangle. The Riemann sum is therefore the area of the rectangles above the x -axis *minus* the area below the axis and above the graph.

Taking the limit over finer and finer partitions, we conclude that

$$\int_a^b f(x)dx = \begin{array}{l} \text{area above the } x\text{-axis, below the graph} \\ \textit{minus} \text{ the area below the } x\text{-axis, above the graph.} \end{array}$$

3. The Fundamental Theorem of Calculus

3.1. Definition. A function F is called an **antiderivative** of f on the interval $[a, b]$ if one has $F'(x) = f(x)$ for all x with $a < x < b$.

For instance, $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$, but so is $G(x) = \frac{1}{2}x^2 + 2008$.

3.2. Theorem. If f is a function whose integral $\int_a^b f(x)dx$ exists, and if F is an antiderivative of f on the interval $[a, b]$, then one has

$$(48) \quad \int_a^b f(x)dx = F(b) - F(a).$$

(a proof was given in lecture.)

Because of this theorem the expression on the right appears so often that various abbreviations have been invented. We will abbreviate

$$F(b) - F(a) \stackrel{\text{def}}{=} [F(x)]_{x=a}^b = [F(x)]_a^b.$$

3.3. Terminology.

In the integral

$$\int_a^b f(x) dx$$

the numbers a and b are called the *bounds of the integral*, the function $f(x)$ which is being integrated is called *the integrand*, and the variable x is *integration variable*.

The integration variable is a *dummy variable*. If you systematically replace it with another variable, the resulting integral will still be the same. For instance,

$$\int_0^1 x^2 dx = \left[\frac{1}{3} x^3 \right]_{x=0}^1 = \frac{1}{3},$$

and if you replace x by φ you still get

$$\int_0^1 \varphi^2 d\varphi = \left[\frac{1}{3} \varphi^3 \right]_{\varphi=0}^1 = \frac{1}{3}.$$

Another way to appreciate that the integration variable is a dummy variable is to look at the Fundamental Theorem again:

$$\int_a^b f(x) dx = F(b) - F(a).$$

The right hand side tells you that the value of the integral depends on a and b , and has absolutely nothing to do with the variable x .

4. Exercises

330. What is a Riemann sum of a function $y = f(x)$?

331. Let f be the function $f(x) = 1 - x^2$.

Draw the graph of $f(x)$ with $0 \leq x \leq 2$.

Compute the Riemann-sum for the partition

$$0 < \frac{1}{3} < 1 < \frac{3}{2} < 2$$

of the interval $[a, b] = [0, 2]$ if you choose each c_k to be the left endpoint of the interval it belongs to. Draw the corresponding rectangles (add them to your drawing of the graph of f).

Then compute the Riemann-sum you get if you choose the c_k to be the right endpoint of the interval it belongs to. Make a new drawing of the graph of f and include the rectangles corresponding to the right endpoint Riemann-sum.

332. Group Problem.

Look at figure 1 (top). Which choice of intermediate points $c_1, \dots, c_6$ leads to the smallest Riemann sum? Which choice would give you the largest Riemann-sum?

(Note: in this problem you're not allowed to change the division points x_i , only the points c_i in between them.)

Find an antiderivative $F(x)$ for each of the following functions $f(x)$. Finding antiderivatives involves a fair amount of guess work, but with experience it gets easier to guess antiderivatives.

333. $f(x) = 2x + 1$

334. $f(x) = 1 - 3x$

335. $f(x) = x^2 - x + 11$

336. $f(x) = x^4 - x^2$

337. $f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$

338. $f(x) = \frac{1}{x}$

339. $f(x) = e^x$

340. $f(x) = \frac{2}{x}$

341. $f(x) = e^{2x}$

342. $f(x) = \frac{1}{2+x}$

343. $f(x) = \frac{e^x - e^{-x}}{2}$

344. $f(x) = \frac{1}{1+x^2}$

345. $f(x) = \frac{e^x + e^{-x}}{2}$

346. $f(x) = \frac{1}{\sqrt{1-x^2}}$

347. $f(x) = \sin x$

348. $f(x) = \frac{2}{1-x}$

349. $f(x) = \cos x$

350. $f(x) = \cos 2x$

351. $f(x) = \sin(x - \pi/3)$

352. $f(x) = \sin x + \sin 2x$

353. $f(x) = 2x(1 + x^2)^5$

In each of the following exercises you should compute the area of the indicated region, and also of the smallest enclosing rectangle with horizontal and vertical sides.

Before computing anything draw the region.

354. The region between the vertical lines $x = 0$ and $x = 1$, and between the x -axis and the graph of $y = x^3$.

355. The region between the vertical lines $x = 0$ and $x = 1$, and between the x -axis and the graph of $y = x^n$ (here $n > 0$, draw for $n = \frac{1}{2}, 1, 2, 3, 4$).

356. The region above the graph of $y = \sqrt{x}$, below the line $y = 2$, and between the vertical lines $x = 0$, $x = 4$.

357. The region above the x -axis and below the graph of $f(x) = x^2 - x^3$.

358. The region above the x -axis and below the graph of $f(x) = 4x^2 - x^4$.

359. The region above the x -axis and below the graph of $f(x) = 1 - x^4$.

360. The region above the x -axis, below the graph of $f(x) = \sin x$, and between $x = 0$ and $x = \pi$.

361. The region above the x -axis, below the graph of $f(x) = 1/(1 + x^2)$ (a curve known as *Maria Agnesi's witch*), and between $x = 0$ and $x = 1$.

362. The region between the graph of $y = 1/x$ and the x -axis, and between $x = a$ and $x = b$ (here $0 < a < b$ are constants, e.g. choose $a = 1$ and $b = \sqrt{2}$ if you have something against either letter a or b .)

363. The region above the x -axis and below the graph of

$$f(x) = \frac{1}{1+x} + \frac{x}{2} - 1.$$

364. Compute

$$\int_0^1 \sqrt{1-x^2} dx$$

without finding an antiderivative for $\sqrt{1-x^2}$ (you can find such an antiderivative, but it's not easy. This integral is the area of some region: which region is it, and what is that area?)

365. **Group Problem.**

Compute these integrals without finding antiderivatives.

$$I = \int_0^{1/2} \sqrt{1-x^2} dx$$

$$J = \int_{-1}^1 |1-x| dx$$

$$K = \int_{-1}^1 |2-x| dx$$

5. The indefinite integral

The fundamental theorem tells us that in order to compute the integral of some function f over an interval $[a, b]$ you should first find an antiderivative F of f . In practice, much of the effort required to find an integral goes into finding the antiderivative. In order to simplify the computation of the integral

$$(49) \quad \int_a^b f(x) dx = F(b) - F(a)$$

the following notation is commonly used for the antiderivative:

$$(50) \quad F(x) = \int f(x) dx.$$

For instance,

$$\int x^2 dx = \frac{1}{3}x^3, \quad \int \sin 5x dx = -\frac{1}{5}\cos 5x, \quad \text{etc} \dots$$

The integral which appears here does not have the integration bounds a and b . It is called an ***indefinite integral***, as opposed to the integral in (49) which is called a ***definite integral***. You use the indefinite integral if you expect the computation of the antiderivative to be a lengthy affair, and you do not want to write the integration bounds a and b all the time.

It is important to distinguish between the two kinds of integrals. Here is a list of differences:

INDEFINITE INTEGRAL	DEFINITE INTEGRAL
$\int f(x)dx$ is a function of x .	$\int_a^b f(x)dx$ is a number.
By definition $\int f(x)dx$ is <i>any function of x whose derivative is $f(x)$</i> .	$\int_a^b f(x)dx$ was defined in terms of Riemann sums and can be interpreted as “area under the graph of $y = f(x)$ ”, at least when $f(x) > 0$.
x is not a dummy variable, for example, $\int 2x dx = x^2 + C$ and $\int 2t dt = t^2 + C$ are functions of different variables, so they are not equal.	x is a dummy variable, for example, $\int_0^1 2x dx = 1$, and $\int_0^1 2t dt = 1$, so $\int_0^1 2x dx = \int_0^1 2t dt$.

5.1. You can always check the answer. Suppose you want to find an antiderivative of a given function $f(x)$ and after a long and messy computation which you don't really trust you get an “answer”, $F(x)$. You can then throw away the dubious computation and differentiate the $F(x)$ you had found. If $F'(x)$ turns out to be equal to $f(x)$, then your $F(x)$ is indeed an antiderivative and your computation isn't important anymore.

For example, suppose that we want to find $\int \ln x dx$. My cousin Louie says it might be $F(x) = x \ln x - x$. Let's see if he's right:

$$\frac{d}{dx}(x \ln x - x) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x.$$

Who knows how Louie thought of this¹, but it doesn't matter: he's right! We now know that $\int \ln x dx = x \ln x - x + C$.

5.2. About “+C”. Let $f(x)$ be a function defined on some interval $a \leq x \leq b$. If $F(x)$ is an antiderivative of $f(x)$ on this interval, then for any constant C the function $\tilde{F}(x) = F(x) + C$ will also be an antiderivative of $f(x)$. So one given function $f(x)$ has many different antiderivatives, obtained by adding different constants to one given antiderivative.

5.3. Theorem. If $F_1(x)$ and $F_2(x)$ are antiderivatives of the same function $f(x)$ on some interval $a \leq x \leq b$, then there is a constant C such that $F_1(x) = F_2(x) + C$.

PROOF. Consider the difference $G(x) = F_1(x) - F_2(x)$. Then $G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, so that $G(x)$ must be constant. Hence $F_1(x) - F_2(x) = C$ for some constant. $\square$

It follows that there is some ambiguity in the notation $\int f(x) dx$. Two functions $F_1(x)$ and $F_2(x)$ can both equal $\int f(x) dx$ without equaling each other. When this happens, they (F_1 and F_2) differ by a constant. This can sometimes lead to confusing situations, e.g. you can check that

$$\begin{aligned} \int 2 \sin x \cos x dx &= \sin^2 x \\ \int 2 \sin x \cos x dx &= -\cos^2 x \end{aligned}$$

are both correct. (Just differentiate the two functions $\sin^2 x$ and $-\cos^2 x$!) These two answers look different until you realize that because of the trig identity $\sin^2 x + \cos^2 x = 1$ they really only differ by a constant: $\sin^2 x = -\cos^2 x + 1$.

To avoid this kind of confusion we will from now on never forget to include the “arbitrary constant +C” in our answer when we compute an antiderivative.

$$\begin{aligned}
\int f(x) dx &= F(x) + C \\
\int x^n dx &= \frac{x^{n+1}}{n+1} + C && \text{for all } n \neq -1 \\
\int \frac{1}{x} dx &= \ln |x| + C && \text{(Note the absolute values)} \\
\int e^x dx &= e^x + C \\
\int a^x dx &= \frac{a^x}{\ln a} + C && \text{(don't memorize: use } a^x = e^{x \ln a} \text{)} \\
\int \sin x dx &= -\cos x + C \\
\int \cos x dx &= \sin x + C \\
\int \tan x dx &= -\ln |\cos x| + C && \text{(Note the absolute values)} \\
\int \frac{1}{1+x^2} dx &= \arctan x + C \\
\int \frac{1}{\sqrt{1-x^2}} dx &= \arcsin x + C
\end{aligned}$$

The following integral is also useful, but not as important as the ones above:

$$\int \frac{dx}{\cos x} = \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + C \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Table 1. The list of the standard integrals everyone should know

Table 1 lists a number of antiderivatives which you should know. All of these integrals should be familiar from the differentiation rules we have learned so far, except for the integrals of $\tan x$ and of $\frac{1}{\cos x}$. You can check those by differentiation (using $\ln \frac{a}{b} = \ln a - \ln b$ simplifies things a bit).

6. Properties of the Integral

Just as we had a list of properties for the limits and derivatives of sums and products of functions, the integral has similar properties.

Suppose we have two functions $f(x)$ and $g(x)$ with antiderivatives $F(x)$ and $G(x)$, respectively. Then we know that

$$\frac{d}{dx} \{F(x) + G(x)\} = F'(x) + G'(x) = f(x) + g(x),$$

in words, $F + G$ is an antiderivative of $f + g$, which we can write as

$$(51) \quad \int \{f(x) + g(x)\} dx = \int f(x) dx + \int g(x) dx.$$

Similarly, $\frac{d}{dx}(cF(x)) = cF'(x) = cf(x)$ implies that

$$(52) \quad \int cf(x) dx = c \int f(x) dx$$

if c is a constant.

¹He took math 222 and learned to integrate by parts.

These properties imply analogous properties for the definite integral. For any pair of functions on an interval $[a, b]$ one has

$$(53) \quad \int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx,$$

and for any function f and constant c one has

$$(54) \quad \int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

Definite integrals have one other property for which there is no analog in indefinite integrals: if you split the interval of integration into two parts, then the integral over the whole is the sum of the integrals over the parts. The following theorem says it more precisely.

6.1. Theorem. *Given $a < c < b$, and a function on the interval $[a, b]$ then*

$$(55) \quad \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

PROOF. Let F be an antiderivative of f . Then

$$\int_a^c f(x) dx = F(c) - F(a) \text{ and } \int_c^b f(x) dx = F(b) - F(c),$$

so that

$$\begin{aligned} \int_a^b f(x) dx &= F(b) - F(a) \\ &= F(b) - F(c) + F(c) - F(a) \\ &= \int_a^c f(x) dx + \int_c^b f(x) dx. \end{aligned}$$

□

So far we have always assumed that $a < b$ in all indefinite integrals $\int_a^b \dots$. The fundamental theorem suggests that when $b < a$, we should define the integral as

$$(56) \quad \int_a^b f(x) dx = F(b) - F(a) = -(F(a) - F(b)) = - \int_b^a f(x) dx.$$

For instance,

$$\int_1^0 x dx = - \int_0^1 x dx = -\frac{1}{2}.$$

7. The definite integral as a function of its integration bounds

Consider the expression

$$I = \int_0^x t^2 dt.$$

What does I depend on? To see this, you calculate the integral and you find

$$I = \left[\frac{1}{3} t^3 \right]_0^x = \frac{1}{3} x^3 - \frac{1}{3} 0^3 = \frac{1}{3} x^3.$$

So the integral depends on x . It does not depend on t , since t is a “dummy variable” (see §3.3 where we already discussed this point.)

In this way you can use integrals to define new functions. For instance, we could define

$$I(x) = \int_0^x t^2 dt,$$

which would be a roundabout way of defining the function $I(x) = x^3/3$. Again, since t is a dummy variable we can replace it by any other variable we like. Thus

$$I(x) = \int_0^x \alpha^2 d\alpha$$

defines the same function (namely, $I(x) = \frac{1}{3}x^3$).

The previous example does not define a new function ($I(x) = x^3/3$). An example of a *new* function defined by an integral is the “error-function” from statistics. It is given by

$$(57) \quad \operatorname{erf}(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

so $\operatorname{erf}(x)$ is the area of the shaded region in figure 3. The integral in (57) cannot be computed in terms of the

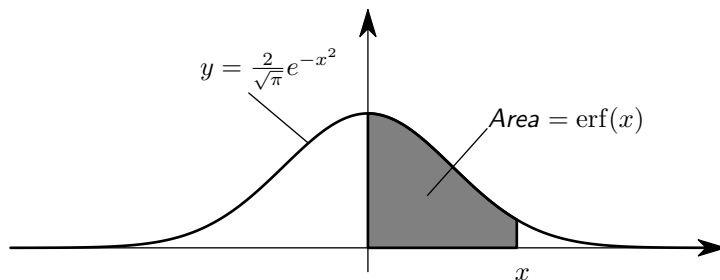


Figure 3. Definition of the Error function.

standard functions (square and higher roots, sine, cosine, exponential and logarithms). Since the integral in (57) occurs very often in statistics (in relation with the so-called normal distribution) it has been given a name, namely, “ $\operatorname{erf}(x)$ ”.

How do you differentiate a function that is defined by an integral? The answer is simple, for if $f(x) = F'(x)$ then the fundamental theorem says that

$$\int_a^x f(t) dt = F(x) - F(a),$$

and therefore

$$\frac{d}{dx} \int_a^x f(t) dt = \frac{d}{dx} \{F(x) - F(a)\} = F'(x) = f(x),$$

i.e.

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

A similar calculation gives you

$$\frac{d}{dx} \int_x^b f(t) dt = -f(x).$$

So what is the derivative of the error function? We have

$$\begin{aligned} \operatorname{erf}'(x) &= \frac{d}{dx} \left\{ \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \right\} \\ &= \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^x e^{-t^2} dt \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2}. \end{aligned}$$

8. Method of substitution

The chain rule says that

$$\frac{dF(G(x))}{dx} = F'(G(x)) \cdot G'(x),$$

so that

$$\int F'(G(x)) \cdot G'(x) dx = F(G(x)) + C.$$

8.1. Example. Consider the function $f(x) = 2x \sin(x^2 + 3)$. It does not appear in the list of standard antiderivatives we know by heart. But we do notice² that $2x = \frac{d}{dx}(x^2 + 3)$. So let's call $G(x) = x^2 + 3$, and $F(u) = -\cos u$, then

$$F(G(x)) = -\cos(x^2 + 3)$$

and

$$\frac{dF(G(x))}{dx} = \underbrace{\sin(x^2 + 3)}_{F'(G(x))} \cdot \underbrace{2x}_{G'(x)} = f(x),$$

so that

$$(58) \quad \int 2x \sin(x^2 + 3) dx = -\cos(x^2 + 3) + C.$$

8.2. Leibniz' notation for substitution. The most transparent way of computing an integral by substitution is by following Leibniz and introduce new variables. Thus to do the integral

$$\int f(G(x))G'(x) dx$$

where $f(u) = F'(u)$, we introduce the substitution $u = G(x)$, and agree to write

$$du = dG(x) = G'(x) dx.$$

Then we get

$$\int f(G(x))G'(x) dx = \int f(u) du = F(u) + C.$$

At the end of the integration we must remember that u really stands for $G(x)$, so that

$$\int f(G(x))G'(x) dx = F(u) + C = F(G(x)) + C.$$

As an example, let's do the integral (58) using Leibniz' notation. We want to find

$$\int 2x \sin(x^2 + 3) dx$$

and decide to substitute $z = x^2 + 3$ (the substitution variable doesn't always have to be called u). Then we compute

$$dz = d(x^2 + 3) = 2x dx \text{ and } \sin(x^2 + 3) = \sin z,$$

so that

$$\int 2x \sin(x^2 + 3) dx = \int \sin z dz = -\cos z + C.$$

Finally we get rid of the substitution variable z , and we find

$$\int 2x \sin(x^2 + 3) dx = -\cos(x^2 + 3) + C.$$

When we do integrals in this calculus class, we always get rid of the substitution variable because it is a variable we invented, and which does not appear in the original problem. But if you are doing an integral which appears in some longer discussion of a real-life (or real-lab) situation, then it may be that the substitution variable actually has a meaning (e.g. "the effective stoichiometric modality of CQF self-inhibition") in which

² You *will* start noticing things like this after doing several examples.

case you may want to skip the last step and leave the integral in terms of the (meaningful) substitution variable.

8.3. Substitution for definite integrals. For definite integrals the chain rule

$$\frac{d}{dx}(F(G(x))) = F'(G(x))G'(x) = f(G(x))G'(x)$$

implies

$$\int_a^b f(G(x))G'(x) dx = F(G(b)) - F(G(a)).$$

which you can also write as

$$(59) \quad \int_{x=a}^b f(G(x))G'(x) dx = \int_{u=G(a)}^{G(b)} f(u) du.$$

8.4. Example of substitution in a definite integral. Let's compute

$$\int_0^1 \frac{x}{1+x^2} dx,$$

using the substitution $u = G(x) = 1 + x^2$. Since $du = 2x dx$, the associated *indefinite* integral is

$$\int \underbrace{\frac{1}{1+x^2}}_{\frac{1}{u}} \underbrace{x dx}_{\frac{1}{2} du} = \frac{1}{2} \int \frac{1}{u} du.$$

To find the definite integral you must compute the new integration bounds $G(0)$ and $G(1)$ (see equation (59).) If x runs between $x = 0$ and $x = 1$, then $u = G(x) = 1 + x^2$ runs between $u = 1 + 0^2 = 1$ and $u = 1 + 1^2 = 2$, so the definite integral we must compute is

$$(60) \quad \int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{1}{u} du,$$

which is in our list of memorable integrals. So we find

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{1}{u} du = \frac{1}{2} [\ln u]_1^2 = \frac{1}{2} \ln 2.$$

Sometimes the integrals in (60) are written as

$$\int_{x=0}^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_{u=1}^2 \frac{1}{u} du,$$

to emphasize (and remind yourself) to which variable the bounds in the integral refer.

9. Exercises

Compute these derivatives:

$$366. \quad \frac{d}{dx} \int_0^x (1+t^2)^4 dt$$

$$367. \quad \frac{d}{dx} \int_x^1 \ln z dz$$

$$368. \quad \frac{d}{dt} \int_0^t \frac{dx}{1+x^2}$$

$$369. \quad \frac{d}{dt} \int_0^{1/t} \frac{dx}{1+x^2}$$

$$370. \quad \frac{d}{dx} \int_x^{2x} s^2 ds$$

$$371. \quad \frac{d}{dq} \int_{-q}^q \frac{dx}{1-x^2} \quad [\text{Which values of } q \text{ are allowed here?}]$$

$$372. \quad \frac{d}{dt} \int_0^{t^2} e^{2x} dx$$

373. Group Problem.

You can see the graph of the error function at http://en.wikipedia.org/wiki/Error_function

(a) Compute the second derivative of the error function. How many inflection points does the graph of the error function have?

(b) The graph of the error function on Wikipedia shows that $\text{erf}(x)$ is negative when $x < 0$. But the error

function is defined as an integral of a positive function so it should be positive. Is Wikipedia wrong? Explain.

Compute the following indefinite integrals:

$$374. \int \{6x^5 - 2x^{-4} - 7x\} dx$$

$$375. \int \{+3/x - 5 + 4e^x + 7^x\} dx$$

$$376. \int (x/a + a/x + x^a + a^x + ax) dx$$

$$377. \int \{\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1\} dx$$

$$378. \int \{2^x + (\frac{1}{2})^x\} dx$$

$$379. \int_{-2}^4 (3x - 5) dx$$

$$380. \int_1^4 x^{-2} dx \quad (\text{hm} \dots)$$

$$381. \int_1^4 t^{-2} dt \quad (!)$$

$$382. \int_1^4 x^{-2} dt \quad (!!!)$$

$$383. \int_0^1 (1 - 2x - 3x^2) dx$$

$$384. \int_1^2 (5x^2 - 4x + 3) dx$$

$$385. \int_{-3}^0 (5y^4 - 6y^2 + 14) dy$$

$$386. \int_0^1 (y^9 - 2y^5 + 3y) dy$$

$$387. \int_0^4 \sqrt{x} dx$$

$$388. \int_0^1 x^{3/7} dx$$

$$389. \int_1^3 \left(\frac{1}{t^2} - \frac{1}{t^4} \right) dt$$

$$390. \int_1^2 \frac{t^6 - t^2}{t^4} dt$$

$$391. \int_1^2 \frac{x^2 + 1}{\sqrt{x}} dx$$

$$392. \int_0^2 (x^3 - 1)^2 dx$$

$$393. \int_0^1 u(\sqrt{u} + \sqrt[3]{u}) du$$

$$394. \int_1^2 (x + 1/x)^2 dx$$

$$395. \int_3^3 \sqrt{x^5 + 2} dx$$

$$396. \int_1^{-1} (x - 1)(3x + 2) dx$$

$$397. \int_1^4 (\sqrt{t} - 2/\sqrt{t}) dt$$

$$398. \int_1^8 \left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} \right) dr$$

$$399. \int_{-1}^0 (x + 1)^3 dx$$

$$400. \int_{-5}^{-2} \frac{x^4 - 1}{x^2 + 1} dx$$

$$401. \int_1^e \frac{x^2 + x + 1}{x} dx$$

$$402. \int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx$$

$$403. \int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4} \right) dx$$

$$404. \int_1^8 \frac{x - 1}{\sqrt[3]{x^2}} dx$$

$$405. \int_{\pi/4}^{\pi/3} \sin t dt$$

$$406. \int_0^{\pi/2} (\cos \theta + 2 \sin \theta) d\theta$$

$$407. \int_0^{\pi/2} (\cos \theta + \sin 2\theta) d\theta$$

$$408. \int_{2\pi/3}^{\pi} \frac{\tan x}{\cos x} dx$$

$$409. \int_{\pi/3}^{\pi/2} \frac{\cot x}{\sin x} dx$$

$$410. \int_1^{\sqrt{3}} \frac{6}{1 + x^2} dx$$

$$411. \int_0^{0.5} \frac{dx}{\sqrt{1 - x^2}}$$

$$412. \int_4^8 (1/x) dx$$

$$413. \int_{\ln 3}^{\ln 6} 8e^x dx$$

$$414. \int_8^9 2^t dt$$

$$415. \int_{-e^2}^{-e} \frac{3}{x} dx$$

$$416. \int_{-2}^3 |x^2 - 1| dx$$

417. $\int_{-1}^2 |x - x^2| dx$

418. $\int_{-1}^2 (x - 2|x|) dx$

419. $\int_0^2 (x^2 - |x - 1|) dx$

420. $\int_0^2 f(x) dx$ where $f(x) = \begin{cases} x^4, & \text{if } 0 \leq x < 1, \\ x^5, & \text{if } 1 \leq x \leq 2. \end{cases}$

421. $\int_{-\pi}^{\pi} f(x) dx$ where $f(x) = \begin{cases} x, & \text{if } -\pi \leq x \leq 0, \\ \sin x, & \text{if } 0 < x \leq \pi. \end{cases}$

422. Compute

$$I = \int_0^2 2x(1 + x^2)^3 dx$$

in two different ways:

(a) Expand $(1 + x^2)^3$, multiply with $2x$, and integrate each term.

(b) Use the substitution $u = 1 + x^2$.

423. Compute

$$I_n = \int 2x(1 + x^2)^n dx.$$

424. If $f'(x) = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

425. Sketch the graph of the curve $y = \sqrt{x+1}$ and determine the area of the region enclosed by the curve, the x -axis and the lines $x = 0$, $x = 4$.

426. Find the area under the curve $y = \sqrt{6x+4}$ and above the x -axis between $x = 0$ and $x = 2$. Draw a sketch of the curve.

427. Graph the curve $y = 2\sqrt{1-x^2}$, $x \in [0, 1]$, and find the area enclosed between the curve and the x -axis. (Don't evaluate the integral, but compare with the area under the graph of $y = \sqrt{1-x^2}$.)

428. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ and between the lines $x = 0$ and $x = a$.

429. Graph the curve $y = 2\sqrt{9-x^2}$ and determine the area enclosed between the curve and the x -axis.

430. Graph the area between the curve $y^2 = 4x$ and the line $x = 3$. Find the area of this region.

431. Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0$ and $y = 3$.

432. Find the area enclosed between the curve $y = \sin 2x$, $0 \leq x \leq \pi/4$ and the axes.

433. Find the area enclosed between the curve $y = \cos 2x$, $0 \leq x \leq \pi/4$ and the axes.

434. Graph $y^2 + 1 = x$, and find the area enclosed by the curve and the line $x = 2$.

435. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

436. Find the area bounded by the curve $y = x(2-x)$ and the line $x = 2y$.

437. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

438. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

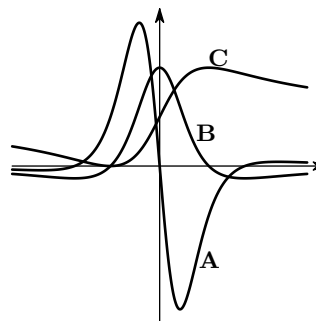
439. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

440. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

441. Group Problem.

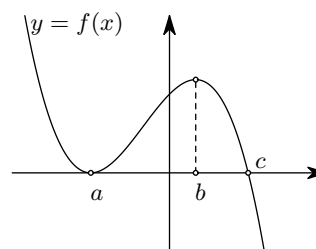
You asked your assistant Joe to produce graphs of a function $f(x)$, its derivative $f'(x)$ and an antiderivative $F(x)$ of $f(x)$.

Unfortunately Joe simply labelled the graphs "A," "B," and "C," and now he doesn't remember which graph is f , which is f' and which is F . Identify which graph is which and explain your answer.



442. Group Problem.

Below is the graph of a function $y = f(x)$.



The function $F(x)$ (graph not shown) is an antiderivative of $f(x)$. Which among the following statements true?

(a) $F(a) = F(c)$

(b) $F(b) = 0$

(c) $F(b) > F(c)$

(d) The graph of $y = F(x)$ has **two** inflection points?

Use a substitution to evaluate the following integrals.

$$443. \int_1^2 \frac{u \, du}{1+u^2}$$

$$444. \int_0^5 \frac{x \, dx}{\sqrt{x+1}}$$

$$445. \int_1^2 \frac{x^2 \, dx}{\sqrt{2x+1}}$$

$$446. \int_0^5 \frac{s \, ds}{\sqrt[3]{s+2}}$$

$$447. \int_1^2 \frac{x \, dx}{1+x^2}$$

$$448. \int_0^\pi \cos(\theta + \frac{\pi}{3}) d\theta$$

$$449. \int \sin \frac{\pi+x}{5} dx$$

$$450. \int \frac{\sin 2x}{\sqrt{1+\cos 2x}} dx$$

$$451. \int_{\pi/4}^{\pi/3} \sin^2 \theta \cos \theta \, d\theta$$

$$452. \int_2^3 \frac{1}{r \ln r}, dr$$

$$453. \int \frac{\sin 2x}{1+\cos^2 x} dx$$

$$454. \int \frac{\sin 2x}{1+\sin x} dx$$

$$455. \int_0^1 z \sqrt{1-z^2} dz$$

$$456. \int_1^2 \frac{\ln 2x}{x} dx$$

$$457. \int_{\xi=0}^{\sqrt{2}} \xi(1+2\xi^2)^{10} d\xi$$

$$458. \int_2^3 \sin \rho (\cos 2\rho)^4 d\rho$$

$$459. \int \alpha e^{-\alpha^2} d\alpha$$

$$460. \int \frac{e^{\frac{1}{t}}}{t^2} dt$$

Applications of the integral

The integral appears as the answer to many different questions. In this chapter we will describe a number of “things which are an integral.” In each example there is a quantity we want to compute, and which we can approximate through Riemann-sums. After letting the partition become arbitrarily fine we then find that the quantity we are looking for is given by an integral. The derivations are an important part of the subject.

1. Areas between graphs

Suppose you have two functions f and g on an interval $[a, b]$, one of which is always larger than the other, i.e. for which you know that $f(x) \leq g(x)$ for all x in the interval $[a, b]$. Then the area of the region between the graphs of the two functions is

$$(61) \quad \text{Area} = \int_a^b (g(x) - f(x)) dx.$$

To get this formula you approximate the region by a large number of thin rectangles. Choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$; choose a number c_k in each interval $[x_{k-1}, x_k]$; form the rectangles

$$x_{k-1} \leq x \leq x_k, \quad f(c_k) \leq y \leq g(c_k).$$

The area of this rectangle is

$$\text{width} \times \text{height} = \Delta x_k \times (g(c_k) - f(c_k)).$$

Hence the combined area of the rectangles is

$$R = (g(c_1) - f(c_1))\Delta x_1 + \cdots + (g(c_n) - f(c_n))\Delta x_n$$

which is just the Riemann-sum for the integral

$$I = \int_a^b (g(x) - f(x)) dx.$$

So,

- (1) since the area of the region between the graphs of f and g is the limit of the combined areas of the rectangles,

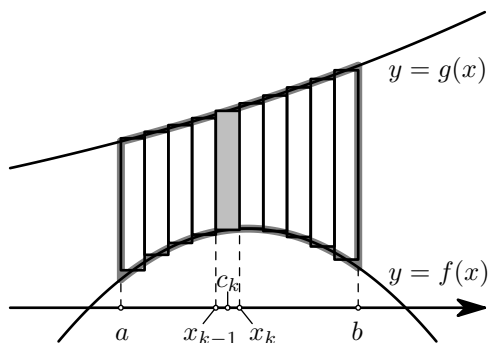


Figure 1. Finding the area between two graphs using Riemann-sums.

- (2) and since this combined area is equal to the Riemann sum R ,
 (3) and since the Riemann-sums R converge to the integral I ,

we conclude that the area between the graphs of f and g is exactly the integral I .

2. Exercises

- 461.** Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.
462. Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.
463. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.
464. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.
465. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.
466. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.
467. Use integration to find the area of the triangular region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.
468. Find the area bounded by the parabola $x^2 - 2 = y$ and the line $x + y = 0$.
469. Where do the graphs of $f(x) = x^2$ and $g(x) = 3/(2 + x^2)$ intersect? Find the area of the region which lies above the graph of g and below the graph of f .

(Hint: if you need to integrate $1/(2 + x^2)$ you could substitute $x = u\sqrt{2}$.)

- 470.** Graph the curve $y = (1/2)x^2 + 1$ and the straight line $y = x + 1$ and find the area between the curve and the line.
471. Find the area of the region between the parabolas $y^2 = x$ and $x^2 = 16y$.
472. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.
473. Find a so that the curves $y = x^2$ and $y = a \cos x$ intersect at the points $(x, y) = (\frac{\pi}{4}, \frac{\pi^2}{16})$. Then find the area between these curves.
474. Group Problem.

Write a definite integral whose value is the area of the region between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$. Find this area. If you cannot evaluate the integral by calculus you may use geometry to find the area. Hint: The part of a circle cut off by a line is a circular sector with a triangle removed.

3. Cavalieri's principle and volumes of solids

You can use integration to derive the formulas for volumes of spheres, cylinder, cones, and many many more solid objects in a systematic way. In this section we'll see the "method of slicing."

3.1. Example – Volume of a pyramid. As an example let's compute the volume of a pyramid whose base is a square of side 1, and whose height is 1. Our strategy will be to divide the pyramid into thin horizontal slices whose volumes we can compute, and to add the volumes of the slices to get the volume of the pyramid.

To construct the slices we choose a partition of the (height) interval $[0, 1]$ into N subintervals, i.e. we pick numbers

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1,$$

and as usual we set $\Delta x_k = x_k - x_{k-1}$, we define the mesh size of the partition to be the largest of the Δx_k .

The k^{th} slice consists of those points on the pyramid whose height is between x_{k-1} and x_k . The intersection of the pyramid with the plane at height x is a square, and by similarity the length of the side of this square is $1 - x$. Therefore the bottom of the k^{th} slice is a square with side $1 - x_{k-1}$, and its top is a square with side $1 - x_k$. The height of the slice is $x_k - x_{k-1} = \Delta x_k$.

Thus the k^{th} slice *contains* a block of height Δx_k whose base is a square with sides $1 - x_k$, and its volume must therefore be larger than $(1 - x_k)^2 \Delta x_k$. On the other hand the k^{th} slice *is contained in* a block of the same height whose base is a square with sides $1 - x_{k-1}$. The volume of the slice is therefore not more than $(1 - x_{k-1})^2 \Delta x_k$. So we have

$$(1 - x_k)^2 \Delta x_k < \text{volume of } k^{\text{th}} \text{ slice} < (1 - x_{k-1})^2 \Delta x_k.$$

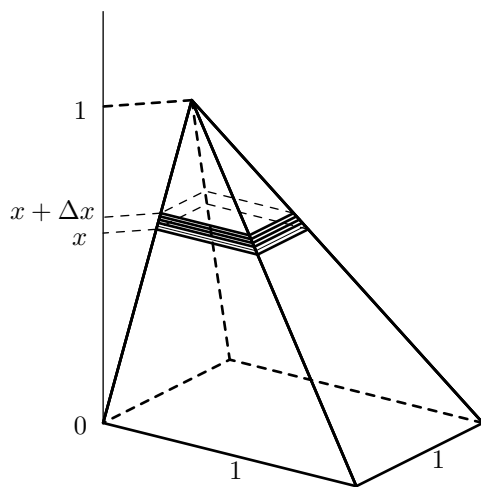


Figure 2. The slice at height x is a square with side $1 - x$.

Therefore there is some c_k in the interval $[x_{k-1}, x_k]$ such that

$$\text{volume of } k^{\text{th}} \text{ slice} = (1 - c_k)^2 \Delta x_k.$$

Adding the volumes of the slices we find that the volume V of the pyramid is given by

$$V = (1 - c_1)^2 \Delta x_1 + \cdots + (1 - c_N)^2 \Delta x_N.$$

The right hand side in this equation is a Riemann sum for the integral

$$I = \int_0^1 (1 - x)^2 dx$$

and therefore we have

$$I = \lim_{N \rightarrow \infty} \{(1 - c_1)^2 \Delta x_1 + \cdots + (1 - c_N)^2 \Delta x_N\} = V.$$

Compute the integral and you find that the volume of the pyramid is

$$V = \frac{1}{3}.$$

3.2. General case. The “method of slicing” which we just used to compute the volume of a pyramid works for solids of any shape. The strategy always consists of dividing the solid into many thin (horizontal) slices, compute their volumes, and recognize that the total volume of the slices is a Riemann sum for some integral. That integral then is the volume of the solid.

To be more precise, let a and b be the heights of the lowest and highest points on the solid, and let $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$ be a partition of the interval $[a, b]$. Such a partition divides the solid into N distinct slices, where slice number k consists of all points in the solid whose height is between x_{k-1} and x_k . The thickness of the k^{th} slice is $\Delta x_k = x_k - x_{k-1}$. If

$$A(x) = \text{area of the intersection of the solid with the plane at height } x.$$

then we can approximate the volume of the k^{th} slice by

$$A(c_k) \Delta x_k$$

where c_k is any number (height) between x_{k-1} and x_k .

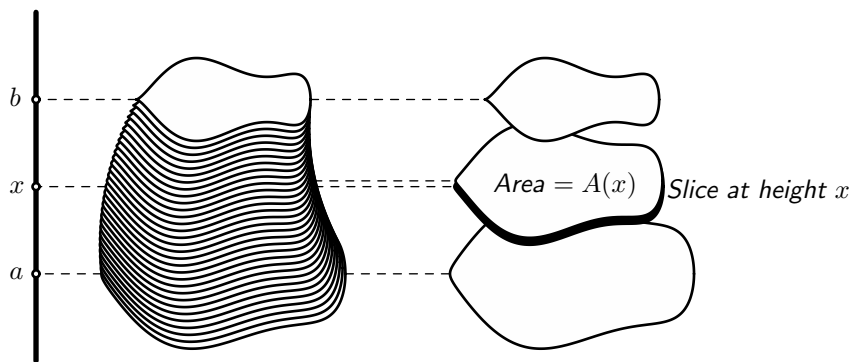


Figure 3. Slicing a solid to compute its volume. The volume of one slice is approximately the product of its thickness (Δx) and the area $A(x)$ of its top. Summing the volume $A(x)\Delta x$ over all slices leads approximately to the integral $\int_a^b f(x)dx$.

The total volume of all slices is therefore approximately

$$V \approx A(c_1)\Delta x_1 + \cdots + A(c_N)\Delta x_N.$$

While this formula only holds approximately, we expect the approximation to get better as we make the partition finer, and thus

$$(62) \quad V = \lim_{N \rightarrow \infty} \{A(c_1)\Delta x_1 + \cdots + A(c_N)\Delta x_N\}.$$

On the other hand the sum on the right is a Riemann sum for the integral $I = \int_a^b A(x)dx$, so the limit is exactly this integral. Therefore we have

$$(63) \quad V = \int_a^b A(x)dx.$$

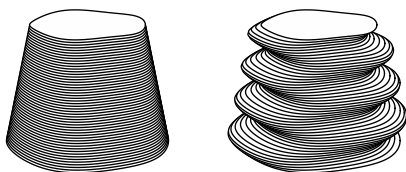


Figure 4. Cavalieri's principle. Both solids consist of a pile of horizontal slices. The solid on the right was obtained from the solid on the left by sliding some of the slices to the left and others to the right. This operation does not affect the volumes of the slices, and hence both solids have the same volume.

3.3. Cavalieri's principle. The formula (63) for the volume of a solid which we have just derived shows that the volume only depends on the areas $A(x)$ of the cross sections of the solid, and not on the particular shape these cross sections may have. This observation is older than calculus itself and goes back at least to Bonaventura Cavalieri (1598 – 1647) who said: *If the intersections of two solids with a horizontal plane always have the same area, no matter what the height of the horizontal plane may be, then the two solids have the same volume.*

This principle is often illustrated by considering a stack of coins: If you put a number of coins on top of each other then the total volume of the coins is just the sum of the volumes of the coins. If you change the shape of the pile by sliding the coins horizontally then the volume of the pile will still be the sum of the volumes of the coins, i.e. it doesn't change.

3.4. Solids of revolution. In principle, formula (63) allows you to compute the volume of any solid, provided you can compute the areas $A(x)$ of all cross sections. One class of solids for which the areas of the cross sections are easy are the so-called “solids of revolution.”

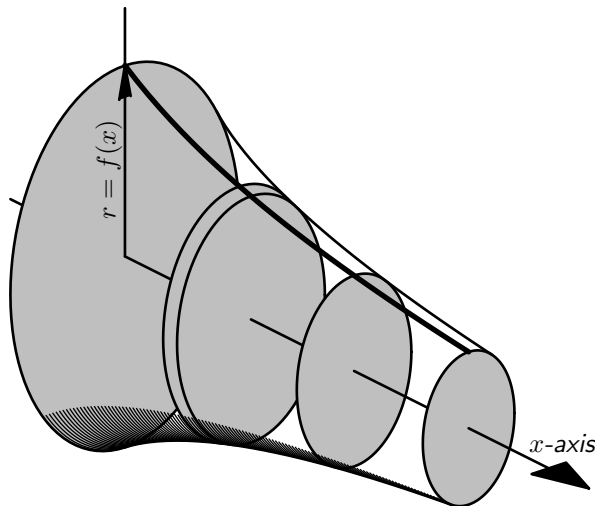


Figure 5. A solid of revolution consists of all points in three-dimensional space whose distance r to the x -axis satisfies $r \leq f(x)$.

A solid of revolution is created by rotating (revolving) the graph of a positive function around the x -axis. More precisely, let f be a function which is defined on an interval $[a, b]$ and which is always positive ($f(x) > 0$ for all x). If you now imagine the x -axis floating in three dimensional space, then the solid of revolution obtained by rotating the graph of f around the x -axis consists of all points in three-dimensional space with $a \leq x \leq b$, and whose distance to the x -axis is no more than $f(x)$.

Yet another way of describing the solid of revolution is to say that the solid is the union of all discs which meet the x -axis perpendicularly and whose radius is given by $r = f(x)$.

If we slice the solid with planes perpendicular to the x -axis, then (63) tells us the volume of the solid. Each slice is a disc of radius $r = f(x)$ so that its area is $A(x) = \pi r^2 = \pi f(x)^2$. We therefore find that

$$(64) \quad V = \pi \int_a^b f(x)^2 dx.$$

4. Examples of volumes of solids of revolution

4.1. Problem 1: Revolve $\mathcal{R}$ around the y -axis . Consider the solid obtained by revolving the region

$$\mathcal{R} = \{(x, y) \mid 0 \leq x \leq 2, \quad (x-1)^2 \leq y \leq 1\}$$

around the y -axis.

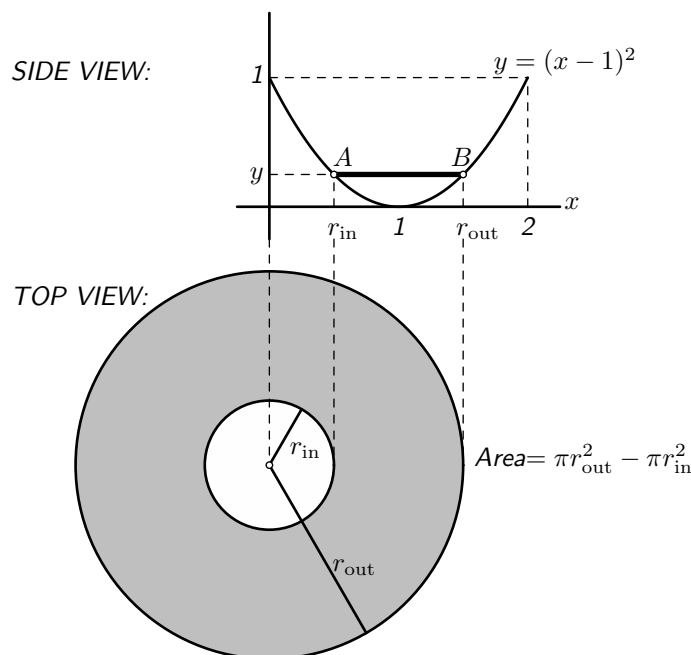
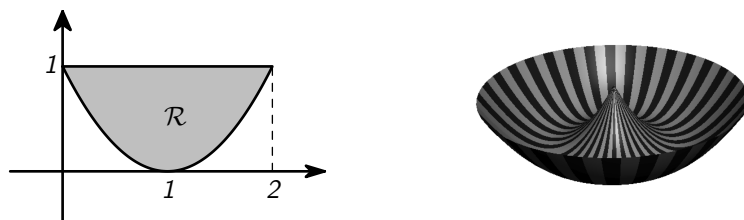


Figure 6. Computing the volume of the solid you get when you revolve the region $\mathcal{R}$ around the y -axis. A horizontal cross section of the solid is a “washer” with inner radius r_{in} , and outer radius r_{out} .



Solution: The region we have to revolve around the y -axis consists of all points above the parabola $y = (x - 1)^2$ but below the line $y = 1$.

If we intersect the solid with a plane at height y then we get a ring shaped region, or “annulus”, i.e. a large disc with a smaller disc removed. You can see it in the figure below: if you cut the region $\mathcal{R}$ horizontally at height y you get the line segment AB , and if you rotate this segment around the y -axis you get the grey ring region pictured below the graph. Call the radius of the outer circle r_{out} and the radius of the inner circle r_{in} . These radii are the two solutions of

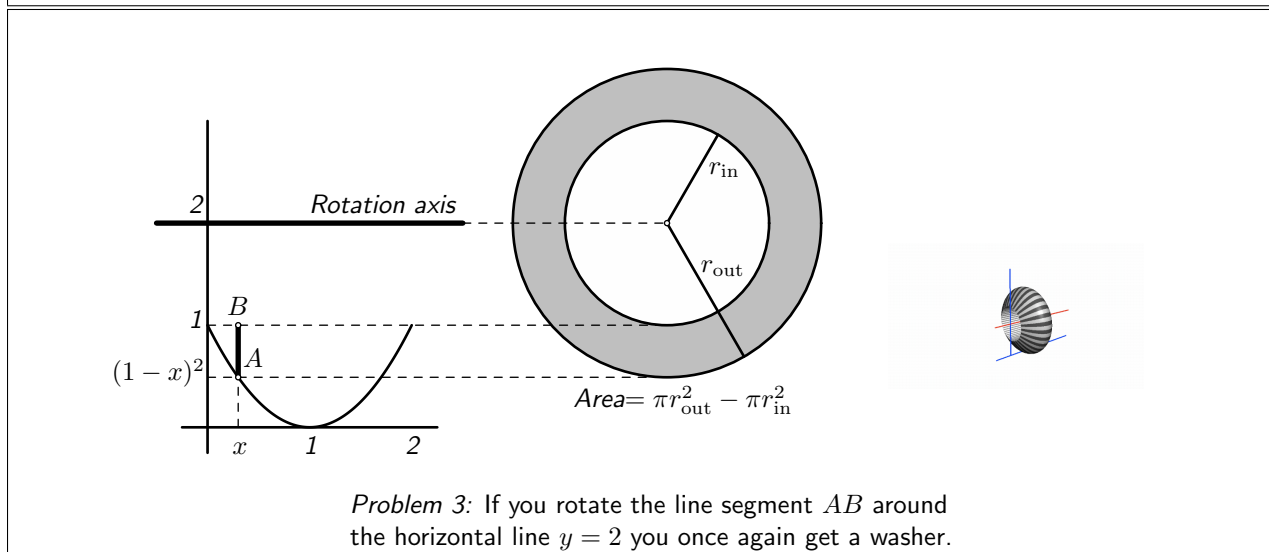
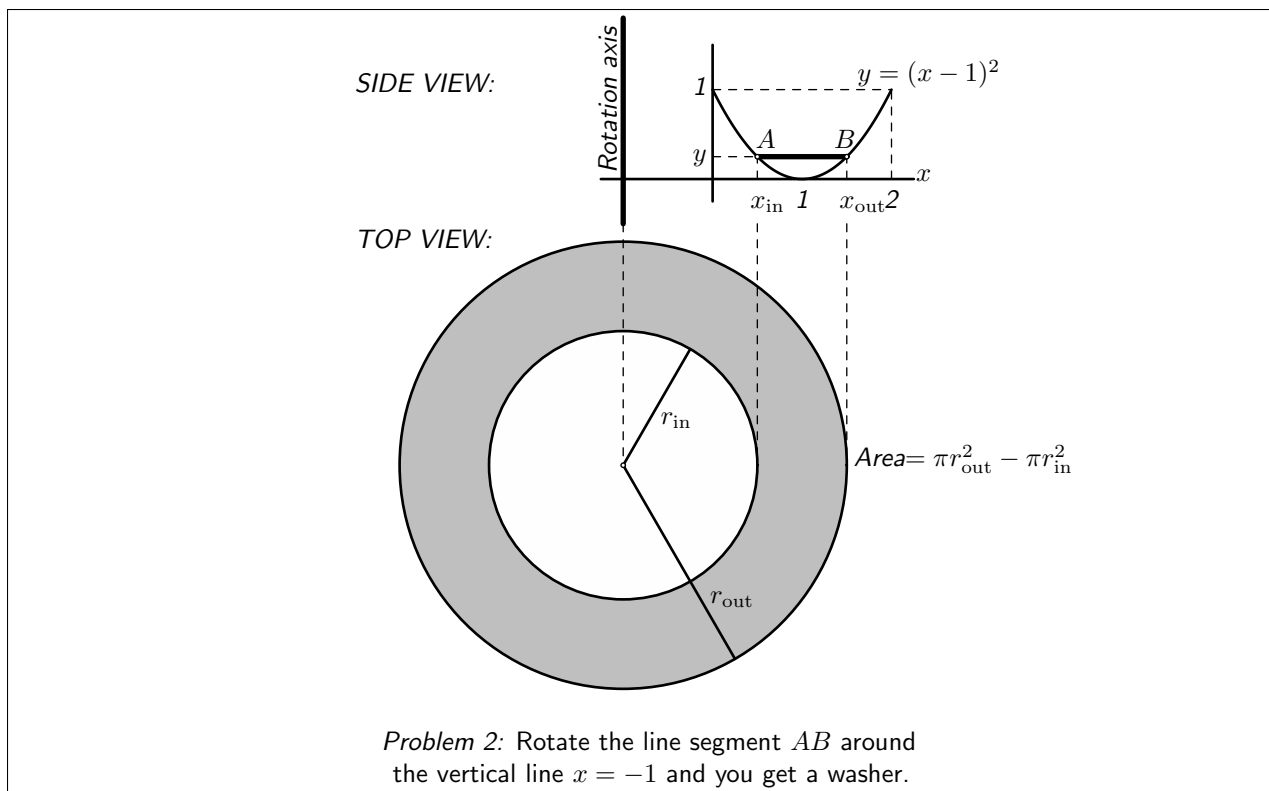
$$y = (1 - r)^2$$

so they are

$$r_{\text{in}} = 1 - \sqrt{y}, \quad r_{\text{out}} = 1 + \sqrt{y}.$$

The area of the cross section is therefore given by

$$A(y) = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2 = \pi(1 + \sqrt{y})^2 - \pi(1 - \sqrt{y})^2 = 4\pi\sqrt{y}.$$



The y -values which occur in the solid are $0 \leq y \leq 1$ and hence the volume of the solid is given by

$$V = \int_0^1 A(y) dy = 4\pi \int_0^1 \sqrt{y} dy = 4\pi \times \frac{2}{3} = \frac{8\pi}{3}.$$

4.2. Problem 2: Revolve $\mathcal{R}$ around the line $x = -1$. Find the volume of the solid of revolution obtained by revolving the same region $\mathcal{R}$ around the line $x = -1$.

Solution: The line $x = -1$ is vertical, so we slice the solid with horizontal planes. The height of each plane will be called y .

As before the slices are ring shaped regions but the inner and outer radii are now given by

$$r_{\text{in}} = 1 + x_{\text{in}} = 2 - \sqrt{y}, \quad r_{\text{out}} = 1 + x_{\text{out}} = 2 + \sqrt{y}.$$

The volume is therefore given by

$$V = \int_0^1 (\pi r_{\text{out}}^2 - \pi r_{\text{in}}^2) dy = \pi \int_0^1 8\sqrt{y} dy = \frac{16\pi}{3}.$$

4.3. Problem 3: Revolve $\mathcal{R}$ around the line $y = 2$. Compute the volume of the solid you get when you revolve the same region $\mathcal{R}$ around the line $y = 2$.

Solution: This time the line around which we rotate $\mathcal{R}$ is horizontal, so we slice the solid with planes perpendicular to the x -axis.

A typical slice is obtained by revolving the line segment AB about the line $y = 2$. The result is again an annulus, and from the figure we see that the inner and outer radii of the annulus are

$$r_{\text{in}} = 1, \quad r_{\text{out}} = 2 - (1 - x)^2.$$

The area of the slice is therefore

$$A(x) = \pi \{2 - (1 - x)^2\}^2 - \pi 1^2 = \pi \{3 - 4(1 - x)^2 + (1 - x)^4\}.$$

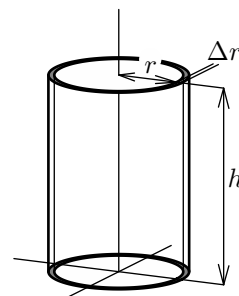
The x values which occur in the solid are $0 \leq x \leq 2$, and so its volume is

$$\begin{aligned} V &= \pi \int_0^2 \{3 - 4(1 - x)^2 + (1 - x)^4\} dx \\ &= \pi \left[3x + \frac{4}{3}(1 - x)^3 - \frac{1}{5}(1 - x)^5 \right]_0^2 \\ &= \frac{56}{15}\pi \end{aligned}$$

5. Volumes by cylindrical shells

Instead of slicing a solid with planes you can also try to decompose it into cylindrical shells. The volume of a cylinder of height h and radius r is $\pi r^2 h$ (height times area base). Therefore the volume of a cylindrical shell of height h , (inner) radius r and thickness Δr is

$$\begin{aligned} \pi h(r + \Delta r)^2 - \pi h r^2 &= \pi h(2r + \Delta r)\Delta r \\ &\approx 2\pi h r \Delta r. \end{aligned}$$



Now consider the solid you get by revolving the region

$$\mathcal{R} = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$$

around the y -axis. By partitioning the interval $a \leq x \leq b$ into many small intervals we can decompose the solid into many thin shells. The volume of each shell will approximately be given by $2\pi x f(x) \Delta x$. Adding the volumes of the shells, and taking the limit over finer and finer partitions we arrive at the following formula for the volume of the solid of revolution:

$$(65) \quad V = 2\pi \int_a^b x f(x) dx.$$

If the region $\mathcal{R}$ is not the region under the graph, but rather the region between the graphs of two functions $f(x) \leq g(x)$, then we get

$$V = 2\pi \int_a^b x \{g(x) - f(x)\} dx.$$

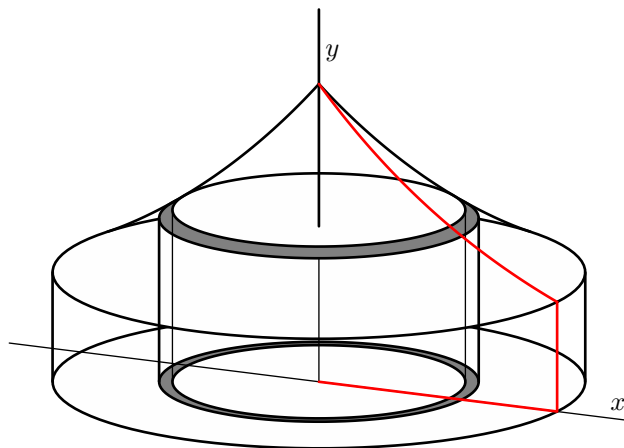


Figure 7. Computing the volume of a circus tent using cylindrical shells. This particular tent is obtained by rotating the graph of $y = e^{-x}$, $0 \leq x \leq 1$ around the y -axis.

5.1. Example – The solid obtained by rotating $\mathcal{R}$ about the y -axis, again. The region $\mathcal{R}$ from §4.1 can also be described as

$$\mathcal{R} = \{(x, y) \mid 0 \leq x \leq 2, f(x) \leq y \leq g(x)\},$$

where

$$f(x) = (x-1)^2 \text{ and } g(x) = 1.$$

The volume of the solid which we already computed in §4.1 is thus given by

$$\begin{aligned} V &= 2\pi \int_0^1 x \{1 - (x-1)^2\} dx \\ &= 2\pi \int_0^2 \{-x^3 + 2x^2\} dx \\ &= 2\pi \left[-\frac{1}{4}x^4 + \frac{2}{3}x^3\right]_0^2 \\ &= 8\pi/3, \end{aligned}$$

which coincides with the answer we found in §4.1.

6. Exercises

475. Group Problem.

What do the dots in “lim...” in equation (62) stand for? (i.e. what approaches what in this limit?)

Draw and describe the solids whose volume you are asked to compute in the following problems:

476. Find the volume enclosed by the paraboloid obtained by rotating the graph of $f(x) = R\sqrt{x/H}$ ($0 \leq x \leq H$) around the x -axis. Here R and H are positive constants. Draw the solid whose volume you are asked to compute, and indicate what R and H are in your drawing.

Find the volume of the solids you get by rotating each of the following graphs around the x -axis:

477. $f(x) = x$, $0 \leq x \leq 2$

478. $f(x) = \sqrt{2-x}$, $0 \leq x \leq 2$

479. $f(x) = (1+x^2)^{-1/2}$, $|x| \leq 1$

480. $f(x) = \sin x$, $0 \leq x \leq \pi$

481. $f(x) = 1-x^2$, $|x| \leq 1$

482. $f(x) = \cos x$, $0 \leq x \leq \pi$ (!!)

483. $f(x) = 1/\cos x$, $0 \leq x \leq \pi/4$

484. Find the volume that results by rotating the semicircle $y = \sqrt{R^2 - x^2}$ about the x -axis.

485. Let $\mathcal{T}$ be triangle $1 \leq x \leq 2$, $0 \leq y \leq 3x - 3$.

(a) Find the volume of the solid obtained by rotating the triangle $\mathcal{T}$ around the x -axis.

(b) Find the volume that results by rotating the triangle $\mathcal{T}$ around the y axis.

(c) Find the volume that results by rotating the triangle $\mathcal{T}$ around the line $x = -1$.

(d) Find the volume that results by rotating the triangle $\mathcal{T}$ around the line $y = -1$.

486. Group Problem.

(a) A spherical bowl of radius a contains water to a depth $h < 2a$. Find the volume of the water in the bowl. (Which solid of revolution is implied in this problem?)

(b) Water runs into a spherical bowl of radius 5 ft at the rate of $0.2 \text{ ft}^3/\text{sec}$. How fast is the water level rising when the water is 4 ft deep?

7. Distance from velocity, velocity from acceleration

7.1. Motion along a line. If an object is moving on a straight line, and if its position at time t is $x(t)$, then we had defined the velocity to be $v(t) = x'(t)$. Therefore the position is an antiderivative of the velocity, and the fundamental theorem of calculus says that

$$(66) \quad \int_{t_a}^{t_b} v(t) \, dt = x(t_b) - x(t_a),$$

or

$$x(t_b) = x(t_a) + \int_{t_a}^{t_b} v(t) \, dt.$$

In words, the integral of the velocity gives you the distance travelled of the object (during the interval of integration).

Equation (66) can also be obtained using Riemann sums. Namely, to see how far the object moved between times t_a and t_b we choose a partition $t_a = t_0 < t_1 < \cdots < t_N = t_b$. Let Δs_k be the distance travelled during the time interval (t_{k-1}, t_k) . The length of this time interval is $\Delta t_k = t_k - t_{k-1}$. During this time interval the velocity $v(t)$ need not be constant, but if the time interval is short enough then we can estimate the velocity by $v(c_k)$ where c_k is some number between t_{k-1} and t_k . We then have

$$\Delta s_k = v(c_k) \Delta t_k$$

and hence the total distance travelled is the sum of the travel distances for all time intervals $t_{k-1} < t < t_k$, i.e.

$$\text{Distance travelled} \approx \Delta s_1 + \cdots + \Delta s_N = v(c_1) \Delta t_1 + \cdots + v(c_N) \Delta t_N.$$

The right hand side is again a Riemann sum for the integral in (66). As one makes the partition finer and finer you therefore get

$$\text{Distance travelled} = \int_{t_a}^{t_b} v(t) \, dt.$$

The return of the dummy. Often you want to write a formula for $x(t) = \cdots$ rather than $x(t_b) = \cdots$ as we did in (66), i.e. you want to say what the position is at time t , instead of at time t_a . For instance, you might want to express the fact that the position $x(t)$ is equal to the initial position $x(0)$ plus the integral of the velocity from 0 to t . To do this you cannot write

$$x(t) = x(0) + \int_0^t v(t) \, dt \quad \Leftarrow \quad \text{BAD FORMULA}$$

because the variable t gets used in two incompatible ways: the t in $x(t)$ on the left, and in the upper bound on the integral ($\int^t$) are the same, but they are not the same as the two t 's in $v(t)dt$. The latter is a dummy variable (see §8 and §3.3). To fix this formula we should choose a different letter or symbol for the integration variable. A common choice in this situation is to decorate the integration variable with a prime (t'), a tilde ($\bar{t}$) or a bar ($\bar{t}$). So you can write

$$x(t) = x(0) + \int_0^t v(\bar{t}) \, d\bar{t}$$

7.2. Velocity from acceleration. The acceleration of the object is by definition the rate of change of its velocity,

$$a(t) = v'(t),$$

so you have

$$v(t) = v(0) + \int_0^t a(\bar{t}) d\bar{t}.$$

Conclusion: *If you know the acceleration $a(t)$ at all times t , and also the velocity $v(0)$ at time $t = 0$, then you can compute the velocity $v(t)$ at all times by integrating.*

7.3. Free fall in a constant gravitational field. If you drop an object then it will fall, and as it falls its velocity increases. The object's motion is described by the fact that *its acceleration is constant*. This constant is called g and is about $9.8\text{m/sec}^2 \approx 32\text{ft/sec}^2$. If we designate the upward direction as positive then $v(t)$ is the upward velocity of the object, and this velocity is actually decreasing. Therefore the constant acceleration is negative: it is $-g$.

If you write $h(t)$ for the height of the object at time t then its velocity is $v(t) = h'(t)$, and its acceleration is $h''(t)$. Since the acceleration is constant you have the following formula for the velocity at time t :

$$v(t) = v(0) + \int_0^t (-g) d\bar{t} = v(0) - gt.$$

Here $v(0)$ is the velocity at time $t = 0$ (the "initial velocity").

To get the height of the object at any time t you must integrate the velocity:

$$\begin{aligned} h(t) &= h(0) + \int_0^t v(\bar{t}) d\bar{t} && \text{(Note the use of the dummy } \bar{t}) \\ &= h(0) + \int_0^t \{v(0) - g\bar{t}\} d\bar{t} && \text{(use } v(\bar{t}) = v(0) - g\bar{t}) \\ &= h(0) + [v(0)\bar{t} - \frac{1}{2}g\bar{t}^2]_{\bar{t}=0}^{\bar{t}=t} \\ &= h(0) + v(0)t - \frac{1}{2}gt^2. \end{aligned}$$

For instance, if you launch the object upwards with velocity 5ft/sec from a height of 10ft, then you have

$$h(0) = 10\text{ft}, \quad v(0) = +5\text{ft/sec},$$

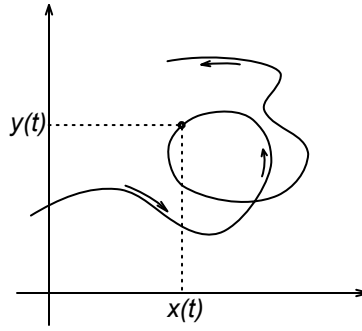
and thus

$$h(t) = 10 + 5t - 32t^2/2 = 10 + 5t - 16t^2.$$

The object reaches its maximum height when $h(t)$ has a maximum, which is when $h'(t) = 0$. To find that height you compute $h'(t) = 5 - 32t$ and conclude that $h(t)$ is maximal at $t = \frac{5}{32}\text{sec}$. The maximal height is then

$$h_{\max} = h(\frac{5}{32}) = 10 + \frac{25}{32} - \frac{25}{64} = 10\frac{25}{64}\text{ft}.$$

7.4. Motion in the plane – parametric curves. To describe the motion of an object in the plane you could keep track of its x and y coordinates at all times t . This would give you *two* functions of t , namely, $x(t)$ and $y(t)$, both of which are defined on the same interval $t_0 \leq t \leq t_1$ which describes the duration of the motion you are describing. In this context a pair of functions $(x(t), y(t))$ is called **a parametric curve**.



As an example, consider the motion described by

$$x(t) = \cos t, \quad y(t) = \sin t \quad (0 \leq t \leq 2\pi).$$

In this motion the point $(x(t), y(t))$ lies on the unit circle since

$$x(t)^2 + y(t)^2 = \cos^2 t + \sin^2 t = 1.$$

As t increases from 0 to 2π the point $(x(t), y(t))$ goes around the unit circle exactly once, in the counter-clockwise direction.

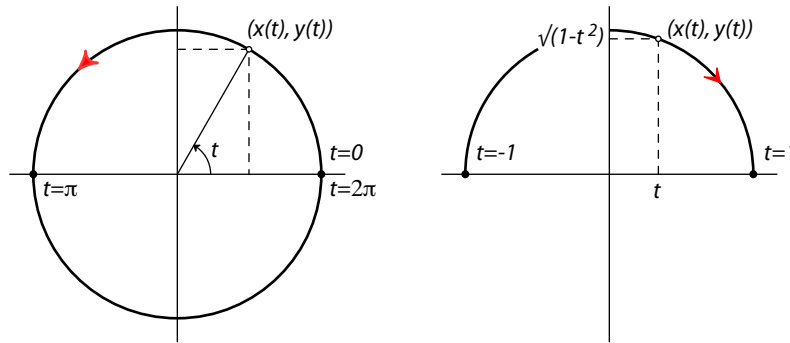


Figure 8. Two motions in the plane. On the left $x(t) = \cos t$, $y(t) = \sin t$ with $0 \leq t \leq 2\pi$, and on the right $x(t) = t$, $y(t) = \sqrt{1-t^2}$ with $-1 \leq t \leq 1$.

In another example one could consider

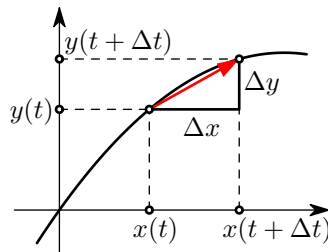
$$x(t) = t, \quad y(t) = \sqrt{1-t^2}, \quad (-1 \leq t \leq 1).$$

Here at all times the x and y coordinates satisfy

$$x(t)^2 + y(t)^2 = 1$$

again so that the point $(x(t), y(t)) = (t, \sqrt{1-t^2})$ again lies on the unit circle. Unlike the previous example we now always have $y(t) \geq 0$ (since $y(t)$ is the square root of something), and unlike the previous example the motion is only defined for $-1 \leq t \leq 1$. As t increases from -1 to $+1$, $x(t) = t$ does the same, and hence the point $(x(t), y(t))$ moves along the *upper half of the unit circle* from the leftmost point to the rightmost point.

7.5. The velocity of an object moving in the plane. We have seen that the velocity of an object which is moving along a line is the derivative of its position. If the object is allowed to move in the plane, so that its motion is described by a parametric curve $(x(t), y(t))$, then we can differentiate both $x(t)$ and $y(t)$, which gives us $x'(t)$ and $y'(t)$, and which leaves us with the following question: *At what speed is a particle moving if it is undergoing the motion $(x(t), y(t))$ ($t_a \leq t \leq t_b$) ?*



To answer this question we consider a short time interval $(t, t + \Delta t)$. During this time interval the particle moves from $(x(t), y(t))$ to $(x(t + \Delta t), y(t + \Delta t))$. Hence it has traveled a distance

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

where

$$\Delta x = x(t + \Delta t) - x(t), \text{ and } \Delta y = y(t + \Delta t) - y(t).$$

Dividing by Δt you get

$$\frac{\Delta s}{\Delta t} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2},$$

for the average velocity over the time interval $[t, t + \Delta t]$. Letting $\Delta t \rightarrow 0$ you find the velocity at time t to be

$$(67) \quad v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

7.6. Example – the two motions on the circle from §7.4. If a point moves along a circle according to $x(t) = \cos t$, $y(t) = \sin t$ (figure 8 on the left) then

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = +\cos t$$

so

$$v(t) = \sqrt{(-\cos t)^2 + (\sin t)^2} = 1.$$

The velocity of this motion is therefore always the same; the point $(\cos t, \sin t)$ moves along the unit circle with constant velocity.

In the second example in §7.4 we had $x(t) = t$, $y(t) = \sqrt{1 - t^2}$, so

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \frac{-t}{\sqrt{1 - t^2}}$$

whence

$$v(t) = \sqrt{1^2 + \frac{t^2}{1 - t^2}} = \frac{1}{\sqrt{1 - t^2}}.$$

Therefore the point $(t, \sqrt{1 - t^2})$ moves along the upper half of the unit circle from the left to the right, and its velocity changes according to $v = 1/\sqrt{1 - t^2}$.

8. The length of a curve

8.1. Length of a parametric curve. Let $(x(t), y(t))$ be some parametric curve defined for $t_a \leq t \leq t_b$. To find the length of this curve you can reason as follows: The length of the curve should be the distance travelled by the point $(x(t), y(t))$ as t increases from t_a to t_b . At each moment in time the velocity $v(t)$ of the point is given by (67), and therefore the distance traveled should be

$$(68) \quad s = \int_{t_a}^{t_b} v(t) dt = \int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Alternatively, you could try to compute the distance travelled by means of Riemann sums. Choose a partition

$$t_a = t_0 < t_1 < \cdots < t_N = t_b$$

of the interval $[t_a, t_b]$. You then get a sequence of points $P_0(x(t_0), y(t_0))$, $P_1(x(t_1), y(t_1))$, $\dots$, $P_N(x(t_N), y(t_N))$, and after “connecting the dots” you get a polygon. You could approximate the length of the curve by computing the length of this polygon. The distance between two consecutive points P_{k-1} and P_k is

$$\begin{aligned} \Delta s_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{\left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta t_k}\right)^2} \Delta t_k \\ &\approx \sqrt{x'(c_k)^2 + y'(c_k)^2} \Delta t_k \end{aligned}$$

where we have approximated the difference quotients

$$\frac{\Delta x_k}{\Delta t_k} \text{ and } \frac{\Delta y_k}{\Delta t_k}$$

by the derivatives $x'(c_k)$ and $y'(c_k)$ for some c_k in the interval $[t_{k-1}, t_k]$.

The total length of the polygon is then

$$\sqrt{x'(c_1)^2 + y'(c_1)^2} \Delta t_1 + \cdots + \sqrt{x'(c_N)^2 + y'(c_N)^2} \Delta t_N$$

This is a Riemann sum for the integral $\int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2} dt$, and hence we find (once more) that the length of the curve is

$$s = \int_{t_a}^{t_b} \sqrt{x'(t)^2 + y'(t)^2} dt.$$

8.2. The length of the graph of a function. The graph of a function ($y = f(x)$ with $a \leq x \leq b$) is also a curve in the plane, and you can ask what its length is. We will now find this length by *representing the graph as a parametric curve* and applying the formula (68) from the previous section.

The standard method of representing the graph of a function $y = f(x)$ by a parametric curve is to choose

$$x(t) = t, \text{ and } y(t) = f(t), \text{ for } a \leq t \leq b.$$

This parametric curve traces the graph of $y = f(x)$ from left to right as t increases from a to b .

Since $x'(t) = 1$ and $y'(t) = f'(t)$ we find that the length of the graph is

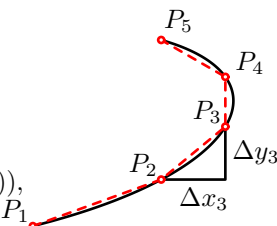
$$L = \int_a^b \sqrt{1 + f'(t)^2} dt.$$

The variable t in this integral is a dummy variable and we can replace it with any other variable we like, for instance, x :

$$(69) \quad L = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

In Leibniz' notation we have $y = f(x)$ and $f'(x) = dy/dx$ so that Leibniz would have written

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$



9. Examples of length computations

9.1. Length of a circle. In §8 we parametrized the unit circle by

$$x(t) = \cos t, \quad y(t) = \sin t, \quad (0 \leq t \leq 2\pi)$$

and computed $\sqrt{x'(t)^2 + y'(t)^2} = 1$. Therefore our formula tells us that the length of the unit circle is

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^{2\pi} 1 dt = 2\pi.$$

This cannot be a PROOF that the unit circle has length 2π since we have already used that fact to define angles in radians, to define the trig functions Sine and Cosine, and to find their derivatives. But our computation shows that the length formula (69) is at least consistent with what we already knew.

9.2. Length of a parabola. Consider our old friend, the parabola $y = x^2$, $0 \leq x \leq 1$. While the area under its graph was easy to compute ($\frac{1}{3}$), its length turns out to be much more complicated.

Our length formula (69) says that the length of the parabola is given by

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx^2}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 4x^2} dx.$$

To find this integral you would have to use one of the following (not at all obvious) substitutions¹

$$x = \frac{1}{4}\left(z - \frac{1}{z}\right) \quad (\text{then } 1 + 4x^2 = \frac{1}{4}(z + 1/z)^2 \text{ so you can simplify the } \sqrt{\cdot})$$

or (if you like hyperbolic functions)

$$x = \frac{1}{2} \sinh w \quad (\text{in which case } \sqrt{1 + 4x^2} = \cosh w.)$$

9.3. Length of the graph of the Sine function. To compute the length of the curve given by $y = \sin x$, $0 \leq x \leq \pi$ you would have to compute this integral:

$$(70) \quad L = \int_0^\pi \sqrt{1 + \left(\frac{d \sin x}{dx}\right)^2} dx = \int_0^\pi \sqrt{1 + \cos^2 x} dx.$$

Unfortunately this is not an integral which can be computed in terms of the functions we know in this course (it's an "elliptic integral of the second kind.") This happens very often with the integrals that you run into when you try to compute the length of a curve. In spite of the fact that we get stuck when we try to compute the integral in (70), the formula is not useless. For example, since $-1 \leq \cos x \leq 1$ we know that

$$1 \leq \sqrt{1 + \cos^2 x} \leq \sqrt{1 + 1} = \sqrt{2},$$

and therefore the length of the Sine graph is bounded by

$$\int_0^\pi 1 dx \leq \int_0^\pi \sqrt{1 + \cos^2 x} dx \leq \int_0^\pi \sqrt{2} dx,$$

i.e.

$$\pi \leq L \leq \pi\sqrt{2}.$$

10. Exercises

487. Find the length of the piece of the graph of $y = \sqrt{1 - x^2}$ where $0 \leq x \leq \frac{1}{2}$.

The graph is a circle, so there are two ways of computing this length. One uses geometry (length of a circular arc = radius $\times$ angle), the other uses an integral.

Use both methods and check that you get the same answer.

488. Compute the length of the part of the *evolute* of the circle, given by

$$x(t) = \cos t - t \sin t, \quad y(t) = \sin t + t \cos t$$

where $0 < t < \pi$.

489. Group Problem.

¹Many calculus textbooks will tell you to substitute $x = \tan \theta$, but the resulting integral is still not easy.

Show that the *Archimedean spiral*, given by

$$x(\theta) = \theta \cos \theta, \quad y(\theta) = \theta \sin \theta, \quad 0 \leq \theta \leq \pi$$

has the same length as the parabola given by

$$y = \frac{1}{2}x^2, \quad 0 \leq x \leq \pi.$$

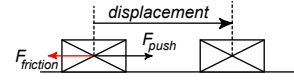
Hint: you can set up integrals for both lengths. If you get the same integral in both cases, then you know the two curves have the same length (even if you don't try to compute the integrals).

11. Work done by a force

11.1. Work as an integral. In Newtonian mechanics a force which acts on an object in motion performs a certain amount of work, i.e. it spends a certain amount of energy. If the force which acts is constant, then the work done by this force is

$$\text{Work} = \text{Force} \times \text{Displacement}.$$

For example if you are pushing a box forward then there will be two forces acting on the box: the force you apply, and the friction force of the floor on the box. The amount of work you do is the product of the force you exert and the length of the displacement. Both displacement and the force you apply are pointed towards the right, so both are positive, and the work you do (energy you provide to the box) is positive.



The amount of work done by the friction is similarly the product of the friction force and the displacement. Here the displacement is still to the right, but the friction force points to the left, so it is negative. The work done by the friction force is therefore negative. Friction extracts energy from the box.

Suppose now that the force $F(t)$ on the box is not constant, and that its motion is described by saying that its position at time t is $x(t)$. The basic formula work = force $\times$ displacement does not apply directly since it assumes that the force is constant. To compute the work done by the varying force $F(t)$ we choose a partition of the time interval $t_a \leq t \leq t_b$ into

$$t_a = t_0 < t_1 < \cdots < t_{N-1} < t_N = t_b$$

In each short time interval $t_{k-1} \leq t \leq t_k$ we assume the force is (almost) constant and we approximate it by $F(c_k)$ for some $t_{k-1} \leq c_k \leq t_k$. If we also assume that the velocity $v(t) = x'(t)$ is approximately constant between times t_{k-1} and t_k then the displacement during this time interval will be

$$x(t_k) - x(t_{k-1}) \approx v(c_k)\Delta t_k,$$

where $\Delta t_k = t_k - t_{k-1}$. Therefore the work done by the force F during the time interval $t_{k-1} \leq t \leq t_k$ is

$$\Delta W_k = F(c_k)v(c_k)\Delta t_k.$$

Adding the work done during each time interval we get the total work done by the force between time t_a and t_b :

$$W = F(c_1)v(c_1)\Delta t_1 + \cdots + F(c_N)v(c_N)\Delta t_N.$$

Again we have a Riemann sum for an integral. If we take the limit over finer and finer partitions we therefore find that the work done by the force $F(t)$ on an object whose motion is described by $x(t)$ is

$$(71) \quad W = \int_{t_a}^{t_b} F(t)v(t)dt,$$

in which $v(t) = x'(t)$ is the velocity of the object.

11.2. Kinetic energy. Newton's famous law relating the force exerted on an object and its motion says $F = ma$, where a is the acceleration of the object, m is its mass, and F is the combination of all forces acting on the object. If the position of the object at time t is $x(t)$, then its velocity and acceleration are $v(t) = x'(t)$ and $a(t) = v'(t) = x''(t)$, and thus the total force acting on the object is

$$F(t) = ma(t) = m \frac{dv}{dt}.$$

The work done by the total force is therefore

$$(72) \quad W = \int_{t_a}^{t_b} F(t)v(t)dt = \int_{t_a}^{t_b} m \frac{dv(t)}{dt} v(t) dt.$$

Even though we have not assumed anything about the motion, so we don't know anything about the velocity $v(t)$, we can still do this integral. The key is to notice that, by the chain rule,

$$m \frac{dv(t)}{dt} v(t) = \frac{d}{dt} \frac{1}{2} m v(t)^2.$$

(Remember that m is a constant.) This says that the quantity

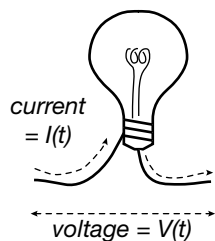
$$K(t) = \frac{1}{2} m v(t)^2$$

is the antiderivative we need to do the integral (72). We get

$$W = \int_{t_a}^{t_b} m \frac{dv(t)}{dt} v(t) dt = \int_{t_a}^{t_b} K'(t) dt = K(t_b) - K(t_a).$$

In Newtonian mechanics the quantity $K(t)$ is called **the kinetic energy** of the object, and our computation shows that *the amount by which the kinetic energy of an object increases is equal to the amount of work done on the object.*

12. Work done by an electric current



If at time t an electric current $I(t)$ (measured in Ampère) flows through an electric circuit, and if the voltage across this circuit is $V(t)$ (measured in Volts) then the energy supplied to the circuit per second is $I(t)V(t)$. Therefore the total energy supplied during a time interval $t_0 \leq t \leq t_1$ is the integral

$$\text{Energy supplied} = \int_{t_0}^{t_1} I(t)V(t)dt.$$

(measured in Joule; the energy consumption of a circuit is defined to be how much energy it consumes per time unit, and the power consumption of a circuit which consumes 1 Joule per second is said to be one Watt.)

If a certain voltage is applied to a simple circuit (like a light bulb) then the current flowing through that circuit is determined by the resistance R of that circuit by *Ohm's law*² which says

$$I = \frac{V}{R}.$$

²http://en.wikipedia.org/wiki/Ohm's_law

12.1. Example. If the resistance of a light bulb is $R = 200\Omega$, and if the voltage applied to it is

$$V(t) = 150 \sin 2\pi ft$$

where $f = 50\text{sec}^{-1}$ is the frequency, then how much energy does the current supply to the light bulb in one second?

To compute this we first find the current using Ohm's law,

$$I(t) = \frac{V(t)}{R} = \frac{150}{200} \sin 2\pi ft = 0.75 \sin 2\pi ft. \quad (\text{Amp})$$

The energy supplied in one second is then

$$\begin{aligned} E &= \int_0^{1 \text{ sec}} I(t)V(t)dt \\ &= \int_0^1 (150 \sin 2\pi ft) \times (0.75 \sin 2\pi ft) dt \\ &= 112.5 \int_0^1 \sin^2(2\pi ft) dt \end{aligned}$$

You can do this last integral by using the double angle formula for the cosine, to rewrite

$$\sin^2(2\pi ft) = \frac{1}{2}\{1 - \cos 4\pi ft\} = \frac{1}{2} - \frac{1}{2} \cos 4\pi ft.$$

Keep in mind that $f = 50$, and you find that the integral is

$$\int_0^1 \sin^2(2\pi ft) dt = \left[\frac{t}{2} - \frac{1}{4\pi f} \sin 4\pi ft \right]_0^1 = \frac{1}{2},$$

and hence the energy supplied to the light bulb during one second is

$$E = 112.5 \times \frac{1}{2} = 56.25(\text{Joule}).$$

Answers and Hints

- 1 The decimal expansion of

$$1/7 = 0.\overline{142857} 142857 142857 \dots$$

repeats after 6 digits. Since $2007 = 334 \times 6 + 3$ the 2007th digit is the same as the 3rd, which happens to be a 2.

- 6
- $100x = 31.313131\dots = 31 + x \implies 99x = 31 \implies x = \frac{31}{99}$
- .

Similarly, $y = \frac{273}{999}$.

In z the initial "2" is not part of the repeating pattern, so subtract it: $z = 0.2 + 0.0154154154\dots$. Now

$$\begin{aligned} 1000 \times 0.0154154154\dots &= 15.4154154154\dots \\ &= 15.4 + 0.0154154154\dots \\ &= 15\frac{2}{5} + 0.0154154154\dots \\ \implies 0.0154154\dots &= \frac{15\frac{2}{5}}{999}. \end{aligned}$$

From this you get

$$z = \frac{1}{5} + \frac{15\frac{2}{5}}{999} = \frac{1076}{4995}.$$

- 8 They are the same function. Both are defined for all real numbers, and both will square whatever number you give them, so they are the same function.

- 14 Both are false:

(a) Since $\arcsin x$ is only defined if $-1 \leq x \leq 1$ and hence not for **all** x , it is not true that $\sin(\arcsin x) = x$ for all x . However, it is true that $\sin(\arcsin x) = x$ for all x in the interval $[-1, 1]$.

(b) $\arcsin(\sin x)$ is defined for all x since $\sin x$ is defined for all x , and $\sin x$ is always between -1 and 1 . However the arcsine function always returns a number (angle) between $-\pi/2$ and $\pi/2$, so $\arcsin(\sin x) = x$ can't be true when $x > \pi/2$ or $x < -\pi/2$.

- 30 (a)

$$\begin{aligned} \Delta y &= (x + \Delta x)^2 - 2(x + \Delta x) + 1 - [x^2 - 2x + 1] \\ &= (2x - 2)\Delta x + (\Delta x)^2 \text{ so that} \end{aligned}$$

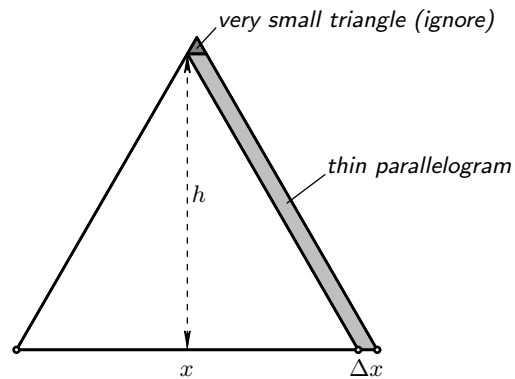
$$\frac{\Delta y}{\Delta x} = 2x - 2 + \Delta x$$

- 31 At
- A
- and
- B
- the graph of
- f
- is tangent to the drawn lines, so the derivative at
- A
- is
- -1
- and the derivative at
- B
- is
- $+1$
- .

- 32
- Δx
- : feet.
- Δy
- pounds.
- $\frac{\Delta y}{\Delta x}$
- and
- $\frac{dy}{dx}$
- are measured in pounds per foot.

- 33 Gallons per second.

- 34 (a)
- $A(x)$
- is an area so it has units square inch and
- x
- is measured in inches, so
- $\frac{dA}{dx}$
- is measured in
- $\frac{\text{inch}^2}{\text{inch}} = \text{inch}$
- .



- (b) Hint: The extra area
- ΔA
- that you get when the side of an equilateral triangle grows from
- x
- to
- $x + \Delta x$
- can be split into a thin parallelogram and a very tiny triangle. Ignore the area of the tiny triangle since the area of the parallelogram will be much larger. What is the area of this parallelogram?

The area of a parallelogram is "base times height" so here it is $h \times \Delta x$, where h is the height of the triangle.

Conclusion: $\frac{\Delta A}{\Delta x} \approx \frac{h\Delta x}{\Delta x} = h$.

The derivative is therefore the height of the triangle.

- 39
- $\delta = \varepsilon/2$
- .

- 40
- $\delta = \min\{1, \frac{1}{6}\varepsilon\}$

- 41
- $|f(x) - (-7)| = |x^2 - 7x + 10| = |x - 2| \cdot |x - 5|$
- . If you choose
- $\delta \leq 1$
- then
- $|x - 2| < \delta$
- implies
- $1 < x < 3$
- , so that
- $|x - 5|$
- is at most
- $|1 - 5| = 4$
- .

So, choosing $\delta \leq 1$ we always have $|f(x) - L| < 4|x - 2|$ and $|f(x) - L| < \varepsilon$ will follow from $|x - 2| < \frac{1}{4}\varepsilon$.

Our choice is then: $\delta = \min\{1, \frac{1}{4}\varepsilon\}$.

- 42
- $f(x) = x^3$
- ,
- $a = 3$
- ,
- $L = 27$
- .

When $x = 3$ one has $x^3 = 27$, so $x^3 - 27 = 0$ for $x = 3$. Therefore you can factor out $x - 3$ from $x^3 - 27$ by doing a long division. You get $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$, and thus

$$|f(x) - L| = |x^3 - 27| = |x^2 + 3x + 9| \cdot |x - 3|.$$

Never choose $\delta > 1$. Then $|x - 3| < \delta$ will imply $2 < x < 4$ and therefore

$$|x^2 + 3x + 9| \leq 4^2 + 3 \cdot 4 + 9 = 37.$$

So if we always choose $\delta \leq 1$, then we will always have

$$|x^3 - 27| \leq 37\delta \quad \text{for } |x - 3| < \delta.$$

Hence, if we choose $\delta = \min \{1, \frac{1}{37}\varepsilon\}$ then $|x - 3| < \delta$ guarantees $|x^3 - 27| < \varepsilon$.

44 $f(x) = \sqrt{x}$, $a = 4$, $L = 2$.

You have

$$\sqrt{x} - 2 = \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} + 2} = \frac{x - 4}{\sqrt{x} + 2}$$

and therefore

$$(73) \quad |f(x) - L| = \frac{1}{\sqrt{x} + 2}|x - 4|.$$

Once again it would be nice if we could replace $1/(\sqrt{x} + 2)$ by a constant, and we achieve this by always choosing $\delta \leq 1$. If we do that then for $|x - 4| < \delta$ we always have $3 < x < 5$ and hence

$$\frac{1}{\sqrt{x} + 2} < \frac{1}{\sqrt{3} + 2},$$

since $1/(\sqrt{x} + 2)$ increases as you decrease x .

So, if we always choose $\delta \leq 1$ then $|x - 4| < \delta$ guarantees

$$|f(x) - 2| < \frac{1}{\sqrt{3} + 2}|x - 4|,$$

which prompts us to choose $\delta = \min \{1, (\sqrt{3} + 2)\varepsilon\}$.

A smarter solution: We can replace $1/(\sqrt{x} + 2)$ by a constant in (73), because for all x in the domain of f we have $\sqrt{x} \geq 0$, which implies

$$\frac{1}{\sqrt{x} + 2} \leq \frac{1}{2}.$$

Therefore $|\sqrt{x} - 2| \leq \frac{1}{2}|x - 4|$, and we could choose $\delta = 2\varepsilon$.

45 Hints:

$$\sqrt{x+6} - 3 = \frac{x+6-9}{\sqrt{x+6}+3} = \frac{x-3}{\sqrt{x+6}+3}$$

so

$$|\sqrt{x+6} - 3| \leq \frac{1}{3}|x - 3|.$$

46 We have

$$\left| \frac{1+x}{4+x} - \frac{1}{2} \right| = \left| \frac{x-2}{4+x} \right|.$$

If we choose $\delta \leq 1$ then $|x - 2| < \delta$ implies $1 < x < 3$ so that

$$\frac{1}{7} < \text{we don't care } \frac{1}{4+x} < \frac{1}{5}.$$

Therefore

$$\left| \frac{x-2}{4+x} \right| < \frac{1}{5}|x - 2|,$$

so if we want $|f(x) - \frac{1}{2}| < \varepsilon$ then we must require $|x - 2| < 5\varepsilon$. This leads us to choose

$$\delta = \min \{1, 5\varepsilon\}.$$

51 The equation (7) already contains a function f , but that is not the right function. In (7) Δx is the variable, and $g(\Delta x) = (f(x + \Delta x) - f(x))/\Delta x$ is the function; we want $\lim_{\Delta x \rightarrow 0} g(\Delta x)$.

67 $A(\frac{2}{3}, -1)$; $B(\frac{2}{5}, 1)$; $C(\frac{2}{7}, -1)$; $D(-1, 0)$; $E(-\frac{2}{5}, -1)$.

68 False! The limit must not only exist *but also be equal to $f(a)$* !

69 There are of course many examples. Here are two: $f(x) = 1/x$ and $f(x) = \sin(\pi/x)$ (see §7.3)

70 False! Here's an example: $f(x) = \frac{1}{x}$ and $g(x) = x - \frac{1}{x}$. Then f and g don't have limits at $x = 0$, but $f(x) + g(x) = x$ *does* have a limit as $x \rightarrow 0$.

71 False again, as shown by the example $f(x) = g(x) = \frac{1}{x}$.

79 $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ so the limit is $\lim_{\alpha \rightarrow 0} \frac{2 \sin \alpha \cos \alpha}{\sin \alpha} = \lim_{\alpha \rightarrow 0} 2 \cos \alpha = 2$.

Other approach: $\frac{\sin 2\alpha}{\sin \alpha} = \frac{\frac{\sin 2\alpha}{2\alpha}}{\frac{\sin \alpha}{\alpha}} \cdot \frac{2\alpha}{\alpha}$. Take the limit and you get 2.

80 $\frac{3}{2}$.

81 Hint: $\tan \theta = \frac{\sin \theta}{\cos \theta}$. Answer: the limit is 1.

82 $\frac{\tan 4\alpha}{\sin 2\alpha} = \frac{\tan 4\alpha}{4\alpha} \cdot \frac{2\alpha}{\sin 2\alpha} \cdot \frac{4\alpha}{2\alpha} = 1 \cdot 1 \cdot 2 = 2$

83 Hint: multiply top and bottom with $1 + \cos x$.

84 Hint: substitute $\theta = \frac{\pi}{2} - \varphi$, and let $\varphi \rightarrow 0$. Answer: -1 .

90 Substitute $\theta = x - \pi/2$ and remember that $\cos x = \cos(\theta + \frac{\pi}{2}) = -\sin \theta$. You get

$$\lim_{x \rightarrow \pi/2} \frac{x - \frac{\pi}{2}}{\cos x} = \lim_{\theta \rightarrow 0} \frac{\theta}{-\sin \theta} = -1.$$

91 Similar to the previous problem, once you use $\tan x = \frac{\sin x}{\cos x}$. The answer is again -1 .

93 Substitute $\theta = x - \pi$. Then $\lim_{x \rightarrow \pi} \theta = 0$, so

$$\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi} = \lim_{\theta \rightarrow 0} \frac{\sin(\pi + \theta)}{\theta} = -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = -1.$$

Here you have to remember from trigonometry that $\sin(\pi + \theta) = -\sin \theta$.

95 Note that the limit is for $x \rightarrow \infty$! As x goes to infinity $\sin x$ oscillates up and down between -1 and $+1$. Dividing by x then gives you a quantity which goes to zero. To give a good proof you use the Sandwich Theorem like this:

Since $-1 \leq \sin x \leq 1$ for all x you have

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}.$$

Since both $-1/x$ and $1/x$ go to zero as $x \rightarrow \infty$ the function in the middle must also go to zero. Hence

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0.$$

97 No. As $x \rightarrow 0$ the quantity $\sin \frac{1}{x}$ oscillates between -1 and $+1$ and does not converge to any particular value. Therefore, no matter how you choose k , it will never be true that $\lim_{x \rightarrow 0} \sin \frac{1}{x} = k$, because the limit doesn't exist.

98 The function $f(x) = (\sin x)/x$ is continuous at all $x \neq 0$, so we only have to check that $\lim_{x \rightarrow 0} f(x) = f(0)$, i.e. $\lim_{x \rightarrow 0} \frac{\sin x}{2x} = A$. This only happens if you choose $A = \frac{1}{2}$.

148 $f'(x) = 2 \tan x / \cos^2 x$ and $f''(x) = 2 / \cos^4 x + 4 \tan x \sin x / \cos^3 x$.

Since $\tan^2 x = \frac{1}{\cos^2 x} - 1$ one has $g'(x) = f'(x)$ and $g''(x) = f''(x)$.

154 $f'(x) = 2 \cos 2x + 3 \sin 3x$.

155 $f'(x) = -\frac{\pi}{x^2} \cos \frac{\pi}{x}$

156 $f'(x) = \cos(\cos 3x) \cdot (-\sin 3x) \cdot 3 = -3 \sin 3x \cos 3x$.

157 $f'(x) = \frac{x^2 \cdot 2x \sin x^2 - 2x \sin x^2}{x^4}$

158 $f'(x) = \frac{1}{(\cos \sqrt{1+x^2})^2} \frac{1}{2\sqrt{1+x^2}} \cdot 2x$

159 $f'(x) = 2(\cos x)(-\sin x) - 2(\cos x) \cdot 2x$

161 $f'(x) = \cos \frac{\pi}{x} + \frac{\pi}{x} \sin \frac{\pi}{x}$. At C one has $x = -\frac{2}{3}$, so $\cos \frac{\pi}{x} = 0$ and $\sin \frac{\pi}{x} = -1$. So at C one has $f'(x) = -\frac{3}{2}\pi$.

162 $v(x) = f(g(x)) = (x+5)^2 + 1 = x^2 + 10x + 26$

$w(x) = g(f(x)) = (x^2 + 1) + 5 = x^2 + 6$

$p(x) = f(x)g(x) = (x^2 + 1)(x + 5) = x^3 + 5x^2 + x + 5$

$q(x) = g(x)f(x) = f(x)g(x) = p(x)$.

165 (a) If $f(x) = \sin ax$, then $f''(x) = -a^2 \sin ax$, so $f''(x) = -64f(x)$ holds if $a^2 = 64$, i.e. $a = \pm 8$. So $\sin 8x$ and $\sin(-8x) = -\sin 8x$ are the two solutions you find this way.

(b) $a = \pm 8$, but A and b can have any value. All functions of the form $f(x) = A \sin(8x + b)$ satisfy (†).

166 (a) $V = S^3$, so the function f for which $V(t) = f(S(t))$ is the function $f(x) = x^3$.

(b) $S'(t)$ is the rate with which Bob's side grows with time. $V'(t)$ is the rate with which the Bob's volume grows with time.

Quantity Units

t minutes

$S(t)$ inch

$V(t)$ inch³

$S'(t)$ inch/minute

$V'(t)$ inch³/minute

(c) Three versions of the same answer:

$V(t) = f(S(t))$ so the chain rule says $V'(t) = f'(S(t))S'(t)$

$V(t) = S(t)^3$ so the chain rule says $V'(t) = 3S(t)^2 S'(t)$

$V = S^3$ so the chain rule says $\frac{dV}{dt} = 3S^2 \frac{dS}{dt}$.

(d) We are given $V(t) = 8$, and $V'(t) = 2$. Since $V = S^3$ we get $S = 2$. From (c) we know $V'(t) = 3S(t)^2 S'(t)$, so $2 = 3 \cdot 2^2 \cdot S'(t)$, whence $S'(t) = \frac{1}{6}$ inch per minute.

208 At $x = 3$.

209 At $x = a/2$.

210 At $x = a + 2a^3$.

211 At $x = a + \frac{1}{2}$.

215 False. If you try to solve $f(x) = 0$, then you get the equation $\frac{x^2 + |x|}{x} = 0$. If $x \neq 0$ then this is the same as $x^2 + |x| = 0$, which has no solutions (both terms are

positive when $x \neq 0$). If $x = 0$ then $f(x)$ isn't even defined. So there is no solution to $f(x) = 0$.

This doesn't contradict the IVT, because the function isn't continuous, in fact it isn't even defined at $x = 0$, so the IVT doesn't have to apply.

223 Not necessarily true, and therefore false. Consider the example $f(x) = x^4$, and see the next problem.

224 An inflection point is a point on the graph of a function where the second derivative changes its sign. At such a point you must have $f''(x) = 0$, but by itself that it is not enough.

227 The first is possible, e.g. $f(x) = x$ satisfies $f'(x) > 0$ and $f''(x) = 0$ for all x .

The second is impossible, since f'' is the derivative of f' , so $f'(x) = 0$ for all x implies that $f''(x) = 0$ for all x .

228 $y = 0$ at $x = -1, 0, 0$. Only sign change at $x = -1$, not at $x = 0$.

$x = 0$ loc min; $x = -\frac{4}{3}$ loc max; $x = -2/3$ inflection point. No global max or min.

229 zero at $x = 0, 4$; sign change at $x = 4$; loc min at $x = \frac{8}{3}$; loc max at $x = 0$; inflection point at $x = 4/3$. No global max or min.

230 sign changes at $x = 0, -3$; global min at $x = -3/4^{1/3}$; no inflection points, the graph is convex.

231 mirror image of previous problem.

232 $x^4 + 2x^2 - 3 = (x^2 - 1)(x^2 + 3)$ so sign changes at $x = \pm 1$. Global min at $x = 0$; graph is convex, no inflection points.

233 Sign changes at $\pm 2, \pm 1$; two global minima, at $\pm\sqrt{5}/2$; one local max at $x=0$; two inflection points, at $x = \pm\sqrt{5}/6$.

234 Sign change at $x = 0$; function is always increasing so no stationary points; inflection point at $x = 0$.

235 sign change at $x = 0, \pm 2$; loc max at $x = 2/5^{1/4}$; loc min at $x = -2/5^{1/4}$. inflection point at $x = 0$.

236 Function not defined at $x = -1$. For $x > -1$ sign change at $x = 0$, no stationary points, no inflection points (graph is concave). Horizontal asymptote $\lim_{x \rightarrow \infty} f(x) = 1$.

For $x < -1$ no sign change, function is increasing and convex, horizontal asymptote with $\lim_{x \rightarrow -\infty} f(x) = 1$.

237 global max (min) at $x = 1$ ($x=-1$), inflection points at $x = \pm\sqrt{3}$; horizontal asymptotes $\lim_{x \rightarrow \pm\infty} f(x) = 0$.

238 $y = 0$ at $x = 0$ but no sign changes anywhere; $x = 0$ is a global min; there's no local or global max; two inflection points at $x = \pm\frac{1}{3}\sqrt{3}$; horizontal asymptotes at height $y = 1$.

239 Not defined at $x = -1$. For $x > -1$ the graph is convex and has a minimum at $x = -1 + \sqrt{2}$; for $x < -1$ the graph is concave with a maximum at $x = -1 - \sqrt{2}$. No horizontal asymptotes.

240 Not def'd at $x = 0$. No sign changes (except at $x = 0$). For $x > 0$ convex with minimum at $x = 1$, for $x < 0$ concave with maximum at $x = -1$.

241 Not def'd at $x = 0$. Sign changes at $x = \pm 1$ and also at $x = 0$. No stationary points. Both branches ($x > 0$ and $x < 0$) are increasing. Non inflection points, no horizontal asymptotes.

242 Zero at $x = 0, -1$ sign only changes at -1 ; loc min at $x = -\frac{1}{3}$; loc max at $x = -1$. Inflection point at $x = -2/3$.

243 Changes sign at $x = -1 \pm \sqrt{2}$ and $x = 0$; loc min at $(-2 + \sqrt{7})/3$, loc max at $(-2 - \sqrt{7})/3$; inflection point at $x = -\frac{2}{3}$.

244 Factor $y = x^4 - x^3 - x = x(x^3 - x^2 - 1)$. One zero is obvious, namely at $x = 0$. For the other(s) you must solve $x^3 - x^2 - 1 = 0$ which is beyond what's expected in this course.

The derivative is $y' = 4x^3 - 3x^2 - 1$. A cubic function whose coefficients add up to 0 so $x = 1$ is a root, and you can factor $y' = 4x^3 - 3x^2 - 1 = (x - 1)(4x^2 + x + 1)$ from which you see that $x = 1$ is the only root. So: one stationary point at $x = 1$, which is a global minimum. The second derivative is $y'' = 12x^2 - 6x$; there are two inflection points, at $x = \frac{1}{2}$ and at $x = 0$.

245 Again one obvious solution to $y = 0$, namely $x = 0$. The other require solving a cubic equation.

The derivative is $y' = 4x^3 - 6x^2 + 2$ which is also cubic, but the coefficients add up to 0, so $x = 1$ is a root. You can then factor $y' = 4x^3 - 6x^2 + 2 = (x - 1)(4x^2 - 2x - 2)$. There are three stationary points: local minima at $x = 1$, $x = -\frac{1}{4} - \frac{1}{2}\sqrt{3}$, local max at $x = -\frac{1}{4} + \frac{1}{2}\sqrt{3}$. one of the two loc min is a global minimum.

246 Global min at $x = 0$, no other stationary points; function is convex, no inflection points. No horizontal asymptotes.

247 The graph is the upper half of the unit circle.

248 Always positive, so no sign changes; global minimum at $x = 0$, no other stationary points; two inflection points at $\pm\sqrt{2}$. No horizontal asymptotes since $\lim_{x \rightarrow \pm\infty} \sqrt[4]{1 + x^2} = \infty$ (DNE).

249 Always positive hence no sign changes; global max at $x = 0$, no other stationary points; two inflection points at $x = \pm\sqrt[4]{3/5}$; second derivative also vanishes at $x = 0$ but this is not an inflection point.

251 Zeroes at $x = 3\pi/4, 7\pi/4$. Absolute max at $x = \pi/4$, abs min at $x = 5\pi/4$. Inflection points and zeroes coincide. Note that $\sin x + \cos x = \sqrt{2} \sin(x + \frac{\pi}{4})$.

252 Zeroes at $x = 0, \pi, 3\pi/2$ but no sign change at $3\pi/2$. Global max at $x = \pi/2$, local max at $x = 3\pi/2$, global min at $x = 7\pi/6, 11\pi/6$.

269 If the length of one side is x and the other y , then the perimeter is $2x + 2y = 1$, so $y = \frac{1}{2} - x$. Thus the area enclosed is $A(x) = x(\frac{1}{2} - x)$, and we're only interested in values of x between 0 and $\frac{1}{2}$.

The maximal area occurs when $x = \frac{1}{4}$ (and it is $A(1/4) = 1/16$.) The minimal area occurs when either $x = 0$ or $x = 1/2$. In either case the "rectangle" is

a line segment of length $\frac{1}{2}$ and width 0, or the other way around. So the minimal area is 0.

270 If the sides are x and y , then the area is $xy = 100$, so $y = 1/x$. Therefore the height plus twice the width is $f(x) = x + 2y = x + 2/x$. This is extremal when $f'(x) = 0$, i.e. when $f'(x) = 1 - 2/x^2 = 0$. This happens for $x = \sqrt{2}$.

271 Perimeter is $2R + R\theta = 1$ (given), so if you choose the angle to be θ then the radius is $R = 1/(2 + \theta)$. The area is then $A(\theta) = \theta R^2 = \theta/(2 + \theta)^2$, which is maximal when $\theta = 2$ (radians). The smallest area arises when you choose $\theta = 0$. Choosing $\theta \geq 2\pi$ doesn't make sense (why? Draw the corresponding wedge!)

You could also say that for any given radius $R > 0$ "perimeter = 1" implies that one has $\theta = (1/R) - 2$. Hence the area will be $A(R) = \theta R^2 = R^2((1/R) - 2) = R - 2R^2$. Thus the area is maximal when $R = \frac{1}{4}$, and hence $\theta = 2$ radians. Again we note that this answer is reasonable because values of $\theta > 2\pi$ don't make sense, but $\theta = 2$ does.

272 (a) The intensity at x is a function of x . Let's call it $I(x)$. Then at x the distance to the big light is x , and the distance to the smaller light is $1000 - x$. Therefore

$$I(x) = \frac{1000}{x^2} + \frac{125}{(1000 - x)^2}$$

(b) Find the minimum of $I(x)$ for $0 < x < 1000$.

$$I'(x) = -2000x^{-3} + 250(1000 - x)^{-3}.$$

$I'(x) = 0$ has one solution, namely, $x = \frac{1000}{3}$. By looking at the signs of $I'(x)$ you see that $I(x)$ must have a minimum. If you don't like looking at signs, you could instead look at the second derivative

$$I''(x) = 6000x^{-4} + 750(1000 - x)^{-4}$$

which is always positive.

273 $r = \sqrt{50/3\pi}$, $h = 100/(3\pi r) = 100/\sqrt{150\pi}$.

284 $dy/dx = e^x - 2e^{-2x}$. Local min at $x = \frac{1}{3} \ln 2$.

$d^2y/dx^2 = e^x + 4e^{-2x} > 0$ always, so the function is convex.

$\lim_{x \rightarrow \pm\infty} y = \infty$, no asymptotes.

285 $dy/dx = 3e^{3x} - 4e^x$. Local min at $x = \frac{1}{2} \ln \frac{4}{3}$.

$d^2y/dx^2 = 9e^{3x} - 4e^x$ changes sign when $e^{2x} = \frac{4}{9}$, i.e. at $x = \frac{1}{2} \ln \frac{4}{9} = \ln \frac{2}{3} = \ln 2 - \ln 3$. Inflection point at $x = \ln 2 - \ln 3$.

$\lim_{x \rightarrow -\infty} f(x) = 0$ so negative x axis is a horizontal asymptote.

$\lim_{x \rightarrow \infty} f(x) = \infty$... no asymptote there.

331 Choosing left endpoints for the c 's gives you

$$f(0)\frac{1}{3} + f(\frac{1}{3})\frac{2}{3} + f(1)\frac{1}{2} + f(\frac{3}{2})\frac{1}{2} = \dots$$

Choosing right endpoints gives

$$f(\frac{1}{3})\frac{1}{3} + f(1)\frac{2}{3} + f(\frac{3}{2})\frac{1}{2} + f(2)\frac{1}{2} = \dots$$

332 The Riemann-sum is the total area of the rectangles, so to get the smallest Riemann-sum you must make the rectangles as small as possible. You can't change their widths, but you can change their heights by changing the c_i . To get the smallest area we make the heights as small as possible. Since f appears to be decreasing, the heights $f(c_i)$ will be smallest when c_i is as large as possible. So we choose the intermediate points c_i all the way to the right of the interval $x_{i-1} \leq c_i \leq x_i$, i.e. $c_1 = x_1, c_2 = x_2, c_3 = x_3, c_4 = x_4, c_5 = x_5, c_6 = b$. To get the *largest* Riemann-sums you choose $c_1 = a, c_2 = x_1, \dots, c_6 = x_5$.

373 (a) The first derivative of $\operatorname{erf}(x)$ is, by definition

$$\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2},$$

so you get the second derivative by differentiating this:

$$\operatorname{erf}''(x) = \frac{-4x}{\sqrt{\pi}} e^{-x^2}.$$

This is negative when $x > 0$, and positive when $x < 0$ so the graph of $\operatorname{erf}(x)$ has an inflection point at $x = 0$.

(b) Wikipedia is not wrong. Let's figure out what sign $\operatorname{erf}(-1)$ has (for instance). By definition you have

$$\operatorname{erf}(-1) = \frac{2}{\sqrt{\pi}} \int_0^{-1} e^{-t^2} dt.$$

Note that in this integral the upper bound (-1) is less than the lower bound (0) . To fix that we switch the upper and lower integration bounds, which introduces a minus sign:

$$\operatorname{erf}(-1) = -\frac{2}{\sqrt{\pi}} \int_{-1}^0 e^{-t^2} dt.$$

The integral we have here is positive because it's an integral of a positive function from a smaller number to a larger number, i.e. it is of the form $\int_a^b f(x) dx$ with $f(x) \geq 0$ and with $a < b$.

With the minus sign that makes $\operatorname{erf}(-1)$ negative.

487 The answer is $\pi/6$.

To get this using the integral you use formula (69) with $f(x) = \sqrt{1-x^2}$. You get $f'(x) = -x/\sqrt{1-x^2}$, so

$$\sqrt{1+f'(x)^2} = \frac{1}{\sqrt{1-x^2}}.$$

The integral of that is $\arcsin x (+C)$, so the answer is $\arcsin \frac{1}{2} - \arcsin 0 = \pi/6$.

GNU Free Documentation License

Version 1.3, 3 November 2008

Copyright © 2000, 2001, 2002, 2007, 2008 Free Software Foundation, Inc.

(<http://fsf.org/>)

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

Preamble

The purpose of this License is to make a manual, textbook, or other functional and useful document “free” in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondly, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of “copyleft”, which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The “**Document**”, below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as “**you**”. You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A “**Modified Version**” of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A “**Secondary Section**” is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document’s overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The “**Invariant Sections**” are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The “**Cover Texts**” are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A “**Transparent**” copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not “Transparent” is called “**Opaque**”.

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The “**Title Page**” means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, “Title Page” means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

The “**publisher**” means any person or entity that distributes copies of the Document to the public.

A section “**Entitled XYZ**” means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as “**Acknowledgements**”, “**Dedications**”, “**Endorsements**”, or “**History**”.) To “**Preserve the Title**” of such a section when you modify the Document means that it remains a section “Entitled XYZ” according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document’s license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

- A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be

listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.

- B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.
- C. State on the Title page the name of the publisher of the Modified Version, as the publisher.
- D. Preserve all the copyright notices of the Document.
- E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
- F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
- G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
- H. Include an unaltered copy of this License.
- I. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.
- J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.
- K. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.
- L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.
- M. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version.
- N. Do not retile any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section.
- O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers. The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique

by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements".

6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License.

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it.

10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See <http://www.gnu.org/copyleft/>.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify

a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of this License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Document.

11. RELICENSING

“Massive Multiauthor Collaboration Site” (or “MMC Site”) means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A “Massive Multiauthor Collaboration” (or “MMC”) contained in the site means any set of copyrightable works thus published on the MMC site.

“CC-BY-SA” means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization.

“Incorporate” means to publish or republish a Document, in whole or in part, as part of another Document.

An MMC is “eligible for relicensing” if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008.

The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing.

THE SIMULTANEOUS EQUATIONS

A. Simultaneous Linear equations with two unknowns

Case I: A pair of linear equations in two variables

$$a_0x + b_0y + c_0 = 0, a_1x + b_1y + c_1 = 0$$

NOTE: x and y are two variables and the unknowns.

Methods to be used

- Solution by substitution
- Solution by eliminating coefficients (Elimination method)
- Matrix method
- Crammer's rule method
- Determinant method
- Graphical method

The last four methods have not been discussed, but the first two are to be reviewed in the subsequent examples below.

Examples

Solve the pair of simultaneous equations

- a) $5x + 2y = 14, 3x - 4y = 24$
 b) $3x + 4y - 9 = 0, 2x + 3y = 8$

Solution

Solution by substitution

a) $5x + 2y = 14$ (1)

$3x - 4y = 24$ (2)

From eqn. (1), making x the subject

$\Rightarrow x = \frac{14-2y}{5}$ (3)

Substituting eqn. (3) in to eqn. (2)

$\Rightarrow 3\left(\frac{14-2y}{5}\right) - 4y = 24$($\times 5$)

$\Rightarrow 3(14 - 2y) - 20y = 120$

$\Rightarrow 42 - 6y - 20y = 120$ or $-26y = 78$

$\therefore y = -3$

Substituting $y = -3$ in two eqn. (3) (either of the original equations)

$\Rightarrow x = \frac{14-2(-3)}{5} = \frac{20}{5} = 4$

$\therefore x = 4, y = -3$

Solving by equating coefficients

$5x + 2y = 14$ (1)

$3x - 4y = 24$ (2)

Multiplying both sides of eqn. (1) by 3 (the coefficient of x in eqn. (2)) and multiplying both sides of eqn. (2) by 5 (the coefficient of y in eqn. (1))

$\Rightarrow 15x + 6y = 42$

$\Rightarrow 15x - 20y = 120$

Now subtracting the equations,

$26y = -78 \therefore y = -3$

Substituting $y = -3$ in either of the original equations

$$5x + 2y = 14, y = -3$$

$$\Rightarrow 5x - 6 = 14 \therefore 5x = 20 \text{ or } x = 4$$

$$\therefore x = 4, y = -3$$

b) Left as an exercise.

ANSWER: $x = -5, y = 6$

Try also to solve simultaneously

$$3x + 2y = 16, 4x - 3y - 10 = 0.$$

ANSWER: $x = -5, y = 6$

Case II: A pair of simultaneous equations in two variables, one Quadratic and one Linear

Example

Solve the pair of simultaneous equations

- a) $4x - 3y = 1, 12xy + 13x^2 = 25$
 b) $x^2 + y^2 - 6x + 4y - 13 = 0, 2x = 3y - 1$

Solution

- a) $4x - 3y = 1$ (1)
 $12xy + 13x^2 = 25$ (2)
 From eqn. (1)
 $y = \frac{4x-1}{3}$ (3)
 Substitute eqn. (3) in to eqn. (2)
 $\Rightarrow 12x\left(\frac{4x-1}{3}\right) + 13x^2 = 25$ ($\times 3$)
 $\Rightarrow 48x^2 - 12x + 39x^2 = 75$
 $\therefore 87x^2 - 12x - 75 = 0$ ($\div 3$)
 $\Rightarrow 29x^2 - 4x - 25 = 0$
 $\Rightarrow (29x + 25)(x - 1) = 0$
 $\therefore x = 1, -\frac{25}{29}$ (4)

Eqn. (4) in to eqn. (3) yields

$$y = \frac{4x-1}{3} = 1, \text{ for } x = 1 \text{ and } y = \frac{-\frac{100}{29}-1}{3} \text{ for } x = -\frac{25}{29}$$

$$\therefore x = 1, y = 1; x = -\frac{25}{29}, y = -\frac{43}{29}$$

- b) $2x = 3y - 1 \Rightarrow x = \frac{3y-1}{2}$ (1)
 $x^2 + y^2 - 6x + 4y - 13 = 0$ (2)
 Substitute eqn. (1) in to eqn. (2)
 $\Rightarrow \left(\frac{3y-1}{2}\right)^2 + y^2 - 6\left(\frac{3y-1}{2}\right) + 4y - 13 = 0$
 $\Rightarrow \frac{9y^2-6y+1}{4} + y^2 + 3 - 9y + 4y - 13 = 0$ ($\times 4$)
 $\Rightarrow 9y^2 - 6y + 1 + 4y^2 + 12 - 36y + 16y - 52 = 0$
 $\Rightarrow 13y^2 - 26y - 39 = 0 \text{ or } y^2 - 2y - 4 = 0$
 $\Rightarrow (y - 3)(y + 1) = 0$
 $\therefore y = -1, 3$
 Substitute eqn. (3) in to eqn. (1)
 For $y = -1, x = \frac{3(-1)-1}{2} = -2$
 For $y = 3, x = \frac{3(3)-1}{2} = 4$
 $\therefore x = -2, y = -1; x = 4, y = 3$

Other techniques of solving specific set of simultaneous equations

Solve:

- a) $x^2 + y^2 = 185, x - y = 3$
- b) $x^3 + y^3 = 4914, x + y = 18$
- c) $x^3 - y^3 = 218, x - y = 2$
- d) $2x + 3y = 5, xy = 1$
- e) $5x - y = 3, y^2 - 6x^2 = 25$
- f) $3x - 2y = 7, xy = 20$

NOTE:

The above equations can also be solved using substitution as in the previous examples.

Solution

- a) $x^2 + y^2 = 185$ (1)
 $x - y = 3$ (2)
 From eqn. (2), squaring both sides
 $(x - y)^2 = 3^2$
 $\Rightarrow x^2 + y^2 - 2xy = 9$
 But from eqn. (1), $x^2 + y^2 = 185$
 $\Rightarrow 185 - 2xy = 9$ or $xy = 88$ (3)
 Again $(x + y)^2 = x^2 + y^2 + 2xy$
 From eqn. (1) and eqn. (3)
 $\Rightarrow (x + y)^2 = 185 + 2 \times 88$
 $\Rightarrow (x + y)^2 = 361$
 $\therefore x + y = \pm 19$ (4)
 Now solving eqn. (2) and eqn. (4) simultaneously
 Adding the equations,
 $\Rightarrow 2x = \pm 19 + 3$
 $\therefore 2x = 19 + 3 = 22$ or $x = 11$
 $\therefore 2x = -19 + 3 = -16$ or $x = -8$
 From eqn. (2), $y = x - 3$
 For $x = 11, y = 11 - 3 = 8$
 For $x = -8, y = -8 - 3 = -11$
 $\therefore x = 11, y = 8; x = -8, y = -11$
- b) $x^3 + y^3 = 4914$ (1)
 $x + y = 18$ (2)
 From eqn. (2), cubing both sides
 $(x + y)^3 = 18^3$
 $\Rightarrow x^3 + y^3 + 3x^2y + 3xy^2 = 5832$
 $\Rightarrow x^3 + y^3 + 3xy(x + y) = 5832$
 But from eqn. (1) and eqn. (2), $x^3 + y^3 = 4914, x + y = 18$
 $\Rightarrow 4914 + 3xy(18) = 5832$
 $\therefore xy = 17$ (3)
 Squaring eqn. (2) on both sides
 $\Rightarrow (x + y)^2 = 18^2$
 $\Rightarrow x^2 + y^2 + 2xy = 324$ (4)
 Substitute eqn. (3) in to eqn. (4)
 $\Rightarrow x^2 + y^2 + 34 = 324$

$$\therefore x^2 + y^2 = 290 \dots\dots\dots (5)$$

$$\text{Also from } (x - y)^2 = x^2 + y^2 - 2xy$$

$$\Rightarrow (x - y)^2 = 290 - 34 = 256$$

$$\therefore x - y = \pm 16 \dots\dots\dots (6)$$

Solving eqn. (2) and eqn. (6) simultaneously

$$x + y = 18, x - y = \pm 16$$

Adding the equations,

$$\Rightarrow 2x = \pm 16 + 18$$

$$\therefore 2x = 16 + 18 = 34 \text{ or } x = 17$$

$$\therefore 2x = -16 + 18 = 2 \text{ or } x = 1$$

Also subtracting the equations,

$$\Rightarrow 2y = 18 - \pm 16$$

$$\therefore 2y = 18 - 16 = 2 \text{ or } y = 1$$

$$\therefore 2y = 18 + 16 = 34 \text{ or } y = 17$$

$$\therefore x = 1, y = 17; x = 17, y = 1$$

NOTE:

The equations have been reduced to a solution depending on the quadratic equation.

c) $x^3 - y^3 = 218 \dots\dots\dots (1)$

$$x - y = 2 \dots\dots\dots (2)$$

From eqn. (2), cubing both sides

$$(x - y)^3 = 2^3$$

$$\Rightarrow x^3 - y^3 - 3x^2y + 3xy^2 = 8$$

$$\Rightarrow x^3 - y^3 - 3xy(x - y) = 8$$

$$\text{But from eqn. (1) and eqn. (2), } x^3 - y^3 = 218, x - y = 2$$

$$\Rightarrow 218 - 3xy(2) = 8$$

$$\therefore xy = 35 \dots\dots\dots (3)$$

Squaring eqn. (2) on both sides

$$\Rightarrow (x - y)^2 = 2^2$$

$$\Rightarrow x^2 + y^2 - 2xy = 4 \dots\dots\dots (4)$$

Substitute eqn. (3) in to eqn. (4)

$$\Rightarrow x^2 + y^2 - 70 = 4$$

$$\therefore x^2 + y^2 = 74 \dots\dots\dots (5)$$

$$\text{Also from } (x + y)^2 = x^2 + y^2 + 2xy$$

$$\Rightarrow (x + y)^2 = 74 + 70 = 144$$

$$\therefore x + y = \pm 12 \dots\dots\dots (6)$$

Solving eqn. (2) and eqn. (6) simultaneously

$$x - y = 2, x + y = \pm 12$$

Adding the equations,

$$\Rightarrow 2x = \pm 12 + 2$$

$$\therefore 2x = 12 + 2 = 14 \text{ or } x = 7$$

$$\therefore 2x = -12 + 2 = -10 \text{ or } x = -5$$

Also subtracting the equations,

$$\Rightarrow 2y = \pm 12 - 2$$

$$\therefore 2y = 12 - 2 = 10 \text{ or } y = 5$$

$$\therefore 2y = -12 - 2 = -14 \text{ or } y = -7$$

$$\therefore x = 7, y = 5; x = -5, y = -7$$

- d) $2x + 3y = 5$ (1)
 $xy = 1$ (2)
 Re-writing eqn. (2) in terms of terms of eqn. (1)
 $\Rightarrow 2x.3y = 6$ (No change in the meaning in eqn. (2))
 Now squaring eqn. (1)
 $(2x + 3y)^2 = 5^2$
 $\Rightarrow 4x^2 + 2.2x.3y + 9y^2 = 25$, but $\Rightarrow 2x.3y = 6$ (*)
 $\therefore 4x^2 + 9y^2 = 25 - 12 = 13$ (3)
 Also from $(2x - 3y)^2 = 4x^2 + 9y^2 - 2.2x.3y$
 But from eqn. (*) and eqn. (3), $2x.3y = 6$, $4x^2 + 9y^2 = 13$
 $\Rightarrow (2x - 3y)^2 = 13 - 12 = 1$
 $2x - 3y = \pm 1$ (4)
 Solving eqn. (1) and eqn. (4) simultaneously
 $2x + 3y = 5$, $2x - 3y = \pm 1$
 Adding the equations,
 $\Rightarrow 4x = 5 \pm 1$
 $\therefore 4x = 5 + 1 = 6$ or $x = \frac{3}{2}$
 $\therefore 4x = 5 - 1 = 4$ or $x = 1$
 Subtracting the equations,
 $\Rightarrow 6y = 5 - \pm 1$
 $\therefore 6y = 5 - 1 = 4$ or $y = \frac{2}{3}$
 $\therefore 6y = 5 - -1 = 6$ or $y = 1$
 $\therefore x = \frac{3}{2}, y = \frac{2}{3}; x = 1, y = 1$
- e) Left as an exercise. **ANSWER:** $x = 2, y = 7$; $x = -\frac{8}{9}, y = -\frac{97}{19}$
- f) Left as an exercise. **ANSWER:** $x = 5, y = 4$; $x = -\frac{8}{3}, y = -\frac{15}{2}$

Task

Solve: $3x - 5y = 2, xy = 8$ **ANSWER:** $x = 4, y = 2$; $x = -3\frac{1}{3}, y = -2\frac{2}{5}$

HOMOGENEOUS EQUATIONS

Homogeneous equations are equations in which all terms have the same degree.

e.g. $ax^3 + bx^2y + cy^2x + dy^3 = f$, is a homogeneous of 3^0

$lx^2 + mxy + ny^2 = p$, is a homogeneous of 2^0

If the powers in every term in variables x and y are added, give a uniform degree.

Solving Homogeneous Equations

The equations to be considered are:

$$\left. \begin{aligned} lx^2 + mxy + ny^2 &= A \\ px^2 + qxy + ry^2 &= B \end{aligned} \right\} \text{Both quadratic}$$

$$\left. \begin{aligned} lx^2 + mxy + ny^2 &= A \\ qxy &= B \end{aligned} \right\}$$

, where l, m, n, p, q, r, A, B are constants.

Examples

- 1) Solve the pair of simultaneous equations.

$$x^2 + xy + 4y^2 = 6 \dots\dots\dots (1)$$

$$3x^2 + 8y^2 = 14 \dots\dots\dots (2)$$

Solution

Re-writing the equations by factoring

$$\Rightarrow x^2 \left[1 + \frac{y}{x} + 4 \left(\frac{y}{x} \right)^2 \right] = 6 \dots\dots\dots (3)$$

$$\Rightarrow x^2 \left[3 + 8 \left(\frac{y}{x} \right)^2 \right] = 14 \dots\dots\dots (4)$$

Eqn. (3) $\div$ eqn. (4)

$$\frac{x^2 \left[1 + \frac{y}{x} + 4 \left(\frac{y}{x} \right)^2 \right]}{x^2 \left[3 + 8 \left(\frac{y}{x} \right)^2 \right]} = \frac{6}{14}; \text{ let } m = \frac{y}{x}$$

$$\Rightarrow \frac{1+m+4m^2}{3+8m^2} = \frac{3}{7} \text{ or } 7 + 7m + 28m^2 = 9 + 24m^2$$

$$\therefore 4m^2 + 7m - 2 = 0 \text{ or } (4m - 1)(m + 2) = 0$$

$$\therefore m = \frac{1}{4}, -2$$

But $m = \frac{y}{x}$ or $y = mx$

$$\Rightarrow y = \frac{1}{4}x \dots\dots\dots (5)$$

$$\Rightarrow y = -2x \dots\dots\dots (6)$$

Using any of the equations (3) or (4)

$$\text{From eqn. (4) } x^2 \left[3 + 8 \left(\frac{y}{x} \right)^2 \right] = 14$$

$$\Rightarrow x^2 [3 + 8m^2] = 14$$

For $m = -2$,

$$x^2 [3 + 8(-2)^2] = 14 \text{ or } x^2 = \frac{14}{35} = \frac{2}{5}$$

$$\therefore x = \pm \sqrt{\frac{2}{5}}$$

$$\text{From eqn. (6), } y = -2x = -2 \left(\pm \sqrt{\frac{2}{5}} \right)$$

$$\therefore x = \sqrt{\frac{2}{5}}, y = -2\sqrt{\frac{2}{5}}$$

$$\therefore x = -\sqrt{\frac{2}{5}}, y = 2\sqrt{\frac{2}{5}}$$

Now for $m = \frac{1}{4}$, equation $x^2 [3 + 8m^2] = 14$ becomes;

$$\Rightarrow x^2 \left[3 + 8 \left(\frac{1}{4} \right)^2 \right] = 14 \text{ or } x^2 = 4$$

$$\therefore x = \pm 2$$

From eqn. (6), $y = -2x = -2(\pm 2)$

$$\therefore x = 2, y = -4$$

$$\therefore x = -2, y = 4$$

$$\therefore x = \sqrt{\frac{2}{5}}, y = -2\sqrt{\frac{2}{5}}; x = -\sqrt{\frac{2}{5}}, y = 2\sqrt{\frac{2}{5}}; x = 2, y = -4; x = -2, y = 4$$

2) Solve the pair of simultaneous equations.

$$x^2 + 4xy + y^2 = 13 \dots\dots\dots (1)$$

$$2x^2 + 3xy = 8 \dots\dots\dots (2)$$

Solution

Re-writing the equations by factoring

$$\Rightarrow x^2 \left[1 + 4 \left(\frac{y}{x} \right) + \left(\frac{y}{x} \right)^2 \right] = 13 \dots\dots\dots (3)$$

$$\Rightarrow x^2 \left[2 + 3 \left(\frac{y}{x} \right) \right] = 8 \dots\dots\dots (4)$$

Eqn. (3) $\div$ eqn. (4)

$$\frac{x^2 \left[1 + 4 \left(\frac{y}{x} \right) + \left(\frac{y}{x} \right)^2 \right]}{x^2 \left[2 + 3 \left(\frac{y}{x} \right) \right]} = \frac{13}{8}; \text{ let } m = \frac{y}{x}$$

$$\Rightarrow \frac{1 + 4m + m^2}{2 + 3m} = \frac{13}{8} \text{ or } 1 + 4m + m^2 = 2 + 3m$$

$$\therefore 8m^2 - 7m - 18 = 0$$

$$\therefore m = \frac{7 \pm \sqrt{49 + 4(8)(18)}}{16} = \frac{7 \pm 25}{16}$$

$$\therefore m = 2, -\frac{9}{8}$$

$$\text{From } m = \frac{y}{x} \Rightarrow y = mx$$

$$\Rightarrow y = 2x \dots\dots\dots (5)$$

$$\Rightarrow y = -\frac{9}{8}x \dots\dots\dots (6)$$

From equation (4)

$$\Rightarrow x^2 \left[2 + 3 \left(\frac{y}{x} \right) \right] = 8 \therefore x^2 [2 + 3m] = 8$$

$$\text{For } m = 2, x^2(2 + 6) = 8$$

$$\therefore x^2 = 1 \text{ or } x = \pm 1$$

$$\text{But } y = 2x$$

$$\therefore \text{For } x = 1, y = 2$$

$$\therefore \text{For } x = -1, y = -2$$

$$\therefore x = 1, y = 2; x = -1, y = -2$$

$$\text{Also for } m = -\frac{9}{8}$$

From equation (4)

$$\Rightarrow x^2 \left[2 + 3 \left(-\frac{9}{8} \right) \right] = 8 \therefore x^2 = -\frac{64}{11}, \text{ hence roots are complex.}$$

Alternatively

Dealing with question (1)

$$x^2 + xy + 4y^2 = 6 \dots\dots\dots (1)$$

$$3x^2 + 8y^2 = 14 \dots\dots\dots (2)$$

Eliminating constants 6 and 14

$$\text{Eqn. (1)} \times 14 - \text{Eqn. (2)} \times 6$$

$$\Rightarrow 14x^2 + 14xy + 56y^2 = 84$$

$$\Rightarrow 18x^2 + 0xy + 48y^2 = 84$$

$$\frac{-4x^2 + 14xy + 8y^2 = 0}{-4x^2 + 14xy + 8y^2 = 0}$$

$$\text{or } \Rightarrow 2x^2 - 7xy - 4y^2 = 0$$

Factorizing to linear factors gives

$$(2x + y)(x - 2y) = 0$$

$$\therefore y = -2x \text{ or } y = \frac{1}{4}x$$

Eqn. (3) in to Eqn. (2) or (1), the original equations

$$3x^2 + 8y^2 = 14, y = -2x, \text{ gives}$$

$$3x^2 + 32x^2 = 14 \therefore x^2 = \frac{2}{5} \text{ or } x = \pm \sqrt{\frac{2}{5}}$$

$$\text{For } x = \sqrt{\frac{2}{5}}, y = -\sqrt{\frac{2}{5}}$$

For $x = -\sqrt{\frac{2}{5}}, y = \sqrt{\frac{2}{5}}$

Also $3x^2 + 8y^2 = 14, y = \frac{1}{4}x$ gives

$3x^2 + 8\left(\frac{1}{4}x\right)^2 = 14$ or $x = \pm 2$

Now for $x = 2, y = \frac{1}{4} \times 2 = \frac{1}{2}$

Also for $x = -2, y = -\frac{1}{4} \times 2 = -\frac{1}{2}$

Dealing with question (2)

$x^2 + 4xy + y^2 = 13$ (1)

$2x^2 + 3xy = 8$ (2)

Eliminating constants 13 and 8

Eqn. (1) $\times 8$ - Eqn. (2) $\times 13$

$8x^2 + 32xy + 8y^2 = 104$

$26x^2 + 39xy + 0y^2 = 104$

$-18x^2 - 7xy + 8y^2 = 0$ or $18x^2 + 7xy - 8y^2 = 0$

Factoring in to linear factors

$(2x - y)(9x + 8y) = 0$

$\therefore y = 2x$ or $y = -\frac{9}{8}x$

Eqn. (3) in to Eqn. (2) or (1), the original equations

$2x^2 + 3xy = 8, y = 2x$

$\Rightarrow 2x^2 + 6x^2 = 8 \therefore x^2 = 1$ or $x = \pm 1$

$\therefore$ For $x = 1, y = 2$; $x = -1, y = -2$

Also $2x^2 + 3xy = 8, y = -\frac{9}{8}x$

$\Rightarrow 2x^2 + \frac{27}{8}x^2 = 8 \therefore x^2 = -\frac{64}{11}$ or $x = \text{complex}$

NOTE:

1. The second alternative gives rise to a quadratic equation in terms of x and y which is factorisable to linear factors after eliminating the constant terms.
2. Substitution method can also be applicable to solve homogeneous simultaneous equations.

Task

Solve the pair of simultaneous equations

a) $2x^2 - xy - y^2 = 8, xy = 6$

b) $x^3 + y^3 = 35, x^2y + xy^2 = 30$

c) $x^2 - 2xy + 8y^2 = 8, 3xy - 2y^2 = 4$

ANSWERS: a) $x = \pm 3; y = \pm 1$ c) $x = \pm 2, y = \pm 1; x = \pm 1, y = \pm 12$

Simultaneous Linear Equations with three unknowns

$$a_1x + b_1y + c_1z = A$$

$$a_2x + b_2y + c_2z = B$$

$$a_3x + b_3y + c_3z = C$$

Methods to be considered are;

- Elimination method
- Reducing to row Echelon form

Elimination method of solving linear set of equations

$$a_1x + b_1y + c_1z = A \dots\dots\dots (1)$$

$$a_2x + b_2y + c_2z = B \dots\dots\dots (2)$$

$$a_3x + b_3y + c_3z = C \dots\dots\dots (3)$$

- a) Eliminating any variable (*i.e.* x or y or z) from any pair of equations

Eliminating x from eqn. (1) and eqn. (2)

Multiply $\frac{a_2}{a_1}$ by eqn. (1) and subtract the result from eqn. (2)

The resulting equation is of the form

$$\boxed{Py + Qz = R} \dots\dots\dots (4)$$

- b) Eliminating the same variable *i.e.* x from any other pair of equations

Eliminating x from eqn. (1) and eqn. (3)

Multiply $\frac{a_3}{a_1}$ by eqn. (1) and subtract the result from eqn. (3)

The resulting equation is of the form

$$\boxed{my + nz = r} \dots\dots\dots (5)$$

Now solving eqn. (4) and eqn. (5) as discussed earlier (*i.e.* solving set of linear equations in two variables), this gives y and z values.

Now substituting y and z values in any of the original equations *i.e.* (1) or (2) or (3), the value of x can be obtained.

Examples

Solve the linear set of equations simultaneously

$$x + 4y + 4z = 7$$

$$3x + 2y + 2z = 6$$

$$9x + 6y + 2z = 14$$

Solution

$$x + 4y + 4z = 7 \dots\dots\dots (1)$$

$$3x + 2y + 2z = 6 \dots\dots\dots (2)$$

$$9x + 6y + 2z = 14 \dots\dots\dots (3)$$

Eliminating x from eqn. (1) and eqn. (2)

$$\text{Eqn. (1)} \times \left(\frac{3}{1}\right) - \text{eqn. (2)}$$

$$3x + 12y + 12z = 21$$

$$3x + 2y + 2z = 6$$

$$10y + 10z = 15 \text{ or } 2y + 2z = 3 \dots\dots\dots (4)$$

Eliminating x from eqn. (1) and eqn. (3)

$$\text{Eqn. (1)} \times \left(\frac{9}{1}\right) - \text{eqn. (3)}$$

$$9x + 36y + 36z = 63$$

$$9x + 6y + 2z = 14$$

$$30y + 34z = 49 \dots\dots\dots (5)$$

Now eliminating y from eqn. (4) and eqn. (5)

$$\text{Eqn. (4)} \times \left(\frac{30}{2}\right) - \text{eqn. (5)}$$

$$30y + 30z = 45$$

$$30y + 34z = 49$$

$$-4z = -4 \therefore z = 1$$

Put $z = 1$ in to eqn. (4) or (5)

$$2y + 2z = 3, z = 1$$

$$\Rightarrow 2y + 2 = 3 \therefore y = \frac{1}{2}$$

Now substitute $z = 1, y = \frac{1}{2}$ in to eqn. (1) or eqn. (2) or eqn. (3) the original equations

Using eqn. (1)

$$x + 4y + 4z = 7 \therefore x + 2 + 4 = 7 \text{ or } x = 1$$

$$\therefore x = 1, y = \frac{1}{2}, z = 1$$

Task

Solve the simultaneous equations

a) $x + 2y - 3z = 3$

$$2x - y - z = 11$$

$$3x + 2y + z = -5$$

ANSWER: $x = 2, y = -4, z = -3$

b) $3x + 2y - z = 19$

$$4x - y + 2z = 4$$

$$2x + 4y - 5z = 32$$

ANSWER: $x = 3, y = 4, z = -2$

Solving Linear set of equations in 3 unknowns by reducing to Row echelon form

Consider the equations to be solved

$$a_1x + b_1y + c_1z = A \dots\dots\dots (1)$$

$$a_2x + b_2y + c_2z = B \dots\dots\dots (2)$$

$$a_3x + b_3y + c_3z = C \dots\dots\dots (3)$$

Writing the equations in matrix form

$$\underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}_{\text{Coefficient}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{Variable matrix}} = \underbrace{\begin{pmatrix} A \\ B \\ C \end{pmatrix}}_{\text{Constant matrix}}, \text{ which may be written in short form } \mathbf{P.V} = \mathbf{K}$$

Writing elements of $\mathbf{K}$ in $\mathbf{P}$ matrix, the augmented matrix is formulated as;

$$Q = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{array} \right)$$

Eliminating elements below the major diagonal in the coefficient matrix gives rise to a triangular matrix

$$\left(\begin{array}{ccc|c} \hat{a}_1 & b_1 & c_1 & A \\ a_2 & \hat{b}_2 & c_2 & B \\ a_3 & b_3 & \hat{c}_3 & C \end{array} \right), \text{ elements to be zeroed/ eliminated are } a_1, a_2, a_3$$

To eliminate a_2 , subtract $\frac{a_2}{a_1}$ times first row from the 2nd row, and to eliminate a_3 , subtract $\frac{a_3}{a_1}$ times first row from 3rd row .i.e.

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{array} \right) \begin{array}{l} \text{Row 1} = R_1 \\ \text{Row 2} = R_2 \\ \text{Row 3} = R_3 \end{array}$$

Now eliminating using the formulae

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 - a_2/a_1 R_1 \\ R_3 \rightarrow R_3 - a_3/a_1 R_1 \end{array}$$

This gives

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ 0 & d & e & D \\ 0 & f & g & E \end{array} \right)$$

Now eliminating **f**, subtract $\frac{f}{d}$ times 2^{nd} row from the 3^{rd} row

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & D \\ a_3 & b_3 & c_3 & H \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 \\ R_3 \rightarrow R_3 \end{array}$$

, which is a triangular matrix

Finally detach the right hand column back to its original position

$$\left(\begin{array}{ccc} a_1 & b_1 & c_1 \\ 0 & d & e \\ 0 & 0 & h \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A \\ D \\ H \end{pmatrix}$$

Multiplying the matrices, the result will be;

$$a_1x + b_1y + c_1z = A \dots\dots\dots (i)$$

$$dy + ez = D \dots\dots\dots (ii)$$

$$hz = H \dots\dots\dots (iii)$$

From equation (iii) and by back substitution, values of x, y and z can be evaluated.

Examples

Solve the linear set of equations by reducing to row echelon form

$$x - 2y - 3z = 0$$

$$3x + 5y + 2z = 0$$

$$2x + 3y - z = 2$$

Solution

$$x - 2y - 3z = 0$$

$$3x + 5y + 2z = 0$$

$$2x + 3y - z = 2$$

Writing the equations in matrix form

$$\left(\begin{array}{ccc} 1 & -2 & -3 \\ 3 & 5 & 2 \\ 2 & 3 & -1 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Formulating the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 3 & 5 & 2 & 0 \\ 2 & 3 & -1 & 2 \end{array} \right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Now eliminating 3 and 2 from column 1

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 3 & 5 & 2 & 0 \\ 2 & 3 & -1 & 2 \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 - 3/1R_1 \\ R_3 \rightarrow R_3 - 2/1R_1 \end{array}$$

This gives

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 7 & 5 & 2 \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 \\ R_3 \rightarrow R_3 - 7/11R_2 \end{array}$$

Now eliminating 7 from 3rd row

$$\begin{pmatrix} 1 & -2 & -3 & | & 0 \\ 0 & 11 & 11 & | & 0 \\ 0 & 0 & -2 & | & 2 \end{pmatrix}$$

Detaching the matrices back to their original positions

$$\begin{pmatrix} 1 & -2 & -3 \\ 0 & 11 & 11 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$x - 2y - 3z = 0 \dots\dots\dots (i)$$

$$11y + 11z = 0 \dots\dots\dots (ii)$$

$$-2z = 2 \dots\dots\dots (iii)$$

From eqn. (iii), $z = 1$

Put $z = 1$ in to eqn. (ii)

$$\Rightarrow 11y - 11 = 0 \therefore y = 1$$

Put $z = 1, y = 1$ in to eqn. (i)

$$x - 2 + 3 = 0 \therefore x = -1$$

$$\therefore x = -1, y = 1, z = 1$$

Task

1. Solve the equations simultaneously using row- echelon method

a) $2x + 3y + 4z = -4$

$$4x + 2y + 3z = -11$$

$$3x + 4y + 2z = -3$$

$$\text{ANSWER: } x = -3, y = 2, z = -1$$

b) $2a + b + 3c = 11$

$$a + 2b - 2c = 3$$

$$4a + 3b + c = 15$$

$$\text{ANSWER: } a = 1, b = 3, c = 2$$

c) $2p + 3q + 4r = 8$

$$3p - 2q - 3r = -2$$

$$5p + 4q + 2r = 3$$

$$\text{ANSWER: } p = 1, q = -2, r = 3$$

2. Simplify and solve the following set of simultaneous equations

a) $4(x + 3y) - 2(4x + 3z) - 3(x - 2y - 4z) = 17$

$$2(4x - 3y) + 5(x - 4z) + 4(x - 3y + 2z) = 23$$

$$3(y + 4z) + 4(2x - y - z) + 2(x + 3y - 2z) = 5$$

$$\text{ANSWER: } x = 1, y = 3, z = -5$$

b) $5(x + 2y) - 4(3x + 4z) - 2(x + 3y - 5z) = 16$

$$2(3x - y) + 3(x - 4z) + 4(2x - 3y + z) = -16$$

$$4(4 - 2z) + 2(2x - 4y - 3) - 3(x + 4y - 2z) = -62$$

$$\text{ANSWER: } x = 2, y = 4, z = -3$$

Other 3 – unknown simultaneous equations

Solve the equations

i). $x + 3 = \frac{y-1}{2} = \frac{z-5}{-2}, 11x - 3y + 7z = 8$

ii). $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z+4y}{2}, 6x + 6y + 2z = 6$ (Uneb 2005)

Solution

i). Let $x + 3 = \frac{y-1}{2} = \frac{z-5}{-2} = \lambda$

$$\left. \begin{aligned} \Rightarrow x + 3 &= \lambda \therefore x = \lambda - 3 \\ \Rightarrow \frac{y-1}{2} &= \lambda \therefore y = 2\lambda + 1 \\ \Rightarrow \frac{z-5}{-2} &= \lambda \therefore z = -2\lambda + 5 \end{aligned} \right\} \dots\dots\dots (1)$$

Substituting eqn. (1) in to the 2nd equation $11x - 3y + 7z = 8$

$$\Rightarrow 11(\lambda - 3) - 3(2\lambda + 1) + 7(-2\lambda + 5) = 8$$

$$\Rightarrow 11\lambda - 6\lambda - 14\lambda - 33 - 3 + 35 = 8$$

$$\Rightarrow -9\lambda = 9 \therefore \lambda = -1$$

Substituting $\lambda = -1$ I to eqn. (1) gives

$$x = -4, y = -1, z = 7$$

ii). Let $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z+4y}{2} = \lambda$

$$\Rightarrow \frac{4x-3y}{4} = \lambda \therefore 4x - 3y = 4\lambda \dots\dots\dots (1)$$

$$\Rightarrow \frac{2y-x}{3} = \lambda \therefore 2y - x = 3\lambda \dots\dots\dots (2)$$

$$\Rightarrow \frac{z+4y}{2} = \lambda \therefore z + 4y = 2\lambda \dots\dots\dots (3)$$

Expressing x, y, z in terms of λ

Eliminating x from eqn. (1) and (2)

$$\text{eqn. (1)} + \text{eqn. (2)} \times 4$$

$$\Rightarrow 4x - 3y + 8y - 4x = 4\lambda + 12\lambda$$

$$\therefore 5y = 16\lambda \Rightarrow y = \frac{16\lambda}{5} \dots\dots\dots (4)$$

Eqn. (4) in to (2) to find x in terms of λ

$$\Rightarrow 2y - x = 3\lambda \therefore x = 2y - 3\lambda \text{ or } x = \frac{32\lambda}{5} - 3\lambda$$

$$\Rightarrow x = \frac{17\lambda}{5} \dots\dots\dots (5)$$

Eqn. (4) in to (3) to find z in terms of λ

$$z + 4y = 2\lambda \Rightarrow z = 2\lambda - 4y$$

$$\Rightarrow z = 2\lambda - \frac{64\lambda}{5} = -\frac{54\lambda}{5} \dots\dots\dots (6)$$

Substituting eqn. (4), (5), (6) in to $6x + 6y + 2z = 6$

$$\Rightarrow 6\left(\frac{17\lambda}{5}\right) + 6\left(\frac{16\lambda}{5}\right) + 2\left(-\frac{54\lambda}{5}\right) = 6$$

$$\Rightarrow 9\lambda = 3 \therefore \lambda = \frac{1}{3}$$

Substituting $\lambda = \frac{1}{3}$ in to eqn. (4), (5) and (6) gives

$$x = \frac{17}{5} \times \frac{1}{3} = \frac{17}{15}$$

$$y = \frac{16}{5} \times \frac{1}{3} = \frac{16}{15}$$

$$x = -\frac{54}{5} \times \frac{1}{3} = -\frac{18}{5}$$

$$\therefore x = \frac{17}{15}, y = \frac{16}{15}, z = -\frac{18}{5}$$

Task

Solve the simultaneous equations

a) $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$ and $3x + 4y + 2z - 25 = 0$	ANSWER: $x = 5, y = 0, z = 5$
b) $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5}$ and $x + y + z = 2$	ANSWER: $x = 1, y = -2, z = 3$
c) $\frac{x+4z}{4} = \frac{y+z}{6} = \frac{3x+y}{5}$ and $4x + 2y + 5z = 15$	ANSWER: $x = 0, y = 5, z = 1$
d) $\frac{x-y}{4} = \frac{z-y}{3} = 2z - x$ and $x + 3y + 2z = 4$	ANSWER: $x = 3, y = -1, z = 2$

SURDS AND INDICES

Indices:

When a quantity **a** multiplies it self n – times .i.e. $a \times a \times a \times \dots \dots \dots n$ – times, the product is called **a power of a** .i.e. a^n

Definition:

Let **a**, x and y be real numbers such that $a^x = y$, then y is a power of **a**, x is called index and **a** is the base.

For example in 2^4 , $3^{\frac{1}{2}}$, $5^{\sqrt{2}}$, 2, 3 and 5 are the bases and 4 , $\frac{1}{2}$ and $\sqrt{2}$ are indices.

Laws of indices:

If m and n are integers then

- i) $a^m \times a^n = a^{m+n}$
- ii) $(a^m)^n = a^{mn}$
- iii) $\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}$
- iv) $(a.b)^m = a^m . b^m$
- v) $a^{-n} = \frac{1}{a^n}$
- vi) $a^0 = 1$
- vii) If $a^q = b$, then $a^p = b^q$ for $q > 0$
- viii) If $a^m = a^n$ then $m = n$

Proof of the laws:

i) $a^m \times a^n = a^{m+n}$

Several cases arise,

Case I: $m > 0$ and $n > 0$.i.e. both m and n are positive.

By definition,

$a^m = a \times a \times a \times \dots \dots \dots m$ – times and

$a^n = a \times a \times a \times \dots \dots \dots n$ – times

$$\begin{aligned} \Rightarrow a^m \times a^n &= (a \times a \times a \times \dots \dots \dots m - times) \times (a \times a \times a \times \dots \dots \dots n - times) \\ &= a \times a \times a \times \dots \dots \dots (m + n) - times \\ &= a^{m+n} \end{aligned}$$

$$\therefore a^m \times a^n = a^{m+n}$$

Case II: $m < 0$ and $n < 0$.i.e. both m and n are negative

Let $m = -k$ and $n = -l$ where $k, n > 0$

$$\begin{aligned} \Rightarrow a^m \times a^n &= a^{-k} \times a^{-l} \\ &= \frac{1}{a^k} \times \frac{1}{a^l} \\ &= \frac{1}{a^{k+l}}, \text{ by first case} \\ \therefore a^{-k} \times a^{-l} &= a^{-(k+l)} = a^{-k-l} = \frac{1}{a^{k+l}} \end{aligned}$$

Case III: $m > 0$ and $n < 0$.i.e. both m – positive and n – negative

Let $n = -l$, $l > 0$

$$\begin{aligned} \Rightarrow a^m \times a^n &= a^m \times a^{-l} \\ &= a^m \times \frac{1}{a^l} \end{aligned}$$

$$\therefore a^m \times a^{-l} = a^m \times \frac{1}{a^l}$$

Now if $m = l$, then

$$a^m \times a^{-l} = a^l \times a^{-l} = a^l \times \frac{1}{a^l} = 1$$

But by first case,

$$a^l \times a^{-l} = a^0 = 1$$

$$\boxed{\therefore a^0 = 1}$$

Also from $a^l \times a^{-l} = 1$, then

$$\boxed{a^{-l} = \frac{1}{a^l}}$$

, showing that a^{-n} is the reciprocal of a^n

Further simplification of $a^m \times a^{-l} = a^m \times \frac{1}{a^l}$, for $m > l$

$$\begin{aligned} \Rightarrow a^m \times a^{-l} &= \left(\left(a \times \frac{1}{a} \right) \times \left(a \times \frac{1}{a} \right) \times \dots \dots \dots l - \text{times} \right) \times a \times a \times \dots \dots \dots (m - l) - \text{times} \\ &= (1 \times 1 \times \dots \dots \dots l - \text{times}) \times a^{m-l} = a^{m-l} \end{aligned}$$

$$\therefore a^m \times a^{-l} = a^m \times \frac{1}{a^l} = a^{m-l}$$

Now again for $m < l$, then

$$\begin{aligned} a^m \times a^{-l} &= \left(\left(a \times \frac{1}{a} \right) \times \left(a \times \frac{1}{a} \right) \times \dots \dots \dots m - \text{times} \right) \times \left(\frac{1}{a} \times \frac{1}{a} \times \dots \dots \dots (l - m) - \text{times} \right) \\ &= (1 \times 1 \times \dots \dots \dots m - \text{times}) \times \frac{1}{a^{l-m}} = a^{-(l-m)} = a^{m-l} \end{aligned}$$

$\therefore$ For any integer m and n , $a^m \times a^n = a^{m+n}$ holds.

ii)

$$\boxed{(a^m)^n = a^{m.n}}$$

Case I: let $m, n > 0$ i.e. m and n are positive.

$$\begin{aligned} (a^m)^n &= a^m \times a^m \times \dots \dots \dots \times n - \text{times} \\ &= (a \times a \times a \times \dots \dots \dots m - \text{times}) \times (a \times a \times a \times \dots \dots \dots m - \text{times}) \times \dots \dots \dots n - \text{times} \\ &= a \times a \times a \times \dots \dots \dots mn - \text{times} \\ &= a^{m.n} \end{aligned}$$

Now if $m > 0, n < 0$, put $n = -l$ where $l > 0$

$$\Rightarrow (a^m)^n = (a^m)^{-l} = \frac{1}{(a^m)^l} = \frac{1}{a^{ml}} = a^{-ml}$$

$$\therefore (a^m)^{-l} = a^{m(-l)} = a^{m.n}, \text{ for } n = -l$$

Also if $m > 0$ and $n = 0$, then

$$(a^m)^n = (a^m)^0 = a^0 = 1$$

iii)

$$\boxed{\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}}$$

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{a \times a \times a \times \dots \dots \dots m - \text{times}}{a \times a \times a \times \dots \dots \dots n - \text{times}} \\ &= a.a.a \dots \dots \dots (m - n) - \text{times} \\ &= a^{m-n} \quad \therefore \frac{a^m}{a^n} = a^{m-n} \end{aligned}$$

iv)

$$\boxed{(a.b)^m = a^m.b^m}$$

$$(a.b)^m = ab \times ab \times ab \dots m - \text{times}$$

$$= (a \times a \times a \times \dots m - \text{times}) \times (b \times b \times b \times \dots m - \text{times})$$

$$\therefore (a.b)^m = a^m.b^m$$

NOTE:

The above laws hold for rational/ fractional indices.

Consider a^r and a^s where $r = \frac{p}{q}$ and $s = \frac{m}{n}$ with p, m, q, n are integers and $p, q > 0$

Let $a^r = b$ and $a^s = c$

$$\Rightarrow a^{\frac{p}{q}} = b \quad \therefore a^p = b^q$$

$$\Rightarrow a^{\frac{m}{n}} = c \quad \therefore a^m = c^n$$

Making powers of b and c the same

$$\Rightarrow (a^p)^n = (b^q)^n \quad \therefore a^{np} = b^{qn}$$

$$\text{Also } (a^m)^q = (c^n)^q \quad \therefore a^{mq} = c^{nq}$$

Now multiplying

$$\Rightarrow b^{qn}.c^{nq} = a^{np}.a^{mq}$$

$$\Rightarrow (b.c)^{nq} = a^{np+mq}$$

$$\therefore b.c = a^{\frac{np+mq}{nq}} = a^{\frac{m}{n} + \frac{p}{q}}$$

$$\Rightarrow a^r.a^s = a^{\frac{m}{n} + \frac{p}{q}} = a^{r+s}$$

Illustrative examples

1. Simplify

a) $\frac{x^{m+2n}.x^{3m-8n}}{x^{5m-6n}}$

b) $b^{-3}.x^{-2} \div 4b^2x$

c) $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n}$

d) $9^{-\left(\frac{1}{2}\right)n} \times 3^{n+2} \times 27^{n+1}$

e) $\left(\frac{a^{-1}b^2}{a^2b^{-4}}\right)^7 \div \left(\frac{a^3b^{-5}}{a^{-2}b^3}\right)^{-5}$

Solution

a) $\frac{x^{m+2n}.x^{3m-8n}}{x^{5m-6n}} = x^{m+2n+(m-8n)-(5m-6n)}$

$$= x^{(m+3m-5m)+(2n-8n+6n)}$$

$$= x^{-m+0} = x^{-m} = \frac{1}{x^m}$$

b) $b^{-3}.x^{-2} \div 4b^2x = \frac{b^{-3}.x^{-2}}{4b^2x}$

$$= \frac{b^{-3}}{b^2} \cdot \frac{x^{-2}}{x} \cdot \frac{1}{4} = \frac{b^{-3-2}.x^{-2-1}}{4}$$

$$= \frac{b^{-5}.x^{-3}}{4} = \frac{1}{4b^5.x^3}$$

c) $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n} = \frac{3(2^n.2^1)-4(2^n.2^{-1})}{(2^n.2^1)-2^n}$, let $2^n = x$

$$= \frac{3(2x)-4\left(\frac{x}{2}\right)}{2x-x} = \frac{6x-2x}{x} = \frac{4x}{x} = 4$$

$\therefore \frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n} = 4$

d) $9^{-\left(\frac{1}{2}\right)n} \times 3^{n+2} \times 27^{n+1} = [3^2]^{-\frac{1}{2}n} \times 3^{n+2} \times (3^3)^{n+1}$

$$= 3^{-2 \times \frac{1}{2}n + n + 2 + 3n + 3} = 3^{3n+5}$$

$$\begin{aligned}
 \text{e) } \left(\frac{a^{-1}b^2}{a^2b^{-4}}\right)^7 \div \left(\frac{a^3b^{-5}}{a^{-2}b^3}\right)^{-5} &= \frac{a^{-7}b^{14}}{a^{14}b^{-28}} \div \frac{a^{-15}b^{25}}{a^{10}b^{-15}} \\
 &= \left(\frac{a^{-7}}{a^{14}}\right) \cdot \left(\frac{b^{14}}{b^{-28}}\right) \div \left(\frac{a^{-15}}{a^{10}}\right) \cdot \left(\frac{b^{25}}{b^{-15}}\right) \\
 &= a^{-7-14} \cdot b^{14-(-28)} \div a^{-15-10} \cdot b^{25-(-15)} \\
 &= \frac{a^{-21} \cdot b^{42}}{a^{-25} \cdot b^{40}} = a^{-21-(-25)} \cdot b^{42-40} = a^4 \cdot b^2
 \end{aligned}$$

2. Simplify

$$\text{a) } \frac{x^2(x^2+1)^{-\frac{1}{2}} - (x^2+1)^{\frac{1}{2}}}{x^2}$$

$$\text{b) } \frac{(1+x)^{\frac{1}{3}} - \frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}}$$

$$\text{c) } \frac{\sqrt{(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} + \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot \sqrt{(1+x)}}}{1-x}$$

$$\text{d) } \frac{x^{\frac{3}{2}} + xy}{xy - y^3} - \frac{\sqrt{x}}{\sqrt{x} - y}$$

$$\text{e) } \frac{(2^{2n} - 3 \cdot 2^{2n-3})(3^n - 2 \cdot 3^{n-2})}{3^{n-4}(4^{n+8} - 2^{2n})}$$

Solution

$$\text{a) } \frac{x^2(x^2+1)^{-\frac{1}{2}} - (x^2+1)^{\frac{1}{2}}}{x^2}, \text{ let } x^2 + 1 = a$$

$$\begin{aligned}
 \Rightarrow \text{Expression} &= \frac{x^2 \cdot a^{-\frac{1}{2}} - a^{\frac{1}{2}}}{x^2} = \frac{a^{-\frac{1}{2}}(x^2 - a^1)}{x^2} \quad (\text{Factoring term with smallest power}) \\
 &= \frac{a^{-\frac{1}{2}}[x^2 - (x^2+1)]}{x^2} = \frac{(x^2+1)^{-\frac{1}{2}} \cdot (-1)}{x^2}
 \end{aligned}$$

$$\therefore \frac{x^2(x^2+1)^{-\frac{1}{2}} - (x^2+1)^{\frac{1}{2}}}{x^2} = -\frac{1}{x^2(x^2+1)^{\frac{1}{2}}}$$

$$\text{b) } \frac{(1+x)^{\frac{1}{3}} - \frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}}, \text{ let } b = 1 + x$$

$$\begin{aligned}
 \text{Expression} &= \frac{b^{\frac{1}{3}} - \frac{1}{3}xb^{-\frac{2}{3}}}{b^{\frac{2}{3}}} = \frac{b^{-\frac{2}{3}}(b^{\frac{1}{3} - \frac{1}{3}x})}{b^{\frac{2}{3}}} \quad (\text{Factoring term with smallest power}) \\
 &= b^{-\frac{2}{3} - \frac{2}{3}} \cdot \left(b - \frac{1}{3}x\right) = b^{-\frac{4}{3}} \cdot \left(b - \frac{1}{3}x\right) \\
 &= (1+x)^{-\frac{4}{3}} \cdot \left(1+x - \frac{1}{3}x\right) = (1+x)^{-\frac{4}{3}} \cdot \left(1 + \frac{2}{3}x\right)
 \end{aligned}$$

$$\therefore \frac{(1+x)^{\frac{1}{3}} - \frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}} = \frac{3+2x}{3(1+x)^{\frac{4}{3}}}$$

$$\text{c) } \frac{\sqrt{(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} + \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot \sqrt{(1+x)}}}{1-x}, \text{ let } 1-x = a, b = 1+x$$

$$\begin{aligned}
 \text{Expression} &= \frac{\sqrt{a^{\frac{1}{2}}b^{-\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}} \cdot \sqrt{b}}}{a} = \frac{\frac{1}{2}a^{\frac{1}{2}}b^{-\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b^{\frac{1}{2}}}{a} \\
 &= \frac{a^{-\frac{1}{2}}b^{-\frac{1}{2}} \cdot (1+b)}{a} = a^{-\frac{1}{2}-1} \cdot b^{-\frac{1}{2}} \cdot (1+b) \\
 &= a^{-\frac{3}{2}} \cdot b^{-\frac{1}{2}} \cdot (1+b) \\
 &= (1-x)^{-\frac{3}{2}} \cdot (1+x)^{-\frac{1}{2}} \cdot (1+1+x) \\
 &= (1-x)^{-\frac{3}{2}} \cdot (1+x)^{-\frac{1}{2}} \cdot (2+x) = \frac{2+x}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}} \\
 &= \frac{2+x}{(1-x)(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}} = \frac{2+x}{(1-x)(1-x^2)^{\frac{1}{2}}}
 \end{aligned}$$

$$\therefore \frac{\sqrt{(1-x)}^{\frac{1}{2}}(1+x)^{-\frac{1}{2}+\frac{1}{2}}(1-x)^{-\frac{1}{2}}\sqrt{(1+x)}}{1-x} = \frac{2+x}{(1-x)\sqrt{1-x^2}}$$

d) $\frac{x^{\frac{3}{2}+xy}}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x}-y} = \frac{x^{\frac{3}{2}+xy}}{xy-y^3} - \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}-y}$, let $x^{\frac{1}{2}} = a, x = a^2$

$$\begin{aligned} \text{Expression} &= \frac{a^3+a^2y}{a^2y-y^3} - \frac{a}{a-y} = \frac{a^2(a+y)}{y(a^2-y^2)} - \frac{a}{a-y} \\ &= \frac{a^2(a+y)}{y(a+y)(a-y)} - \frac{a}{a-y} \\ &= \frac{a^2}{y(a-y)} - \frac{a}{a-y} = \frac{a}{a-y} \left(\frac{a}{y} - 1 \right) \\ &= \frac{a}{a-y} \cdot \frac{(a-y)}{y} = \frac{a}{y} \\ &= \frac{x^{\frac{1}{2}}}{y} = \frac{\sqrt{x}}{y} \end{aligned}$$

$$\therefore \frac{x^{\frac{3}{2}+xy}}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x}-y} = \frac{\sqrt{x}}{y}$$

e) $\frac{(2^{2n}-3 \cdot 2^{2n-3})(3^n-2 \cdot 3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})} = \frac{[(2^n)^2-3((2^n)^2 \cdot 2^{-3})][3^n-2 \cdot 3^{n-2}]}{3^n \cdot 3^{-4} \cdot ((2^2)^{n+2}-(2^n)^2)}$ let $2^n = a, 3^n = b$

$$\begin{aligned} &= \frac{\left[a^2-3 \left(a^2 \frac{1}{8} \right) \right] \left[b-2 \cdot b \frac{1}{9} \right]}{b \frac{1}{81} ((2^2)^{n+2}-a^2)} = \frac{\left(\frac{8a^2-3a^2}{8} \right) \cdot \left(\frac{9b-2b}{9} \right)}{\frac{b}{81} ((2^n)^2 \cdot 2^4-a^2)} \\ &= \frac{\frac{5a^2}{8} \cdot \frac{7b}{9}}{\frac{b}{81} (a^2 \cdot 16-a^2)} = \frac{\frac{5a^2}{8} \cdot \frac{7b}{9}}{\frac{b}{81} \cdot 15a^2} \\ &= \frac{5a^2}{8} \cdot \frac{7b}{9} \cdot \frac{81}{15a^2b} = \frac{5 \times 7 \times 81}{8 \times 9 \times 15} = \frac{21}{8} \end{aligned}$$

$$\therefore \frac{(2^{2n}-3 \cdot 2^{2n-3})(3^n-2 \cdot 3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})} = \frac{21}{8}$$

3. Simplify

a) $\left(\frac{81}{256} \right)^{\frac{5}{4}}$

b) $\frac{(0.3)^{\frac{1}{3}} \cdot \left(\frac{1}{27} \right)^{\frac{1}{4}} \cdot (9)^{\frac{1}{6}} \cdot (0.81)^{\frac{2}{3}}}{(0.9)^{\frac{2}{3}} \cdot (3)^{-\frac{1}{2}} \cdot \left(\frac{1}{3} \right)^{-2} \cdot (243)^{-\frac{1}{4}}}$

c) $\frac{3^{-2} \times 6^2 \times \sqrt{48}}{5^2 \times \sqrt[3]{\frac{1}{25}} \times (15)^{-\frac{4}{3}} \times (3)^{\frac{1}{3}}}$

d) $\frac{8^{\frac{2}{3}+4^{\frac{2}{3}}}}{16^{\frac{4}{3}}}$

e) $\frac{(8)^{\frac{1}{6}} \times (4)^{\frac{1}{3}}}{(32)^{\frac{1}{6}} \times (16)^{\frac{1}{2}}}$

Solution

a) $\left(\frac{81}{256} \right)^{\frac{5}{4}} = \left[\frac{3^4}{4^4} \right]^{\frac{5}{4}} = \frac{3^{4 \times \frac{5}{4}}}{4^{4 \times \frac{5}{4}}} = \frac{3^{-5}}{4^{-5}} = \frac{4^5}{3^5} = \frac{1024}{243}$

b) $\frac{(0.3)^{\frac{1}{3}} \cdot \left(\frac{1}{27} \right)^{\frac{1}{4}} \cdot (9)^{\frac{1}{6}} \cdot (0.81)^{\frac{2}{3}}}{(0.9)^{\frac{2}{3}} \cdot (3)^{-\frac{1}{2}} \cdot \left(\frac{1}{3} \right)^{-2} \cdot (243)^{-\frac{1}{4}}} = \frac{\left(\frac{3}{10} \right)^{\frac{1}{3}} \cdot \left(\frac{1}{3^3} \right)^{\frac{1}{4}} \cdot (3^2)^{\frac{1}{6}} \cdot \left(\frac{81}{100} \right)^{\frac{2}{3}}}{\left(\frac{9}{10} \right)^{\frac{2}{3}} \cdot (3)^{-\frac{1}{2}} \cdot (3^{-1})^{-2} \cdot (3^5)^{-\frac{1}{4}}}$

$$\begin{aligned} &= \frac{(3)^{\frac{1}{3}} \cdot (10)^{-\frac{1}{3}} \cdot (3)^{-\frac{3}{4}} \cdot (3)^{\frac{1}{3}} \cdot \left(\frac{3^4}{10^2} \right)^{\frac{2}{3}}}{(3^2)^{\frac{2}{3}} \cdot (10^{-1})^{\frac{2}{3}} \cdot (3)^{-\frac{1}{2}} \cdot (3^2)^{-2} \cdot (3)^{-\frac{5}{4}}} = \frac{3^{\frac{1}{3} + \frac{1}{3} + \frac{8}{3} - \frac{1}{3} + \frac{4}{3}}}{3^{\frac{2}{3} + \frac{1}{2} + 2 + \frac{5}{4} - \frac{2}{3}}} \\ &= \frac{3^{\frac{31}{12}}}{3^{\frac{31}{12}}} \times \frac{10^{-\frac{5}{3}}}{10^{-\frac{5}{3}}} = 3^{\frac{31}{12} - \frac{31}{12}} \times 10^{-\frac{5}{3} + \frac{5}{3}} \\ &= 3^1 \times 10^{-1} = \frac{3}{10} \end{aligned}$$

$$\begin{aligned}
 \text{c) } \frac{3^{-2} \times 6^2 \times \sqrt{48}}{5^2 \times \sqrt[3]{\frac{1}{25}} \times (15)^{-\frac{4}{3}} \times (3)^{\frac{1}{3}}} &= \frac{3^{-2} \times (3 \times 2)^2 \times (3 \times 16)^{\frac{1}{2}}}{5^2 \times \sqrt[3]{\frac{1}{25}} \times (15)^{-\frac{4}{3}} \times (3)^{\frac{1}{3}}} \\
 &= \frac{3^{-2} \times (3 \times 2)^2 \times (3 \times 16)^{\frac{1}{2}}}{5^2 \times (5^{-2})^{\frac{1}{3}} \times (3 \times 5)^{-\frac{4}{3}} \times (3)^{\frac{1}{3}}} = \frac{3^{-2} \times 3^2 \times 2^2 \times 3^{\frac{1}{2}} \times (2^4)^{\frac{1}{2}}}{5^2 \times 5^{-\frac{2}{3}} \times 3^{\frac{4}{3}} \times 5^{-\frac{4}{3}} \times (3)^{\frac{1}{3}}} \\
 &= \frac{3^{-2+2+\frac{1}{2} \times 2} \times 2^{2+2}}{5^{2-\frac{2}{3}-\frac{4}{3}+\frac{4}{3}} \times 3^{-\frac{4}{3}+\frac{1}{3}}} = \frac{3^{\frac{1}{2}}}{5^{-1} \times 3^{-1}} \times 2^4 \times \frac{1}{5^0} \\
 &= 3^{\frac{1}{2}+1} \times 16 = 16\sqrt{27} \\
 \text{d) } \frac{\frac{8^{\frac{2}{3}} + 4^{\frac{3}{2}}}{16^{\frac{3}{4}}}}{\frac{(2^3)^{\frac{2}{3}} + (2^2)^{\frac{3}{2}}}{(2^4)^{\frac{3}{4}}}} &= \frac{2^2 + 2^3}{2^3} = \frac{4+8}{8} = \frac{12}{8} = \frac{3}{2} \\
 \text{e) } \frac{(8)^{\frac{1}{6}} \times (4)^{\frac{1}{3}}}{(32)^{\frac{1}{6}} \times (16)^{\frac{1}{2}}} &= \frac{(2^3)^{\frac{1}{6}} \times (2^2)^{\frac{1}{3}}}{(2^5)^{\frac{1}{6}} \times (2^4)^{\frac{1}{2}}} = \frac{2^{\frac{3}{6} + \frac{2}{3}}}{2^{\frac{5}{6} + \frac{4}{2}}} = \frac{2^{\frac{5}{6}}}{2^{\frac{17}{6}}} = 2^{\frac{5}{6} - \frac{17}{6}} = 2^{-1} = \frac{1}{2}
 \end{aligned}$$

PROOFS IN INDICES

1. If $a = 2, b = 3$, show that $(a^3 b^2 c^3)^2 \sqrt{a^{-3} b^{-4} c^6} = 144\sqrt{2}.c^9$

Solution

$$\begin{aligned}
 (a^3 b^2 c^3)^2 \sqrt{a^{-3} b^{-4} c^6} &= a^6 b^4 c^6 \cdot (a^{-3} b^{-4} c^6)^{\frac{1}{2}} \\
 &= a^6 b^4 c^6 \cdot a^{-\frac{3}{2}} b^{-2} c^3 \\
 &= a^{6-\frac{3}{2}} b^{4-2} c^{3+6} \\
 &= a^{\frac{9}{2}} b^2 c^9 \\
 \Rightarrow a^{\frac{9}{2}} b^2 c^9 &= (2^{\frac{9}{2}}) (3)^2 c^9 = (2^9)^{\frac{1}{2}} \cdot 9c^9 \\
 &= (2^8 \times 2)^{\frac{1}{2}} \cdot 9c^9 = 2^4 \times 2^{\frac{1}{2}} \cdot 9c^9 = 16 \times 9\sqrt{2} c^9 = 144\sqrt{2}.c^9
 \end{aligned}$$

2. If $x = [a + \sqrt{a^2 + b^3}]^{\frac{1}{3}} + [a - \sqrt{a^2 + b^3}]^{\frac{1}{3}}$, show that $x^3 + 3bx - 2a = 0$

Solution

$$\text{Let } \sqrt{a^2 + b^3} = B$$

$$\Rightarrow x = [a + B]^{\frac{1}{3}} + [a - B]^{\frac{1}{3}}$$

Cubing both sides

$$\begin{aligned}
 x^3 &= \left[(a + B)^{\frac{1}{3}} + (a - B)^{\frac{1}{3}} \right]^3 \\
 &= \left[(a + B)^{\frac{1}{3}} \right]^3 + 3 \left[(a + B)^{\frac{1}{3}} \right]^2 (a - B)^{\frac{1}{3}} + 3(a + B)^{\frac{1}{3}} (a - B)^{\frac{2}{3}} + \left[(a - B)^{\frac{1}{3}} \right]^3 \\
 &= a + B + 3(a + B)^{\frac{2}{3}} (a - B)^{\frac{1}{3}} + 3(a + B)^{\frac{1}{3}} (a - B)^{\frac{2}{3}} + a - B \\
 &= 2a + 3(a + B)^{\frac{1}{3}} (a - B)^{\frac{1}{3}} \left[(a + B)^{\frac{1}{3}} + (a - B)^{\frac{1}{3}} \right] \\
 &= 2a + 3[(a + B)(a - B)]^{\frac{1}{3}} x \\
 &= 2a + 3(a^2 - B^2)^{\frac{1}{3}} x \\
 &= 2a + 3(a^2 - (a^2 + b^3))^{\frac{1}{3}} x \quad [\because \sqrt{a^2 + b^3} = B] \\
 &= 2a + 3(-b^3)^{\frac{1}{3}} x \\
 &= 2a - 3bx \\
 \therefore x^3 - 2a + 3bx &= 0
 \end{aligned}$$

3. If $x = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}}$, show that $9x^3 - 27x = 82$

Solution

$x = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}}$, cubing both sides,

$$\Rightarrow x^3 = \left(3^{\frac{2}{3}} + 3^{-\frac{2}{3}}\right)^3 = \left(3^{\frac{2}{3}}\right)^3 + \left(3^{-\frac{2}{3}}\right)^3 + 3 \cdot 3^{\frac{2}{3}} \cdot 3^{-\frac{2}{3}} \left(3^{\frac{2}{3}} + 3^{-\frac{2}{3}}\right)$$

$$[\because (\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)]$$

$$\Rightarrow x^3 = 3^2 + 3^{-2} + 3x = 9 + \frac{1}{9} + 3x$$

$$\therefore x^3 = \frac{82}{9} + 3x \text{ or } 9x^3 - 27x = 82$$

4. If $2^x = 4^y = 8^z$ and $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{22}{7}$, show that $x = \frac{7}{16}$, $y = \frac{7}{32}$ and $z = \frac{7}{48}$

Solution

$$2^x = 4^y \Rightarrow 2^x = 2^{2y} \therefore x = 2y \dots\dots\dots (1)$$

$$\text{Also } 2^x = 8^z \Rightarrow 2^x = 2^{3z} \therefore x = 3z \dots\dots\dots (2)$$

Eqn. (1) = eqn. (2)

$$\Rightarrow 2y = 3z \text{ or } z = \frac{2y}{3} \dots\dots\dots (3)$$

$$\text{From } \frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{22}{7}$$

$$\Rightarrow \frac{1}{2x} + \frac{1}{4\left(\frac{x}{2}\right)} + \frac{1}{8\left(\frac{2y}{3}\right)} = \frac{22}{7}$$

$$\Rightarrow \frac{1}{2x} + \frac{1}{4\left(\frac{x}{2}\right)} + \frac{1}{8\left(\frac{2}{3} \times \frac{x}{2}\right)} = \frac{22}{7}$$

$$\Rightarrow \frac{1}{2x} + \frac{1}{2x} + \frac{3}{8x} = \frac{22}{7}$$

$$\Rightarrow \frac{11}{8x} = \frac{22}{7} \therefore x = \frac{7}{22} \times \frac{11}{8} = \frac{7}{16}$$

From eqn. (1), $x = 2y$

$$\Rightarrow y = \frac{x}{2} = \frac{7}{16} \times \frac{1}{2} = \frac{7}{32}$$

From eqn. (2), $x = 3z$

$$\Rightarrow z = \frac{x}{3} = \frac{7}{16} \times \frac{1}{3} = \frac{7}{48}$$

$$\therefore x = \frac{7}{16}, y = \frac{7}{32}, z = \frac{7}{48}$$

5. If $\frac{x}{y} = \frac{z}{w}$, show that $\frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = (wxyz)^{\frac{m}{2}}$

Solution

$$\text{Let } \frac{x}{y} = \frac{z}{w} = k$$

$$\Rightarrow x = yk, z = wk$$

$$\Rightarrow \frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = \frac{y^m k^m + y^m + w^m k^m + w^m}{y^{-m} k^{-m} + y^{-m} + w^{-m} k^{-m} + w^{-m}}$$

$$= \frac{y^m (k^m + 1) + w^m (k^m + 1)}{y^{-m} (k^{-m} + 1) + w^{-m} (k^{-m} + 1)}$$

$$= \frac{(k^m + 1)(y^m + w^m)}{(k^{-m} + 1)(y^{-m} + w^{-m})}$$

$$= \frac{(k^m + 1)}{\left(\frac{1}{k^m} + 1\right)} \cdot \frac{(y^m + w^m)}{\left(\frac{1}{y^m} + \frac{1}{w^m}\right)}$$

$$= \frac{k^m (k^m + 1)}{(1 + k^m)} \cdot \frac{(y^m + w^m)(y^m \cdot w^m)}{(y^m + w^m)}$$

$$= k^m \cdot y^m \cdot w^m = (k \cdot y \cdot w)^m$$

But from $\frac{x}{y} = \frac{z}{w} = k \Rightarrow \frac{x}{y} = k, \frac{z}{w} = k$, multiplying the equations

$$\Rightarrow k^2 = \frac{xz}{yw} \text{ or } k = \left(\frac{xz}{yw}\right)^{\frac{1}{2}}$$

$$\begin{aligned}\therefore (k.y.w)^m &= \left(\left(\frac{xz}{yw} \right)^{\frac{1}{2}} . y . w \right)^m = \left((x.z.y^{-1}.w^{-1})^{\frac{1}{2}} . y . w \right)^m \\ &= \left(x^{\frac{1}{2}} . z^{\frac{1}{2}} . y^{1-\frac{1}{2}} . w^{1-\frac{1}{2}} \right)^m = \left(x^{\frac{1}{2}} . z^{\frac{1}{2}} . y^{\frac{1}{2}} . w^{\frac{1}{2}} \right)^m = (wxyz)^{\frac{m}{2}}\end{aligned}$$

6. If $a^x = b^y = c^z = d^w$ and $ab = cd$, show that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z} + \frac{1}{w}$

Solution

Let $a^x = b^y = c^z = d^w = k$

$$\Rightarrow a^x = k \text{ or } a = k^{\frac{1}{x}}$$

$$\Rightarrow b^y = k \text{ or } b = k^{\frac{1}{y}}$$

$$\Rightarrow c^z = k \text{ or } c = k^{\frac{1}{z}}$$

$$\Rightarrow d^w = k \text{ or } d = k^{\frac{1}{w}}$$

From $a.b = c.d$

$$\Rightarrow k^{\frac{1}{x}} . k^{\frac{1}{y}} = k^{\frac{1}{z}} . k^{\frac{1}{w}} \text{ or } k^{\frac{1}{x} + \frac{1}{y}} = k^{\frac{1}{z} + \frac{1}{w}}$$

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{z} + \frac{1}{w}$$

7. If $3^x = 5^y = 75^z$, show that $xz = z(2x + y)$

Solution

Let $3^x = 5^y = 75^z = k$

$$\Rightarrow 3^x = k \text{ or } 3 = k^{\frac{1}{x}}$$

$$\Rightarrow 5^y = k \text{ or } 5 = k^{\frac{1}{y}}$$

$$\Rightarrow 75^z = k \text{ or } 75 = k^{\frac{1}{z}}$$

But $75 = 25 \times 3 = 5^2 \times 3$

$$\Rightarrow k^{\frac{1}{z}} = \left(k^{\frac{1}{y}} \right)^2 \times k^{\frac{1}{x}}$$

$$\Rightarrow k^{\frac{1}{z}} = k^{\frac{1}{x} + \frac{2}{y}}$$

$$\therefore \frac{1}{z} = \frac{1}{x} + \frac{2}{y} \text{ or } \frac{1}{z} = \frac{y+2}{xy}$$

$$\therefore xz = z(2x + y)$$

8. If $\frac{x^{n+1}}{a^n} = \frac{y^{n+1}}{b^n} = \frac{z^{n+1}}{c^n} = a + b + c$, prove that $\left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}} \right)^{\frac{n}{n+1}} = a + b + c$

Solution

Let $a + b + c = k$

$$\Rightarrow \frac{x^{n+1}}{a^n} = k \text{ or } a^n = \frac{x^{n+1}}{k}$$

$$\therefore a = \left(\frac{x^{n+1}}{k} \right)^{\frac{1}{n}} = \frac{x^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}$$

Similarly

$$\therefore b = \frac{y^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}, \quad c = \frac{z^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}$$

$$\Rightarrow \left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}} \right)^{\frac{n}{n+1}} = \left(a.k^{\frac{1}{n}} + b.k^{\frac{1}{n}} + c.k^{\frac{1}{n}} \right)^{\frac{n}{n+1}}$$

$$= \left[(a + b + c).k^{\frac{1}{n}} \right]^{\frac{n}{n+1}} = (a + b + c)^{\frac{n}{n+1}} . k^{\frac{1}{n+1}}$$

But $a + b + c = k$

$$\therefore \left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}} \right)^{\frac{n}{n+1}} = (a + b + c)^{\frac{n}{n+1}} . (a + b + c)^{\frac{1}{n+1}}$$

$$= (a + b + c)^{\frac{n}{n+1} + \frac{1}{n+1}} = a + b + c$$

9. If $a = xy^{p-1}$, $b = xy^{q-1}$, $c = xy^{r-1}$, prove that $a^{q-r} \times b^{r-p} \times c^{p-q} = 1$

Solution

$$\begin{aligned} a^{q-r} \times b^{r-p} \times c^{p-q} &= (xy^{p-1})^{q-r} \times (xy^{q-1})^{r-p} \times (xy^{r-1})^{p-q} \\ &= x^{(q-r)} \cdot y^{(p-1)(q-r)} \times x^{(r-p)} \cdot y^{(q-1)(r-p)} \times x^{(p-q)} \cdot y^{(r-1)(p-q)} \\ &= x^{q-r+r-p+p-q} \times y^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)} \\ &= x^0 \cdot y^{pq-pr-q+r+qr-qp-r+p+rp-rq-p+q} \\ &= x^0 \cdot y^0 = 1 \end{aligned}$$

Solving equations involving indices

1. Find x if $x^{x\sqrt{x}} = (x\sqrt{x})^x$, where x is a positive integer.

Solution

$$\begin{aligned} x^{x\sqrt{x}} &= (x\sqrt{x})^x \Rightarrow x^{x \cdot \frac{1}{2}} = \left(x \cdot \frac{1}{2}\right)^x \\ \Rightarrow x^{\frac{3}{2}} &= \left(x^{\frac{3}{2}}\right)^x = x^{\frac{3}{2}x} \\ \Rightarrow x^{\frac{3}{2}} &= \frac{3}{2}x, \text{ squaring both sides} \\ \Rightarrow x^3 &= \frac{9}{4}x^2 \text{ or } 4x^3 - 9x^2 = 0 \\ \Rightarrow x^2(4x - 9) &= 0 \therefore x = 0, \frac{9}{4} \end{aligned}$$

$$\text{But } x \neq 0 \therefore x = \frac{9}{4}$$

2. Solve $4^x \cdot 2^y = 128$ and $3^{2x+2y} = 9^{xy}$

Solution

$$\begin{aligned} 4^x \cdot 2^y &= 128 \Rightarrow 2^{2x+y} = 2^7 \\ \therefore 2x + y &= 7 \dots\dots\dots (1) \\ \text{Also } 3^{2x+2y} &= 9^{xy} \Rightarrow 3^{2x+2y} = 3^{2xy} \\ \therefore 2x + 2y &= 2xy \text{ or } x + y = xy \dots\dots\dots (2) \\ \text{From eqn. (1), } y &= 7 - 2x \dots\dots\dots (3) \\ \text{Eqn. (3) in to eqn. (2)} \\ \Rightarrow x + 7 - 2x &= x(7 - 2x) \\ \Rightarrow 7 - x &= 7x - 2x^2 \text{ or } 2x^2 - 8x + 7 = 0 \\ \therefore x &= \frac{8 \pm \sqrt{(-8)^2 - 4 \times 2 \times 7}}{2 \times 2} \\ &= \frac{8 \pm \sqrt{8}}{4} = \frac{8 \pm 2\sqrt{2}}{4} = \frac{4 \pm \sqrt{2}}{2} \end{aligned}$$

$$\text{From eqn. (3) } y = 7 - 2x$$

$$\text{For } x = \frac{4 \pm \sqrt{2}}{2}, y = 7 - 2\left(\frac{4 \pm \sqrt{2}}{2}\right) = 7 - (4 \pm \sqrt{2})$$

$$\therefore x = \frac{4 + \sqrt{2}}{2}, y = 3 - \sqrt{2}; x = \frac{4 - \sqrt{2}}{2}, y = 11 + \sqrt{2}$$

3. Solve

- $a^{x-3} \cdot a^{y+2} = a^2 \cdot a^x$, $a^x \cdot a^y = a$, where $a \neq 0, 1$
- $(\sqrt{a})^{x+y} = (\sqrt[3]{a})^{y+z-1}$, $(\sqrt{b})^{x+z-2} = (\sqrt[5]{b})^{y+z}$, $(\sqrt[4]{c})^y = (\sqrt[7]{c})^{x+y+z}$
- $2^{x+y+z} = 8^{x+z-y}$, $5^{3y+2} = 25^{x+z}$, $3^{2x+y+2z} = 9^{3x+y}$

Solution

$$\begin{aligned} \text{i) } a^{x-3} \cdot a^{y+2} &= a^2 \cdot a^x \\ \Rightarrow a^{x-3+y+2} &= a^{2+x} \\ \Rightarrow x + y - 1 &= 2 + x \therefore y = 3 \end{aligned}$$

$$\text{Also } a^x \cdot a^y = a \Rightarrow a^{x+y} = a$$

$$\Rightarrow x + y = 1, \text{ but } y = 3$$

$$\Rightarrow x = 1 - 3 = -2$$

$$\therefore x = -2, y = 3$$

$$\text{ii) } (\sqrt{a})^{x+y} = (\sqrt[3]{a})^{y+z-1}$$

$$\Rightarrow a^{\frac{x+y}{2}} = a^{\frac{y+z-1}{3}}$$

$$\Rightarrow \frac{x+y}{2} = \frac{y+z-1}{3} \text{ or } 3x + 3y = 2y + 2z - 2$$

$$\therefore 3x + y - 2z = -2 \dots\dots\dots (1)$$

$$(\sqrt{b})^{x+z-2} = (\sqrt[5]{b})^{y+z}$$

$$\Rightarrow b^{\frac{x+z-2}{2}} = b^{\frac{y+z}{5}}$$

$$\Rightarrow \frac{x+z-2}{2} = \frac{y+z}{5} \text{ or } 5x + 5z - 10 = 2y + 2z$$

$$\therefore 5x - 2y + 3z = 10 \dots\dots\dots (2)$$

$$(\sqrt[4]{c})^y = (\sqrt[5]{c})^{x+y+z}$$

$$\Rightarrow c^{\frac{y}{4}} = c^{\frac{x+y+z}{5}}$$

$$\Rightarrow \frac{y}{4} = \frac{x+y+z}{5} \text{ or } 7y = 4x + 4y + 4z$$

$$\therefore 4x - 3y + 4z = 0 \dots\dots\dots (3)$$

Eliminating y from eqn. (1) and eqn. (2)

$$2 \times \text{eqn. (1)} + \text{eqn. (2)}$$

$$\Rightarrow 6x + 2y - 4z + 5x - 2y + 3z = -4 + 10$$

$$\therefore 11x - z = 6 \dots\dots\dots (4)$$

Eliminating y from eqn. (1) and eqn. (3)

$$3 \times \text{eqn. (1)} + \text{eqn. (3)}$$

$$\Rightarrow 9x + 3y - 6z + 4x - 3y + 4z = -6 + 0$$

$$\therefore 13x - 2z = -6 \dots\dots\dots (5)$$

Eliminating z from eqn. (4) and eqn. (5)

$$2 \times \text{eqn. (4)} - \text{eqn. (5)}$$

$$\Rightarrow 22x - 2z - (13x - 2z) = 12 - -6$$

$$\Rightarrow 9x = 18 \therefore x = 2$$

$$\text{From eqn. (4), } z = 11x - 6$$

$$\Rightarrow z = 22 - 6 = 16$$

$$\text{From eqn. (1), } y = -2 + 2z - 3x$$

$$\Rightarrow y = -2 + 32 - 6 = 24$$

$$\therefore x = 2, y = 24, z = 16$$

$$\text{iii) } 2^{x+y+z} = 8^{x+z-y}$$

$$\Rightarrow 2^{x+y+z} = 2^{3x+3z-3y} \text{ or } x + y + z = 3x + 3z - 3y$$

$$\Rightarrow 2x - 4y + 2z = 0$$

$$\therefore x - 2y + z = 0 \dots\dots\dots (1)$$

$$5^{3y+2} = 25^{x+z}$$

$$\Rightarrow 5^{3y+2} = 5^{2x+2z} \text{ or } 3y + 2 = 2x + 2z$$

$$\therefore 2x - 3y + 2z = 2 \dots\dots\dots (2)$$

$$3^{2x+y+2z} = 9^{3x+y}$$

$$\Rightarrow 3^{2x+y+2z} = 3^{6x+2y} \text{ or } 2x + y + 2z = 6x + 2y$$

$$\therefore 4x + y - 2z = 0 \dots\dots\dots (3)$$

Eliminating z from eqn. (1) and eqn. (2)

$$2 \times \text{eqn. (1)} - \text{eqn. (2)}$$

$$\Rightarrow 2x - 4y + 2z - (2x - 3y + 2z) = 0 - 2$$

$$\Rightarrow -y = -2 \therefore y = 2$$

Eliminating z from eqn. (2) and eqn. (3)

eqn. (2) + eqn. (3)

$$\Rightarrow 6x - 2y = 2, \text{ but } y = 2$$

$$\Rightarrow 6x - 4 = 2 \therefore x = 1$$

From eqn. (1), $z = 2y - x$

$$\Rightarrow z = 4 - 1 = 3$$

$$\therefore x = 1, y = 2, z = 3$$

Task

1. Show that

i) $\frac{\sqrt{x^3+xy}}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x-y}} = \sqrt{\left(\frac{x}{y}\right)}$

ii) If $x = \sqrt[3]{p+q} + \sqrt[3]{p-q}$ and $p^2 - q^2 = r$, then $x^3 - 3rx - 2p = 0$

iii) If $(2.381)^x = (0.2381)^y = 10^z$, then $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$

iv) If $x = 3^{\frac{1}{4}} + 3^{-\frac{1}{4}}, y = 3^{\frac{1}{4}} - 3^{-\frac{1}{4}}$, then $3(x^2 + y^2)^2 = 64$

v) If $2^x = 3^y = 6^{-z}$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

2. Prove that

a) $\frac{16(32)^m - 2^{3m-2} \cdot 4^{m+1}}{15 \cdot (2^{m-1})(16)^m} - \frac{5^m}{\sqrt{5^{2m}}} = 1$

b) $\frac{(9^{n+\frac{1}{4}}) \cdot \sqrt{3 \times 3^n}}{3\sqrt{3^{-n}}} = 27$

c) If $pqr = 1$, then $\frac{1}{1+p+q^{-1}} + \frac{1}{1+q+r^{-1}} + \frac{1}{1+r+p^{-1}} = 1$

d) $\frac{(81)^n \cdot 3^5 - 3^{4n-1} \cdot (243)}{9^{2n} \cdot 3^2} - \frac{4 \cdot 3^n}{3^{n+1} - 3^n} = 4$

e) If $x = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}}$, then $9x^3 - 27x = 82$

3. Simplify

a) $\sqrt[3]{a^6 b^{-2} c^{-4}} \times \sqrt[4]{a^{-6} b^4 c^3}$

ANSWER: $a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{3}{2}}$

b) $\left(\frac{5^{-1} \cdot 7^2}{5^2 \cdot 7^{-4}}\right)^{\frac{7}{2}} \div \left(\frac{5^3 \cdot 7^{-5}}{5^{-2} \cdot 7^3}\right)^{\frac{7}{2}}$

ANSWER: 175

LOGARITHMS

Definition:

The logarithm of a *positive* quantity N to any given base a is defined as the index of the power to which the base a must be raised to make it equal to the given quantity N . i.e. if $a^x = N$, then x is called the logarithm of N to the base a and is written as

$$\log_a N = x$$

NOTE:

1. We notice that a logarithm is just a power, thus it can also be called an *index* or an *exponent*. This implies that in the derivation of laws of logarithms, laws of indices come in to use.
2. Logarithms are defined for only *positive numbers*.

PROPERTIES/LAWS/THEOREMS OF LOGARITHMS

1. *The logarithm of unity to any base is zero.*

$$\log_a 1 = 0$$

Proof:

From $a^0 = 1$ [Setting $x = 0$ for $a^x = N$], then
 $0 = \log_a 1$ [From definition]
 $\therefore \log_a 1 = 0$

2. *Logarithm of any number to the same base is 1*

$$\log_a a = 1$$

Proof:

From $a^1 = a$ [Setting $x = 1$ for $a^x = N$], then
 $\log_a a = 1$ [From definition $x = \log_a N$]
 $\therefore \log_a a = 1$

3. *A number a power logarithm of another number N to base of a number a is N*

$$a^{\log_a N} = N$$

Proof:

Let $\log_a N = x \Rightarrow a^x = N$
 $\therefore a^{\log_a N} = N$
 In general,

$$1. a^{\log_a b^{\log_b N}} = N$$

Proof:

Let $a^{\log_a b} = y \Rightarrow y = b$
 Now $y^{\log_b N} = b^{\log_b N} = N$
 $1. a^{\log_a b^{\log_b N}} = N$

4. *The logarithm of a product is equal to the sum of logarithms of its factors. i.e.*

$$\log_a (M.N) = \log_a M + \log_a N$$

Proof:

From **LHS** = $\log_a(M.N)$

Let $\log_a(M.N) = b \Rightarrow a^b = M.N$ (1)

Let $\log_a M = x \Rightarrow a^x = M$ (2)

Let $\log_a N = y \Rightarrow a^y = N$ (3)

Eqn. (2) , (3) in to eqn. (1)

$$\Rightarrow a^b = a^x \cdot a^y = a^{x+y}$$

$$\Rightarrow b = x + y$$

$$\therefore \log_a(M.N) = \log_a M + \log_a N$$

NOTE:

- $\log_a(M + N) \neq \log_a M + \log_a N$
- $\log_a(M.N) = \log_a M + \log_a N$

5. The logarithm of a fraction is equal to the logarithm of numerator diminished by the logarithm of the denominator .i.e.

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

Let $\log_a(M.N) = b \Rightarrow a^b = \frac{M}{N}$ (1)

Let $\log_a M = x \Rightarrow a^x = M$ (2)

Let $\log_a N = y \Rightarrow a^y = N$ (3)

Eqn. (2), (3) in to eqn. (1)

$$\Rightarrow a^b = a^x \div a^y = a^{x-y}$$

$$\Rightarrow b = x - y$$

$$\therefore \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

NOTE:

- $\log_a\left(\frac{M}{N}\right) \neq \log_a M \div \log_a N$
- $\log_a(M - N) = \log_a M - \log_a N$

6. The logarithm of the p^{th} power of a positive number is p times the logarithm of that number.i.e.

$$\log_a M^p = p \log_a M$$

Proof:

Let $\log_a M = x \Rightarrow a^x = M$

Now powering by p on both sides

$$(a^x)^p = M^p \Rightarrow a^{xp} = M^p$$

$$\Rightarrow \log_a M^p = xp = p \log_a M$$

$$\therefore \log_a M^p = p \log_a M$$

Similarly if $p = \frac{1}{r}$, then

$$\log_a M^{\frac{1}{r}} = \frac{1}{r} \log_a M$$

7. To transform a system of logarithms having a given base to a new system with a different base. i. e.

$$\log_{mn} N = \frac{\log_n N}{1 + \log_n M}$$

Proof:

Let $\log_a N = x$ be given and required $\log_b N = y \Rightarrow b^y = N$

$\Rightarrow \log_a N = x$, becomes

$$\log_a b^y = x = \log_a N$$

$$\Rightarrow y \log_a b = \log_a N \quad \text{or} \quad y = \frac{1}{\log_a b} \times \log_a N$$

$$\therefore \log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N$$

NOTE:

1. To transform a logarithm from base a to base b we have only to multiply it by $\frac{1}{\log_a b}$. This is known as *Base changing formula*
2. From the above base change of bases, if $N = a$,

$$\log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N, \text{ becomes}$$

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b}$$

$$\log_b a = \frac{1}{\log_a b}$$

Proof:

Let $\log_b a = x \Rightarrow b^x = a$

$\Rightarrow b = a^{\frac{1}{x}}$, introducing logarithm to base a on both sides

$$\Rightarrow \log_a b = \log_a a^{\frac{1}{x}} = \frac{1}{x} \log_a a = \frac{1}{x}$$

$$\therefore \log_b a = \frac{1}{\log_a b} \quad \text{or} \quad \log_a b = \frac{1}{\log_b a}$$

8. A base and its reciprocal inverse as a base

$$\log_b N = -\log_{\frac{1}{b}} N$$

Proof:

Let $\log_b N = x \Rightarrow b^x = N$ or $b = N^{\frac{1}{x}}$

Introducing logarithm to base $\frac{1}{b}$

$$\Rightarrow \log_{\frac{1}{b}} b = \log_{\frac{1}{b}} N^{\frac{1}{x}} = \frac{1}{x} \log_{\frac{1}{b}} N$$

$$\Rightarrow x \log_{\frac{1}{b}} b = \log_{\frac{1}{b}} N \quad \text{or} \quad x = \frac{\log_{\frac{1}{b}} N}{\log_{\frac{1}{b}} b}$$

From change of base formula

$$\Rightarrow \log_{\frac{1}{b}} b = \frac{\log_b b}{\log_b \frac{1}{b}} = \frac{1}{\log_b b^{-1}} = \frac{1}{-\log_b b} = -1$$

$$\Rightarrow x = \frac{\log_{\frac{1}{b}} N}{\log_{\frac{1}{b}} b} = \frac{\log_{\frac{1}{b}} N}{-1} = -\log_{\frac{1}{b}} N$$

$$\therefore \log_b N = -\log_{\frac{1}{b}} N$$

9. Logarithm of a number with a composite base

$$\log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N$$

Proof:

Let $\log_{mn} N = x \Rightarrow (mn)^x = N$ or $mn = N^{\frac{1}{x}}$

Introducing logarithm to base n on both sides

$$\Rightarrow \log_n(mn) = \log_n N^{\frac{1}{x}} = \frac{1}{x} \log_n N$$

$$\Rightarrow x = \frac{\log_n N}{\log_n(mn)} = \frac{\log_n N}{\log_n m + \log_n n} = \frac{\log_n N}{\log_n m + 1}$$

$$\therefore \log_{mn} N = \frac{\log_n N}{1 + \log_n m}$$

Application of laws of logarithms
A. Solving equations

1. Solve the equations:

i) $\log(x^2 + 2x) = 0.9031$

ii) $\log_a(5x - 6) + \log_a(2x + 3) = \log_a(10x^2 - 3x - 6)$

iii) $2\log_a(x - 2) = \log_a(2x - 5)$

iv) $\log_x 3 + \log_x 9 + 729 = 9$

Solution

i) $\log(x^2 + 2x) = 0.9031$

$$\Rightarrow \log_{10}(x^2 + 2x) = 0.9031$$

$$\Rightarrow 10^{0.9031} = x^2 + 2x \quad (\text{From the definition})$$

$$\Rightarrow x^2 + 2x - 8 = 0 \quad \text{or} \quad (x + 4)(x - 2) = 0$$

$$\therefore x = -4, 2$$

ii) $\log_a(5x - 6) + \log_a(2x + 3) = \log_a(10x^2 - 3x - 6)$

$$\Rightarrow \log_a(5x - 6)(2x + 3) = \log_a(10x^2 - 3x - 6)$$

Since the bases are the same, then also the numbers must be equal.

$$\Rightarrow (5x - 6)(2x + 3) = 10x^2 - 3x - 6$$

$$\Rightarrow 10x^2 + 3x - 18 = 10x^2 - 3x - 6$$

$$\Rightarrow 6x = 12 \therefore x = 2$$

iii) $\log_a(x - 2) = \log_a(2x - 5)$

$$\Rightarrow \log_a(x - 2)^2 = \log_a(2x - 5)$$

$$\Rightarrow (x - 2)^2 = 2x - 5$$

$$\Rightarrow x^2 - 4x + 3 = 2x - 5$$

$$\Rightarrow x^2 - 6x + 9 = 0 \quad \text{or} \quad (x - 3)^2 = 0$$

$$\therefore x = 3$$

iv) $\log_x 3 + \log_x 9 + 729 = 9$

$$\Rightarrow \log_x(3 \times 9 \times 729) = 9$$

$$\Rightarrow x^9 = 19683$$

$$\Rightarrow (x^9)^{\frac{1}{9}} = (19683)^{\frac{1}{9}}$$

$$\therefore x = 3$$

Task

1. Solve the equations

i) $\ln(x^2 - 1) - \ln(x - 1) = \ln 5 \quad (x > 0)$

ii) $\ln 4 + 2 \ln x - \ln(x^2 + x) = \ln 2 \quad (x > 0)$ **ANSWER:** i) $\frac{3}{2}$ ii) $\frac{3}{2}$

NOTE: $\log_e N = \ln N$

2. Solve that equations

- i) $3^{\log_3 2} + 5^{\log_5 7} = 6^{\log_6 x}$
- ii) $11^{\log_x 7} = 7$
- iii) $\log_{10} x - 2 + \log_x 10 = 0$
- iv) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$
- v) $\log_2 x - \log_x 8 = 2$ (Uneb 2007)
- vi) $\log_4(6 - x) = \log_2 x$ (Uneb 2003)
- vii) $\log_n 4 + \log_4 n^2 = 3$
- viii) $\log_2 x + \log_x 64 = 5$
- ix) $\log_5 x - \log_x 5 = \frac{3}{2}$

Solution

i) $3^{\log_3 2} + 5^{\log_5 7} = 6^{\log_6 x}$

Using $a^{\log_a x} = x$,

$$\Rightarrow 2 + 7 = x \therefore x = 9$$

ii) $11^{\log_x 7} = 7$

$$\text{Let } \log_x 7 = y \Rightarrow 11^y = 7$$

Introducing logarithm to base 11

$$\Rightarrow \log_{11} 11^y = \log_{11} 7 \text{ or } y = \log_{11} 7$$

$$\text{But } \log_x 7 = y \therefore \log_x 7 = \log_{11} 7$$

Since numbers are the same, then $x = 11$

Alternatively

From $a^{\log_a x} = x$, since $11^{\log_x 7} = 7$, then $x = 11$

$$\Rightarrow a^{\log_a 7} = 7, \text{ also } a \text{ must be } 11$$

$$\therefore 11^{\log_{11} 7} = 7 \Rightarrow x = 11$$

iii) $\log_{10} x - 2 + \log_x 10 = 0$

$$\Rightarrow \log_{10} x + \log_x 10 = 2$$

Changing to base 10

$$\Rightarrow \log_{10} x + \frac{\log_{10} 10}{\log_{10} x} = 2$$

$$\Rightarrow \log_{10} x + \frac{1}{\log_{10} x} = 2; \text{ let } \log_{10} x = y$$

$$\Rightarrow y + \frac{1}{y} = 2 \text{ or } y^2 - 2y + 1 = 0 \therefore y = 1$$

$$\therefore \log_{10} x = 1 \text{ or } x = 10$$

iv) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

Since 4 and 16 are multiples of 2, changing the base to 2

$$\Rightarrow \log_2 x + \frac{\log_2 x}{\log_2 4} + \frac{\log_2 x}{\log_2 16} = \frac{21}{4}$$

$$\Rightarrow \log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{4} = \frac{21}{4}$$

$$\Rightarrow \log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{4} = \frac{21}{4}$$

$$\Rightarrow \left(1 + \frac{1}{2} + \frac{1}{4}\right) \log_2 x = \frac{21}{4}$$

$$\Rightarrow \frac{7}{4} \log_2 x = \frac{21}{4} \text{ or } \log_2 x = 3$$

$$\therefore x = 2^3 = 8$$

v) $\log_2 x - \log_x 8 = 2$, changing to base 2

$$\begin{aligned} \Rightarrow \log_2 x - \frac{\log_2 8}{\log_2 x} &= 2 \\ \Rightarrow \log_2 x - \frac{3 \log_2 2}{\log_2 x} &= 2 \\ \Rightarrow \log_2 x - \frac{3}{\log_2 x} &= 2, \text{ let } \log_2 x = y \\ \Rightarrow y - \frac{3}{y} &= 2 \text{ or } y^2 - 2y - 3 = 0 \\ \Rightarrow (y+1)(y-3) &= 0 \text{ or } y = -1, 3 \\ \Rightarrow \log_2 x = -1 \text{ or } 2^{-1} &= \frac{1}{2} = x \\ \Rightarrow \log_2 x = 3 \text{ or } 2^3 &= 8 = x \\ \therefore x &= \frac{1}{2}, 3 \end{aligned}$$

vi) Left as an exercise (**Hint:** Change to base 2)

ANSWER: $x = 2, -3$

vii) $\log_n 4 + \log_4 n^2 = 3$

$$\begin{aligned} \Rightarrow \log_n 4 + 2 \log_4 n &= 3 \\ \Rightarrow \frac{\log_4 4}{\log_4 n} + 2 \log_4 n &= 3, \text{ let } \log_4 n = x \\ \Rightarrow \frac{1}{x} + 2x &= 3 \text{ or } 2x^2 - 3x + 1 = 0 \\ \Rightarrow (2x-1)(x-1) &= 0 \therefore x = \frac{1}{2}, 1 \\ \Rightarrow \log_4 n = \frac{1}{2} \text{ or } 4^{1/2} &= 2 = n \\ \Rightarrow \log_4 n = 1 \text{ or } 4^1 &= 4 = n \\ \therefore n &= 2, 4 \end{aligned}$$

viii) Left as an exercise.

ANSWER: 4, 8

ix) Left as an exercise.

ANSWER: $x = \frac{\sqrt{5}}{5}, 25$

Task

- a) Find x if $\log_x 8 - \log_{x^2} 16 = 1$ **ANSWER:** $x = 2$ [**HINT:** Change to base 2 or x]
 b) Find x if $\log_x 3 + \log_3 x = 2.5$ **ANSWER:** $x = \sqrt{3}, 9$
 c) Find x if $\log_x 3 + \log_3 x = \frac{10}{3}$ **ANSWER:** $x = 27, \sqrt[3]{3}$
 d) Solve for x : $\log_{10} \left(\frac{x^2+24}{x} \right) = 1$ **ANSWER:** $x = 4, 6$

B. In proof numbers.

1. Show that $\log_{10} \frac{1}{n}$ is negative where $n \geq 2$

Solution

$$\log_{10} \frac{1}{n} = \log_{10} 1 - \log_{10} n = -\log_{10} n, \text{ which is negative for } n \geq 2$$

2. If $a^2 + b^2 = 7ab$, prove that $\log \left[\frac{1}{3}(a+b) \right] = \frac{1}{2}(\log a + \log b)$

Solution

$$\begin{aligned} a^2 + b^2 &= 7ab \Rightarrow (a+b)^2 - 2ab = 7ab \\ \therefore (a+b)^2 &= 9ab \text{ or } \left(\frac{a+b}{3} \right)^2 = ab \\ \Rightarrow \log \left[\frac{1}{3}(a+b) \right]^2 &= \log a \cdot b = \log a + \log b \\ \Rightarrow 2 \log \left[\frac{1}{3}(a+b) \right] &= \log a + \log b \\ \therefore \log \left[\frac{1}{3}(a+b) \right] &= \frac{1}{2}(\log a + \log b) \end{aligned}$$

3. Prove that $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$ (Assuming that the base of each logarithm is 10)

Solution

L.H.S:

$$\begin{aligned} \log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} &= \log 2 + 16(\log 16 - \log 15) + 12(\log 25 - \log 24) \\ &\quad + 7(\log 81 - \log 80) \\ &= \log 2 + 16 \log 2^4 - 16 \log(3 \times 5) + 12 \log 5^2 - 12 \log(3 \times 2^3) + 7 \log 3^4 - 7 \log(2^4 \times 5) \\ &= \log 2 + 64 \log 2 - 16 \log 3 - 16 \log 5 + 24 \log 5 - 12 \log 3 - 36 \log 2 + 28 \log 3 - 28 \log 2 - 7 \log 5 \\ &= (1 + 64 - 36 - 28) \log 2 + (-16 - 12 + 28) \log 3 + (-16 + 24 - 7) \log 5 \\ &= \log 2 + \log 5 \\ &= \log 2 + \log 5 = \log(2 \times 5) \\ &= \log 10 = 1 \end{aligned}$$

4. Prove that $\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$

Solution

$$\text{Let } \log_a(ab) = x \Rightarrow ab = a^x \text{ or } a = (ab)^{1/x}$$

$$\log_b(ab) = y \Rightarrow ab = a^y \text{ or } b = (ab)^{1/y}$$

$$\text{Now } (ab)^1 = (ab)^{1/x} \cdot (ab)^{1/y} = (ab)^{\frac{1}{x} + \frac{1}{y}}$$

$$\therefore 1 = \frac{1}{x} + \frac{1}{y}$$

$$\Rightarrow \frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$$

5. Prove that $\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)} = 1$

Solution

Proceeding as in the previous example,

$$\text{Let } \log_a(abc) = x \Rightarrow abc = a^x \text{ or } a = (abc)^{1/x}$$

$$\log_b(abc) = y \Rightarrow abc = a^y \text{ or } b = (abc)^{1/y}$$

$$\log_c(abc) = z \Rightarrow abc = a^z \text{ or } c = (abc)^{1/z}$$

$$\text{Now } (abc)^1 = (abc)^{1/x} \cdot (abc)^{1/y} \cdot (abc)^{1/z} = (abc)^{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$$

$$\therefore 1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\Rightarrow \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)} = 1$$

6. Prove that

$$\text{i) } \log_a b \times \log_b c \times \log_c d = \log_a d$$

$$\text{ii) } \log_a b \times \log_b c \times \log_c d \times \log_d a = 1$$

$$\text{iii) } \log_a b \times \log_b a = 1$$

Solution

Changing to another base, say k

$$\text{i) } \text{L.H.S: } \log_a b \times \log_b c \times \log_c d = \frac{\log_k b}{\log_k a} \times \frac{\log_k c}{\log_k b} \times \frac{\log_k d}{\log_k c} = \frac{\log_k d}{\log_k a}$$

Now changing to base a

$$\Rightarrow \frac{\log_k d}{\log_k a} = \left(\frac{\log_a d}{\log_a k} \right) \times \left(\frac{\log_a a}{\log_a k} \right) = \log_a d$$

$$\therefore \log_a b \times \log_b c \times \log_c d = \log_a d$$

- ii) **L.H.S:** $\log_a b \times \log_b c \times \log_c d \times \log_d a = \frac{\log_k b}{\log_k a} \times \frac{\log_k c}{\log_k b} \times \frac{\log_k d}{\log_k c} \times \frac{\log_k a}{\log_k d} = 1$
- iii) **L.H.S:** $\log_a b \times \log_b a = \frac{\log_k b}{\log_k a} \times \frac{\log_k a}{\log_k b} = 1$

Task

Prove that:

- a) $\log_a x \times \log_b y = \log_b x \times \log_a y$
- b) $\log_b a \times \log_a b \times \log_a c = \frac{1}{\log_c a}$
- c) $\log_2 3 \times \log_3 8 = 1$

More conditional proofs

- a) If $\log(x^3 \cdot y^3) = a$ and $\log \frac{x}{y} = b$, show that $\log x = \frac{1}{6}(3b + a)$ and $\log y = \frac{1}{6}(a - 3b)$
- b) If $a^3 + b^3 = 0$, prove that $\log(a + b) = \frac{1}{2}(\log a + \log b + \log 3)$ given that $a + b \neq 0$.
- c) If $a^x = b^y = c^z$ and $b^2 = ac$, prove that $y = \frac{2xz}{x+z}$
- d) If $\log_{2a} a = x$, $\log_{3a} 2a = y$, $\log_{4a} 3a = z$, show that $xyz + 1 = 2yz$
- e) If $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$, prove that $1 + abc = 2bc$
- f) If $z = \log_r(mn)$, $x = \log_m(rn)$ and $y = \log_n(mr)$, prove that $x + y + z = xyz - 2$
- g) If $\log_a N = x$ and $\log_b N = y$, prove that $\log_{ab} N = \frac{xy}{x+y}$

Solution

- a) $\log(x^3 \cdot y^3) = a$, $\log \frac{x}{y} = b$, to show that: $\log x = \frac{1}{6}(3b + a)$ and $\log y = \frac{1}{6}(a - 3b)$

$$\text{From } \log(x^3 \cdot y^3) = a \Rightarrow \log(xy)^3 = a \quad \therefore \log(xy) = \frac{a}{3}$$

$$\Rightarrow \log x + \log y = \frac{a}{3} \dots\dots\dots (1)$$

$$\text{Also } \log \frac{x}{y} = b \text{ gives}$$

$$\Rightarrow \log x - \log y = b \dots\dots\dots (2)$$

$$\text{eqn. (1) + eqn. (2)}$$

$$\Rightarrow 2 \log x = \frac{a}{3} + b = \frac{a+3b}{3}$$

$$\therefore \log x = \frac{1}{6}(3b + a)$$

$$\text{eqn. (1) - eqn. (2)}$$

$$\Rightarrow 2 \log y = \frac{a}{3} - b = \frac{a-3b}{3}$$

$$\therefore \log y = \frac{1}{6}(a - 3b)$$

- b) $a^3 + b^3 = 0$

$$\text{But } (a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

$$\Rightarrow a^3 + b^3 = (a + b)^3 - 3ab(a + b)$$

$$\Rightarrow (a + b)^3 - 3ab(a + b) = 0$$

$$\Rightarrow (a + b)[(a + b)^2 - 3ab] = 0$$

$$\therefore (a + b)^2 - 3ab = 0 \text{ or } (a + b)^2 = 3ab$$

Introducing logarithms

$$\Rightarrow \log(a + b)^2 = \log(3ab)$$

$$\Rightarrow 2 \log(a + b) = \log 3 + \log a + \log b$$

$$\therefore \log(a + b) = \frac{1}{2}(\log a + \log b + \log 3)$$

- c) Let $a^x = b^y = c^z = k$

$$\Rightarrow a^x = k \text{ or } a = k^{1/x}$$

$$\Rightarrow b^y = k \text{ or } b = k^{1/y}$$

$$\Rightarrow c^z = k \text{ or } c = k^{1/z}$$

$$\text{Given: } b^2 = ac$$

$$\Rightarrow \left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}} \cdot k^{\frac{1}{z}}$$

$$\Rightarrow k^{\frac{2}{y}} = k^{\frac{1}{x} + \frac{1}{z}}$$

$$\therefore \frac{2}{y} = \frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz}$$

$$\therefore y = \frac{2xz}{x+z}$$

Note: the number portrays the use of laws of indices.

$$\text{d) } \log_{2a} a = x \Rightarrow (2a)^x = a \text{ or } 2a = a^{\frac{1}{x}}$$

$$\log_{3a} 2a = y \Rightarrow (3a)^y = 2a \text{ or } 3a = (2a)^{\frac{1}{y}}$$

$$\log_{4a} 3a = z \Rightarrow (4a)^z = 3a \text{ or } 4a = (3a)^{\frac{1}{z}}$$

$$\text{From } 2a = a^{\frac{1}{x}} \Rightarrow 2 = a^{\frac{1}{x}-1} \dots\dots\dots (1)$$

$$\text{From } 3a = (2a)^{\frac{1}{y}} \Rightarrow 3a = \left(a^{\frac{1}{x}}\right)^{\frac{1}{y}} = a^{\frac{1}{xy}}$$

$$\therefore 3 = a^{\frac{1}{xy}-1} \dots\dots\dots (2)$$

$$\text{From } 4a = (3a)^{\frac{1}{z}} \Rightarrow 2^2 \cdot a = \left(a^{\frac{1}{xy}}\right)^{\frac{1}{z}} = a^{\frac{1}{xyz}}$$

$$\therefore \left(a^{\frac{1}{x}-1}\right)^2 \cdot a = a^{\frac{1}{xyz}}$$

$$\Rightarrow a^{\left(\frac{2}{x}-2+1\right)} = a^{\frac{1}{xyz}}$$

$$\therefore \frac{2}{x} - 1 = \frac{1}{xyz} \text{ or } 2yz - xyz = 1$$

$$\therefore xyz + 1 = 2yz$$

$$\text{e) } a = \log_{24} 12, b = \log_{36} 24, c = \log_{48} 36, \text{ to prove that } 1 + abc = 2bc$$

$$a = \log_{24} 12 \Rightarrow (24)^a = 12 \text{ or } 24 = (12)^{\frac{1}{a}} \dots\dots\dots (1)$$

$$b = \log_{36} 24 \Rightarrow (36)^b = 24 \text{ or } 36 = (24)^{\frac{1}{b}} \dots\dots\dots (2)$$

$$c = \log_{48} 36 \Rightarrow (48)^c = 36 \text{ or } 48 = (36)^{\frac{1}{c}} \dots\dots\dots (3)$$

From eqn. (1)

$$24 = (12)^{\frac{1}{a}} \Rightarrow 2 = \frac{(12)^{\frac{1}{a}}}{12} = (12)^{\frac{1}{a}-1} \dots\dots\dots (4)$$

From eqn. (2)

$$36 = (24)^{\frac{1}{b}} \Rightarrow 3 = \frac{(24)^{\frac{1}{b}}}{12} = \frac{(12)^{\frac{1}{b}} \cdot (2)^{\frac{1}{b}}}{12} = (12)^{\frac{1}{b}-1} \cdot (2)^{\frac{1}{b}} \dots\dots\dots (5)$$

eqn. (4) in to eqn. (5)

$$\Rightarrow 3 = (12)^{\frac{1}{b}-1} \cdot \left((12)^{\frac{1}{a}-1}\right)^{\frac{1}{b}}$$

$$\Rightarrow 3 = (12)^{\frac{1}{ab} - \frac{1}{b}} \cdot (12)^{\frac{1}{b}-1} = (12)^{\frac{1}{ab} - \frac{1}{b} + \frac{1}{b} - 1}$$

$$\therefore 3 = (12)^{\frac{1}{ab}-1} \dots\dots\dots (6)$$

eqn. (4), (6) in to eqn. (3)

$$48 = (36)^{\frac{1}{c}} \Rightarrow 2^2 \times 12 = [3(12)]^{\frac{1}{c}}$$

$$\Rightarrow \left[(12)^{\frac{1}{a}-1}\right]^2 \cdot (12) = \left[(12)^{\frac{1}{ab}-1} \cdot (12)^{\frac{1}{c}}\right]^{\frac{1}{c}}$$

$$\Rightarrow (12)^{\frac{2}{a}-2+1} = \left[(12)^{\frac{1}{ab}-1+1} \right]^{\frac{1}{c}}$$

$$\therefore \frac{2}{a} - 1 = \frac{1}{abc} \text{ or } 2bc - abc = 1$$

$$\therefore 1 + abc = 2bc$$

- f) Given: $z = \log_r(mn)$, $x = \log_m(rn)$, $y = \log_n(mr)$ to prove: $x + y + z = xyz - 2$

$$z = \log_r(mn) \Rightarrow r^z = mn \text{ or } r = (mn)^{\frac{1}{z}} \dots\dots\dots (1)$$

$$x = \log_m(rn) \Rightarrow m^x = rn \text{ or } m = (nr)^{\frac{1}{x}} \dots\dots\dots (2)$$

$$y = \log_n(mr) \Rightarrow n^y = mr \text{ or } n = (mr)^{\frac{1}{y}} \dots\dots\dots (3)$$

eqn. (1) $\div$ eqn. (2), to eliminate n

$$\Rightarrow \frac{r^z}{m^x} = \frac{m}{r} \Rightarrow r^{z+1} = m^{x+1}$$

$$\therefore m = r^{\left(\frac{z+1}{x+1}\right)} \dots\dots\dots (4)$$

eqn. (2) $\div$ eqn. (3), to eliminate r

$$\Rightarrow \frac{m^x}{n^y} = \frac{n}{m} \Rightarrow m^{x+1} = n^{y+1}$$

$$\therefore n = m^{\left(\frac{x+1}{y+1}\right)} \dots\dots\dots (5)$$

eqn. (4) in to eqn. (5)

$$\Rightarrow n = \left[r^{\left(\frac{z+1}{x+1}\right)} \right]^{\left(\frac{x+1}{y+1}\right)} = r^{\left(\frac{z+1}{y+1}\right)} \dots\dots\dots (6)$$

Now eqn. (4), (6) in to eqn. (1)

$r^z = mn$, becomes

$$\Rightarrow r^z = r^{\left(\frac{z+1}{x+1}\right)} \cdot r^{\left(\frac{z+1}{y+1}\right)}$$

$$\Rightarrow z = \left(\frac{z+1}{x+1}\right) + \left(\frac{z+1}{y+1}\right)$$

$$\therefore z = \frac{yz+y+z+xz+x+z+1}{xy+x+y+1}$$

$$\Rightarrow xyz + xz + yz + z = yz + y + z + 1 + xz + x + z + 1$$

$$\therefore xyz + z = x + y + 2z + 2$$

$$\therefore x + y + z = xyz - 2$$

- g) $\log_a N = x \Rightarrow a^x = N \text{ or } a = N^{\frac{1}{x}} \dots\dots\dots (1)$

$$\log_b N = y \Rightarrow b^y = N \text{ or } b = N^{\frac{1}{y}} \dots\dots\dots (2)$$

Multiplying the equations

$$\Rightarrow a \cdot b = N^{\frac{1}{x}} \cdot N^{\frac{1}{y}} = N^{\left(\frac{1}{x} + \frac{1}{y}\right)}$$

Introducing logarithm to base ab

$$\Rightarrow \log_{ab} ab = \log_{ab} N^{\left(\frac{1}{x} + \frac{1}{y}\right)}$$

$$\Rightarrow 1 = \left(\frac{1}{x} + \frac{1}{y}\right) \log_{ab} N \text{ or } 1 = \left(\frac{x+y}{xy}\right) \log_{ab} N$$

$$\therefore \log_{ab} N = \frac{xy}{x+y}$$

Task

1. Prove that
 - i) $\log_{b^2} a \times \log_{x^2} b = \frac{1}{4} \log_x a$
 - ii) If $\log_a \left(1 + \frac{1}{8}\right) = l$, $\log_a \left(1 + \frac{1}{15}\right) = m$ and $\log_a \left(1 + \frac{1}{24}\right) = n$, then $\log_a \left(1 + \frac{1}{80}\right) = l - m - n$
 - iii) If $\log_a n = x$ and $\log_c n = y$ where $n \neq 1$ then $\frac{x-y}{x+y} = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$
2. a) Given that $\log_3 x = p$ and $\log_{18} x = q$, show that $\log_6 3 = \frac{q}{p-q}$ (**Uneb 2002**).
 b) Show that $\log_8 x = \frac{2}{3} \log_4 x$. Hence without using tables/calculator, evaluate $\log_8 6$ to 3 d.p, if $\log_4 3 = 0.7925$. (**Uneb 2000**)
 c) Prove that $\log_6 x = \frac{\log_3 x}{1 + \log_3 2}$. Hence given that $\log_3 2 = 0.631$, find without using tables/calculator $\log_6 4$ correct to 3 s.f. (**Uneb 1998 March**)
 d) If x and y are positive numbers and each is greater than 1,
 $x^{x+y} = y^{12}$ and $y^{x+y} = x^3$
 Prove that $\log x = 2 \log y$. Hence find x and y

ANSWER:

a) $\log_3 x = p \Rightarrow 3^p = x$ or $3 = x^{\frac{1}{p}}$ (1)

$\log_{18} x = q \Rightarrow 18^q = x$ or $18 = x^{\frac{1}{q}}$ (2)

eqn. (2) $\div$ eqn. (1)

$$\Rightarrow 6 = x^{\frac{1}{q}} \div x^{\frac{1}{p}} = x^{\frac{1}{q} - \frac{1}{p}} = x^{\left(\frac{p-q}{pq}\right)}$$

Now introducing logarithm to base 6

$$\Rightarrow \log_6 6 = \log_6 x^{\left(\frac{p-q}{pq}\right)} = \left(\frac{p-q}{pq}\right) \log_6 x$$

$$\Rightarrow 1 = \left(\frac{p-q}{pq}\right) \log_6 x \therefore \log_6 x = \frac{pq}{p-q}, \text{ dividing through by } p$$

$$\Rightarrow \frac{\log_6 x}{p} = \frac{q}{p-q}$$

$$\Rightarrow \frac{q}{p-q} = \frac{\log_6 x}{\log_3 x} = \frac{\log_6 x}{\frac{\log_6 x}{\log_6 3}}$$

$$\therefore \log_6 3 = \frac{q}{p-q}$$

Alternatively

From $\log_{18} x = q$

$$q = \frac{\log_3 x}{\log_3 18} = \frac{\log_3 x}{\log_3 (3 \times 6)}$$

$$q = \frac{\log_3 x}{\log_3 3 + \log_3 6} = \frac{p}{1 + \log_3 6}$$

$$\Rightarrow q(1 + \log_3 6) = p$$

$$\Rightarrow q + q \log_3 6 = p \text{ or } p - q = q \log_3 6$$

$$\Rightarrow \log_3 6 = \frac{p-q}{q}, \text{ but } \log_3 6 = \frac{\log_6 6}{\log_6 3} = \frac{1}{\log_6 3}$$

$$\therefore \frac{p-q}{q} = \frac{1}{\log_6 3} \text{ or } \log_6 3 = \frac{q}{p-q}$$

b) To show that $\log_8 x = \frac{2}{3} \log_4 x$

LHS: $\log_8 x = \frac{\log_4 x}{\log_4 8} = \log_4 x \cdot \frac{1}{\log_4 8}$

But $\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3 \log_2 2}{2 \log_2 2} = \frac{3}{2}$

$$\therefore \log_8 x = \log_4 x \cdot \frac{1}{\log_4 8} = \frac{2}{3} \log_4 x$$

Putting $x = 6$ in to the proof,

$$\Rightarrow \log_8 6 = \frac{2}{3} \log_4 6 = \frac{2}{3} \log_4 (2 \times 3)$$

$$= \frac{2}{3} [\log_4 2 + \log_4 3]$$

$$= \frac{2}{3} \left[\frac{\log_2 2}{\log_2 4} + \log_4 3 \right] = \frac{2}{3} \left[\frac{1}{2 \log_2 2} + \log_4 3 \right] = \frac{2}{3} [0.5 + 0.7925] = 0.862$$

c) To prove $\log_6 x = \frac{\log_3 x}{1 + \log_3 2}$

Let $\log_6 x = y \Rightarrow 6^y = x$ or $6 = x^{\frac{1}{y}}$

Introducing logarithm to base 3

$$\Rightarrow \log_3 6 = \log_3 x^{\frac{1}{y}} = \frac{1}{y} \log_3 x$$

$$\Rightarrow \log_3 x = y \log_3 6 \text{ or } y = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3 (2 \times 3)}$$

$$\Rightarrow y = \frac{\log_3 x}{\log_3 2 + \log_3 3} = \frac{\log_3 x}{1 + \log_3 2}$$

$$\therefore \log_6 x = \frac{\log_3 x}{1 + \log_3 2}$$

Now setting $x = 4$

$$\Rightarrow \log_6 4 = \frac{\log_3 4}{1 + \log_3 2} = \frac{2 \log_3 2}{1 + \log_3 2}$$

$$\therefore \log_6 4 = \frac{2 \times 0.631}{1 + 0.631} = \frac{1.262}{1.631} = 0.774$$

d) $x^{x+y} = y^{12} \Rightarrow \log x^{x+y} = \log y^{12}$

$$\Rightarrow (x+y) \log x = 12 \log y \dots\dots\dots (1)$$

$$y^{x+y} = x^3 \Rightarrow \log y^{x+y} = \log x^3$$

$$\Rightarrow (x+y) \log y = 3 \log x \dots\dots\dots (2)$$

eqn. (1) $\div$ eqn. (2)

$$\Rightarrow \frac{(x+y) \log x}{(x+y) \log y} = \frac{12 \log y}{3 \log x} \text{ or } \frac{\log x}{\log y} = \frac{4 \log x}{\log y}, \text{ let } \log x = a, \log y = b$$

$$\Rightarrow \frac{a}{b} = \frac{4b}{a} \text{ or } a^2 = 4b^2$$

$$\therefore a = 2b$$

$$\therefore \log x = 2 \log y$$

$$\text{Now from } \log x = 2 \log y \Rightarrow x = y^2 \dots\dots\dots (3)$$

$x^{x+y} = y^{12}$, becomes

$$\Rightarrow (y^2)^{(y^2+y)} = y^{12} \text{ or } y^{2y^2+2y} = y^{12}$$

$$\Rightarrow 2y^2 + 2y = 12 \text{ or } y^2 + y - 6 = 0$$

$$\Rightarrow (y+3)(y-2) = 0 \therefore y = -3, 2$$

From eqn. (3), $x = y^2$

$$\text{For } y = -3, x = (-3)^2 = 9$$

$$y = 2, x = (2)^2 = 4$$

$$\therefore x = 9, y = -3; x = 4, y = 2$$

Task

1. Prove that $abc = 1$ if $a = \log_b c$, $b = \log_a c$, $\log_a b$
2. Show that $\frac{\log \sqrt{27} + \log \sqrt{8} - \log \sqrt{125}}{\log 6 - \log 5} = \frac{3}{2}$ without using tables/calculator.
3. Prove that $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$
4. Prove that $\log_4 2 \cdot \log_2 6 \cdot \log_6 4 = 1$
5. Prove that $\log_2 [\log_2 \{\log_3 (\log 27^3)\}] = 0$

SIMULTANEOUS EQUATIONS INVOLVING LOGARITHMS

1) Solve the following simultaneous equations

- a) $2 \log_x y = 1$
 $xy = 64$
- b) $6 \log_3 x + 6 \log_{27} y = 7$
 $4 \log_9 x + 4 \log_3 y = 9$
- c) $y \log_2 8 = x$
 $2^x + 8^y = 8192$
- d) $\log x - \log 2 = 2 \log y$
 $x - 5y + 2 = 0$
- e) $\log_2 x + 2 \log_4 y = 4$
 $x + 12y = 52$
- f) $\ln 6 + \ln(x - 3) = 2 \ln y$
 $2y - x = 3$
- g) $2 + \log_2(2x + 1) = 2 \log_2 y$
 $x = 22 - y$

Solution

- a) $2 \log_x y = 1 \Rightarrow \log_x y^2 = 1 \therefore y^2 = x$ (1)
 $xy = 64$ (2)
 eqn. (1) in to eqn. (2)
 $\Rightarrow y \cdot y^2 = 64$ or $y^3 = 64 \therefore y = 4$ (3)
 From eqn. (1), $x = 4^2 = 16$
 $\therefore x = 16, y = 4$

- b) $6 \log_3 x + 6 \log_{27} y = 7$
 Changing to the same base
 $\Rightarrow 6 \log_3 x + 6 \frac{\log_3 y}{\log_3 27} = 7$
 $\Rightarrow 6 \log_3 x + 6 \frac{\log_3 y}{\log_3 3^3} = 7$ or $\Rightarrow 6 \log_3 x + 6 \frac{\log_3 y}{3} = 7$
 $\therefore 6 \log_3 x + 2 \log_3 y = 7$ (1)
 $4 \log_9 x + 4 \log_3 y = 9 \Rightarrow 4 \frac{\log_3 x}{\log_3 9} + 4 \log_3 y = 9$
 $\therefore 2 \log_3 x + 4 \log_3 y = 9$ (2)
 Eliminating $\log_3 x$ from eqn. (1) and eqn. (2)
 eqn. (1) $- 3 \times$ eqn. (2)
 $\Rightarrow 6 \log_3 x + 2 \log_3 y = 7$
 $\underline{6 \log_3 x + 12 \log_3 y = 27}$
 $-10 \log_3 y = -20 \Rightarrow \log_3 y = 2$
 $\Rightarrow y = 3^2 = 9$ (3)
 eqn. (3) in to eqn. (2)
 $\Rightarrow 2 \log_3 x + 4 \log_3 9 = 9$
 $\Rightarrow 2 \log_3 x + 8 = 9$
 $\Rightarrow 2 \log_3 x = 1$ or $\log_3 x^2 = 1$
 $\Rightarrow x^2 = 3 \therefore x = +\sqrt{3}$ (4)
 $\therefore x = +\sqrt{3}; y = 9$

c) $y \log_2 8 = x \Rightarrow y \log_2 2^3 = x$ or $x = 3y$ (1)
 $2^x + 8^y = 8192$ (2)
 eqn. (1) in to eqn. (2)
 $\Rightarrow 2^{3y} + 8^y = 8192$
 $\Rightarrow 8^y + 8^y = 8192$
 $\Rightarrow 2(8^y) = 8192$ or $8^y = 4096 = 8^4$
 $\therefore y = 4$ (3)
 eqn. (3) in to eqn. (1)
 $\Rightarrow x = 3 \times 4 = 12$
 $\therefore x = 12, y = 4$

d) Left as an exercise **ANSWER:** $x = \frac{1}{2}, y = \frac{1}{4}; x = 8, y = 2$

e) Left as an exercise **ANSWER:** $x = 4, y = 4; x = 48, y = \frac{1}{3}$

f) Left as an exercise **ANSWER:** $x = 9, y = 6$

g) Left as an exercise **ANSWER:** $x = 12, y = 10$

SURDS

Definition:

A surd is an irrational root of a rational number.

A rational number is a number with exact n^{th} root .i.e. $\sqrt[n]{b} = \text{Real number}$
e.g. $\sqrt{4} = 2$, $\sqrt[4]{16} = 2$, $\sqrt[3]{8} = 2$, these are rational numbers.

If $a = \sqrt[n]{b}$ for some rational number b such that a itself is an irrational number, then $a = \sqrt[n]{b}$ is a surd.

When $a = \sqrt[n]{b}$ is a surd, n is called *order of a* and b is called *a radicand of a* .
Surd of 2^{nd} , 3^{rd} , 4^{th} and higher orders are respectively called *quadratic*, *cubic*, *quartic* and *quintic*.

Compound, Simple and Similar surds

A *compound surd* is a surd with coefficient different from

± 1 . e.g. $2\sqrt{2}$, $3\sqrt{5}$, $-3\sqrt{7}$ etc

A *simple surd* is a surd with coefficient ± 1 . e.g. $\sqrt{2}$, $\sqrt[4]{\frac{1}{4}}$, $\sqrt[3]{7}$ etc

Similar surds are surds if in reduced form (simplified form) contain same surds. e.g. $\sqrt{45}$ and $\sqrt{80}$ are similar surds, since $\sqrt{45} = 3\sqrt{5}$ and $\sqrt{80} = 4\sqrt{5}$.
Similar surds can be added or subtracted easily after being reduced to their simplest forms.

e.g. $\sqrt{75} + \sqrt{27} = 5\sqrt{3} + 3\sqrt{3} = 8\sqrt{3}$ and $2\sqrt{128} - \sqrt{162} = 16\sqrt{2} - 9\sqrt{2} = 7\sqrt{2}$.

Rules of simplifying surds

- i) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{a \cdot b}$; surds of the same order can be multiplied or divided
- ii) $\sqrt[n]{a} \div \sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}$
- iii) $m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$
- iv) $(m\sqrt{a} + n\sqrt{b})(m\sqrt{a} - n\sqrt{b}) = (m\sqrt{a})^2 - (n\sqrt{b})^2$
- v) $N\sqrt[n]{a} \times M\sqrt[n]{b} = MN\sqrt[n]{a \cdot b}$
- vi) If $a + \sqrt{b} = 0$ if and only if $a = b = 0$
- vii) If $a + \sqrt{b} = \sqrt{c}$ if and only if $a = 0$, $b = c$
- viii) If $a + \sqrt{b} = c + \sqrt{d}$ if and only if $a = c$, $b = d$
- ix) If $\sqrt{a + \sqrt{b}} = \sqrt{x} - \sqrt{y}$ then $\sqrt{a - \sqrt{b}} = \pm(\sqrt{x} - \sqrt{y})$

Rules /properties from (viii) to (ix) represent quadratic mixed surds properties.

RATIONALIZATION

If the product of two surds is a rational number and if one of them is multiplied by the other, the process is called *Rationalization*.

Thus $\sqrt{3} \times \sqrt{3} = 3$, so multiplication by $\sqrt{3}$ is rationalization of $\sqrt{3}$.

Similarly $\sqrt{5} - \sqrt{3}$ then multiplied by $\sqrt{5} + \sqrt{3}$ is rationalized to 2 as

$$(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = 5 - 3 = 2$$

NOTE:

Rationalization can be in the numerator/denominator of a given surd.

Rationalization of Denominator

Case I: Two quadratic surds Denominator. $\frac{a}{\sqrt{b}+\sqrt{c}}$

$$\begin{aligned}\frac{a}{\sqrt{b}+\sqrt{c}} &= \frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})} = \frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b})^2-(\sqrt{c})^2} \\ &= \frac{a(\sqrt{b}-\sqrt{c})}{b-c} = \frac{a}{b-c}\sqrt{b} - \frac{a}{b-c}\sqrt{c}\end{aligned}$$

Note:

$\sqrt{b} - \sqrt{c}$ is a surd conjugate of $\sqrt{b} + \sqrt{c}$

Case II: Three quadratic surds Denominator. $\frac{a}{\sqrt{b}+\sqrt{c}+\sqrt{d}}$

First multiply both numerator and denominator by $(\sqrt{b} + \sqrt{c}) - \sqrt{d}$

Denominator becomes $(\sqrt{b} + \sqrt{c})^2 - (\sqrt{d})^2$ or $b + c - d + 2\sqrt{bc}$

Then multiply both numerator and denominator by $(b + c - d) - 2\sqrt{bc}$

Denominator becomes $(b + c - d)^2 - (2\sqrt{bc})^2 = (b + c - d)^2 - 4bc$ which is a rational quantity.

Example

1. Express as equivalent fraction with a rational denominator

- $\frac{1}{\sqrt{2}+\sqrt{3}}$
- $\frac{7+4\sqrt{3}}{7-4\sqrt{3}}$
- $\frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{2\sqrt{3}-\sqrt{2}}$

Solution

- $\frac{1}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2})^2-(\sqrt{3})^2} = \frac{\sqrt{2}-\sqrt{3}}{-1} = \sqrt{3} - \sqrt{2}$
- $\frac{7+4\sqrt{3}}{7-4\sqrt{3}} = \frac{(7+4\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})} = \frac{49+28\sqrt{3}+28\sqrt{3}+16\sqrt{9}}{7^2-(4\sqrt{3})^2}$
 $= \frac{49+48+56\sqrt{3}}{49-48} = 74 + 56\sqrt{3}$
- $\frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{2\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} + \frac{2\sqrt{3}+\sqrt{2}}{(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})}$
 $= \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2} + \frac{2\sqrt{3}+\sqrt{2}}{(2\sqrt{3})^2-(\sqrt{2})^2} = \sqrt{3} - \sqrt{2} + \frac{2\sqrt{3}+\sqrt{2}}{12-2}$
 $= \frac{10\sqrt{3}-10\sqrt{2}+2\sqrt{3}+\sqrt{2}}{10} = \frac{12\sqrt{3}-9\sqrt{2}}{10}$
 $= \frac{3}{10}(4\sqrt{3} - 3\sqrt{2})$

2. Express as equivalent fraction with a rational denominator

- $\frac{1}{1+\sqrt{2}-\sqrt{3}}$
- $\frac{1}{2+\sqrt{3}+\sqrt{5}}$

Solution

- $\frac{1}{1+\sqrt{2}-\sqrt{3}} = \frac{1}{(1+\sqrt{2})-\sqrt{3}} = \frac{(1+\sqrt{2})+\sqrt{3}}{[(1+\sqrt{2})-\sqrt{3}](1+\sqrt{2})+\sqrt{3}}$
 $= \frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})^2-(\sqrt{3})^2} = \frac{1+\sqrt{2}+\sqrt{3}}{1+2\sqrt{2}+2-3}$
 $= \frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}} = \frac{(1+\sqrt{2}+\sqrt{3})\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}$
 $= \frac{\sqrt{2}(1+\sqrt{2}+\sqrt{3})}{4} = \frac{\sqrt{2}+2+\sqrt{6}}{4}$

$$\begin{aligned}
 \text{b) } \frac{1}{2+\sqrt{3}+\sqrt{5}} &= \frac{1}{(2+\sqrt{3})+\sqrt{5}} = \frac{(2+\sqrt{3})-\sqrt{5}}{[(2+\sqrt{3})+\sqrt{5}](2+\sqrt{3})-\sqrt{5}} \\
 &= \frac{2+\sqrt{3}-\sqrt{5}}{(2+\sqrt{3})^2-(\sqrt{5})^2} = \frac{2+\sqrt{3}-\sqrt{5}}{4+4\sqrt{3}+3-5} \\
 &= \frac{2+\sqrt{3}-\sqrt{5}}{2+4\sqrt{3}} = \frac{(2+\sqrt{3}-\sqrt{5})(2-4\sqrt{3})}{(2+4\sqrt{3})(2-4\sqrt{3})} \\
 &= \frac{4-8\sqrt{3}+2\sqrt{3}-4\sqrt{9}-2\sqrt{5}+4\sqrt{15}}{2^2-(4\sqrt{3})^2} = \frac{-8-6\sqrt{3}-2\sqrt{5}+4\sqrt{15}}{4-48} \\
 &= \frac{1}{44} (8 + 6\sqrt{3} + 2\sqrt{5} - 4\sqrt{15})
 \end{aligned}$$

Task

Express as equivalent fractions with rational denominators

$$\text{a) } \frac{12}{3+\sqrt{5}-2\sqrt{2}} \quad \text{ANSWER: } 1 + \sqrt{5} + \sqrt{10} - \sqrt{2}$$

$$\text{b) } \frac{1}{2+\sqrt{3}-\sqrt{5}} \quad \text{ANSWER: } \frac{1}{44} (8 + 6\sqrt{3} + 2\sqrt{5} - 4\sqrt{15})$$

3. Express as equivalent fractions with rational denominators

$$\text{a) } \frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$$

$$\text{b) } \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$$

Solution

$$\begin{aligned}
 \text{a) } \frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}} &= \frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}} = \frac{\sqrt{2}[(\sqrt{2}+\sqrt{3})+\sqrt{5}]}{[(\sqrt{2}+\sqrt{3})-\sqrt{5}][(\sqrt{2}+\sqrt{3})+\sqrt{5}]} \\
 &= \frac{\sqrt{4}+\sqrt{6}+\sqrt{10}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2} = \frac{2+\sqrt{6}+\sqrt{10}}{2+2\sqrt{6}+3-5} \\
 &= \frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}} = \frac{\sqrt{6}(2+\sqrt{6}+\sqrt{10})}{2\sqrt{6} \times \sqrt{6}} \\
 &= \frac{2\sqrt{6}+6+\sqrt{60}}{12} = \frac{6+2\sqrt{6}+2\sqrt{15}}{12} \\
 &= \frac{3+\sqrt{6}+\sqrt{15}}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}} &= \frac{1}{(\sqrt{2}+\sqrt{3})+\sqrt{5}} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{[(\sqrt{2}+\sqrt{3})+\sqrt{5}](\sqrt{2}+\sqrt{3})-\sqrt{5}} \\
 &= \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2+2\sqrt{6}+3-5} \\
 &= \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}} = \frac{\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{2\sqrt{6} \times \sqrt{6}} \\
 &= \frac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{12} = \frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}
 \end{aligned}$$

Task

Express as equivalent fractions with rational denominators

$$\frac{\sqrt{10}-\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{10}-\sqrt{5}} \quad \text{ANSWER: } \frac{3\sqrt{30}+5\sqrt{15}-10\sqrt{2}-12}{7}$$

4. If $x = \frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$, find the value of;

$$\text{i) } \frac{x+\sqrt{8}}{x-\sqrt{8}}$$

$$\text{ii) } \frac{x+\sqrt{12}}{x-\sqrt{12}}$$

$$\text{iii) } \frac{x+\sqrt{8}}{x-\sqrt{8}} + \frac{x+\sqrt{12}}{x-\sqrt{12}}$$

Solution

$$x = \frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}} = \frac{2\sqrt{6 \times 4}}{\sqrt{2}+\sqrt{3}} = \frac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$$

$$= \frac{4\sqrt{6}(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{4\sqrt{12}-4\sqrt{18}}{(\sqrt{2})^2-(\sqrt{3})^2}$$

$$= \frac{4(2\sqrt{3})-4(3\sqrt{2})}{2-3} = 12\sqrt{2} - 8\sqrt{3}$$

i) Now $\frac{x+\sqrt{8}}{x-\sqrt{8}} = \frac{12\sqrt{2}-8\sqrt{3}+\sqrt{8}}{12\sqrt{2}-8\sqrt{3}-\sqrt{8}} = \frac{12\sqrt{2}-8\sqrt{3}+2\sqrt{2}}{12\sqrt{2}-8\sqrt{3}-2\sqrt{2}}$

$$= \frac{14\sqrt{2}-8\sqrt{3}}{10\sqrt{2}-8\sqrt{3}} = \frac{7\sqrt{2}-4\sqrt{3}}{5\sqrt{2}-4\sqrt{3}}$$

$$= \frac{(7\sqrt{2}-4\sqrt{3})(5\sqrt{2}+4\sqrt{3})}{(5\sqrt{2}-4\sqrt{3})(5\sqrt{2}+4\sqrt{3})}$$

$$= \frac{70-48+18\sqrt{6}}{(5\sqrt{2})^2-(4\sqrt{3})^2} = \frac{22+8\sqrt{6}}{50-48} = 11 + 4\sqrt{6}$$

ii) $\frac{x+\sqrt{12}}{x-\sqrt{12}} = \frac{12\sqrt{2}-8\sqrt{3}+\sqrt{12}}{12\sqrt{2}-8\sqrt{3}-\sqrt{12}} = \frac{12\sqrt{2}-8\sqrt{3}+2\sqrt{3}}{12\sqrt{2}-8\sqrt{3}-2\sqrt{3}}$

$$= \frac{12\sqrt{2}-6\sqrt{3}}{12\sqrt{2}-10\sqrt{3}} = \frac{6\sqrt{2}-3\sqrt{3}}{6\sqrt{2}-5\sqrt{3}}$$

$$= \frac{(6\sqrt{2}-3\sqrt{3})(6\sqrt{2}+5\sqrt{3})}{(6\sqrt{2}-5\sqrt{3})(6\sqrt{2}+5\sqrt{3})}$$

$$= \frac{72-45+30\sqrt{6}-18\sqrt{6}}{(6\sqrt{2})^2-(5\sqrt{3})^2} = \frac{27+12\sqrt{6}}{72-75}$$

$$= \frac{27+12\sqrt{6}}{-3} = -9 - 4\sqrt{6}$$

iii) $\frac{x+\sqrt{8}}{x-\sqrt{8}} + \frac{x+\sqrt{12}}{x-\sqrt{12}} = 11 + 4\sqrt{6} + -9 - 4\sqrt{6} = (11 - 9) + (4\sqrt{6} - 4\sqrt{6}) = 2$

5. If $x = \sqrt{3} + \frac{1}{\sqrt{3}}$, calculate;

i) $x - \frac{\sqrt{126}}{\sqrt{42}}$

ii) $x - \frac{1}{x - \frac{2\sqrt{3}}{3}}$

iii) $\left(x - \frac{\sqrt{126}}{\sqrt{42}}\right) \cdot \left(x - \frac{1}{x - \frac{2\sqrt{3}}{3}}\right)$, correct to 2 d.p

Solution

$$x = \sqrt{3} + \frac{1}{\sqrt{3}} = \frac{\sqrt{9}+1}{\sqrt{3}} = \frac{3+1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

i) $x - \frac{\sqrt{126}}{\sqrt{42}} = x - \sqrt{\left(\frac{126}{42}\right)} = x - \sqrt{3}$

$$\therefore x - \frac{\sqrt{126}}{\sqrt{42}} = \frac{4}{\sqrt{3}} - \sqrt{3} = \frac{4-\sqrt{9}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

ii) $x - \frac{1}{x - \frac{2\sqrt{3}}{3}} = \frac{4}{\sqrt{3}} - \frac{1}{\frac{4}{\sqrt{3}} - \frac{2\sqrt{3}}{3}}$

$$= \frac{4\sqrt{3}}{3} - \frac{1}{\frac{4\sqrt{3}-2\sqrt{3}}{3}} = \frac{4\sqrt{3}}{3} - \frac{3}{2\sqrt{3}}$$

$$= \frac{8\sqrt{9}-9}{6\sqrt{3}} = \frac{15}{6\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6}$$

iii) $\left(x - \frac{\sqrt{126}}{\sqrt{42}}\right) \cdot \left(x - \frac{1}{x - \frac{2\sqrt{3}}{3}}\right) = \frac{1}{\sqrt{3}} \times \frac{5\sqrt{3}}{6} = \frac{5}{6} = 0.83$

Proofs in surds

1. If $x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, $y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, show that $3x^2 - 5xy + 3y^2 = 289$

Solution

$$\begin{aligned}
 x &= \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{(\sqrt{3}-\sqrt{2})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{3+2-2\sqrt{6}}{(\sqrt{3})^2-(\sqrt{2})^2} \\
 &= 5 - 2\sqrt{6} \\
 x^2 &= (5 - 2\sqrt{6})^2 = 25 - 20\sqrt{6} + 24 \\
 &= 49 - 20\sqrt{6} \\
 x \cdot y &= \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right) \left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) = 1 \\
 y &= \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{1}{x} = \frac{1}{5-2\sqrt{6}} \\
 &= \frac{5+2\sqrt{6}}{(5-2\sqrt{6})(5+2\sqrt{6})} = \frac{5+2\sqrt{6}}{5^2-(2\sqrt{6})^2} \\
 &= 5 + 2\sqrt{6} \\
 y^2 &= (5 + 2\sqrt{6})^2 = 25 + 20\sqrt{6} + 24 \\
 &= 49 + 20\sqrt{6} \\
 \therefore 3x^2 - 5xy + 3y^2 &= 3(49 - 20\sqrt{6}) - 5(1) + 3(49 + 20\sqrt{6}) \\
 &= 147 - 60\sqrt{6} - 5 + 147 + 60\sqrt{6} = 289
 \end{aligned}$$

2. If $x = \frac{5-\sqrt{21}}{2}$, prove that $x^3 + \frac{1}{x^3} - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 0$

Solution

$$\begin{aligned}
 \text{Let } u &= x + \frac{1}{x} = \frac{5-\sqrt{21}}{2} + \frac{2}{5-\sqrt{21}} \\
 &= \frac{(5-\sqrt{21})^2+2}{2(5-\sqrt{21})} = \frac{25-10\sqrt{21}+21+4}{2(5-\sqrt{21})} \\
 &= \frac{50-10\sqrt{21}}{2(5-\sqrt{21})} = \frac{10(5-\sqrt{21})}{2(5-\sqrt{21})} = \frac{10}{2} = 5 \\
 \text{Now } x^2 + \frac{1}{x^2} &= \left(x + \frac{1}{x}\right)^2 - 2 \cdot x \cdot \frac{1}{x} = u^2 - 2 \\
 &= 5^2 - 2 = 23 \\
 x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) \\
 &= 5^3 - 3 \times 5 = 125 \\
 \therefore x^3 + \frac{1}{x^3} - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) &= 125 - 5(23) + 5 = 0
 \end{aligned}$$

3. If $x = \frac{-3+\sqrt{5}}{2}$, show that $\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0$

Solution

$$\begin{aligned}
 x &= \frac{-3+\sqrt{5}}{2} \Rightarrow \frac{1}{x} = \frac{2}{-3+\sqrt{5}} = \frac{2(-3-\sqrt{5})}{(-3+\sqrt{5})(-3-\sqrt{5})} \\
 \frac{1}{x} &= \frac{-6-2\sqrt{5}}{(-3)^2-(\sqrt{5})^2} = \frac{-2(3+\sqrt{5})}{9-5} = -\frac{3+\sqrt{5}}{2} \\
 \text{Now } x + \frac{1}{x} &= \frac{-3+\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2} = -3 \\
 x^2 + \frac{1}{x^2} &= \left(x + \frac{1}{x}\right)^2 - 2 \\
 \therefore x^2 + \frac{1}{x^2} &= (-3)^2 - 2 = 7 \\
 \therefore \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 &= 7 - 3 - 4 = 0
 \end{aligned}$$

Task

1. If $x = 7 - \sqrt{48}$, prove that $x^3 + \frac{1}{x^3} - 15\left(x^2 + \frac{1}{x^2}\right) + 20\left(x + \frac{1}{x}\right) - 72 = 0$
2. If $x = 7 - \sqrt{5}$, prove that $x^4 - x^3 - 20x^2 - 16x + 39 = 15$
3. It is given that $\sqrt{3} = 1.732$ and $\sqrt{5} = 2.236$, evaluate $\frac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$

HINT:
$$\frac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}} = \frac{(\sqrt{5}+\sqrt{3})(4-\sqrt{15})}{(4+\sqrt{15})(4-\sqrt{15})} = \frac{4\sqrt{5}+4\sqrt{3}-\sqrt{75}-\sqrt{45}}{16-15}$$
$$= 4\sqrt{5} + 4\sqrt{3} - 5\sqrt{3} - 3\sqrt{5} = \sqrt{5} - \sqrt{3}$$
$$= 2.236 - 1.732 = \mathbf{0.504}$$

4. If $2\sqrt{54} + 5\sqrt{294} + \frac{19}{30}\sqrt{6} - \sqrt{\frac{27}{50}} - \sqrt{\frac{2}{3}} = a\sqrt{b}$, find a .

ANSWER: $a = 41$

TO FIND THE SQUARE ROOT OF A MIXED QUADRATIC SURD

A surd of the form $a + \sqrt{b}$ is a mixed quadratic surd. To find $\sqrt{a + \sqrt{b}}$, proceed as follows.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, squaring both sides

$$(\sqrt{a + \sqrt{b}})^2 = (\sqrt{x} + \sqrt{y})^2$$

$$\Rightarrow a + \sqrt{b} = x + y + 2\sqrt{xy}$$

$$\Rightarrow a = x + y \dots\dots\dots (1)$$

$$\Rightarrow \sqrt{b} = 2\sqrt{xy} \text{ or } b = 4xy$$

$$\Rightarrow b = 4xy \dots\dots\dots (2)$$

$$\text{From } (x - y)^2 = x^2 + y^2 - 2xy$$

$$\Rightarrow (x - y)^2 = (x + y)^2 - 2xy - 2xy$$

$$\Rightarrow (x - y)^2 = (x + y)^2 - 4xy$$

$$\therefore x - y = \pm\sqrt{a^2 - b} \dots\dots\dots (3)$$

Now taking the positive sign in the equation

$$\Rightarrow x - y = +\sqrt{a^2 - b} \dots\dots\dots (4)$$

Adding eqn. (1) and (4)

$$\Rightarrow x = \frac{1}{2}(a + \sqrt{a^2 - b})$$

$$\Rightarrow y = \frac{1}{2}(a - \sqrt{a^2 - b})$$

Hence $\sqrt{a + \sqrt{b}}$ can be determined

NOTE:

If we take a $-ve$ sign from eqn. (3), the value of x and y will be interchanged and we shall get a second square root of $\sqrt{a + \sqrt{b}}$

Example

1. Find the square root of $31 + 4\sqrt{21}$

Solution

Let $\sqrt{31 + 4\sqrt{21}} = \sqrt{x} + \sqrt{y}$, squaring both sides

$$\Rightarrow 31 + 4\sqrt{21} = x + y + 2\sqrt{xy}$$

$$\Rightarrow x + y = 31 \dots\dots\dots (1)$$

$$\begin{aligned}
 2\sqrt{xy} &= 4\sqrt{21}, \text{ squaring} \\
 \Rightarrow 4xy &= 336 \text{ or } xy = 84 \dots\dots\dots (2) \\
 \text{But } (x-y)^2 &= (x+y)^2 - 4xy \\
 \Rightarrow x-y &= \sqrt{31^2 - 336} = \pm 25 \\
 \text{Taking the +ve sign} \\
 \Rightarrow x-y &= 25 \dots\dots\dots (3) \\
 \text{Adding eqn. (1) and (3)} \\
 2x &= 56 \Rightarrow x = 28 \\
 \text{From eqn. (1) } y &= 31 - x = 31 - 28 = 3 \\
 \therefore \sqrt{31 + 4\sqrt{21}} &= \pm(\sqrt{28} + \sqrt{3}) = \pm(2\sqrt{7} + \sqrt{3})
 \end{aligned}$$

2. Find the square root of $\sqrt{32} - \sqrt{24}$

Solution

$$\begin{aligned}
 \sqrt{32} - \sqrt{24} &= 4\sqrt{2} - 2\sqrt{6} = \sqrt{2}(4 - 2\sqrt{3}) \\
 \text{Then } \sqrt{\sqrt{32} - \sqrt{24}} &= \sqrt{\sqrt{2}(4 - 2\sqrt{3})} = 2^{\frac{1}{4}}\sqrt{4 - 2\sqrt{3}} \\
 \text{Evaluating } \sqrt{4 - 2\sqrt{3}} \\
 \text{Let } \sqrt{4 - 2\sqrt{3}} &= \sqrt{x} - \sqrt{y}, \text{ squaring both sides} \\
 \Rightarrow 4 - 2\sqrt{3} &= x + y - 2\sqrt{xy} \\
 \Rightarrow x + y &= 4 \dots\dots\dots (1) \\
 2\sqrt{xy} &= 2\sqrt{3} \text{ or } \sqrt{xy} = \sqrt{3}, \text{ squaring} \\
 \Rightarrow xy &= 3 \dots\dots\dots (2) \\
 \text{But } (x-y)^2 &= (x+y)^2 - 4xy \\
 \Rightarrow x-y &= \sqrt{4^2 - 3} = \pm 2 \\
 \text{Taking the +ve sign} \\
 \Rightarrow x-y &= 2 \dots\dots\dots (3) \\
 \text{Adding eqn. (1) and (3)} \\
 2x &= 6 \Rightarrow x = 3 \\
 \text{From eqn. (1) } y &= 4 - x = 4 - 3 = 1 \\
 \therefore \sqrt{\sqrt{32} - \sqrt{24}} &= 2^{\frac{1}{4}}\sqrt{4 - 2\sqrt{3}} = \pm 2^{\frac{1}{4}}(\sqrt{3} - 1)
 \end{aligned}$$

3. If $x = \sqrt{\left(\frac{7+4\sqrt{3}}{7-4\sqrt{3}}\right)}$, show that $x^2(x-14)^2 = 1$

Solution

$$\begin{aligned}
 x &= \sqrt{\left(\frac{7+4\sqrt{3}}{7-4\sqrt{3}}\right)} \Rightarrow x = \sqrt{\frac{(7+4\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})}} \\
 \Rightarrow x &= \frac{7+4\sqrt{3}}{49-48} = 7 + 4\sqrt{3} \\
 \text{Now } (x-14)^2 &= (7 + 4\sqrt{3} - 14)^2 = (-7 + 4\sqrt{3})^2 \\
 \Rightarrow (x-14)^2 &= 49 - 56\sqrt{3} + 48 \\
 &= 97 - 56\sqrt{3} \\
 \text{Also } x^2 &= (7 + 4\sqrt{3})^2 \\
 &= 49 + 56\sqrt{3} + 48 = 97 + 56\sqrt{3} \\
 \therefore x^2(x-14)^2 &= (97 + 56\sqrt{3})(97 - 56\sqrt{3}) = 97^2 - (56\sqrt{3})^2 = 1
 \end{aligned}$$

4. If $2x = \sqrt{a} + \frac{1}{\sqrt{a}}$, show that $\frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = \frac{a-1}{2}$

Solution

$$2x = \sqrt{a} + \frac{1}{\sqrt{a}} \Rightarrow x = \frac{a+1}{2\sqrt{a}}$$

$$\text{Now } x^2 = \frac{(a+1)^2}{4a} = \frac{a^2+2a+1}{4a}$$

$$\text{Also } x^2 - 1 = \frac{a^2+2a+1}{4a} - 1 = \frac{a^2-2a+1}{4a} = \frac{(a-1)^2}{4a}$$

$$\begin{aligned} \Rightarrow \frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}} &= \frac{\sqrt{\frac{(a-1)^2}{4a}}}{\frac{a+1}{2\sqrt{a}} - \sqrt{\frac{(a-1)^2}{4a}}} \\ &= \frac{\frac{a-1}{2\sqrt{a}}}{\frac{a+1}{2\sqrt{a}} - \frac{a-1}{2\sqrt{a}}} = \frac{a-1}{2\sqrt{a}} \div \frac{2a}{2\sqrt{a}} = \frac{a-1}{2} \end{aligned}$$

Task

- Simplify $\frac{\sqrt{3}-2}{2\sqrt{3}+3}$ (Uneb 2007)
- Express as an equivalent fraction with a rational denominator
 $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ (Uneb 1996) **ANSWER:** $\frac{6+2\sqrt{6}+2\sqrt{15}}{12}$
- Express $\frac{1+\sqrt{3}}{(\sqrt{3}-1)^3}$ in the form $a + b\sqrt{c}$ **ANSWER:** $\frac{7}{2} + 2\sqrt{3}$
- Find the square root of $14 + 6\sqrt{5}$ **ANSWER:** $\pm (3 + \sqrt{5})$
- Simplify $\frac{2\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$
- Find the rational numbers a and b such that
 $3 + \sqrt{2} = (a + b\sqrt{2})(6 - 2\sqrt{3})$
ANSWER: $a + b\sqrt{2} = \frac{13}{14} + \frac{3}{7}\sqrt{2} \Rightarrow a = \frac{13}{14}, b = \frac{3}{7}, c = 2$

THE THEORY OF QUADRATIC EQUATIONS

An equation of degree two is a quadratic equation and generally takes the form

$$ax^2 + bx + c = 0$$

, where a, b, c are rational numbers.

Types of Quadratic Equations

Are of two types .i.e.

a) **Pure Quadratic Equation**

Is an equation of the form

$$ax^2 + c = 0$$

, where $b = 0$ in $ax^2 + bx + c = 0$

b) **Affected Quadratic Equation**

Is an equation which is not a pure quadratic equation, and takes on the form, with $ab \neq 0 \Leftrightarrow a \neq 0, b \neq 0$

$$ax^2 + bx + c = 0$$

Methods of solving a Quadratic Equation $ax^2 + bx + c = 0$

I) **Method of Factorization**

If the expression $ax^2 + bx + c$ can be factored in to linear factors, then each of the factors put to zero, provides us with a root of the quadratic equation. Thus if

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

, then the roots of the equation $ax^2 + bx + c = 0$ are

$$x = \alpha \text{ and } x = \beta$$

Example

Solve:

a) $x^2 - 5x + 6 = 0$

b) $2x^2 - x - 3 = 0$

c) $4x^2 - x - 3 = 0$

Solution

a) $x^2 - 5x + 6 = 0$

Sum of roots = -5 , (coefficient of middle term)

Product = 6 , (product of the constant term and coefficient of x^2)

Numbers = $(-2, -3)$

$$\therefore x^2 - 2x - 3x + 6 = 0$$

$$\Rightarrow x(x - 2) - 3(x - 2) = 0$$

$$\Rightarrow (x - 3)(x - 2) = 0$$

$$\therefore x - 3 = 0 \text{ or } x = 3$$

$$\therefore x - 2 = 0 \text{ or } x = 2$$

Hence roots are 3 and 2

b) $2x^2 - x - 3 = 0$

Sum of roots = -5

$$\text{Product} = 2 \times -3 = -6$$

Numbers = $(-3, 2)$

$$\therefore 2x^2 - 3x + 2x - 3 = 0$$

$$\Rightarrow x(2x - 3) + 1(2x - 3) = 0$$

$$\Rightarrow (x + 1)(2x - 3) = 0$$

$$\therefore x + 1 = 0 \text{ or } x = -1$$

$$\therefore 2x - 3 = 0 \text{ or } x = \frac{3}{2}$$

Hence roots are -1 and $\frac{3}{2}$

c) Left as an exercise:

$$\text{ANSWER: } x = 1; -\frac{3}{4}$$

II) Method of Perfect Square

Let $ax^2 + bx + c = 0$ be given equation.

Step 1: Divide both sides by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2: Transpose the constant term (i.e. term independent of x) on the

R.H.S

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 3: Add $\frac{b^2}{4a^2}$ to both sides

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\text{or } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{NOTE: } \frac{b^2}{4a^2} = \left(\frac{b}{a} \div 2\right)^2 = \left(\frac{\text{coeff. of } x}{2}\right)^2$$

Step 4: Find the square root on both sides

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Now solving for x

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This is a quadratic formula for solving quadratic equations.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ for } ax^2 + bx + c = 0$$

Example

Solve:

a) $x^2 - 5x + 6 = 0$

b) $2x^2 - x - 3 = 0$

Solution

By quadratic formula

a) $x^2 - 5x + 6 = 0 \Rightarrow a = 1, b = -5, c = 6$

$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = \frac{5 \pm 1}{2}$$

$$\therefore x = \frac{5+1}{2} = 3$$

$$\therefore x = \frac{5-1}{2} = 2$$

$\therefore$ Roots are 3 and 2

b) $2x^2 - x - 3 = 0 \Rightarrow a = 2, b = -1, c = -3$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)} = \frac{1 \pm 5}{4}$$

$$\therefore x = \frac{1+5}{4} = \frac{3}{2}$$

$$\therefore x = \frac{1-5}{2} = -2$$

$\therefore$ Roots are $\frac{3}{2}$ and -2

Nature of Roots of a Quadratic Equation

The equation $ax^2 + bx + c = 0$ has roots $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Nature of roots depend on $(b^2 - 4ac)$ inside the radical, called

DISCRIMINANT

Let the roots be α and β

Let $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Deduction

Case I:

If $b^2 - 4ac$ is positive. i.e. $b^2 - 4ac > 0$, α and β are real and unequal or real and distinct

Case II:

If $b^2 - 4ac$ is zero. i.e. $b^2 - 4ac = 0$, α and β are real and equal each reducing in this case to $-\frac{b}{2a}$ or roots are coincident/double/repeated.

Case III:

If $b^2 - 4ac$ is negative. i.e. $b^2 - 4ac < 0$, α and β are imaginary and unequal

Case IV:

If $b^2 - 4ac$ is a perfect square, α and β are rational and unequal

By applying these tests the nature of roots of any quadratic equation may be determined without solving.

Graphical Interpretation of $y = ax^2 + bx + c$

The graph $y = ax^2 + bx + c$ is a parabola with vertical axis of symmetry.

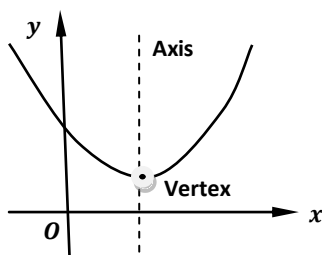


Fig. $y = ax^2 + bx + c$, $a > 0$
(Opens upwards and has a minimum values)

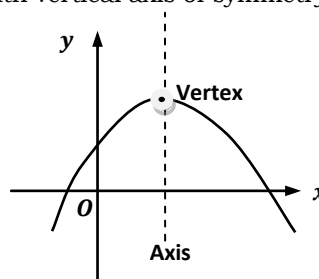
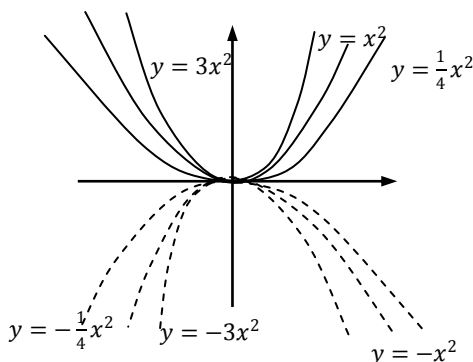


Fig. $y = ax^2 + bx + c$, $a < 0$
(Opens down wards and has a minimum values)



Note: The bigger $|a|$ the narrower the parabola.

For an equation $ax^2 + bx + c = 0$

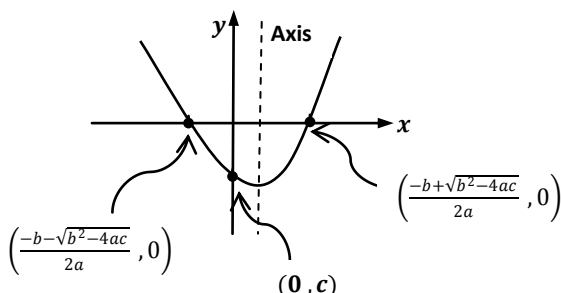


Fig I. $b^2 - 4ac > 0$ There are two intercepts

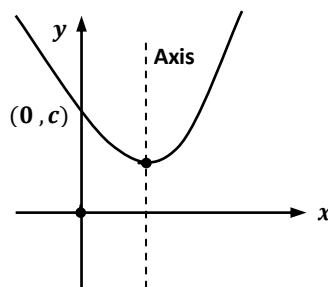


Fig II. $b^2 - 4ac < 0$. No x - intercepts

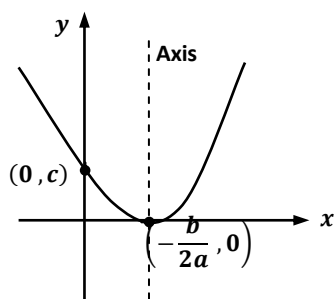


Fig III. $b^2 - 4ac = 0$. There is one intercept. (at the point where the parabola meets the x -axis and x -axis in a

Illustrative examples on deduced cases on roots of a quadratic equation $ax^2 + bx + c = 0$

1. Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real value of x .

Solution

From discriminate, $b^2 - 4ac$, $a = 2$, $b = -6$, $c = 7$
 $\Rightarrow (-6)^2 - 4(2)(7) = -20$. Thus roots are imaginary.

2. If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k .

Solution

For equal roots $b^2 - 4ac = 0$, $a = 1$, $b = 2(k+2)$, $c = 9k$
 $\Rightarrow [2(k+2)]^2 - 4(1)(9k) = 0 \dots\dots\dots \div (4)$
 $\Rightarrow (k+2)^2 - 9k = 0$
 $\Rightarrow k^2 + 4k + 4 - 9k = 0$ or $k^2 - 5k + 4 = 0$
 $\Rightarrow (k-1)(k-4) = 0 \therefore k = 1, 4$

3. Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

Solution

For rational roots, $b^2 - 4ac$ is a perfect square
 $a = 1$, $b = -2p$, $c = p^2 - q^2 + 2qr - r^2$
 $\Rightarrow b^2 - 4ac = (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$
 $= 4p^2 - 4p^2 + 4(q^2 + 2qr + r^2) = 4(q^2 + 2qr + r^2)$
 $= 4(q+r)^2 = [2(q+r)]^2$, hence roots are rational.

4. Prove that the roots of the equation $x^2 - 2ax + a^2 - b^2 - c^2 = 0$ are real.

Solution

For real roots, $b^2 - 4ac \geq 0$

Note: for real and distinct roots, $b^2 - 4ac > 0$

$$\Rightarrow (-2a)^2 - 4(1)(a^2 - b^2 - c^2) \geq 0$$

$$\Rightarrow 4a^2 - 4a^2 + 4(b^2 + c^2) \geq 0$$

$$\Rightarrow 4(b^2 + c^2) \geq 0$$

R.H.S will always be true for any value of b and c .

5. Prove that the roots of the equation $(a - b - c)x^2 + ax + b + c = 0$ are real if a, b, c are real

Solution

For real roots $B^2 - 4AC \geq 0$

$$\Rightarrow a^2 - 4(a - b - c)(b + c) \geq 0$$

Let $b + c = n$

$$\Rightarrow a^2 - 4(a - n)n \geq 0$$

$$\Rightarrow a^2 - 4an + 4n^2 \geq 0$$

$$\Rightarrow (a - 2n)^2 \geq 0$$

$$\therefore [a - 2(b + c)]^2 \geq 0$$

This inequality will be satisfied if a, b and c are real, hence a, b and c are real.

Task

- Prove that the roots of the equations are real rational
 - $(a + c - b)x^2 + 2cx + (b + c - a) = 0$
 - $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$
- Prove that the roots of the equation $(a - b + c)x^2 + 4(a - b)x + (a - b - c) = 0$ are real
- For what values of m will the equations
 - $x^2 - 15 - m(2x - 8) = 0$
 - $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$, have equal roots?

ANSWER: a) 3, 5 b) 2, $-\frac{10}{9}$

Relationship between roots of a quadratic equation .i.e. α and β and its coefficients in $ax^2 + bx + c = 0$

From $ax^2 + bx + c = 0$, with roots α and β ,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \left[\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right]$$

$$\text{Then } \alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}$$

$$\boxed{\alpha + \beta = -\frac{b}{a}}$$

$$\begin{aligned} \text{Also } \alpha \cdot \beta &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2} \\ &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} \\ &= \frac{4ac}{4a^2} = \frac{c}{a} \end{aligned}$$

$$\boxed{\alpha \cdot \beta = \frac{c}{a}}$$

From $ax^2 + bx + c = 0$, re writing in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

From

$$\alpha + \beta = -\frac{b}{a} \text{ or } \frac{b}{a} = -(\alpha + \beta) \text{ and } \alpha \cdot \beta = \frac{c}{a}$$

$\therefore$ the quadratic equation becomes

$$x^2 - x(\alpha + \beta) + \alpha \cdot \beta = 0$$

In general, if a quadratic equation $ax^2 + bx + c = 0$ can be written with coefficient of the first term as unity $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then

- Sum of roots = coefficient of x with its sign changed
- Product of roots = the third term

Note:

Re writing $x^2 - x(\alpha + \beta) + \alpha \cdot \beta = 0$ by factorization,

$$(x - \alpha) \cdot (x - \beta) = 0$$

, is a quadratic equation with coefficient

of x^2 being unity.

In general, the quadratic equation with given roots is

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Illustrative examples

- Form the equation whose roots are;

- 3 and -2
- $2 + \sqrt{3}$ and $2 - \sqrt{3}$

Solution

Required equation is $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

- Sum of roots = $3 + -2 = 1$

Product of roots = $2(-3) = -6$

$\therefore$ Required equations is $x^2 - x - 6 = 0$

- Sum of roots = $(2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$

Product of roots = $(2 + \sqrt{3}) \cdot (2 - \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 1$

$\therefore$ Required equations is $x^2 - 4x + 1 = 0$

- If α and β are the roots of the equation $x^2 - px + q = 0$, find;

- Values of $\alpha^2 + \beta^2$
- Values of $\alpha^3 + \beta^3$
- Values of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
- Value of $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$

Solution

$x^2 - px + q = 0$, α, β are roots

$\Rightarrow \alpha + \beta = -(-p) = p$ [i.e. sign of coeff. of x changed]

$\Rightarrow \alpha \cdot \beta = q$

- i) From $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$
 $\therefore \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
 $= p^2 - 2q$
- ii) From $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$
 $\therefore \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
 $= p^3 - 3pq$
- iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{p^2 - 2q}{q^2}$
- iv) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2 = \frac{\alpha^2}{\beta^2} - 2\frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} + \frac{\beta^2}{\alpha^2}$
 $= \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} - 2 = \frac{\alpha^4 + \beta^4}{(\alpha\beta)^2} - 2$
 From $(\alpha + \beta)^4 = \alpha^4 + 4\alpha^3\beta + 6\alpha^2\beta^2 + 4\alpha\beta^3 + \beta^4$
 $\therefore \alpha^4 + \beta^4 = (\alpha + \beta)^4 - (4\alpha^3\beta + 6\alpha^2\beta^2 + 4\alpha\beta^3)$
 $= (\alpha + \beta)^4 - [4\alpha\beta(\alpha^2 + \beta^2) + 6(\alpha\beta)^2]$
 $= p^4 - [4q(p^2 - 2q) + 6q^2]$
 $= p^4 - 4q(p^2 - 2q) - 6q^2$
 $= p^4 - 4qp^2 + 8q^2 - 6q^2$
 $= p^4 - 4qp^2 + 2q^2$
3. If α and β are the roots of the equation $x^2 - 7x + 2 = 0$, evaluate;
 i) $\alpha^2 + \beta^2$
 ii) $\frac{2-\alpha}{3+\beta} + \frac{2-\beta}{3+\alpha}$

Solution

$x^2 - 7x + 2 = 0$, with roots α and β , then $\alpha + \beta = 7$, $\alpha\beta = 2$

- i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
 $= 7^2 - 2(2) = 45$
- ii) $\frac{2-\alpha}{3+\beta} + \frac{2-\beta}{3+\alpha} = \frac{(2-\alpha)(3+\beta) + (2-\beta)(3+\alpha)}{(3+\beta)(3+\alpha)}$
 $= \frac{6+2\alpha-3\alpha-\alpha^2+6+2\beta-3\beta-\beta^2}{9+3\alpha+3\beta+\alpha\beta}$
 $= \frac{12+2(\alpha+\beta)-3(\alpha+\beta)-(\alpha^2+\beta^2)}{9+3(\alpha+\beta)+\alpha\beta}$
 $= \frac{12+2(\alpha+\beta)-3(\alpha+\beta)-[(\alpha+\beta)^2-2\alpha\beta]}{9+3(\alpha+\beta)+\alpha\beta}$
 $= \frac{12+14-21-(7^2-2(2))}{9+21+2} = -\frac{28}{32} = -\frac{7}{8}$
4. If α and β are the roots of the equation $x^2 - px + q = 0$, form an equation whose roots are $\alpha\beta + \alpha + \beta$ and $\alpha\beta - \alpha - \beta$

Solution

Required equation is $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

From $x^2 - px + q = 0$ with roots α and β ,

$$\alpha + \beta = -(-p) = p, \quad \alpha\beta = q$$

$$\text{Sum of roots} = \alpha\beta + \alpha + \beta + \alpha\beta - \alpha - \beta = 2\alpha\beta = 2q$$

$$\begin{aligned} \text{Product of roots} &= (\alpha\beta + \alpha + \beta)(\alpha\beta - \alpha - \beta) \\ &= (q + p)(q - p) = q^2 - p^2 \end{aligned}$$

$$\therefore \text{Equation is } x^2 - 2qx + q^2 - p^2 = 0$$

5. If α and β are the roots of the equation $2x^2 - 4x + 1 = 0$, form an equation whose roots are $\alpha^2 + \beta$ and $\beta^2 + \alpha$

Solution

From $2x^2 - 4x + 1 = 0$ with roots α and β ,

$$\Rightarrow \alpha + \beta = -\left(-\frac{4}{2}\right) = 2, \quad \alpha\beta = \frac{1}{2}$$

Required equation is $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$$\begin{aligned} \text{Sum of roots} &= \alpha^2 + \beta + \beta^2 + \alpha = \alpha^2 + \beta^2 + \alpha + \beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta + \alpha + \beta \\ &= 2^2 - 2 \cdot \frac{1}{2} + 2 = 4 - 1 + 2 = 5 \end{aligned}$$

$$\begin{aligned} \text{Product of roots} &= (\alpha^2 + \beta) \cdot (\beta^2 + \alpha) \\ &= \alpha^2\beta^2 + \alpha^3 + \beta^3 + \alpha\beta \\ &= (\alpha\beta)^2 + \alpha\beta + (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= \frac{1}{4} + 2 + 8 - 3 \cdot \frac{1}{2} \cdot 2 = \frac{23}{4} \end{aligned}$$

$$\therefore x^2 - 5x + \frac{23}{4} = 0 \text{ or } 4x^2 - 20x + 23 = 0$$

6. If α and β are the roots of the equation $ax^2 + bx + c = 0$, find an equation whose roots are $a\alpha + b\beta$ and $b\alpha + a\beta$

Solution

$$\begin{aligned} \text{Sum of roots} &= a\alpha + b\beta + b\alpha + a\beta \\ &= \alpha(a + b) + \beta(a + b) = (\alpha + \beta)(a + b) \\ &= -\frac{b}{a} \times (a + b) = \frac{-b(a+b)}{a} \end{aligned}$$

$$\begin{aligned} \text{Product of roots} &= (a\alpha + b\beta) \cdot (b\alpha + a\beta) \\ &= ab\alpha^2 + a^2\beta\alpha + b^2\alpha\beta + ab\beta^2 \\ &= ab(\alpha^2 + \beta^2) + \alpha\beta(a^2 + b^2) \\ &= ab[(\alpha + \beta)^2 - 2\alpha\beta] + \alpha\beta(a^2 + b^2) \\ &= ab\left(\frac{b^2}{a^2} - \frac{2c}{a}\right) + \frac{c(a^2 + b^2)}{a} \\ &= \frac{b(b^2 - 2ac) + c(a^2 + b^2)}{a} \end{aligned}$$

Hence the required equation is

$$\begin{aligned} x^2 - \frac{b(a+b)}{a}x + \frac{b(b^2 - 2ac) + c(a^2 + b^2)}{a} &= 0 \\ \text{or } ax^2 - b(a + b)x + b(b^2 - 2ac) + c(a^2 + b^2) &= 0 \\ \text{or } ax^2 - bax - b^2x + b^3 - 2abc + a^2c + b^2c &= 0 \\ \text{or } ax(x - b) - b^2(x - b) + c(a^2 - 2ab + b^2) &= 0 \\ \text{or } (x - b)(ax - b^2) + c(a - b)^2 &= 0 \end{aligned}$$

7. The roots of a quadratic equation $x^2 - px + q = 0$ are α and β . Determine the equation whose roots are $\alpha^2 + \beta^{-2}$ and $\beta^2 + \alpha^{-2}$ expressing the coefficients in terms of p and q . prove further that if p and q are both real, then this equation can have roots if $p = 0$ or $p^2 = 4q$.

Solution

Required equation: $x^2 - (A + B)x + A.B = 0$

$$\text{Let } A = \alpha^2 + \beta^{-2} = \alpha^2 + \frac{1}{\beta^2} \text{ and } B = \beta^2 + \alpha^{-2} = \beta^2 + \frac{1}{\alpha^2}$$

$$\begin{aligned} A + B &= \alpha^2 + \frac{1}{\beta^2} + \beta^2 + \frac{1}{\alpha^2} \\ &= (\alpha^2 + \beta^2) + \frac{1}{\alpha^2} + \frac{1}{\beta^2} \end{aligned}$$

$$= (\alpha + \beta)^2 - 2\alpha\beta + \frac{\alpha^2 + \beta^2}{\alpha^2 \cdot \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta + \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha \cdot \beta)^2}}$$

But from the equation $x^2 - px + q = 0$, α and β are roots then $\alpha + \beta = p, q = \alpha\beta$

$$\begin{aligned}\therefore A + B &= p^2 - 2q + \frac{p^2 - 2q}{q^2} \\ &= (p^2 - 2q) \left(1 + \frac{1}{q^2}\right) = \frac{(p^2 - 2q)(q^2 + 1)}{q^2}\end{aligned}$$

$$\begin{aligned}A \cdot B &= (\alpha^2 + \beta^2) \cdot \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) \\ &= \alpha^2 \cdot \beta^2 + 1 + 1 + \frac{1}{\alpha^2 \cdot \beta^2} \\ &= (\alpha \cdot \beta)^2 + 2 + \frac{1}{(\alpha \cdot \beta)^2} \\ &= q^2 + 2 + \frac{1}{q^2} = \frac{q^4 + 2q^2 + 1}{q^2} \\ &= \frac{(q^2 + 1)^2}{q^2}\end{aligned}$$

Required equation is; $x^2 - \frac{(p^2 - 2q)(q^2 + 1)}{q^2}x + \frac{(q^2 + 1)^2}{q^2} = 0$

If p and q are real then x must also be real . i. e.

$$\left[-\frac{(p^2 - 2q)(q^2 + 1)}{q^2}\right]^2 \geq 4 \frac{(q^2 + 1)^2}{q^2} \Rightarrow \frac{(p^2 - 2q)^2 (q^2 + 1)^2}{q^4} \geq 4 \frac{(q^2 + 1)^2}{q^2}$$

Dividing both sides by $\frac{(q^2 + 1)^2}{q^4}$

$$\Rightarrow (p^2 - 2q)^2 \geq 4q^2 \text{ or } (p^2 - 2q)^2 - 4q^2 \geq 0$$

By factoring using a difference of two squares identity,

$$\Rightarrow (p^2 - 2q + 2q)(p^2 - 2q - 2q) \geq 0 \text{ or } p^2(p^2 - 4q) \geq 0$$

$$\Rightarrow p^2 \geq 0 \text{ or } p^2 - 4q \geq 0$$

$$\therefore p = 0 \text{ or } p = 4q$$

8. If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $p:q$, prove that $ac(p + q)^2 = b^2pq$

Solution

Let the roots of the equation be α and β

$$\Rightarrow \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a} \dots\dots\dots (1)$$

$$\text{But } \alpha:\beta = \frac{p}{q} \text{ or } \frac{\alpha}{\beta} = \frac{p}{q}$$

$$\therefore \alpha = \frac{p}{q}\beta \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow \frac{p}{q}\beta + \beta = -\frac{b}{a}, \text{ also } \beta \cdot \frac{p}{q}\beta = \frac{c}{a}$$

$$\therefore \beta \left(\frac{p}{q} + 1\right) = -\frac{b}{a} \text{ or } \beta = -\frac{bq}{a(p+q)}, \beta^2 = \frac{qc}{ap}$$

Solving these equations

$$\Rightarrow \left[-\frac{bq}{a(p+q)}\right]^2 = \frac{qc}{ap} \text{ or } \frac{b^2q^2}{a^2(p+q)^2} = \frac{qc}{ap}$$

$$\therefore b^2q^2 \cdot ap = a^2(p+q)^2 \cdot qc, \text{ simplifying}$$

$$\Rightarrow ac(p+q)^2 = b^2pq$$

Task

- a) If α and β are the roots of the equation $ax^2 + bx + c = 0$, and r is the ratio of the roots, show that $ac(r + q)^2 = b^2r$

- b) If the roots of the equation $lx^2 + nx + n = 0$ are in the ratio $p:q$ prove that

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$$

9. If the roots of the equation $ax^2 + 2bx + c = 0$ are α and β and those of the equation $Ax^2 + 2Bx + C = 0$ are $\alpha + \delta$ and $\beta + \delta$, prove that $\frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2}$

Solution

Given: $ax^2 + 2bx + c = 0$, α, β are roots, then

$$\alpha + \beta = -\frac{2b}{a}, \alpha\beta = \frac{c}{a} \dots\dots\dots (1)$$

Given: $Ax^2 + 2Bx + C = 0$, $\alpha + \delta, \beta + \delta$ are roots, then

$$\alpha + \delta + \beta + \delta = -\frac{2B}{A}$$

$$\Rightarrow \alpha + \beta + 2\delta = -\frac{2B}{A}$$

$$\text{Or } 2\delta = -\frac{2B}{A} - (\alpha + \beta), \text{ from eqn. (1)}$$

$$\Rightarrow 2\delta = -\frac{2B}{A} - \left(-\frac{2b}{a}\right)$$

$$\therefore \delta = \frac{b}{a} - \frac{B}{A} \dots\dots\dots (2)$$

$$\text{Also } (\alpha + \delta)(\beta + \delta) = \frac{C}{A}$$

$$\Rightarrow \alpha\beta + \alpha\delta + \beta\delta + \delta^2 = \frac{C}{A} \quad \text{or} \quad \alpha\beta + \delta(\alpha + \beta) + \delta^2 = \frac{C}{A}$$

$$\Rightarrow \frac{c}{a} + \delta\left(-\frac{2b}{a}\right) + \delta^2 = \frac{C}{A}$$

$$\therefore \delta\left(-\frac{2b}{a}\right) + \delta^2 = \frac{C}{A} - \frac{c}{a} \dots\dots\dots (3)$$

Eqn. (2) in to eqn. (3)

$$\Rightarrow \left(\frac{b}{a} - \frac{B}{A}\right)\left(-\frac{2b}{a}\right) + \left(\frac{b}{a} - \frac{B}{A}\right)^2 = \frac{C}{A} - \frac{c}{a}$$

$$\text{Let } \frac{b}{a} = p, \frac{B}{A} = q$$

$$\Rightarrow (p - q)(-2p) + (p - q)^2 = \frac{C}{A} - \frac{c}{a}$$

$$\Rightarrow -2p^2 + 2pq + p^2 - 2pq + q^2 = \frac{C}{A} - \frac{c}{a}$$

$$\Rightarrow q^2 - p^2 = \frac{C}{A} - \frac{c}{a}$$

$$\Rightarrow \frac{B^2}{A^2} - \frac{b^2}{a^2} = \frac{C}{A} - \frac{c}{a} \quad \text{or} \quad \frac{B^2}{A^2} - \frac{C}{A} = \frac{b^2}{a^2} - \frac{c}{a}$$

$$\therefore \frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2} \text{ as required.}$$

To find the condition that the roots of the equation $ax^2 + bx + c = 0$ should be;

- i) Equal in magnitude and opposite in sign
 - ii) Reciprocals
 - iii) With one of the roots as zero
 - iv) Both positive
- i) The roots will be *equal in magnitude* and *opposite in sign* if their *sum is zero*. Hence the required condition is:
- $$\alpha + \beta = -\frac{b}{a} = 0 \quad \text{or} \quad \boxed{b = 0}, \text{ where } \alpha \text{ and } \beta \text{ are roots of the equation.}$$
- ii) The roots will be *reciprocals* if their *product is unity*. Hence the required condition is:
- $$\alpha\beta = \frac{c}{a} = 1 \quad \text{or} \quad \boxed{c = a}$$

- iii) When one of the roots is zero then the product must be zero. Hence the required condition is:

$$\alpha\beta = \frac{c}{a} = 0 \text{ or } \boxed{c = 0}$$

- iv) The roots will be both positive if;

- a) The *product* of the roots is *positive*. i.e. $\alpha\beta = \frac{c}{a} = +ve$

This is only true if c and a are of *like signs*

- b) The *sum* of the roots is also *positive*. i.e. $\alpha + \beta = -\frac{b}{a} = +ve$

This is possible if b and a are of *unlike signs*. Thus the required condition is that the signs of a and c are alike and opposite to that of b .

10. The roots of the equation $x^2 + ax + b = 0$ are α and β . Find the equation whose roots are $p\alpha + q\beta$ and $p\beta + q\alpha$.

If the original equation is $x^2 - 4x - 5 = 0$, find the values of $\frac{p}{q}$ in order that the new equation shall have one zero root.

Solution

Required equation: $x^2 - (A + B)x + A.B = 0$

Let $A = p\alpha + q\beta$ and $B = p\beta + q\alpha$

$$A + B = p\alpha + q\beta + p\beta + q\alpha$$

$$= \alpha(p + q) + \beta(p + q) = (p + q)(\alpha + \beta)$$

But $\alpha + \beta = -a$, $\alpha\beta = b$

$$\therefore A + B = -a(p + q)$$

$$A.B = (p\alpha + q\beta)(p\beta + q\alpha)$$

$$= p^2\alpha\beta + pq\alpha^2 + pq\beta^2 + q^2\alpha\beta$$

$$= (p^2 + q^2)\alpha\beta + pq(\alpha^2 + \beta^2)$$

$$= (p^2 + q^2).b + pq[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= (p^2 + q^2).b + pq(a^2 - 2b)$$

$$\therefore \text{Required equation is: } x^2 + a(p + q)x + (p^2 + q^2).b + pq(a^2 - 2b) = 0$$

Now to find the values of $\frac{p}{q}$ in original equation $x^2 - 4x - 5 = 0$, so that the new equation shall have one zero root,

Comparing the equations $x^2 - 4x - 5 = 0$ and $x^2 + ax + b = 0$,

$$\Rightarrow a = -4, b = -5$$

Substituting these values in the obtained equation gives;

$$x^2 - 4(p + q)x - 5(p^2 + q^2) + pq((-4)^2 - 2 \times -5) = 0$$

$$\Rightarrow x^2 - 4(p + q)x - 5(p^2 + q^2) + 26pq = 0$$

This equation will have one zero root if the product is zero

$$\Rightarrow -5(p^2 + q^2) + 26pq = 0 \text{ or } 5p^2 - 26pq + 5q^2 = 0$$

Factoring

$$\Rightarrow 5p^2 - 25pq - pq + 5q^2 = 0$$

$$\Rightarrow 5p(p - 5q) - q(p - 5q) = 0$$

$$\Rightarrow (p - 5q)(5p - q) = 0$$

$$\therefore p - 5q = 0 \text{ or } \frac{p}{q} = 5$$

$$\therefore 5p - q = 0 \text{ or } \frac{p}{q} = \frac{1}{5}$$

$$\therefore \frac{p}{q} = 5, \frac{1}{5}$$

Other examples on quadratic equation

11. If α and β are roots of the equation $ax^2 + 2bx + c = 0$ while δ and γ are the roots of the equation $a_1x^2 + 2b_1x + c_1 = 0$, show that if $(\alpha - \gamma)(\beta - \delta) + (\alpha - \delta)(\beta - \gamma) = 0$, then $ac_1 + a_1c = 2bb_1$.

Solution

$$\text{From } ax^2 + 2bx + c = 0, \alpha + \beta = -\frac{2b}{a}, \alpha\beta = \frac{c}{a}$$

$$\text{From } a_1x^2 + 2b_1x + c_1 = 0, \gamma + \delta = -\frac{2b_1}{a_1}, \gamma\delta = \frac{c_1}{a_1}$$

$$(\alpha - \gamma)(\beta - \delta) + (\alpha - \delta)(\beta - \gamma) = 0$$

$$\Rightarrow \alpha\beta - \alpha\delta - \gamma\beta + \gamma\delta + \alpha\beta - \alpha\gamma - \delta\beta + \gamma\delta = 0$$

$$\Rightarrow 2\alpha\beta + 2\gamma\delta - \delta(\alpha + \beta) - \gamma(\delta + \gamma) = 0$$

$$\Rightarrow \frac{2c}{a} + \frac{2c_1}{a_1} - (\alpha + \beta)(\delta + \gamma) = 0$$

$$\Rightarrow \frac{2c}{a} + \frac{2c_1}{a_1} + \frac{2b}{a} \cdot -\frac{2b_1}{a_1} = 0$$

$$\Rightarrow \frac{2c}{a} + \frac{2c_1}{a_1} - \frac{4bb_1}{aa_1} = 0 \dots\dots\dots \left(\times \frac{aa_1}{2} \right)$$

$$\Rightarrow a_1c + ac_1 - 2bb_1 = 0$$

$$\therefore ac_1 + a_1c = 2bb_1$$

12. If x is a real number, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ does not lie strictly between 2 and 6.

Solution

$$\text{Let } y = \frac{x^2+2x-11}{2(x-3)}$$

$$\Rightarrow 2yx - 6y = x^2 + 2x - 11 \text{ or } x^2 + (2 - 2y)x + 6y - 11 = 0$$

$$\begin{aligned} \Rightarrow x &= \frac{-(2-2y) \pm \sqrt{[-(2-2y)]^2 - 4(6y-11)}}{2} \\ &= \frac{-(2-2y) \pm \sqrt{4-8y+4y^2-24+44}}{2} \\ &= \frac{-(2-2y) \pm \sqrt{4y^2-32y+48}}{2} \end{aligned}$$

$$x \text{ will be real if } 4y^2 - 32y + 48 \geq 0 \text{ or } y^2 - 8y + 12 \geq 0$$

$$\Rightarrow (y-2)(y-6) \geq 0$$

Between 2 and 6 . i. e. $2 < y < 6$, one of the factors on L.H.S will be +ve and the other -ve, so the product will be negative. Hence $\frac{x^2+2x-11}{2(x-3)}$ will not lie between 2 and 6

13. If x is a real number, prove that the expression $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$ and 1.

Solution

$$\text{Let } y = \frac{x}{x^2-5x+9} \text{ or } yx^2 + (-5y-1)x + 9y = 0$$

$$\Rightarrow x = \frac{(5y+1) \pm \sqrt{(5y+1)^2 - 36y^2}}{2y}$$

$$\text{For real values of } x, (5y+1)^2 - 36y^2 \geq 0$$

$$\Rightarrow 25y^2 + 10y + 1 - 36y^2 \geq 0$$

$$\Rightarrow -11y^2 + 10y + 1 \geq 0 \text{ or } 11y^2 - 10y - 1 \leq 0$$

$$\Rightarrow 11y^2 - 11y + y - 1 \leq 0$$

$$\Rightarrow 11y(y-1) + (y-1) \leq 0$$

$$\Rightarrow (11y+1)(y-1) \leq 0$$

Between $-\frac{1}{11} < y < 1$, the inequality is true for any value of y . Hence $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$ and 1.

14. For real x , $f(x) = \frac{x^2-k}{x-2}$ can take any real value. Find the range of values of k can take.

Solution

$$\text{Let } \frac{x^2-k}{x-2} = y \text{ or } yx - 2y = x^2 - k$$

$$\therefore x^2 - yx + 2y - k = 0$$

$$\Rightarrow x = \frac{y \pm \sqrt{y^2 - 4(2y-k)}}{2}$$

$$\text{For real value of } x, y^2 - 4(2y - k) \geq 0$$

$$\Rightarrow y^2 - 8y + 4k \geq 0$$

$$\Rightarrow y^2 - 8y \geq -4k, \text{ completing squares,}$$

$$\Rightarrow y^2 - 8y + 16 \geq -4k + 16$$

$$\therefore (y - 4)^2 \geq 16 - 4k$$

Since L.H.S is always +ve, taking its smallest value say 0 i.e. $y = 4$, then

$$0 \geq 16 - 4k \Rightarrow k \geq 4$$

$$\therefore k \geq 4$$

15. Find the range of values of k can take given that for real x , $f(x) = \frac{x^2+3x}{x+k}$ can take any real value.

Solution

$$\text{Let } \frac{x^2+3x}{x+k} = y \text{ or } yx + ky = x^2 + 3x$$

$$\therefore x^2 + (3 - y)x - ky = 0$$

$$\Rightarrow x = \frac{-(3-y) \pm \sqrt{(3-y)^2 + 4ky}}{2}$$

$$= \frac{(y-3) \pm \sqrt{(3-y)^2 + 4ky}}{2}$$

$$\text{For real values of } x, (3 - y)^2 + 4ky \geq 0$$

$$\Rightarrow 9 - 6y + y^2 + 4ky \geq 0$$

$$\Rightarrow y^2 + (4k - 6)y + 9 \geq 0$$

$$\Rightarrow y^2 + (4k - 6)y \geq -9, \text{ completing squares}$$

$$\Rightarrow y^2 + (4k - 6)y + (2k - 3)^2 \geq -9 + (2k - 3)^2$$

$$\therefore (y + (2k - 3))^2 \geq -9 + (2k - 3)^2$$

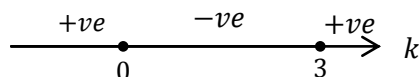
Since y can take on any real value, let its smallest value be $-(2k - 3)^2$

$$\Rightarrow 0 \geq -9 + (2k - 3)^2$$

$$\Rightarrow 4k^2 - 12k + 9 - 9 \leq 0$$

$$\therefore 4k^2 - 12k \leq 0 \text{ or } k^2 - 3k \leq 0$$

For $0 \leq k \leq 3$, the inequality is true.



16. Given that $y = \frac{3x+k}{x^2-1}$, where x is real and k is a constant, show that y can take all real values if $|k| < 3$.

Solution

$$y = \frac{3x+k}{x^2-1} \text{ or } yx^2 - y = 3x + k$$

$$\therefore yx^2 - 3x - y - k = 0$$

$$\text{For real values of } x, (-3)^2 \geq 4y(-y - k)$$

$$\Rightarrow 9 + 4y^2 + 4ky \geq 0$$

$$\Rightarrow 4y^2 + 4ky \geq -9 \quad \text{or} \quad y^2 + ky \geq -\frac{9}{4}, \text{ completing squares,}$$

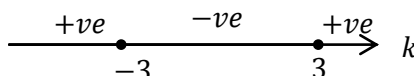
$$\Rightarrow y^2 + ky + \frac{k^2}{4} \geq -\frac{9}{4} + \frac{k^2}{4}$$

$$\therefore \left(y + \frac{k}{2}\right)^2 \geq \frac{k^2 - 9}{4}$$

Since y can take on any real value, let its smallest value be $-\frac{k}{2}$

$$\Rightarrow 0 \geq \frac{k^2 - 9}{4} \quad \text{or} \quad k^2 - 9 \leq 0$$

$$\therefore (k + 3)(k - 3) \leq 0$$



For interval $-3 \leq k \leq 3$, the inequality holds. Thus $|k| < 3$ is the required interval.

Task

1. If α and β are the roots of the equation $ax^2 + bx + c = 0$, find the value of;

a) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

ANSWER: $\frac{b^2 - 2ac}{ac}$

b) $\alpha^4 + \beta^4$

ANSWER: $\frac{b^4 - 4ab^2c + 2a^2c^2}{a^4}$

c) $(\alpha^2 - \beta)^2 + (\beta^2 - \alpha)^2$

ANSWER: $\frac{b^4 - 4ab^2c + 2a^2c^2 + a^2b^2 - 2a^3c + 2a^2bc}{a^4}$

d) $\alpha^4\beta^7 + \alpha^7\beta^4$

ANSWER: $\frac{bc^4(3ac - b^2)}{a^7}$

2. The roots of the equation are α and β

i) Show that if $\alpha - \beta = 1$, then $a^2 = 4(b^2 - ac)$

ii) Find the quadratic equation whose roots are $\alpha + \alpha\beta$ and β

3. If α and β are the roots of the equation $x^2 - px + q = 0$, show that $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} =$

$$\frac{p^4}{q^2} - \frac{4p^2}{q} + 2$$

4. If p, q are the roots of the equation $3x^2 + 6x + 2 = 0$, form an equation whose roots are $-\frac{p^2}{q}$ and $-\frac{q^2}{p}$.

ANSWER: $3x^2 - 18x + 2 = 0$

5. If α and β are the roots of the equation $ax^2 + bx + c = 0$, form an equation whose roots are;

i) $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$

ANSWER: $(a - b + c)x^2 - 2x(a - c) +$

$a + b + c = 0$

ii) $\alpha^2 + \beta, \alpha^{-2} + \beta^{-2}$

ANSWER: $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$

6. If the roots of the equation $ax^2 + 8(b - a)x + 4(4a - 8b + c) = 0$ are $(4 - 2\alpha)$ and $(4 - 2\beta)$, find the equation whose roots are α and β .

ANSWER: $x^2 - \frac{4b}{a}x + \frac{c}{a} = 0$

7. If $(a + b + c)x^2 + x(b + 2c) + c = 0$ has roots α and β , find the equation whose roots are $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ expressing the coefficients in terms of a, b and c .

ANSWER: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

8. Given that the roots of the equation $x + \frac{1}{x} = 4$ are α and β , evaluate

i) $(\alpha - \beta)^2$

ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

- iii) $\alpha^3 + \beta^3$
 iv) $2\alpha^2 + 2\beta^2 + 5\alpha\beta$
9. Given that the roots of the equation $ax^2 + bx + c = 0$ are β and $n\beta$, show that $(n+1)^2ac = nb^2$
10. Given that α and β are the roots of the equation $x^2 - px + q = 0$, obtain the quadratic equation whose roots are α^{-2} , β^{-2} ,
ANSWER: $q^2x^2 - (p^2 - 2p)qx + 1 = 0$
11. If the roots of a quadratic equation $x^2 + 2bx + c = 0$ differ by 4, show that $8a = -c \pm \sqrt{(c^2 + 16b^2)}$. Hence find two values of a for which the equation $ax^2 + 2x + 3 = 0$ has roots differing by 4. **ANSWER:** $a = -1, \frac{1}{4}$
12. The roots of the equation $9x^2 + 6x + 1 = 4kx$ where k is a real constant, are denoted as α and β .
 a) Show that the equation whose roots are α^{-1} and β^{-1} is $x^2 + 6x + 9 = 4kx$
 b) Find the set of values of k for which α and β are real.
ANSWER: $k \leq 0, k \geq 3$
 c) Find also the set of values of k for which α and β are real and positive. **ANSWER:** $k \geq 3$
13. If α and β are the roots of the equation $x^2 + bx + c = 0$, express $(\alpha - \beta^2)(\alpha^2 - \beta)$ in terms of b and c . hence or otherwise show that if one root of the equation is the square of the other, then $b^3 + c^3 + c = 3bc$
14. Given that the roots of the equation $x^2 + px + q = 0$ has roots α and β , express p and q in terms of α and β . Find the equation, the coefficients expressed in terms of p and q , whose roots are $3\alpha - \beta$ and $3\beta - \alpha$.
ANSWER: $p = -\alpha - \beta$, $q = \alpha\beta$, $x^2 + 2px + 16q - 3q^2 = 0$
15. Given that the roots of the equation $x^2 + px + q = 0$ are α and β , express $(\alpha - 2\beta)(\beta - 2\alpha)$ in terms of p and q . hence/otherwise show that the condition for one root of the equation to be double the other is $2p^2 = 9q$.
ANSWER: $9q - 2p^2$
16. Given that α and β are the roots of the equation $2x^2 - 7x - 17 = 0$, show that the equation with roots $(\alpha - 4)$, $(\beta - 4)$ is $2x^2 + 9x - 13 = 0$.
17. Given that α and β are the roots of the equation $x^2 - px + q = 0$, prove that $\alpha + \beta = p$, $\alpha\beta = q$. Prove also that;
 a) $\alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n)^2 - 2q^n$ [**Hint:** Let $\alpha^n = A$, $\beta^n = B$]
 b) $\alpha^4 + \beta^4 = p^4 - 4p^2q + 2q^2$
 Hence/otherwise form a quadratic equation whose roots are the fourth powers of those of the equation $x^2 - 3x + 1 = 0$
18. The roots of the equation $x^2 + 6x + c = 0$ differ by $2n$, where n is real and non zero, show that $n^2 = 9 - c$. Given that the roots also have opposite signs, find set of possible values of n .
ANSWER: $n > 3$, $n < -3$
19. If the roots of the equation $x^2 - px + q = 0$ are α and β , find the value of;
 i) $\alpha^2(\alpha^2\beta^{-1} - \beta) + \beta^2(\beta^2\alpha^{-1} - \alpha)$ **ANSWER:** $\frac{p(p^2-4q)(p^2-q)}{q}$
 ii) $(\alpha - p)^{-4} + (\beta - p)^{-4}$ **ANSWER:** $\frac{p^4-4p^2q+2q^2}{q^4}$

20. a) For what values of m will the equation $\frac{x^2-bx}{ax-c} = \frac{m-1}{m+1}$ have roots equal in magnitude but opposite in sign? **ANSWER:** $m = \frac{a-b}{a+b}$
- b) If x is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ cannot lie strictly between 5 and 9
- c) Show that for real x , $-\frac{1}{4} \leq \frac{x}{x^2+4} \leq \frac{1}{4}$
- d) The roots of a quadratic equation $x^2 + px + q = 0$ are α and β . Show that the equation whose roots are $\alpha^2 - q\alpha$ and $\beta^2 - q\beta$ is given by $x^2 - (p^2 + pq - 2q)x + q^2(p + q + 1) = 0$ **Ubeb 2011**

CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE A COMMON ROOT

Let $ax^2 + bx + c = 0$ and

$$lx^2 + mx + n = 0$$

, be the two equations with a common root $x = \alpha \neq 0$

Substituting $x = \alpha$ in to the equations

$$\Rightarrow a\alpha^2 + b\alpha + c = 0 \dots\dots\dots (1)$$

$$\Rightarrow l\alpha^2 + m\alpha + n = 0 \dots\dots\dots (2)$$

Eliminating α^2 from eqn. (1) and eqn. (2)

$$l \times \text{eqn. (1)} - m \times \text{eqn. (2)}, \text{ gives}$$

$$\alpha(lb - am) + lc - an = 0$$

$$\therefore \alpha = \frac{an - lc}{bl - am} \dots\dots\dots (3)$$

Now eliminating constants c and n from eqn. (1) and eqn. (2)

$$n \times \text{eqn. (1)} - c \times \text{eqn. (2)}, \text{ gives}$$

$$\alpha^2(an - lc) + \alpha(bn - mc) = 0$$

$$\Rightarrow \alpha[\alpha(an - lc) + (bn - mc)] = 0$$

Either $\alpha = 0$, but $\alpha \neq 0$

, or $\alpha(an - lc) + (bn - mc)$

$$\Rightarrow \alpha = \frac{mc - bn}{an - lc} \dots\dots\dots (4)$$

Equating equations (3) and (4)

$$\Rightarrow \frac{an - lc}{bl - am} = \frac{mc - bn}{an - lc}$$

$$\therefore (an - lc)^2 = (mc - bn)(bl - am)$$

Illustrative examples

- 1) Given that $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ share a common root $\alpha \neq 0$, prove that $\frac{ps - qr}{q - s} = \frac{q - s}{r - p}$

Solution

$$x^2 + px + q = 0, x = \alpha, \alpha^2 + p\alpha + q = 0 \dots\dots\dots (1)$$

$$x^2 + rx + s = 0, x = \alpha, \alpha^2 + r\alpha + s = 0 \dots\dots\dots (2)$$

Eliminating α^2 from eqn. (1) and eqn. (2)

$$\text{eqn. (1)} - \text{eqn. (2)}, \text{ gives}$$

$$\Rightarrow (p - r)\alpha + q - s = 0$$

$$\therefore \alpha = \frac{s - q}{p - r} = \frac{q - s}{r - p} \dots\dots\dots (3)$$

Now eliminating constants q and s from eqn. (1) and eqn. (2)

$$q \times \text{eqn. (2)} - s \times \text{eqn. (1)}, \text{ gives}$$

$$\Rightarrow \alpha^2(s - q) + \alpha(ps - qr) = 0$$

$$\Rightarrow \alpha[\alpha(s - q) + (ps - qr)] = 0$$

Either $\alpha = 0$, but $\alpha \neq 0$

, or $\alpha(s - q) + (ps - qr)$

$$\Rightarrow \alpha = \frac{qr - ps}{s - q} = \frac{ps - qr}{q - s} \dots\dots\dots (4)$$

Equating equations (3) and (4)

$$\Rightarrow \frac{ps - qr}{q - s} = \frac{q - s}{r - p}, \text{ as required}$$

- 2) Given that $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then
 $a^3 + b^3 + c^3 = 3abc$

Solution

Let $x = \alpha \neq 0$ be the common root.

$$\Rightarrow a\alpha^2 + b\alpha + c = 0 \dots\dots\dots (1)$$

$$\Rightarrow b\alpha^2 + c\alpha + a = 0 \dots\dots\dots (2)$$

Eliminating α^2 from eqn. (1) and eqn. (2)

$b \times \text{eqn. (1)} - a \times \text{eqn. (2)}$, gives

$$\Rightarrow (b^2 - ac)\alpha + bc - a^2 = 0$$

$$\therefore \alpha = \frac{a^2 - bc}{b^2 - ac} \dots\dots\dots (3)$$

Now eliminating constants a and c from eqn. (1) and eqn. (2)

$a \times \text{eqn. (1)} - c \times \text{eqn. (2)}$, gives

$$\Rightarrow \alpha^2(a^2 - bc) + \alpha(ab - c^2) = 0$$

$$\Rightarrow \alpha[a(a^2 - bc) + (ab - c^2)] = 0$$

Either $\alpha = 0$, but $\alpha \neq 0$

, or $\alpha(a^2 - bc) + (ab - c^2) = 0$

$$\Rightarrow \alpha = \frac{c^2 - ab}{a^2 - bc} \dots\dots\dots (4)$$

Equating equations (3) and (4)

$$\Rightarrow \frac{a^2 - bc}{b^2 - ac} = \frac{c^2 - ab}{a^2 - bc}$$

$$\Rightarrow (a^2 - bc)^2 = (b^2 - ac)(c^2 - ab)$$

$$\Rightarrow a^4 - 2a^2bc + b^2c^2 = b^2c^2 - ac^3 - ab^3 + a^2bc$$

$$\Rightarrow a^4 - 2a^2bc = -ac^3 - ab^3 + a^2bc \dots\dots\dots (\div a)$$

$$\Rightarrow a^3 - 2abc = -c^3 - b^3 + abc$$

$$\therefore a^3 + b^3 + c^3 = 3abc$$

- 3) If $x^2 - px + q = 0$ and $x^2 - qx + p = 0$ have a common root, prove that
 $p + q + 1 = 0$

Solution

Let $x = \alpha \neq 0$ be the common root.

$$\Rightarrow \alpha^2 - p\alpha + q = 0 \dots\dots\dots (1)$$

$$\Rightarrow \alpha^2 - q\alpha + p = 0 \dots\dots\dots (2)$$

Eliminating α^2 from eqn. (1) and eqn. (2)

$\text{eqn. (1)} - \text{eqn. (2)}$, gives

$$(-p + q)\alpha + q - p = 0$$

$$\Rightarrow \alpha = -\frac{q - p}{q - p} = -1$$

$$\therefore \alpha = -1 \dots\dots\dots (3)$$

eqn. (3) in to eqn. (1) or (2)

$$\Rightarrow (-1)^2 - p(-1) + q = 0$$

$$\therefore 1 + p + q = 0. \text{ As required}$$

- 4) Find p such that $x^2 - 5x + 6 = 0$ and $x^2 + px - 10 = 0$ have a common root.

Solution

Let $x = \alpha \neq 0$ be the common root.

$$\Rightarrow \alpha^2 - 5\alpha + 6 = 0 \dots\dots\dots (1)$$

$$\Rightarrow \alpha^2 + p\alpha - 10 = 0 \dots\dots\dots (2)$$

Eliminating α^2 from eqn. (1) and eqn. (2)

eqn. (1) - eqn. (2), gives

$$(-5 - p)\alpha + 16 = 0$$

$$\Rightarrow \alpha = \frac{-16}{-5-p} = \frac{16}{p+5} \dots\dots\dots (3)$$

eqn. (3) in to (1)

$$\Rightarrow \left(\frac{16}{p+5}\right)^2 - 5\left(\frac{16}{p+5}\right) + 6 = 0$$

$$\Rightarrow 256 - 80(p+5) + 6(p+5)^2 = 0$$

$$\Rightarrow 256 - 80p - 400 + 6p^2 + 60p + 150 = 0$$

$$\Rightarrow 6 - 20p + 6p^2 = 0 \dots\dots\dots (\div 2)$$

$$\therefore 3p^2 - 10p + 3 = 0$$

$$\Rightarrow 3p^2 - 9p - p + 3 = 0$$

$$\Rightarrow 3p(p-3) - (p-3) = 0$$

$$\Rightarrow (3p-1)(p-3) = 0$$

$$\therefore p = 3, \frac{1}{3}$$

Alternatively

Eliminating constants 6 and -10 from eqn. (1) and eqn. (2)

$10 \times$ eqn. (1) + $6 \times$ eqn. (2), gives

$$16\alpha^2 + (6p-50)\alpha = 0$$

$$\Rightarrow \alpha = \frac{50-6p}{16} = \frac{25-3p}{8} \dots\dots\dots (4)$$

Equating equations (3) and (4)

$$\Rightarrow \frac{16}{p+5} = \frac{25-3p}{8}, \text{ simplifying the equation gives}$$

$$3p^2 - 10p + 3 = 0, \text{ solving the equation}$$

$$\therefore p = 3, \frac{1}{3}$$

- 5) Prove that if the equations $x^2 + ax + b = 0$ and $cx^2 + 2ax - 3b = 0$ have a common root and neither a nor b is zero, then $b = \frac{5a^2(c-2)}{(c+3)^2}$

Solution

Let $x = \alpha \neq 0$ be the common root.

$$\Rightarrow \alpha^2 + a\alpha + b = 0 \dots\dots\dots (1)$$

$$\Rightarrow c\alpha^2 + 2a\alpha - 3b = 0 \dots\dots\dots (2)$$

Eliminating α^2 from eqn. (1) and eqn. (2)

$c \times$ eqn. (1) - eqn. (2), gives

$$\alpha(ac - 2a) + bc + 3b = 0$$

$$\therefore \alpha = -\frac{(bc+3b)}{(ac-2a)} = \frac{-b(c+3)}{a(c-2)} \dots\dots\dots (3)$$

Eliminating constants $b, -3b$ from eqn. (1) and eqn. (2)

$3 \times$ eqn. (1) + eqn. (2), gives

$$\alpha^2(3+c) + 5a\alpha = 0$$

$$\Rightarrow \alpha[\alpha(3+c) + 5a] = 0$$

$$\therefore \alpha = -\frac{5a}{c+3} \dots\dots\dots (4)$$

Equating equations (3) and (4)

$$\begin{aligned}\Rightarrow \frac{-b(c+3)}{a(c-2)} &= -\frac{5a}{c+3} \\ \Rightarrow b(c+3)^2 &= 5a^2(c-2) \\ \therefore b &= \frac{5a^2(c-2)}{(c+3)^2}, \text{ as required.}\end{aligned}$$

Task

1. Show that if the equations $x^2 + px + q = 0$ and $x^2 + 2Px + Q = 0$ have a common root then $(q - Q)^2 + 4(P - p)(Pp - pQ) = 0$
2. If the equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root, then $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (a_1c_2 - a_2c_1)^2$
3. Show that if the equations $x^2 + bx + c = 0$, $x^2 + px + q = 0$ have a common root, then $(c - q)^2 = (b - p)(cp - bq)$
4. Show that if the equations $x^2 + ax + 1 = 0$ and $x^2 + x + b = 0$ have a common root, then $(b - 1)^2 = (a - 1)(1 - ab)$

COMMON FACTORS

If two functions $f(x)$ and $g(x)$ have a common factor $(x - a)$, then

$$f(x) = (x - a).F(x) \dots\dots\dots (1)$$

$$g(x) = (x - a).G(x) \dots\dots\dots (2)$$

For any constant k ,

$$f(x) + kg(x) = (x - a).F(x) + k(x - a).G(x)$$

$$f(x) + kg(x) = (x - a)[F(x) + kG(x)]$$

Thus for any constant k , $(x - a)$ is a common factor of $f(x) + kg(x)$

NOTE:

Equations (1) and (2) are identities. Thus ,

$$f(a) = 0, \quad g(a) = 0$$

, which gives a pair of simultaneous

equations which provides another approach to problems involving a common factor.

Examples

1. Find constants p and q such that $(x - 2)$ is a common factor of $x^3 - x^2 - 2px + 3q$ and $qx^3 - px^2 + x + 2$

Solution

$$\text{Let } f(x) = x^3 - x^2 - 2px + 3q$$

$$g(x) = qx^3 - px^2 + x + 2$$

Since $(x - 2)$ is a factor to both $f(x)$ and $g(x)$, then $f(2) = 0$ $g(2) = 0$

$$\Rightarrow 8 - 4 - 4p + 3q = 0 \quad \text{or} \quad -4p + 3q = -4 \dots\dots\dots (1)$$

$$\Rightarrow 8q - 4p + 2 + 2 = 0 \quad \text{or} \quad -4p + 8q = -4 \dots\dots\dots (2)$$

Solving eqn. (1) and eqn. (2) simultaneously,

$$\text{eqn. (1)} - \text{eqn. (2)}$$

$$\Rightarrow -5q = 0 \therefore q = 0$$

From eqn. (1), for $q = 0$, $p = 1$

$$\therefore p = 1, q = 0$$

2. Find constants p and q such that $(x - 1)$ is a common factor of $x^4 + x^2 - q$ and $x^4 - 2px^2 + 2$

Solution

$$\text{Let } f(x) = x^4 + x^2 - q, f(1) = 1 + 1 - q = 0 \therefore q = 2$$

$$g(x) = qx^3 - px^2 + x + 2, g(1) = 1 - 2p + 2 = 0 \therefore p = \frac{3}{2}$$

$$\therefore p = \frac{3}{2}, q = 2$$

3. Show that if $(x + 1)$ is a common factor of $x^3 - ax^2 + b$ and $x^4 - ax^3 + bx^2 + c$, then $2a = 2b - 2 = -2 - c$.

Solution

$$\text{Let } f(x) = x^3 - ax^2 + b$$

$$g(x) = x^4 - ax^3 + bx^2 + c$$

For any constant k , $(x + 1)$ is a factor of $f(x) + kg(x)$

$$\Rightarrow f(-1) + kg(-1) = 0, \text{ for any value of } k.$$

$$\therefore -1 - a - b + k(1 + a + b + c) = 0$$

$$\text{For } k = 0, a + b = -1 \dots\dots\dots (1)$$

$$\text{For } k = 1, 2b + c = 0 \dots\dots\dots (2)$$

From eqn. (1), multiplying through by 2

$$\Rightarrow 2a + 2b = -2$$

$$\text{From eqn. (2), } 2b + c = 0 \text{ or } b = -\frac{c}{2} \dots\dots\dots (3)$$

Eqn. (3) in to eqn. (1)

$$a + \frac{c}{2} = -1 \text{ or } 2a + c = -2 \text{ or } 2a = -2 - c$$

$$\therefore 2a = 2b - 2 = -2 - c$$

4. Find constant a for which the function $f(x) = ax^2 + 2x - 1$ and $g(x) = x^2 + 4x + a$ have a common factor.

Solution

Let $x - \alpha$ be a common factor

$$\Rightarrow f(x) + kg(x) \text{ has a factor of } x - \alpha$$

$$\Rightarrow f(\alpha) + kg(\alpha) = 0$$

$$\Rightarrow a\alpha^2 + 2\alpha - 1 + k(\alpha^2 + 4\alpha + a) = 0$$

For $k = -a$, to eliminate α^2

$$\Rightarrow a\alpha^2 + 2\alpha - 1 - a\alpha^2 - 4a\alpha - a^2 = 0$$

$$\Rightarrow \alpha(2 - 4a) - 1 - a^2 = 0$$

$$\therefore \alpha = \frac{a^2 + 1}{2 - 4a} \dots\dots\dots (1)$$

For $k = \frac{1}{a}$, to eliminate constants

$$\Rightarrow a\alpha^2 + 2\alpha - 1 + \frac{\alpha^2}{a} + \frac{4\alpha}{a} + 1 = 0$$

$$\Rightarrow \alpha^2 \left(a + \frac{1}{a} \right) + \alpha \left(2 + \frac{4}{a} \right) = 0$$

$$\Rightarrow \alpha \left[\alpha \left(a + \frac{1}{a} \right) + \left(2 + \frac{4}{a} \right) \right] = 0$$

$$\Rightarrow \alpha \left(a + \frac{1}{a} \right) + \left(2 + \frac{4}{a} \right) = 0$$

$$\therefore \alpha = -\frac{\left(2 + \frac{4}{a} \right)}{\left(a + \frac{1}{a} \right)} = \frac{-(2a + 4)}{a^2 + 1} \dots\dots\dots (2)$$

Equating equations (1) and (2)

$$\Rightarrow \frac{a^2 + 1}{2 - 4a} = \frac{-(2a + 4)}{a^2 + 1}$$

$$\Rightarrow a^4 + 2a^2 + 1 = -(2a + 4)(2 - 4a)$$

$$\Rightarrow a^4 + 2a^2 + 1 = -(4a - 8a^2 + 8 - 16a) = 12a + 8a^2 - 8$$

$$\therefore a^4 - 6a^2 - 12a + 9 = 0$$

$$\text{Let } h(a) = a^4 - 6a^2 - 12a + 9$$

For $a = 3$, $h(3) = 0 \therefore a - 3$ is a factor of $h(a)$

$$\begin{array}{r}
 a^3 + 3a^2 + 3a - 3 \\
 \underline{a - 3} \quad \begin{array}{l} a^4 + 0a^3 - 6a^2 - 12a + 9 \\ a^4 - 3a^3 \end{array} \\
 \hline
 3a^3 - 6a^2 - 12a + 9 \\
 \underline{3a^3 - 9a^2} \\
 3a^2 - 12a + 9 \\
 \underline{3a^2 - 9a} \\
 3a + 9 \\
 \underline{3a + 9} \\
 0
 \end{array}$$

$$\therefore h(a) = (a - 3)(a^3 + 3a^2 + 3a - 3) = 0$$

$$\Rightarrow a = 3, 0.26$$

$$\therefore a = 3$$

5. Determine the value of m such that the equations $x^3 + mx - 1 = 0$ and $x^3 - 3x + m = 0$ have a common root.

Solution

Let common root be $x = \alpha \neq 0 \Rightarrow (x - \alpha)$ is a common factor.

$$\text{Let } f(x) = x^3 + mx - 1, g(x) = x^3 - 3x + m$$

$\Rightarrow f(x) + kg(x)$, has a common factor $(x - \alpha)$

$$\Rightarrow f(\alpha) + kg(\alpha) = 0$$

$$\Rightarrow \alpha^3 + m\alpha - 1 + k(\alpha^3 - 3\alpha + m) = 0$$

For $k = -1$ to eliminate α^3

$$\Rightarrow \alpha^3 + m\alpha - 1 - \alpha^3 + 3\alpha - m = 0$$

$$\Rightarrow \alpha(m + 3) = m + 1 \therefore \alpha = \frac{m+1}{m+3}$$

Now from the equation $x^3 + mx - 1 = 0$ with $x = \alpha$ as a root, then $\alpha^3 + m\alpha - 1 = 0$ must have the root $\alpha = \frac{m+1}{m+3}$

$$\Rightarrow \left(\frac{m+1}{m+3}\right)^3 + m\left(\frac{m+1}{m+3}\right) - 1 = 0 \dots\dots\dots \times (m+3)^3$$

$$\Rightarrow (m+1)^3 + (m^2 + m)(m+3)^2 - (m+3)^3 = 0$$

$$\Rightarrow m^3 + 3m^2 + 3m + 1 + (m^2 + m)(m^2 + 6m + 9) - (m^3 + 9m^2 + 27m + 27) = 0$$

$$\Rightarrow m^3 + 3m^2 + 3m + 1 + m^4 + 6m^3 + 9m^2 + m^3 + 6m^2 + 9m - m^3 - 9m^2 - 27m - 27 = 0$$

$$\Rightarrow m^4 + 7m^3 + 9m^2 - 15m - 26 = 0$$

Solving by factor theorem,

$$\text{Let } f(m) = m^4 + 7m^3 + 9m^2 - 15m - 26$$

$$\text{For } m = -2, f(-2) = (-2)^4 + 7(-2)^3 + 9(-2)^2 - 15(-2) - 26 = 0$$

$\Rightarrow m + 2$, is a factor of $f(m)$

$$\begin{array}{r}
 m^3 + 5m^2 - m - 13 \\
 \underline{m + 2} \quad \begin{array}{l} m^4 + 7m^3 + 9m^2 - 15m - 26 \\ m^4 + 2m^3 \end{array} \\
 \hline
 5m^3 + 9m^2 - 15m - 26 \\
 \underline{5m^3 + 10m^2} \\
 -m^2 - 15m - 26 \\
 \underline{-m^2 - 2m} \\
 -13m - 26 \\
 \underline{-13m - 26} \\
 0
 \end{array}$$

$$\therefore (m+2)(m^3+5m^2-m-13)=0$$

$$\therefore m = -2, 1.49, -4.60, -1.89$$

6. Show that If the cubic functions $x^3 + ax^2 + b$ and $ax^3 + bx^2 + x - a$ have a common factor, it is also a common factor of the quadratic function $(b - a^2)x^2 + x - a(1 + b) = 0$

Solution

Let the common factor be $(x - \alpha)$

Let $f(x) = x^3 + ax^2 + b$, $g(x) = ax^3 + bx^2 + x - a$

$\Rightarrow f(x) + kg(x)$, has a common factor $(x - \alpha)$

$$\Rightarrow f(\alpha) + kg(\alpha) = 0$$

$$\Rightarrow \alpha^3 + a\alpha^2 + b + k(a\alpha^3 + b\alpha^2 + \alpha - a) = 0$$

For $k = -\frac{1}{a}$ to eliminate α^3

$$\Rightarrow \alpha^3 + a\alpha^2 + b - \frac{1}{a}(a\alpha^3 + b\alpha^2 + \alpha - a) = 0$$

$$\Rightarrow a\alpha^3 + a^2\alpha^2 + ab - a\alpha^3 - b\alpha^2 - \alpha + a = 0$$

$$\Rightarrow \alpha^2(a^2 - b) - \alpha + a(1 + b) = 0$$

$\therefore (b - a^2)\alpha^2 + \alpha - a(1 + b) = 0$. this equation is as a result of $x = \alpha$ being a true root of the quadratic equation $(b - a^2)x^2 + x - a(1 + b) = 0$. Hence the common factor of the above two equations is also a common factor of the quadratic equation.

Task

Show that if $(x + a)$ is a common factor of $x^3 + px^2 + q$ and $ax^3 + bx + c$, then it is also a factor of $apx^2 - bx + aq - c$.

REPEATED FACTORS

Let $(x - a)$ be a repeated factor of $f(x)$

$$\Rightarrow f(x) = (x - a)^2 \cdot g(x) \text{ .i.e. repeated twice}$$

Differentiating with respect to x

$$f'(x) = g(x) \cdot 2(x - a) + (x - a)^2 \cdot g'(x)$$

Thus if $(x - a)$ is a repeated factor of $f(x)$, then it is also a factor of $f'(x)$

NOTE:

$(x - a)$ is a repeated factor of $f(x)$ if $f'(a) = 0$

Examples

1. Determine whether $f(x) \equiv 3x^4 - 8x^3 - 6x^2 + 24x - 13$ has any repeated factors, and if so find them.

Solution

$$f'(x) \equiv 12x^3 - 24x^2 - 12x + 24$$

$$\equiv 12(x^3 - 2x^2 - x + 2)$$

$$\equiv 12[x^2(x - 2) - (x - 2)]$$

$$\equiv 12(x - 2)(x^2 - 1) \equiv 12(x - 1)(x + 1)(x - 2)$$

Now $f'(x) = 0$ if $x = \pm 1, 2$

Checking the value of $f(x)$ for these values of x

$$\Rightarrow f(1) = 0, f(-1) \neq 0, f(2) \neq 0$$

So $(x + 1)$ and $(x - 2)$ are not factors of $f(x)$

Hence $(x - 1)$ is the only repeated factor of $f(x)$

2. If the equation $3x^4 + 2x^3 - 6x^2 - 6x + p = 0$ has equal roots, find the possible values of p .

Solution

$$\text{Let } f(x) = 3x^4 + 2x^3 - 6x^2 - 6x + p$$

The equation $f(x) = 0$ has equal roots if $f(x)$ has repeated factors.

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$

$$\begin{aligned}\Rightarrow f'(x) &\equiv 12x^3 + 6x^2 - 12x - 6 \\ &\equiv 6(2x^3 + x^2 - 2x - 1) \\ &\equiv 6[x^2(2x + 1) - (2x + 1)] \\ &\equiv 6(2x + 1)(x^2 - 1) \\ &\equiv 6(x - 1)(x + 1)(2x + 1)\end{aligned}$$

If $(x - 1)$ is a repeated factor of $f(x)$, then $f(1) = 0$

$$\Rightarrow 3 + 2 - 6 - 6 + p = 0 \therefore p = 7$$

Similarly $(x + 1)$ is a repeated factor of $f(x)$, then $f(-1) = 0$

$$\Rightarrow 3 - 2 - 6 + 6 + p = 0 \therefore p = -1$$

Also $(2x + 1)$ is a repeated factor of $f(x)$, then $f\left(-\frac{1}{2}\right) = 0$

$$\Rightarrow 3\left(-\frac{1}{2}\right)^4 + 2\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) + p = 0 \therefore p = -\frac{23}{16}$$

3. If the equation $2x^3 - 9x^2 + 12x + p = 0$ has two equal roots, find the possible values of p .

Solution

$$\text{Let } f(x) = 2x^3 - 9x^2 + 12x + p = 0$$

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$

$$\begin{aligned}\Rightarrow f'(x) &\equiv 6x^2 - 18x + 12 \\ &\equiv 6(x^2 - 3x + 2) \\ &\equiv 6(x - 2)(x - 1)\end{aligned}$$

If $(x - 2)$ is a repeated factor of $f(x)$, then $f(2) = 0$

$$\Rightarrow 16 - 36 + 24 + p = 0 \therefore p = -4$$

Similarly $(x - 1)$ is a repeated factor of $f(x)$, then $f(1) = 0$

$$\Rightarrow 2 - 9 + 12 + p = 0 \therefore p = -5$$

$$\therefore p = -4, -5$$

Task

- Find the possible values of a for which the equation $2x^3 - 3x^2 - 12x + a = 0$ has repeated roots. **ANSWER:** $-7, 20$
- Find the value of a for which the function $2x^3 - ax^2 - 12x - 7$ has a repeated factor. **ANSWER:** 3

MISCELLANEOUS EQUATIONS

Special quadratic equations

Some equations do not appear quadratic but they can be transformed in to quadratic equations using a *suitable substitution*.

TYPE I: Equation of the form $\frac{u}{c} + \frac{a}{u} = b$ where a and b are constants and u is a suitable substitution.

Example

Solve the equations:

- a) $x^2 + \frac{9}{x^2} = 10$
- b) $x^{4/3} + 16x^{-4/3} = 17$
- c) $x^2 + 2x = 34 + \frac{35}{x^2+2x}$
- d) $y(y+1) + \frac{12}{y(y+1)} = 8$
- e) $9x^{2/3} + 16x^{-2/3} = 37$
- f) $x^2 + 2x + \frac{12}{x^2+2x} = 7$

Solution

- a) $x^2 + \frac{9}{x^2} = 10$, let $x^2 = u$
 $\Rightarrow u + \frac{9}{u} = 10$ (Multiplying both sides by u and re-arranging)
 $\Rightarrow u^2 - 10u + 9 = 0$ or $(u-9)(u-1) = 0$
 $\therefore u = 9, 1$
 But $x^2 = u \Rightarrow x^2 = 9, x^2 = 1$
 $\therefore x = \pm 1, \pm 3$

Note:

The values of x are *four* because the equation is of degree 4. i.e. 4^0

- b) $x^{4/3} + 16x^{-4/3} = 17$ or $x^{4/3} + \frac{16}{x^{4/3}} = 17$, let $x^{4/3} = u$
 $\Rightarrow u + \frac{16}{u} = 17$ or $u^2 - 17u + 16 = 0$
 $\Rightarrow (u-16)(u-1) = 0 \therefore u = 16, 1$
 But $x^{4/3} = u \Rightarrow x^{4/3} = 16, x^{4/3} = 1$
 Multiplying the powers by $\frac{3}{4}$
 $x = 16^{3/4} = 8; x = (1)^{3/4} = 1$
 $\therefore x = 1, 8$
- c) $x^2 + 2x = 34 + \frac{35}{x^2+2x}$, let $x^2 + 2x = u$
 $\Rightarrow u = 34 + \frac{35}{u}$ or $u^2 - 34u - 35 = 0$
 $\Rightarrow (u-35)(u+1) = 0 \therefore u = 35, -1$
 But $x^2 + 2x = u$,
 $\Rightarrow x^2 + 2x = 35$ or $x^2 + 2x - 35 = 0$
 $\Rightarrow (x+7)(x-5) = 0 \therefore x = 5, -7$
 Also $x^2 + 2x = -1$ or $x^2 + 2x + 1 = 0$
 $\Rightarrow (x+1)^2 = 0 \therefore x = -1, -1$
 $\therefore x = 5, -1, -1, -7$
- d) $y(y+1) + \frac{12}{y(y+1)} = 8$, let $y(y+1) = u$
 $\Rightarrow u + \frac{12}{u} = 8$ or $u^2 - 8u + 12 = 0$
 $\Rightarrow (u-6)(u-2) = 0 \therefore u = 6, 2$

$$\Rightarrow y(y+1) = 2 \text{ or } y^2 + y - 2 = 0$$

$$\Rightarrow (y+2)(y-1) = 0 \therefore y = 1, -2$$

$$\text{Also } y(y+1) = 6 \text{ or } y^2 + y - 6 = 0$$

$$\Rightarrow (y+3)(y-2) = 0 \therefore y = 2, -3$$

$$\therefore y = 1, \pm 2, -3$$

$$\text{e) } 9x^{2/3} + 16x^{-2/3} = 37 \text{ or } 9x^{2/3} + \frac{16}{x^{2/3}} = 37, \text{ let } x^{2/3} = u$$

$$\Rightarrow 9u + \frac{16}{u} = 37 \text{ or } 9u^2 - 37u + 16 = 0$$

$$\Rightarrow 9u^2 - 36u - u + 16 = 0$$

$$\Rightarrow 9u(u-4) - (u-4) = 0$$

$$\Rightarrow (9u-1)(u-4) = 0 \therefore u = \frac{1}{9}, 4$$

$$\text{Now } x^{2/3} = u$$

$$\Rightarrow x^{2/3} = \frac{1}{9} \text{ or } x = \left(\frac{1}{9}\right)^{3/2} = \frac{1}{27}$$

$$\Rightarrow x^{2/3} = 4 \text{ or } x = (4)^{3/2} = 8$$

$$\therefore x = 8, \frac{1}{27}$$

$$\text{f) Left as an exercise.}$$

$$\text{ANSWER: } \pm 1 - 3, \pm\sqrt{5}$$

Task

Solve the equations:

$$\text{a) } \sqrt{3x} + \frac{3}{\sqrt{3x}} = 4 \text{ ANSWER: } 0, 2$$

$$\text{b) } (x^2 - 2x)^2 + 24 = 11(x^2 - 2x) \text{ ANSWER: } -2, -1, 3, 4$$

Note: the equation can be expressed in the form $x^2 - 2x + \frac{24}{x^2 - 2x} = 11$

$$\text{c) } \frac{x^2 + 4x}{3} + \frac{84}{x^2 + 4x} = 11 \text{ ANSWER: } -7, -6, 2, 3$$

$$\text{d) } (3x^2 + 2x)^2 + 8 = 9(3x^2 + 2x) \text{ ANSWER: } -2, -1, \frac{1}{3}, \frac{4}{3}$$

TYPE II: Indicial equations reducible to quadratic equation . i. e. $ax^2 + bx + c = 0$

Example

1. Solve the equations:

$$\text{i) } 3^{2x} + 9 = 10 \cdot 3^x$$

$$\text{ii) } 4^x - 3 \cdot 2^{x+3} = -128$$

$$\text{iii) } 5(5^x + 5^{-x}) = 26$$

$$\text{iv) } 5^x + 5^{2-x} = 26$$

$$\text{v) } 2 - 5e^{-x} + 2e^{-2x} = 0$$

$$\text{vi) } 3^{2(x+1)} - 10 \times 3^x + 1 = 0$$

Solution

$$\text{i) } 3^{2x} + 9 = 10 \cdot 3^x$$

$$\Rightarrow (3^x)^2 + 9 = 10 \times 3^x, \text{ let } 3^x = u$$

$$\Rightarrow u^2 + 9 = 10u \text{ or } u^2 - 10u + 9 = 0$$

$$\Rightarrow (u-9)(u-1) = 0 \therefore u = 9, 1$$

$$\text{But } 3^x = u$$

$$\Rightarrow 3^x = 9 = 3^2 \therefore x = 2$$

$$\Rightarrow 3^x = 1 = 3^0 \therefore x = 0$$

$$\therefore x = 2, 0$$

- ii) $4^x - 3 \cdot 2^{x+3} = -128$
 $\Rightarrow (2^x)^2 - 3 \times 2^x \times 2^3 = -128$ or $(2^x)^2 - 24 \times 2^x = -128$, let $2^x = u$
 $\Rightarrow u^2 - 24u = -128$ or $u^2 - 24u + 128 = 0$
 $\Rightarrow u = \frac{24 \pm \sqrt{24^2 - 4 \times 128}}{2} = \frac{24 \pm 8}{2}$
 $\therefore u = 16, 8$
 But $2^x = u$
 $\Rightarrow 2^x = 16 = 2^4 \therefore x = 4$
 $\Rightarrow 2^x = 8 = 2^3 \therefore x = 3$
 $\therefore x = 3, 4$
- iii) $5(5^x + 5^{-x}) = 26$ or $5\left(5^x + \frac{1}{5^x}\right) = 26$, let $5^x = u$
 $\Rightarrow 5\left(u + \frac{1}{u}\right) = 26$ or $5u^2 - 26u + 5 = 0$
 $\Rightarrow 5u^2 - 25u - u + 5 = 0$
 $\Rightarrow (5u - 1)(u - 5) = 0 \therefore u = 5, \frac{1}{5}$
 Now $5^x = u$
 $\Rightarrow 5^x = 5 \therefore x = 1$
 $\Rightarrow 5^x = \frac{1}{5} = 5^{-1} \therefore x = -1$
 $\therefore x = \pm 1$
- v) $2 - 5e^{-x} + 2e^{-2x} = 0$ or $2 - 5e^{-x} + 2(e^{-x})^2 = 0$, let $e^{-x} = u$
 $\Rightarrow 2 - 5u + 2u^2 = 0$
 $\Rightarrow (2u - 1)(u - 2) = 0 \therefore u = 2, \frac{1}{2}$
 Now $e^{-x} = u$
 $\Rightarrow e^{-x} = 2$, introducing logarithm to base e
 $\Rightarrow \ln e^{-x} = \ln 2 \therefore x = -\ln 2$
 Also $e^{-x} = \frac{1}{2}$
 $\Rightarrow \ln e^{-x} = \ln\left(\frac{1}{2}\right) \therefore x = +\ln 2$
 $\therefore x = \pm \ln 2$
- vi) $3^{2(x+1)} - 10 \times 3^x + 1 = 0$ or $3^{2x} \cdot 3^2 - 10 \times 3^x + 1 = 0$, let $3^x = u$
 $\Rightarrow 9u^2 - 10u + 1 = 0$
 $\Rightarrow (9u - 1)(u - 1) = 0 \therefore u = 1, \frac{1}{9}$
 But $3^x = u$
 $\Rightarrow 3^x = 1 = 3^0 \therefore x = 0$
 $\Rightarrow 3^x = \frac{1}{9} = 3^{-2} \therefore x = -2$
 $\therefore x = 0, -2$

Task

- Solve: i) $4^x - 5 \cdot 2^x + 4 = 0$ **ANSWER:** 0, 2
 ii) $3^{2x+1} - 3^{x+1} - 3^x + 1 = 0$ **ANSWER:** -1, 0
 iii) $5^{2x} - 5^{x+1} + 4 = 0$ **ANSWER:** 0, 0.861
 iv) $2^{2x+8} - 32(2^x) + 1 = 0$ **ANSWER:** -4
 v) $2^{2x+3} - 57 = 65(2^x - 1) = 0$ **ANSWER:** ± 3

2. Solve the equation

- a) $t - 1.324\sqrt{t} - 2.896 = 0$
 b) $x + \sqrt{x} = \frac{6}{25}$

Solution

a) $t - 1.324\sqrt{t} - 2.896 = 0$, let $\sqrt{t} = u \Rightarrow u^2 = t$

$$\therefore u^2 - 1.324u - 2.896 = 0$$

$$\therefore u = \frac{1.324 \pm \sqrt{(-1.324)^2 + 4 \times 2.896}}{2} = \frac{1.324 \pm 3.652}{2}$$

$$\therefore u = 2.488, -1.164$$

$$\Rightarrow t = (2.488)^2 = 6.19; (-1.164)^2 = 1.35$$

$$\therefore t = 6.19, 1.35$$

b) $x + \sqrt{x} = \frac{6}{25}$, let $\sqrt{x} = u \Rightarrow x = u^2$

$$\Rightarrow u^2 + u = \frac{6}{25} \text{ or } 25u^2 + 25u - 6 = 0$$

$$\Rightarrow u = \frac{-25 \pm \sqrt{25^2 + 4 \times 6 \times 25}}{2 \times 25} = \frac{-25 \pm 5}{50}$$

$$\therefore u = -\frac{2}{5}, -\frac{3}{5}$$

$$\therefore x = \frac{4}{25}, \frac{9}{25}$$

Task

Solve the equations:

a) $6\sqrt{x} = 5x^{\frac{1}{2}} - 13$; **ANSWER:** $\frac{1}{9}, \frac{25}{4}$

b) $\sqrt{2x} + \frac{1}{\sqrt{2x}} = 2$; **ANSWER:** 0

TYPE III: Equation of the form $Ax^4 + Bx^2 + C = 0$. This equation contains only 3 - terms

Solve the equation:

i) $x^4 - 13x^2 + 36 = 0$

ii) $2x^4 - x^2 = 6$

Solution

i) $x^4 - 13x^2 + 36 = 0$, let $x^2 = u$

$$\Rightarrow u^2 - 13u + 36 = 0 \text{ or } (u - 9)(u - 4) = 0$$

$$\therefore u = 9, 4$$

But $x^2 = u$

$$\Rightarrow x^2 = 9 \therefore x = \pm 3$$

$$\Rightarrow x^2 = 4 \therefore x = \pm 2$$

$$\therefore x = \pm 2, \pm 3$$

ii) $2x^4 - x^2 = 6$, let $x^2 = u$

$$\Rightarrow 2u^2 - u - 6 = 0 \text{ or } (2u + 3)(u - 2) = 0$$

$$\therefore u = -\frac{3}{2}, 2$$

But $x^2 = u$

$$\Rightarrow x^2 = 2 \therefore x = \pm\sqrt{2}$$

$$\Rightarrow x^2 = -\frac{3}{2} \therefore x = \sqrt{\left(-\frac{3}{2}\right)}, \text{ no real values of } x$$

$$\therefore x = \pm\sqrt{2}$$

TYPE IV: Equation of the form $(x + a)(x + b)(x + c)(x + d) = n$, where $a + b = c + d$ and a, b, c, d and n are constants

Example:

Solve the equations:

- i) $(x + 1)(x + 3)(x + 4)(x + 6) = 72$
- ii) $(x - 7)(x - 3)(x + 5)(x + 1) = 1680$
- iii) $(x - 5)(x - 7)(x + 6)(x + 4) = 504$

Solution

- i) $(x + 1)(x + 3)(x + 4)(x + 6) = 72$
 Since $1 + 6 = 3 + 4 = 7$, then re-grouping
 $\Rightarrow (x + 1)(x + 6) \cdot (x + 3)(x + 4) = 72$
 $\Rightarrow (x^2 + 7x + 6) \cdot (x^2 + 7x + 12) = 72$
 Now let $x^2 + 7x = u$
 $\Rightarrow (u + 6)(u + 12) = 72$
 $\Rightarrow u^2 + 18u + 72 = 72$ or $u(u + 18) = 0$
 $\therefore u = 0, -18$
 But $x^2 + 7x = u$
 $\Rightarrow x^2 + 7x = 0 \therefore x = 0, -7$
 $\Rightarrow x^2 + 7x = -18$ or $x^2 + 7x + 18 = 0$
 $\therefore x = \frac{-7 \pm \sqrt{7^2 - 4 \times 18}}{2} = \frac{-7 \pm \sqrt{-23}}{2} = \frac{-7 \pm i\sqrt{23}}{2}$
 $\therefore x = 0, -7, \frac{-7 \pm i\sqrt{23}}{2}$
- ii) $(x - 7)(x - 3)(x + 5)(x + 1) = 1680$
 Since $-7 + 5 = -3 + 1 = -2$, then re-grouping
 $\Rightarrow (x - 7)(x + 5) \cdot (x - 3)(x + 1) = 1680$
 $\Rightarrow (x^2 - 2x - 35) \cdot (x^2 - 2x - 3) = 1680$
 Now let $x^2 - 2x = u$
 $\Rightarrow (u - 35)(u - 3) = 1680$
 $\Rightarrow u^2 - 38u + 105 = 1680$ or $u^2 - 38u - 1575 = 0$
 $\Rightarrow u = \frac{38 \pm \sqrt{38^2 + 4 \times 1575}}{2} = \frac{38 \pm 88}{2}$
 $\therefore u = 63, -25$
 But $x^2 - 2x = u$
 $\Rightarrow x^2 - 2x = 63$ or $x^2 - 2x - 63 = 0$
 $\Rightarrow (x - 9)(x + 7) = 0$
 $\therefore x = 9, -7$
 $\Rightarrow x^2 - 2x = -25$ or $x^2 - 2x + 25 = 0$
 $\therefore x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 25}}{2} = \frac{2 \pm \sqrt{-96}}{2} = \frac{2 \pm i\sqrt{96}}{2} = \frac{2 \pm i4\sqrt{6}}{2}$
 $\therefore x = 9, -7, 1 \pm i2\sqrt{6}$
- iii) Left as an exercise.

ANSWER: $3, -2, 8, -7$

Task

Solve the equations:

- i) $x(2x + 1)(x - 2)(2x - 3) = 63$
- ii) $16x(x + 1)(x + 2)(x + 3) = 9$

ANSWER: $3, -\frac{3}{2}, \frac{3 \pm \sqrt{-47}}{4}$

ANSWER: $-\frac{3}{2}, -\frac{3}{2}, \frac{3 \pm \sqrt{10}}{2}$

TYPE IV: equation of the form $ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0$, where coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of the quadratic.

Equations of this form are known as *Reciprocal Equations*, and so are named because they are not altered when x is changed in to its $\frac{1}{x}$.

Example:

Solve the equations:

- a) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$
- b) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$
- c) $4x^4 + 17x^3 + 8x^2 + 17x + 4 = 0$
- d) $x^4 + x^3 - 4x^2 + x + 1 = 0$
- e) $x^4 + 1 - 3(x^3 + x) = 2x^2$

Solution

a) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$

Dividing through by x^2

$$\Rightarrow 12x^2 - 56x + 89 - \frac{56}{x} + \frac{12}{x^2} = 0$$

$$\Rightarrow 12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0$$

Now let $\left(x + \frac{1}{x}\right) = u$

$$\Rightarrow u^2 = x^2 + \frac{1}{x^2} + 2 \text{ or } x^2 + \frac{1}{x^2} = u^2 - 2$$

The substitutions that solve reciprocal equations are;

$$x + \frac{1}{x} = u, \quad x^2 + \frac{1}{x^2} = u^2 - 2$$

$$\Rightarrow 12(u^2 - 2) - 56u + 89 = 0$$

$$\Rightarrow 12u^2 - 56u + 65 = 0$$

$$\Rightarrow u = \frac{56 \pm \sqrt{(56)^2 - 4 \times 12 \times 65}}{2 \times 12} = \frac{56 \pm 4}{48}$$

$$\therefore u = \frac{5}{2}, \frac{13}{6} \quad (\text{Note: left as fractions})$$

But $\left(x + \frac{1}{x}\right) = u$

$$\Rightarrow x + \frac{1}{x} = \frac{13}{6} \text{ or } 6x^2 - 13x + 6 = 0$$

$$\Rightarrow (3x - 2)(2x - 3) = 0 \therefore x = \frac{3}{2}, \frac{2}{3}$$

$$\text{Also } x + \frac{1}{x} = \frac{5}{2} \text{ or } 2x^2 - 5x + 2 = 0$$

$$\Rightarrow (2x - 1)(x - 2) = 0 \therefore x = 2, \frac{1}{2}$$

$$\therefore x = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}$$

b) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$

Dividing through by x^2

$$\Rightarrow x^2 + 2x - 1 + \frac{2}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 1 = 0$$

Now let $x + \frac{1}{x} = u, x^2 + \frac{1}{x^2} = u^2 - 2$ (as derived as before)

$$\Rightarrow u^2 - 2 + 2u - 1 = 0$$

$$\Rightarrow u^2 + 2u - 3 = 0$$

$$\Rightarrow (u+3)(u-1) = 0 \quad \therefore u = -1, -3$$

$$\text{But } x + \frac{1}{x} = u$$

$$\Rightarrow x + \frac{1}{x} = -1 \text{ or } x^2 + x + 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\text{Also } x + \frac{1}{x} = -3 \text{ or } x^2 + 3x + 1 = 0$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{-1 \pm i\sqrt{3}}{2}, \frac{-3 \pm \sqrt{5}}{2}$$

c) Left as an exercise.

$$\text{ANSWER: } \frac{1}{3}, \frac{1}{2}, 2, 3$$

d) Left as an exercise.

$$\text{ANSWER: } -\frac{1}{4}, -4$$

e) Left as an exercise.

$$\text{ANSWER: } 1, \frac{-3 \pm \sqrt{5}}{2}$$

f) $x^4 + 1 - 3(x^3 + x) = 2x^2$

Re arranging the equation,

$$\Rightarrow x^4 - 3x^3 - 2x^2 - 3x + 1 = 0$$

Dividing through by x^2

$$\Rightarrow x^2 - 3x - 2 - \frac{3}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 2 = 0$$

$$\text{Now let } x + \frac{1}{x} = u, x^2 + \frac{1}{x^2} = u^2 - 2 \text{ (as derived as before)}$$

$$\Rightarrow u^2 - 2 - 3u - 2 = 0$$

$$\Rightarrow u^2 - 3u - 4 = 0$$

$$\Rightarrow (u+1)(u-4) = 0 \quad \therefore u = -1, 4$$

$$\text{But } x + \frac{1}{x} = u$$

$$\Rightarrow x + \frac{1}{x} = -1 \text{ or } x^2 + x + 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\text{Also } x + \frac{1}{x} = 4 \text{ or } x^2 - 4x + 1 = 0$$

$$\Rightarrow x = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$\therefore x = \frac{-1 \pm i\sqrt{3}}{2}, 2 \pm \sqrt{3}$$

Solving equations which are not reciprocal but solved in a similar manner as reciprocal

Example

Solve the equation:

a) $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$

b) $x^2 + \frac{9}{x^2} - 4\left(x - \frac{3}{x}\right) - 2 = 0$

Solution

a) $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$

Dividing through by x^2

$$\Rightarrow 6x^2 - 25x + 12 + \frac{25}{x} + \frac{6}{x^2} = 0$$

$$\Rightarrow 6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) + 12 = 0$$

Now let $x - \frac{1}{x} = u$

$$u^2 = x^2 + \frac{1}{x^2} - 2 \Rightarrow u^2 + 2 = x^2 + \frac{1}{x^2} \Rightarrow u^2 - 2 - 3u - 2 = 0$$

$$\Rightarrow 6(u^2 + 2) - 25u + 12 = 0 \text{ or } 6u^2 - 25u + 24 = 0$$

$$\Rightarrow u = \frac{25 \pm \sqrt{25^2 - 4 \times 6 \times 24}}{2 \times 6} = \frac{25 \pm 7}{12} \therefore u = \frac{32}{12} = \frac{8}{3}, \frac{18}{12} = \frac{3}{2}$$

$$\text{Now } x - \frac{1}{x} = \frac{8}{3} \text{ or } 3x^2 - 8x - 3 = 0$$

$$\Rightarrow 3x^2 - 9x + x - 3 = 0 \text{ or } 3x(x - 3) + 1(x - 3) = 0$$

$$\Rightarrow (3x + 1)(x - 3) = 0 \therefore x = -\frac{1}{3}, 3$$

$$\text{Also } x - \frac{1}{x} = \frac{3}{2} \text{ or } 2x^2 - 3x - 2 = 0$$

$$\Rightarrow 2x^2 - 4x + x - 2 = 0 \text{ or } 2x(x - 2) + 1(x - 2) = 0$$

$$\Rightarrow (2x + 1)(x - 2) = 0 \therefore x = -\frac{1}{2}, 2$$

$$\therefore x = -\frac{1}{3}, 3, 2, -\frac{1}{2}$$

b) $x^2 + \frac{9}{x^2} - 4\left(x - \frac{3}{x}\right) - 2 = 0$

$$\text{Let } x - \frac{3}{x} = u, u^2 = x^2 + \frac{9}{x^2} - 6 \Rightarrow x^2 + \frac{9}{x^2} = u^2 + 6$$

$$\Rightarrow u^2 + 6 - 4u - 2 = 0 \text{ or } u^2 - 4u + 4 = 0$$

$$\Rightarrow (u - 2)^2 = 0 \therefore u = 2$$

$$\Rightarrow x - \frac{3}{x} = 2 \text{ or } x^2 - 2x - 3 = 0$$

$$\Rightarrow (x + 1)(x - 3) = 0 \therefore x = -1, 3$$

Task

$$\text{Solve the equation: } x^2 + \frac{9}{x^2} - 4\left(x - \frac{3}{x}\right) - 6 = 0$$

EQUATIONS INVOLVING SQUARE ROOTS

Steps followed:

1. Isolate one square root on either L.H.S or R.H.S of the equation and square both sides
2. If there is another remaining square root that persists, isolate it again and square both sides
3. Continue to square both sides until there is no more square root, then solve the resulting equation.

NOTE:

These equations yield *extraneous* roots which do not satisfy the equation being solved. Thus for correct roots, test the roots in the original equation, that which satisfy is the correct root.

TYPE A: $P\sqrt{bx+c} + Q\sqrt{mx+n} + R\sqrt{ax+\beta} = k$, where P, Q, R are constants.

Case I: For $R = 0$. i. e. $P\sqrt{bx+c} + Q\sqrt{mx+n} = k$

Example:

Solve the equations:

a) $\sqrt{x+1} + \sqrt{x-2} = 3$

b) $\sqrt{3x-3} - \sqrt{x} = 1$

c) $\sqrt{x-5} + \sqrt{x} = 5$

d) $\sqrt{x-1} + 2\sqrt{x-4} = 4$

Solution

a) $\sqrt{x+1} + \sqrt{x-2} = 3$; let $A = x+1, B = x-2$

$$\Rightarrow \sqrt{A} + \sqrt{B} = 3$$

Isolating $\sqrt{A}$ and squaring both sides

$$\Rightarrow (\sqrt{A})^2 = (3 - \sqrt{B})^2$$

$$\Rightarrow A = 9 - 6\sqrt{B} + B$$

Isolating $\sqrt{B}$ and squaring both sides

$$\Rightarrow 6\sqrt{B} = 9 + B - A$$

$$\Rightarrow 6\sqrt{x-2} = 9 + x - 2 - x - 1$$

$$\Rightarrow 6\sqrt{x-2} = 6 \text{ or } \sqrt{x-2} = 1$$

$$\therefore (\sqrt{x-2})^2 = 1 \Rightarrow x-2 = 1$$

$$\therefore x = 3$$

Testing $x = 3$ in to the original equation,

$$\sqrt{3+1} + \sqrt{3-2} = 3 = R.H.S$$

$\therefore x = 3$ is a root.

b) $\sqrt{3x-3} - \sqrt{x} = 1$; let $A = 3x+3$

$$\Rightarrow \sqrt{A} + \sqrt{x} = 1$$

Isolating $\sqrt{A}$ and squaring both sides

$$\Rightarrow (\sqrt{A})^2 = (1 + \sqrt{x})^2$$

$$\Rightarrow A = 1 + 2\sqrt{x} + x$$

$$\Rightarrow 3x - 3 = 1 + 2\sqrt{x} + x$$

$$\text{Now } 2\sqrt{x} = 2x - 4 \text{ or } \sqrt{x} = x - 2$$

Squaring both sides again,

$$\Rightarrow x = (x-2)^2 = x^2 - 4x + 4$$

$$\Rightarrow x^2 - 5x + 4 = 0 \text{ or } (x-1)(x-4) = 0$$

$$\Rightarrow x = 1, 4$$

Testing the values,

$$\text{For } x = 1, \sqrt{3-3} - \sqrt{1} = 0 \neq 1$$

Hence $x = 1$ is not a correct root *i.e.* it is an extraneous root.

$$\text{For } x = 4, \sqrt{12-3} - \sqrt{4} = 1 = R.H.S.$$

Only $x = 4$ is the correct root.

c) $\sqrt{x-5} + \sqrt{x} = 5$, let $B = x-5$

$$\Rightarrow \sqrt{B} + \sqrt{x} = 5$$

Isolating $\sqrt{B}$ and squaring both sides

$$\Rightarrow (\sqrt{B})^2 = (5 - \sqrt{x})^2$$

$$\Rightarrow B = 25 - 10\sqrt{x} + x$$

$$\Rightarrow x - 5 = 25 - 10\sqrt{x} + x \text{ or } 10\sqrt{x} = 30$$

$$\Rightarrow \sqrt{x} = 3, \text{ squaring both sides,}$$

$$\Rightarrow x = 9$$

$$\text{Testing } x = 9, \sqrt{9-4} + \sqrt{9} = 5 = R.H.S$$

$\therefore x = 9$ is the correct root.

d) $\sqrt{x-1} + 2\sqrt{x-4} = 4$, let $x-1 = A, x-4 = B$

$$\Rightarrow \sqrt{A} + 2\sqrt{B} = 4$$

Isolating $2\sqrt{B}$ and squaring both sides,

$$\Rightarrow (2\sqrt{B})^2 = (4 - \sqrt{A})^2$$

$$\Rightarrow 4B = 16 - 8\sqrt{A} + A$$

Isolating $8\sqrt{A}$ and squaring both sides,

$$\Rightarrow (8\sqrt{A})^2 = (A - 4B + 16)^2$$

$$\Rightarrow 64A = (x - 1 - 4(x - 4) + 16)^2$$

$$\Rightarrow 64(x - 1) = (x - 1 - 4x + 16 + 16)^2 = (-3x + 31)^2$$

$$\Rightarrow 64x - 64 = 9x^2 - 186x + 961 \text{ or } 9x^2 - 250x + 1025 = 0$$

$$\Rightarrow x = \frac{250 \pm \sqrt{(-250)^2 - 4 \times 9 \times 1025}}{18} = \frac{250 \pm 160}{18}$$

$$\Rightarrow x = \frac{205}{9}, 5$$

Testing the values,

$$\text{For } x = 5, \sqrt{5 - 1} + 2\sqrt{5 - 4} = 4 = \text{R.H.S}$$

Hence $x = 5$ is root.

$$\text{For } x = \frac{205}{9}, \sqrt{\left(\frac{205}{9} - 1\right)} + 2\sqrt{\left(\frac{205}{9} - 4\right)} = \frac{40}{3} \neq \text{R.H.S.}$$

Only $x = 5$ is the correct root.

Task

Solve the equations:

a) $2\sqrt{2x - 12} - \sqrt{2x - 3} = 3$

ANSWER: 14 not 6

b) $2\sqrt{x + 4} - \sqrt{x - 1} = 4$

ANSWER: 5 not $1\frac{4}{9}$

c) $\sqrt{x - 5} + 2 = \sqrt{x + 7}$

ANSWER: 9

d) $\sqrt{3x + 4} - \sqrt{x - 3} = 4$

ANSWER: 4, 7

Case II: For $R \neq 0$. i.e. $P\sqrt{bx + c} + Q\sqrt{mx + n} + R\sqrt{ax + \beta} = k$

Example:

Solve the equations:

a) $2\sqrt{x + 1} - 3\sqrt{2x - 5} = \sqrt{x - 2}$

b) $\sqrt{3 - x} - \sqrt{7 + x} = \sqrt{16 + 2x}$

c) $\sqrt{x + 6} - \sqrt{x + 3} = \sqrt{2x + 5}$

Solution

a) $2\sqrt{x + 1} - 3\sqrt{2x - 5} = \sqrt{x - 2}$

$$\text{Let } x + 1 = A, 2x - 5 = B, x - 2 = C$$

$$\Rightarrow 2\sqrt{A} - 3\sqrt{B} = \sqrt{C}, \text{ squaring both sides,}$$

$$\Rightarrow (2\sqrt{A} - 3\sqrt{B})^2 = C$$

$$\Rightarrow 4A + 9B - 12\sqrt{AB} = C$$

Isolating $12\sqrt{AB}$ and squaring both sides,

$$\Rightarrow (12\sqrt{AB})^2 = (4A + 9B - C)^2$$

$$\Rightarrow 144AB = (4A + 9B - C)^2, \text{ substituting for A, B and C}$$

$$\Rightarrow 144(x + 1)(2x - 5) = (4(x + 1) + 9(2x - 5) - (x - 2))^2$$

$$\Rightarrow 144(2x^2 - 3x - 5) = (4x + 4 + 18x - 45 - x + 2)^2$$

$$\Rightarrow 288x^2 - 432x - 720 = (21x - 39)^2 = 441x^2 - 1638x + 1521$$

$$\Rightarrow 153x^2 - 1206x + 2241 = 0 \text{ or } 17x^2 - 134x + 249 = 0$$

$$\Rightarrow x = \frac{134 \pm \sqrt{(134)^2 - 4 \times 17 \times 249}}{34} = \frac{134 \pm 32}{34}$$

$$\Rightarrow x = 3, \frac{83}{17}$$

Testing the values,

For $x = 3$, $2\sqrt{3+1} - 3\sqrt{6-5} = \sqrt{3-2} = 1$, hence $x = 3$ is a root

For $x = \frac{83}{17}$, $2\sqrt{\frac{83}{17}+1} - 3\sqrt{\frac{166}{17}-5} \neq \sqrt{\frac{83}{17}-2} = 1$, hence $x = \frac{83}{17}$ is not a root

$\therefore x = 3$ is the only correct root.

b) $\sqrt{3-x} - \sqrt{7+x} = \sqrt{16+2x}$

Let $3-x = A$, $7+x = B$, $16+2x = C$

$\Rightarrow \sqrt{A} - \sqrt{B} = \sqrt{C}$, squaring both sides,

$$\Rightarrow (\sqrt{A} - \sqrt{B})^2 = C$$

$$\Rightarrow A + B - 2\sqrt{AB} = C$$

Isolating $2\sqrt{AB}$ and squaring both sides,

$$= (A + B - C)^2 = 4AB, \text{ substituting for A, B and C}$$

$$\Rightarrow 4(3-x)(7+x) = (3-x+7+x-(16+2x))^2$$

$$\Rightarrow 4(21-4x-x^2) = (-6-2x)^2$$

$$\Rightarrow 84-16x-4x^2 = 36+24x+4x^2$$

$$\Rightarrow 8x^2+40x-48=0 \text{ or } x^2+5x-6=0$$

$$\Rightarrow (x+6)(x-1)=0 \Rightarrow x=1, -6$$

Testing the values,

For $x = 1$, $\sqrt{3-1} - \sqrt{7+1} \neq \sqrt{16+2}$, hence $x = 1$ is not a root.

For $x = -6$, $\sqrt{3+6} - \sqrt{7-6} = \sqrt{16-12}$, hence $x = -6$ is a root.

$\therefore x = -6$ is the correct root.

c) Left as an exercise.

ANSWER: $x = -2$

TYPE B: Any equation which can be thrown in to the form

$$ax^2 + bx + c + p\sqrt{ax^2 + bx + c} = q$$

This is solved by putting $y = \sqrt{ax^2 + bx + c}$

$$\Rightarrow y^2 = ax^2 + bx + c$$

$\therefore$ the equation being solved is $y^2 + py - q = 0$

Let α and β be roots of this equation.

$$\Rightarrow \alpha = \sqrt{ax^2 + bx + c}, \beta = \sqrt{ax^2 + bx + c}$$

From these equations we obtain *four* values of x .

NOTE:

If *no sign* is prefixed to a *radical* it is usually understood that it is to be taken as a *positive*. Hence if both α and β are both *positive*, all the four values of x satisfy the original equation.

If however, α or β is *negative*, the roots found from the resulting quadratic equation will satisfy the equation $ax^2 + bx + c - p\sqrt{ax^2 + bx + c} = q$ but not the original equation.

Example

Solve the equations:

- a) $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$
 b) $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$
 c) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$
 d) $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$
 e) $3x^2 - 18 + \sqrt{3x^2 - 4x + 6} = 4x$

Solution

a) $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$

Adding 3 on both sides

$$\Rightarrow x^2 - 5x + 3 + 2\sqrt{x^2 - 5x + 3} = 15$$

$$\text{Now let } y = \sqrt{x^2 - 5x + 3} \Rightarrow y^2 = x^2 - 5x + 3$$

$$\Rightarrow y^2 + 2y = 15 \text{ or } y^2 + 2y - 15 = 0$$

$$\Rightarrow (y + 5)(y - 3) = 0 \therefore y = -5, 3$$

$$\text{But } y = \sqrt{x^2 - 5x + 3}$$

$$\Rightarrow \sqrt{x^2 - 5x + 3} = -5 \dots\dots\dots (1)$$

$$\Rightarrow \sqrt{x^2 - 5x + 3} = 3 \dots\dots\dots (2)$$

From eqn. (1), squaring both sides,

$$\Rightarrow x^2 - 5x + 3 = 25 \text{ or } x^2 - 5x - 22 = 0$$

$$\therefore x = \frac{5 \pm \sqrt{25 + 88}}{2} = \frac{5 \pm \sqrt{113}}{2}$$

From eqn. (2), squaring both sides,

$$\Rightarrow x^2 - 5x + 3 = 9 \text{ or } x^2 - 5x - 6 = 0$$

$$\Rightarrow (x - 6)(x + 1) = 0 \therefore x = -1, 6$$

From the discussion above,

$x = -1, 6$ satisfy the equation $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$, because the radical $2\sqrt{x^2 - 5x + 3}$ is *positive* and $y = +3$

But for $x = \frac{5 \pm \sqrt{113}}{2}$ satisfy the equation $x^2 - 5x - 2\sqrt{x^2 - 5x + 3} = 12$, because the radical $-2\sqrt{x^2 - 5x + 3}$ is *negative* and $y = -5$

b) $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$, re arranging

$$\Rightarrow x^2 + 6x + 2\sqrt{x^2 + 6x} = 24$$

$$\text{Let } y = \sqrt{x^2 + 6x} \Rightarrow y^2 = x^2 + 6x$$

$$\Rightarrow y^2 + 2y = 24 \text{ or } y^2 + 2y - 24 = 0$$

$$\Rightarrow (y + 6)(y - 4) = 0 \therefore y = -6, 4$$

$$\text{But } y = \sqrt{x^2 + 6x}$$

$$\Rightarrow \sqrt{x^2 + 6x} = 4 \dots\dots\dots (1)$$

$$\Rightarrow \sqrt{x^2 + 6x} = -6 \dots\dots\dots (2)$$

From eqn. (1), squaring both sides,

$$\Rightarrow x^2 + 6x = 16 \text{ or } x^2 + 6x - 16 = 0$$

$$\Rightarrow (x + 8)(x - 2) = 0 \therefore x = -8, 2$$

From eqn. (2), squaring both sides,

$$\Rightarrow x^2 + 6x = 36 \text{ or } x^2 + 6x - 36 = 0$$

$$\therefore x = \frac{-6 \pm \sqrt{36 + 4 \times 36}}{2} = -3 \pm 3\sqrt{5}$$

$\therefore$ Roots to $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$ are $\therefore x = -8, 2$ for $y = +4$ and roots of

$x^2 - 2\sqrt{x^2 + 6x} = 24 - 6x$ are $-3 \pm 3\sqrt{5}$ for $y = -4$. The roots of the latter equation are extraneous.

c) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$

Subtracting 3 from both sides,

$$\Rightarrow x^2 - 6x + 6 = 4\sqrt{x^2 - 6x + 6} - 3$$

$$\text{Now let } y = \sqrt{x^2 - 6x + 6} \Rightarrow y^2 = x^2 - 6x + 6$$

$$\Rightarrow y^2 = 4y - 3 \text{ or } y^2 - 4y + 3 = 0$$

$$\Rightarrow (y - 1)(y - 3) = 0 \therefore y = 1, 3$$

$$\text{But } y = \sqrt{x^2 - 6x + 6}$$

$$\Rightarrow \sqrt{x^2 - 6x + 6} = 1 \dots\dots\dots (1)$$

$$\Rightarrow \sqrt{x^2 - 6x + 6} = 3 \dots\dots\dots (2)$$

From eqn. (1), squaring both sides,

$$\Rightarrow x^2 - 6x + 6 = 1 \text{ or } x^2 - 6x + 5 = 0$$

$$\Rightarrow (x - 5)(x - 1) = 0 \therefore x = 1, 5$$

From eqn. (2), squaring both sides,

$$\Rightarrow x^2 - 6x + 6 = 9 \text{ or } x^2 - 6x - 3 = 0$$

$$\therefore x = \frac{6 \pm \sqrt{36 + 4 \times 3}}{2} = 3 \pm 2\sqrt{3}$$

$\therefore x = 1, 5, 3 \pm 2\sqrt{3}$ satisfy the original equation.

d) $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$, re arranging the equation,

$$x^2 + 3x - 40 + 10\sqrt{x^2 + 3x + 16} = 0$$

Adding 56 on both sides,

$$\Rightarrow x^2 + 3x + 16 + 10\sqrt{x^2 + 3x + 16} = 56$$

$$\text{Let } y = \sqrt{x^2 + 3x + 16} \Rightarrow y^2 = x^2 + 3x + 16$$

$$\Rightarrow y^2 + 10y = 56 \text{ or } y^2 + 10y - 56 = 0$$

$$\Rightarrow (y + 14)(y - 4) = 0 \therefore y = 4, -14$$

$$\text{But } y = \sqrt{x^2 + 3x + 16}$$

$$\Rightarrow \sqrt{x^2 + 3x + 16} = 4 \dots\dots\dots (1)$$

$$\Rightarrow \sqrt{x^2 + 3x + 16} = -14 \dots\dots\dots (2)$$

From eqn. (1), squaring both sides,

$$\Rightarrow x^2 + 3x + 16 = 16 \text{ or } x^2 + 3x = 0$$

$$\Rightarrow x(x + 3) = 0 \therefore x = 0, -3$$

From eqn. (2), squaring both sides,

$$\Rightarrow x^2 + 3x + 16 = 196 \text{ or } x^2 + 3x - 180 = 0$$

$$\therefore x = \frac{-3 \pm \sqrt{9 + 4 \times 180}}{2} = \frac{3 \pm 27}{2} \Rightarrow x = -15, 12$$

Since $y = +4$ and $10\sqrt{x^2 + 3x + 16}$ is also $+ve$, then $x = 0, -3$ satisfy the equation $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$.

Since $y = -14$ and $10\sqrt{x^2 + 3x + 16}$ is $+ve$, then $x = -15, 12$ satisfy the equation $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$. These are extraneous roots.

e) Left as an exercise.

ANSWER: $x = 3, -\frac{5}{3}$ and $\frac{2 \pm \sqrt{70}}{3}$

NOTE:

- In general some equations of degree 4⁰ can be converted in to a quadratic
- Before clearing an equation of a radical, it is advisable whether any common factor can be removed by division.

Example:

Solve: $2\sqrt{x^2 - x - 2} - 3\sqrt{2x^2 - 9x + 10} = x - 2$

$\Rightarrow 2\sqrt{(x+1)(x-2)} - 3\sqrt{(2x-5)(x-2)} = x - 2$

Dividing through by $\sqrt{x-2}$

$\Rightarrow 2\sqrt{x+1} - 3\sqrt{2x-5} = \sqrt{x-2}$, which is as case II for $R \neq 0$

Solving the equation, $x = 3$.

NOTE:

When one root of a quadratic equation is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots of a quadratic equation as illustrated below.

Example:

Solve the equation:

i) $(1 - a^2)(x + a) - 2a(1 - x^2) = 0$

ii) $\frac{(1-x)^2}{2-x^2} = \frac{(1-a)^2}{2-a^2}$

Solution

i) $(1 - a^2)(x + a) - 2a(1 - x^2) = 0$

By inspection, clearly $x = a$ is a root to the equation.

Re writing the equation,

$\Rightarrow x + a - a^2x - a^3 - 2a + 2ax^2 = 0$

$\therefore 2ax^2 + (1 - a^2)x - a(1 + a^2) = 0$

Let α and β be roots of this equation, then

$\Rightarrow \alpha\beta = -\frac{a(1+a^2)}{2a} = -\frac{(1+a^2)}{2}$

Since one root of the equation is a . i . e . say $\alpha = a$, then the other root β is;

$\beta = -\frac{(1+a^2)}{2a} = -\frac{(1+a^2)}{2a}$

ii) Left as an exercise.

Task

Solve the equation:

a) $\sqrt{4x^2 - 7x - 15} - \sqrt{x^2 - 3x} = \sqrt{x^2 - 9}$

ANSWER: 1, 3

b) $\sqrt{2x^2 - 9x + 4} + 3\sqrt{2x - 1} = \sqrt{2x^2 + 21x - 11}$

ANSWER: 5, $\frac{1}{2}$

c) $\sqrt{x^2 - 7ax + 10a^2} - \sqrt{x^2 + ax - 6a^2} = x - 2a$

ANSWER: $2a, -\frac{10a}{3}$

SERIES

Arithmetic and Geometric Progressions

(A. ps and G. ps)

Basic concepts:

- a) A *Sequence* is a set of quantities/numbers $a_1, a_2, a_3, \dots$, stated in a definite order and each term formed according to a fixed pattern.
i.e. $a_r = f(r)$.

Example

- $1, 3, 5, 7, \dots$, is a sequence (the next term would be 9)
 - $2, 6, 18, 54, \dots$, is a sequence (the next term would be 162)
 - $1^2, -2^2, 3^2, -4^2, \dots$, is a sequence (the next term would be $+5^2$)
- i) A *finite sequence* contains only a finite number of terms.
ii) An *infinite sequence* is unending.

NOTE:

The numbers in a sequence are called *terms*.

- b) A *series* is formed by the *sum* of the terms of a sequence.

Example

For $1, 3, 5, 7, \dots$ is a sequence
 $1 + 3 + 5 + 7, \dots$ is a series.

- c) **The Arithmetic Progression (A.P)**

A series in which each term is obtained from the proceeding one by adding or subtracting a constant quantity is called the *Arithmetic Progression*.

Example

- $1 + 3 + 5 + 7, \dots$ is an A.P
- $3 - 2 - 7 - \dots$ is an A.P
- $a + (a + d) + (a + 2d) + \dots$ is an A.P

The constant quantity is called *a common difference, d*.

Consider the A.P below;

$$a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$$

This is an A.P if the common difference

$$d = a_n - a_{n-1}, \text{ for all } n > 1$$

From the above examples of series,

- $1 + 3 + 5 + 7, \dots, d = 3 - 1 = 5 - 3 = 7 - 5 = 2$
- $3 - 2 - 7 - \dots, d = -2 - 3 = -7 - -2 = -5$
- $a + (a + d) + (a + 2d) + \dots$,
common difference $= (a + d) - a = (a + 2d) - (a + d) = d$

To find General term(n^{th} - term) of an A.P.

Let $a_1 + a_2 + a_3 + \dots$, be an A.P.

Let $d = \text{common difference}$, then

$$d = a_n - a_{n-1}, \text{ for all } n > 1$$

$$\Rightarrow a_n = d + a_{n-1}$$

Now for $n = 2, a_2 = d + a_1$

$$n = 3, a_3 = d + a_2 = d + (d + a_1) = a_1 + 2d$$

$$n = 4, a_4 = d + a_3 = d + (2d + a_1) = a_1 + 3d$$

Observation of the terms;

$$a_2 = a_1 + d = a_1 + (2 - 1)d$$

$$a_3 = a_2 + d = a_1 + (3 - 1)d$$

$$a_4 = a_3 + d = a_1 + (4 - 1)d$$

$$\vdots$$

$$a_n = a_{n-1} + d = a_1 + (n - 1)d$$

Thus the general term of an A.P is

$$a_n = a + (n - 1)d$$

From the formula,

$$a = 1^{st} \text{ term}$$

$$n = \text{no. of terms}$$

$$a_n = \text{last term}$$

$$d = \text{common difference}$$

NOTE:

1. The general term of an A.P is $a_n = a_1 + (n - 1)d$, has four terms. If the *three* are given then the 4^{th} can be evaluated.
2. From the general term of an A.P, the standard form of an A.P is stated as;

$$a + (a + d) + (a + 2d) + \dots \dots \dots + [a + (n - 1)d]$$

3. The most convenient terms in an A.P with given sum and product:

- a) If **three terms** are in an A.P, then use

$$(a - d), a, (a + d)$$

- b) If **four terms** are in an A.P, then use

$$(a - 3d), (a - d), (a + d), (a + 3d)$$

NOTICE:

- a) If you sum up the terms, the common difference, **d**, vanishes, and the sum is in terms of only the first term, **a**.
- b) In case of **odd number** of terms, the common difference is taken as **d** and the middle term as **a**.
- c) In case of **even number** of terms, the common difference is taken as **2d** and the two middle terms are $(a - d)$ and $(a + d)$.

Properties of an A.P

- a) If the constant is *added* to or *subtracted* from every term in an A.P, the resulting series is an A.P.
- b) If every term is *multiplied* or *divided* by a constant, the resulting series is also in an A.P.

Examples

1. Find the 17th – term of the series

- a) $4 + 6 + 8 + \dots$
 b) $-8 - 7 - 6 - 5 - \dots$

Solution

- a) $4 + 6 + 8 + \dots$

Since $d = 6 - 4 = 8 - 6 = 2$, the series is an A.P

Using $a_n = a + (n - 1)d$

Given: $a = 4, n = 17, d = 2$

Required: a_n

$$\Rightarrow a_n = 4 + (17 - 1) \times 2 = 36$$

$\therefore$ the 17th – term is 36

- b) $-8 - 7 - 6 - 5 - \dots$

$$d = -7 - -8 = -6 - -7 = 1$$

Given: $a = -8, n = 17, d = 1$

Required: a_{17}

$$\Rightarrow a_{17} = -8 + (17 - 1) \times 1 = 8$$

$\therefore$ the 17th – term is 8

2. State the 20th – term and the number of terms of the following series;

- a) $2 + 4 + 6 + \dots + 100$
 b) $-8 - 6 - 4 - \dots + 200$

Solution

- a) $2 + 4 + 6 + \dots + 100$

Given: $a = 2, n = ?, d = 4 - 2 = 2, a_n = 100$

Required: $a_{20} = ?, n = ?$

Using $a_n = a + (n - 1)d$

$$100 = 2 + (n - 1) \times 2 \text{ or } 50 = 1 + n - 1$$

$$\therefore n = 50$$

$$\text{Now } a_{20} = a + (20 - 1)d = a + 19d$$

$$a_{20} = 2 + 19 \times 2 = 40$$

$\therefore$ No. of terms is 50, 20th – term is 40

- b) $-8 - 6 - 4 - \dots + 200$

Given: $a = -8, n = ?, d = -6 - -8 = 2, a_n = 200$

Required: $a_{20} = ?, n = ?$

Using $a_n = a + (n - 1)d$

$$200 = -8 + (n - 1) \times 2 \text{ or } 100 = -4 + n - 1$$

$$\therefore n = 105$$

$$\text{Now } a_{20} = a + (20 - 1)d = a + 19d$$

$$a_{20} = -8 + 19 \times 2 = 30$$

$\therefore$ No. of terms is 105, 20th – term is 30

3. The 3rd – term of an A.P is 18 and the 7th – term is 30. Find the 17th – term.

Solution

Using $a_n = a + (n - 1)d$

Given: $a_3 = 18, a_7 = 30$

Required: $a_{17} = ?$

$$\Rightarrow a_3 = a + (3 - 1)d = a + 2d = 18 \dots\dots\dots (1)$$

$$\Rightarrow a_7 = a + (7 - 1)d = a + 6d = 30 \dots\dots\dots (2)$$

Solving the equations simultaneously

Eqn. (1) – eqn. (2)

$$\Rightarrow -4d = -12, d = 3$$

Subst. $d = 3$ in to eqn. (1) or (2)

$$\text{For } a + 2d = 18 \Rightarrow a = 18 - 2 \times 3 = 12$$

$$\text{Now } a_{17} = a + (17 - 1)d = a + 16d = 12 + 16 \times 3 = 60$$

$\therefore$ the 17th – term is 60

4. The 3rd and 13th term of an A.P are respectively equal to –40 and 0. Find the A.P and its 20th – term.

Solution

Using $a_n = a + (n - 1)d$

$$\text{Given: } a_3 = -40, a_{13} = 0$$

Required: $a_{20} = ?$

$$\Rightarrow a_3 = a + (3 - 1)d = a + 2d = -40 \dots\dots\dots (1)$$

$$\Rightarrow a_{13} = a + (13 - 1)d = a + 12d = 0 \dots\dots\dots (2)$$

Solving the equations simultaneously

Eqn. (1) – eqn. (2)

$$\Rightarrow -10d = -40, d = 4$$

Subst. $d = 4$ in to eqn. (1) or (2)

$$\text{For } a + 2d = -40 \Rightarrow a = -40 - 2 \times 4 = -48$$

$$\text{Now } a_{20} = a + (20 - 1)d = a + 19d = -48 + 19 \times 4 = -12$$

$\therefore$ the 20th – term is – 12

The A.P is $a, (a + d), (a + 2d), \dots\dots\dots$

$\therefore$ the A.P is – 48, –44, –40, $\dots\dots\dots$, 28

5. In an A.P $u_1 + u_2 + \dots\dots\dots + u_4 = 15$ and $u_{11} = -3$. Find the greatest integer N such that $u_N \geq 0$

Solution

Using $a_n = a + (n - 1)d$

$$u_1 = a, u_2 = a + d, u_3 = a + 2d, u_4 = a + 3d$$

$$\Rightarrow a + a + d + a + 2d + a + 3d = 15$$

$$\therefore 4a + 6d = 15 \dots\dots\dots (1)$$

$$\text{Given: } u_{16} = -3 \Rightarrow a + 15d = -3 \dots\dots\dots (2)$$

Solving (1) and (2) simultaneously,

$$\text{From (1) } a = -3 - 15d \dots\dots\dots (3)$$

Eqn. (3) in to (1)

$$\Rightarrow 4[-3 - 15d] + 6d = 15$$

$$\Rightarrow -12 - 60d + 6d = 15 \text{ or } d = -\frac{27}{54} = -\frac{1}{2}$$

Subst. $d = -\frac{1}{2}$ in to (3)

$$\Rightarrow a = -3 - 15\left(-\frac{1}{2}\right) = \frac{-6+15}{2} = \frac{9}{2}$$

$$\text{Now } u_N = a + (N - 1)d \geq 0$$

$$\Rightarrow \frac{9}{2} + (N - 1) \times \left(-\frac{1}{2}\right) \geq 0 \dots\dots\dots (\times 2)$$

$$\Rightarrow 9 + (1 - N) \geq 0 \text{ or } 10 \geq N$$

$\therefore$ the greatest integer is 10

6. The n^{th} term of a series is $U_n = a 3^n + b n + c$. Given that $U_1 = 4, U_2 = 13$ and $U_3 = 46$. Find the values of a, b and c .

Solution

Given: $U_n = a 3^n + b n + c$

For $U_1 = 4, n = 1 \Rightarrow 4 = 3a + b + c$ (1)

For $U_2 = 13, n = 2 \Rightarrow 13 = 9a + 2b + c$ (2)

For $U_3 = 46, n = 3 \Rightarrow 46 = 27a + 3b + c$ (3)

Solving (1), (2) and (3) simultaneously

eqn. (1) – eqn. (2), to eliminate c

$\Rightarrow -9 = -6a - b$ or $6a + b = 9$ (4)

eqn. (1) – eqn. (3), to eliminate c

$\Rightarrow -42 = -24a - 2b$ or $12a + b = 21$ (5)

Now eliminating b from (4) and (5)

eqn. (4) – eqn. (5), to eliminate b

$\Rightarrow -6a = -12 \therefore a = 2$

Subst. $a = 2$ in to eqn. (5) or (4)

$6a + b = 9, a = 2 \Rightarrow b = 9 - 6 \times 2 = -3$

Subst. $a = 2, b = -3$ in to (1) or (2) or (3)

From $\Rightarrow c = 4 - (3a + b) = 4 - (6 - 3) = 1$

$\therefore a = 2, b = -3, c = 1$

Examples on sum and product of a certain number of terms in an A.P

1. Find three terms in an A.P such that;

a) Their sum is 21 and the product is 315

b) Their sum is 33 and the sum of their squares is 563

Solution

Let the numbers be $a - d, a, a + d$

- a) Given: sum = 21, $a - d + a + a + d = 21$

$\Rightarrow 3a = 21 \therefore a = 7$

Product = 315, $a(a - d)(a + d) = 315$

$\Rightarrow a(a^2 - d^2) = 315$

$\Rightarrow 7(7^2 - d^2) = 315$ or $49 - d^2 = 45$

$\therefore d = \sqrt{4} = \pm 2$

The numbers are: $[7 - (\pm 2)], 7, [7 + (\pm 2)]$

$\therefore 5, 7, 9$, for $d = +2$

$\therefore 9, 7, 5$, for $d = -2$

- b. sum = 33, $a - d + a + a + d = 33$

$\Rightarrow 3a = 33 \therefore a = 11$

sum of squares = 563, $(a - d)^2 + a^2 + (a + d)^2 = 563$

$\Rightarrow (11 - d)^2 + 11^2 + (11 + d)^2 = 563$

$\Rightarrow 121 - 22d + d^2 + 121 + 121 + 22d + d^2 = 563$

$\Rightarrow 2d^2 + 363 = 563, d^2 = 100 \therefore d = \pm 10$

Now numbers are:

For $d = 10, a = 11 \Rightarrow (11 - 10), 11, (11 + 10)$ or $1, 11, 21$

For $d = -10, a = 11 \Rightarrow (11 - (-10)), 11, (11 - 10)$ or $21, 11, 1$

2. Find four numbers in an A.P such that their sum is 20 and sum of the squares is 120.

Solution

Let the numbers be: $(a - 3d), (a - d), (a + d), (a + 3d)$

$$\text{sum} = 20 \Rightarrow a - 3d + a - d + a + d + a + 3d = 20$$

$$\Rightarrow 4a = 20 \therefore a = 5$$

$$\text{sum of squares} = 120, (a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120$$

$$\Rightarrow (5 - 3d)^2 + (5 - d)^2 + (5 + d)^2 + (5 + 3d)^2 = 120$$

$$\Rightarrow 25 - 30d + 9d^2 + 25 + 30d + 9d^2 + 25 - 10d + d^2 + 25 + 10d + d^2 = 120$$

$$\Rightarrow 100 + 20d^2 = 120 \therefore d = \pm 1$$

The numbers are:

For $d = 1, a = 5, (5 - 3), (5 - 1), (5 + 1), (5 + 3)$

$\therefore 2, 4, 6, 8$

For $d = -1, a = 5, (5 + 3), (5 + 1), (5 - 1), (5 - 3)$

$\therefore 8, 6, 4, 2$

3. Find five numbers in an A.P such that;

a) Their sum is 20 and product of the first and last term is 15

b) Their sum is 25 and sum of their squares is 135

Solution

- a) Let the numbers be: $a - 2d, a - d, a, a + d, a + 2d$

$$\text{Now sum} = 20 = a - 2d + a - d + a + a + d + a + 2d$$

$$\Rightarrow 20 = 5a \therefore a = 4$$

$$\text{Product of the first and last} = (a - 2d) \times (a + 2d) = 15$$

$$\Rightarrow (4 - 2d) \times (4 + 2d) = 15 \text{ or } 16 - 4d^2 = 15$$

$$\Rightarrow d = \pm \frac{1}{2}$$

$\therefore$ numbers are: $3, 3\frac{1}{2}, 4, 4\frac{1}{2}$ and 5

- b) $\text{Sum} = 5a = 25 \therefore a = 5$

$$\text{Sum of squares} = 135 = (a - 2d)^2 + (a - d)^2 + a^2 + (a + d)^2 + (a + 2d)^2$$

$$135 = a^2 - 4ad + 4d^2 + a^2 - 2ad + d^2 + a^2 + a^2 + 2ad + d^2 + a^2 + 4ad + 4d^2$$

$$\Rightarrow 135 = 5a^2 + 10d^2, \text{ but } a = 5$$

$$\Rightarrow 135 = 5 \times 5^2 + 10d^2 \text{ or } 135 = 125 + 10d^2$$

$$\therefore d = \pm 1$$

$\therefore$ numbers are: 3, 4, 5, 6 and 7

Task

1. Find three numbers in an A.P such that

a) Sum is 27 and product is 648

b) Sum is 27 and sum of their squares is 275

c) Sum is 12 and sum of their cubes is 408

d) Sum is 15 and sum of squares of its first and third term is 58

e) Sum is 9 and the sum of their squares is 77

f) Sum is 6 and their product is -90

ANSWER: (a) 6, 9, 12, $d = \pm 3, a = 9$ (b) 5, 9, 13, $d = \pm 4, a = 9$ (c) 1, 4, 7, $d = \pm 3, a = 4$

(d) 3, 5, 7 (e) -2, 3, 8 (f) -5, 2, 9

2. The sum of four integers of an A.P is 24 and their product is 945. Find the integers.

ANSWER: 3, 5, 7, 9

Hint: Use $(a - 3d), (a - d), (a + d), (a + 3d)$

PROOFS IN A.P.S

- 1) If a, b and c are respectively the p^{th} , q^{th} and r^{th} terms of an A.P, prove that $a(q-r) + b(r-p) + c(p-q) = 0$

Solution

Using $U_N = A + (N-1)D$ for an A.P

Given: $U_p = a, U_q = b, U_r = c$

$$\Rightarrow a = A + (p-1)D \dots\dots\dots (1)$$

$$\Rightarrow b = A + (q-1)D \dots\dots\dots (2)$$

$$\Rightarrow c = A + (r-1)D \dots\dots\dots (3)$$

Note:

A and D are the unknowns, hence solving for them,

eqn. (1) - eqn. (2)

$$\Rightarrow a - b = [p-1 - (q-1)]D = (p-q)D$$

$$\Rightarrow D = \frac{a-b}{p-q} \dots\dots\dots (4)$$

eqn. (4) in to (1) to find A

From (1), $A = a - (p-1)D$

$$\Rightarrow A = a - (p-1) \left(\frac{a-b}{p-q} \right) = \frac{ap-aq-(ap-pb-a+b)}{p-q}$$

$$\therefore A = \frac{-aq+pb+a-b}{p-q} \dots\dots\dots (5)$$

Subst. (4) and (5) in to (3)

From $c = A + (r-1)D$

$$\Rightarrow c = \frac{-aq+pb+a-b}{p-q} + (r-1) \left(\frac{a-b}{p-q} \right)$$

$$\Rightarrow c(p-q) = -aq + pb + a - b + ar - br - a + b$$

$$= -aq + pb + ar - br$$

$$\therefore c(p-q) = a(r-q) + b(p-r) \text{ or } a(q-r) + b(r-p) + c(p-q) = 0$$

- 2) Prove that the n^{th} term is $p+q-n$ when the p^{th} term of an A.P is q and the q^{th} term is p .

Solution

Using $U_N = A + (N-1)D$ for an A.P

Given: $U_p = q, U_q = p$

$$\Rightarrow p = A + (p-1)D \dots\dots\dots (1)$$

$$\Rightarrow q = A + (q-1)D \dots\dots\dots (2)$$

$$\text{Required: } U_n = p+q-n \dots\dots\dots (3)$$

Now solving for A and D

eqn. (1) - eqn. (2), to eliminate A

$$p - q = (q - p)D = -D(p - q)$$

$$\therefore D = -1$$

From (1) and $D = -1$

$$A = p - (q-1)D = p + q - 1$$

Required: $U_n = A + (n-1)D$

$$U_n = p + q - 1 + (n-1) \times -1 = p + q - 1 - n + 1 = p + q - n$$

$$\therefore n^{th} \text{ term} = p + q - n$$

3. If p times the p^{th} term of an A.P is equal to q times the q^{th} , prove that the $(p+q)^{th}$ term is zero.

Solution

Using $U_N = A + (N - 1)D$ for an A.P

Given: $p \times U_p = q \times U_q$

Required: to prove $U_{p+q} = 0$

$$U_p = A + (p - 1)D, U_q = A + (q - 1)D$$

$\Rightarrow p \times U_p = q \times U_q$ becomes

$$p \times [A + (p - 1)D] = q \times [A + (q - 1)D] \dots\dots\dots (1)$$

$$\text{Now } U_{p+q} = A + (p + q - 1)D \dots\dots\dots (2)$$

Expressing eqn. (2) in terms of only D

$$\text{From (1), } pA + (p^2 - p)D = Aq + (q^2 - q)D$$

$$\Rightarrow A(p - q) = (q^2 - q)D - (p^2 - p)D$$

$$A(p - q) = (q^2 - p^2 - q + p)D$$

$$= -[p^2 - q^2 - (p - q)]D$$

$$= -[(p - q)(p + q) - (p - q)]D \dots\dots\dots [\div (p - q)]$$

$$A = -[p + q - 1]D$$

$$\therefore A = [1 - (p + q)]D \dots\dots\dots (3)$$

Subst. (3) in to (2), must give zero for the proof to have been proved as required.

$$\begin{aligned} U_{p+q} &= A + (p + q - 1)D \\ &= [1 - (p + q)]D + (p + q - 1)D \\ &= D[1 - p - q + p + q - 1] = 0, \text{ as required.} \end{aligned}$$

Task

- The m^{th} term of an A.P is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, prove that the $(mn)^{\text{th}}$ term is unity. i.e. $U_{mn} = 1$
- If p, q, r, s are any four consecutive terms of an A.P, show that $p^2 - 3q^2 + 3r^2 - s^2 = 0$.
[Hint: Let $p = \alpha - 3\beta, q = \alpha - \beta, r = \alpha + \beta, s = \alpha + 3\beta$]
- The 8^{th} term of an A.P is double the 13^{th} term. Prove that the 2^{nd} term is double the 10^{th} term.

Other forms of proofs in A.Ps

- If a, b and c are in an A.P, prove that;
 - $a^3 + c^3 + 6abc = 8b^3$
 - $a^2 + 4ac + c^2 = 2(ab + bc + ca)$

Solution

For a, b and c to be in A.P, then $b - a = c - b = d$ (common difference)

$$\Rightarrow b = a + d, c = b + d = a + d + d = a + 2d, 2b = a + c$$

- $$\begin{aligned} \text{L.H.S} &= a^3 + c^3 + 6abc = a^3 + (a + 2d)^3 + 6a(a + d)(a + 2d) \\ &= a^3 + a^3 + 6a^2d + 12ad^2 + 8d^3 + 6a(a^2 + 2ad + ad + 2d^2) \\ &= 8a^3 + 6a^2d + 12ad^2 + 8d^3 + 18a^2d + 12ad^2 \\ &= 8a^3 + 24a^2d + 24ad^2 + 8d^3 = 8(a^3 + 3a^2d + 3ad^2 + d^3) = 8(a + d)^3 = 8b^3 \end{aligned}$$
- $$\begin{aligned} \text{L.H.S} &= a^2 + 4ac + c^2 = a^2 + c^2 + 4ac \\ &= (a + c)^2 - 2ac + 4ac = (a + c)^2 + 2ac \\ &= (a + c)(a + c) + 2ac \\ &= (2b)(a + c) + 2ac = 2(ab + bc + ac) \end{aligned}$$

- If a, b and c are in an A.P, show that

- i) $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in an A.P
 ii) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in an A.P

Solution

- i) if a, b, c are in an A.P, then

$$b - a = c - b \dots\dots\dots (1)$$

Now $\frac{1}{ab}, \frac{1}{ac}, \frac{1}{ab}$ are in an A.P if $\frac{1}{ac} - \frac{1}{bc} = \frac{1}{ab} - \frac{1}{ac}$

$$\Rightarrow \frac{1}{c} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{a} \left(\frac{1}{b} - \frac{1}{c} \right)$$

$$\Rightarrow \frac{b-a}{abc} = \frac{c-b}{abc}$$

$\therefore b - a = c - b \Leftrightarrow a, b, c$ are in an A.P .hence the result follows.

- ii) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in an A.P if $\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}}$

$$\Rightarrow \frac{\sqrt{b}+\sqrt{c}-\sqrt{c}-\sqrt{a}}{(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})} = \frac{\sqrt{c}+\sqrt{a}-\sqrt{a}-\sqrt{b}}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$$

$$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{b}}{\sqrt{b}+\sqrt{a}} \therefore b - a = c - b \Leftrightarrow a, b, c \text{ are in an A.P. Hence the result.}$$

3. If a^2, b^2 and c^2 are in an A.P, show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$

Solution

If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$, are in an A.P, then $\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b} \therefore b^2 - a^2 = c^2 - b^2, \text{ hence } a^2, b^2, c^2 \text{ are in an A.P}$$

Task:

- a. If $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in an A.P, show that a^2, b^2, c^2 are also in an A.P
 b. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in an A.P, show that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in an A.P

TO FIND THE SUM OF A FINITE NUMBER OF QUANTITIES IN AN A.P

Let $a_1, a_2, a_3, \dots, a_n$ be n - quantities in an A.P, and let the last term a_n be denoted as l . If their common difference, then

$$a_n = a_1 + (n - 1)d = l$$

Now summing up the n - terms of an A.P

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \dots + a_1 + (n - 1)d$$

$$= a_1 + (a_1 + d) + (a_1 + 2d) + \dots + (l - d) + l \dots\dots\dots (1)$$

Re-writing this sum in a reverse order

$$\Rightarrow S_n = l + (l - d) + (l - 2d) + \dots + (a_1 + d) + a_1 \dots\dots\dots (2)$$

Adding the equations (1) and (2)

$$\Rightarrow 2S_n = (a_1 + l) + (a_1 + l) + (a_1 + l) + \dots + (a_1 + l), (n - \text{times})$$

$$\Rightarrow 2S_n = n(a_1 + l)$$

$$\therefore S_n = \frac{n}{2}(a_1 + l) = \frac{n}{2}[a_1 + a_1 + (n - 1)d]$$

$$= \frac{n}{2}[2a_1 + (n - 1)d]$$

Consequently,

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

In general,

$$S_n = \frac{n}{2}[\text{1}^{\text{st}} \text{ term} + \text{last term}]$$

NOTE:

The above formula contains four quantities. *i.e.* S_n, a, n, d . If three of them are known the fourth can be evaluated

Examples

1. Find the sum of the series

a) $3 + 8 + 13 + \dots + 12^{th} \text{ term}$

b) $\frac{3}{4} + \frac{2}{3} + \frac{7}{12} + \dots + 19^{th} \text{ term}$

Solution

- a) $3 + 8 + 13 + \dots + 12^{th} \text{ term}$

Given: $a_1 = 3, d = 8 - 3 = 13 - 8 = 5, n = 12, a_{12} = ?$

Using $a_n = a_1 + (n - 1)d$

$a_{12} = 3 + (12 - 1) \times 5 = 58$

Using $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

$\Rightarrow S_{12} = \frac{12}{2}[2 \times 3 + (12 - 1) \times 5] = 366$

Alternatively

Using $S_n = \frac{n}{2}[1^{st} \text{ term} + \text{last term}]$

$\Rightarrow S_{12} = \frac{12}{2}[3 + 58] = 366$

- b) $\frac{3}{4} + \frac{2}{3} + \frac{7}{12} + \dots + 19^{th} \text{ term}$

Given: $a_1 = \frac{3}{4}, d = \frac{2}{3} - \frac{3}{4} = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12}, n = 19, a_{19} = ?$

$a_{19} = \frac{3}{4} + (19 - 1) \times \left(-\frac{1}{12}\right) = -\frac{3}{4}$

Now $S_{19} = \frac{19}{2}\left[2 \times \frac{3}{4} + (19 - 1) \times \left(-\frac{1}{12}\right)\right] = \frac{19}{2}\left[\frac{3}{2} - \frac{18}{12}\right] = 0$

Alternatively

Using $S_n = \frac{n}{2}[1^{st} \text{ term} + \text{last term}]$

$\Rightarrow S_{19} = \frac{19}{2}\left[\frac{3}{4} + -\frac{3}{4}\right] = 0$

2. How many terms of the following series may be taken to;

i) Their sum is 66 if $-9, -6, 3, \dots$?

ii) Their sum is -80 if $15 + 13 + 11 + \dots$?

iii) Their sum is $n(2n + 4)$ if $6 + 10 + 14 + \dots$?

Solution

- i) $-9, -6, 3, \dots$

Let $S_n = 66, a_1 = -9, d = -6 - (-9) = 3$

Using $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

$\Rightarrow 66 = \frac{n}{2}[2 \times -9 + (n - 1) \times 3]$

$132 = -18n + 3(n^2 - n)$

$\therefore 3n^2 - 21n - 132 = 0$ or $n^2 - 7n - 44 = 0$

$\Rightarrow (n - 11)(n + 4) = 0 \therefore n = 11, -4$

Since n is a positive integer, $n = -4$ is rejected.

$\therefore$ number of terms is 11

- ii) $15 + 13 + 11 + \dots$

Let $S_n = -80, a_1 = 15, d = 13 - 15 = -2$

Using $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

$$\Rightarrow -80 = \frac{n}{2} [2 \times 15 + (n-1) \times -2]$$

$$-160 = 30n - 2(n^2 - n)$$

$$\therefore 2n^2 - 32n - 160 = 0 \text{ or } n^2 - 16n - 80 = 0$$

$$\Rightarrow (n-20)(n+4) = 0 \therefore n = 20, -4$$

Since n is a positive integer, $n = -4$ is rejected.

$\therefore$ number of terms is 20

iii) $6 + 10 + 14 + \dots \dots \dots$

Let $S_N = n(2n+4)$, $a_1 = 6$, $d = 10 - 6 = 4$

Using $S_N = \frac{N}{2} [2a_1 + (N-1)d]$

$$\Rightarrow n(2n+4) = \frac{N}{2} [2 \times 6 + (N-1) \times 4]$$

$$\Rightarrow 4n^2 + 8n = 12N + 4N^2 - 4N \text{ or } n^2 + 2n = 3N + N^2 - N$$

$$\therefore N^2 + 2N - (n^2 + 2n) = 0, \text{ solving for } N$$

$$\Rightarrow N = \frac{-2 \pm \sqrt{2^2 + 4(n^2 + 2n)}}{2}$$

$$= \frac{-2 \pm 2\sqrt{1+n^2+2n}}{2}$$

$$= -1 \pm \sqrt{(n+1)^2} = -1 \pm n + 1$$

$$\therefore N = -1 + n + 1 = n \text{ or } N = -1 - n - 1 = -2 - n \text{ (Reject)}$$

$\therefore$ number of terms is n

3. The sum of the series is 525. Its last term is 39 and the first term is 3. If the series is an A.P, find its common difference.

Solution

Given: $S_n = 525$, $a_n = 39$, $a_1 = 3$

Using $S_n = \frac{n}{2} [1^{\text{st}} \text{ term} + \text{last term}] = \frac{n}{2} [a_1 + a_n]$

$$\Rightarrow 525 = \frac{n}{2} [3 + 39] = 21n \therefore n = 25$$

From $a_n = a_1 + (n-1)d$ or from $S_n = \frac{n}{2} [2a_1 + (n-1)d]$

From $S_n = \frac{n}{2} [2a_1 + (n-1)d]$

$$\Rightarrow 525 = \frac{25}{2} [2 \times 3 + (25-1)d]$$

$$\Rightarrow 1050 = 150 + 600d \therefore d = \frac{3}{2}$$

$\therefore$ Common difference is $\frac{3}{2}$

4. The first and last terms of an A.P are -4 and 146 respectively. The sum of the A.P is 7171. Find the number of terms and sum of the first 20 terms.

Solution

Given: $S_n = 7171$, $a_n = 146$, $a_1 = -4$

Required: $n = ?$, $d = ?$

Using $S_n = \frac{n}{2} [1^{\text{st}} \text{ term} + \text{last term}] = \frac{n}{2} [a_1 + a_n]$

$$\Rightarrow 7171 = \frac{n}{2} [-4 + 146] = 71n$$

$$\Rightarrow 7171 = 71n \therefore n = 101$$

$\therefore$ No. of terms is 101

Now finding d using either $a_n = a_1 + (n-1)d$ or $S_n = \frac{n}{2} [2a_1 + (n-1)d]$

This time let us use $a_n = a_1 + (n-1)d$

$$\Rightarrow 146 = -4 + (101-1)d$$

$$\Rightarrow 150 = 100d \therefore d = \frac{3}{2}$$

$$S_{20} = \frac{20}{2} \left[2 \times -4 + (20 - 1) \times \frac{3}{2} \right] = 205$$

$$\therefore S_{20} = 205$$

5. The sum of a certain number of terms in an A.P is 5500. The first and the last terms are 100 and 1000 respectively. Find the number of terms and the sum of the last 5 terms of the progression.

Solution

$$\text{Given: } S_n = 5500, a_1 = 100, a_n = 1000$$

$$\text{Required: } n = ?, d = ?, S_{\text{last 5-terms}} = ?$$

$$\text{Using } S_n = \frac{n}{2} [1^{\text{st}} \text{ term} + \text{last term}] = \frac{n}{2} [a_1 + a_n]$$

$$\Rightarrow 5500 = \frac{n}{2} [100 + 1000] = \frac{1100}{2} n$$

$$\Rightarrow 5500 = \frac{1100}{2} n \therefore n = 10$$

$$\therefore \text{No. of terms is 10}$$

$$\text{Now finding } d \text{ using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow 1000 = 100 + (10 - 1)d$$

$$\therefore d = 100$$

The A.P can be stated as; 100, 200, 300, 400, ... 900, 1000

Sum of last 5 - terms = Sum of all the terms - Sum of the 1st 4 - terms

$$\Rightarrow S_{\text{last 5-terms}} = S_{10} - S_{1^{\text{st}} \text{ five}}$$

$$\begin{aligned} \Rightarrow S_{\text{last 5-terms}} &= 5500 - \left[\frac{5}{2} (2 \times 100 + (5 - 1) \times 100) \right] \\ &= 5500 - [500 + 1000] = 4000 \end{aligned}$$

Note:

The last five terms are; 600, 700, 800, 900, 1000

6. The sum of the last three terms of an A.P having n - terms is 150 times the sum of the first three terms of the progressions. If the third term and the second term are 15 and 5. Find the last term and the number of terms of the progression.

Solution

$$\text{Given: } S_{3(\text{last})} = 150 \times S_{3(\text{first})}, a_3 = 15, a_2 = 5$$

$$\text{Required: } n = ?, d = ?$$

$$\text{For an A.P, the last term is } a_n = a_1 + (n - 1)d$$

$$\text{For the } 2^{\text{nd}} \text{ last tem} = a_n - d$$

$$= a_1 + (n - 1)d - d = a_1 + (n - 2)d$$

$$\text{For the } 3^{\text{rd}} \text{ last tem} = a_n - 2d$$

$$= a_1 + (n - 1)d - 2d = a_1 + (n - 3)d$$

$$\text{Now } S_{3(\text{last})} = [a_1 + (n - 1)d] + [a_1 + (n - 2)d] + [a_1 + (n - 3)d]$$

$$\therefore S_{3(\text{last})} = 3a_1 + (3n - 6)d$$

$$\text{Also } S_{3(\text{first})} = \frac{3}{2} [2a_1 + (3 - 1)d] = 3a_1 + 3d$$

$$\text{From } S_{3(\text{last})} = 150 \times S_{3(\text{first})}$$

$$\Rightarrow 3a_1 + (3n - 6)d = 150(3a_1 + 3d)$$

$$\Rightarrow a_1 + (n - 2)d = 150(a_1 + d)$$

$$\therefore 149a_1 + (152 - n)d = 0 \dots\dots\dots (1)$$

$$\text{Given: } a_3 = a_1 + (3 - 1)d = a_1 + 2d$$

$$\therefore 15 = a_1 + 2d \dots\dots\dots (2)$$

$$a_2 = a_1 + d = 5 \dots\dots\dots (3)$$

$$\text{Eqn. (2) - eqn. (3)}$$

$$\Rightarrow 10 = d$$

$$\text{From } a_1 + d = 5 \Rightarrow a_1 = 5 - 10 = -5$$

Now subst. $d = 10$, $a_1 = -5$ in to eqn. (1)

$$\Rightarrow 149a_1 + (152 - n)d = 0 \text{ yields}$$

$$149(-5) + (152 - n) \times 10 = 0$$

$$\Rightarrow 152 - n = \frac{149 \times 5}{10} = 74.5$$

$\therefore$ No. of terms is 78

$$a_{78} = -5 + (78 - 1) \times 10 = 760$$

7. The sum of the first 13-terms of an A.P is 21 and the sum of the first 21 terms is 13. Find the sum of the first 34 - terms.

Solution

$$\text{Given: } S_{13} = 21, S_{21} = 13$$

$$\text{Required: } S_{34} = ?, d = ?, a = ?$$

$$\text{Using } S_n = \frac{n}{2} [2a_1 + (n - 1)d]$$

$$\Rightarrow S_{13} = \frac{13}{2} [2a + (13 - 1)d] = 21 \dots\dots\dots (1)$$

$$\Rightarrow S_{21} = \frac{21}{2} [2a + (21 - 1)d] = 13 \dots\dots\dots (2)$$

$$\text{From (1), } 2a + 12d = \frac{42}{13} \dots\dots\dots (3)$$

$$\text{From (2), } 2a + 20d = \frac{26}{21} \dots\dots\dots (4)$$

Eqn. (3) - eqn. (4) gives

$$-8d = \frac{544}{273} \therefore d = -\frac{68}{273}$$

Substitute $d = -\frac{68}{273}$ in to eqn. (3) to find a

$$\Rightarrow 2a = \frac{42}{13} - 2 \times \left(-\frac{68}{273}\right) \therefore a = \frac{566}{91}$$

$$\therefore S_{34} = \frac{34}{2} \left[2 \times \frac{566}{91} + (34 - 1) \left(-\frac{68}{273}\right) \right] = -34$$

8. The fifth term of an A.P is 12 and the sum of the first five terms is 80. Determine the first term and the common difference. (Uneb 2007)

Solution

$$\text{Given: } a_5 = 12, S_5 = 80$$

$$\text{Required: } a_1 = ?, d = ?$$

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow a_5 = 12 = a_1 + 4d \dots\dots\dots (1)$$

$$\text{Using } S_n = \frac{n}{2} [2a_1 + (n - 1)d]$$

$$\Rightarrow 80 = \frac{5}{2} [2a_1 + 4d]$$

$$\Rightarrow 80 = 5a_1 + 10d \text{ or } 16 = a_1 + 2d \dots\dots\dots (2)$$

Eqn. (1) - eqn. (2) gives

$$-4 = 2d \therefore d = -2$$

Substitute $d = -2$ in to eqn. (2) to find a_1

$$a_1 = 16 - 2d = 16 + 4 = 20$$

1st - term is 20, common difference = -2

9. The 10th - term of an A.P is 29 and the 15th - term is 44. Find the value of the common difference and the first term. Hence find the sum of the first 60 terms. (Uneb 2002)

Solution

Given: $a_{10} = 29, a_{15} = 44$

Required: $a_1 = ?, d = ?, S_{60} = ?$

Using $a_n = a_1 + (n - 1)d$

$$\Rightarrow a_{10} = 29 = a_1 + 9d \dots\dots\dots (1)$$

$$\Rightarrow a_{15} = 44 = a_1 + 14d \dots\dots\dots (2)$$

Eqn. (1) – eqn. (2) gives

$$-15 = -5d \therefore d = 3$$

Substitute $d = 3$ in to eqn. (1) to find a_1

$$a_1 = 29 - 9d = 29 - 27 = 2$$

$$\text{Now } S_{60} = \frac{60}{2} [2 \times 2 + (60 - 1) \times 3] = 5\,430$$

$\therefore 1^{\text{st}} - \text{term is } 2, \text{common difference} = 3, S_{60} = 5\,430$

Other examples on summation of the terms in an A.P

1. a) Find the sum of all natural numbers between 500 and 1000 which are divisible by 13
- b) Obtain the sum of all natural numbers up to 1000 which are:
 - i) Divisible by 5 but not divisible by 2
 - ii) Not divisible by 2 and not divisible by 5

Solution

- a) By inspection; $\frac{500}{13} \approx 38; \frac{1000}{13} \approx 76.9$

Now multiples of 13 after 38 are;

507, 520, 533, ..., 988

Note: $13 \times 76 = 988, 13 \times 77 = 1001$

This is an A.P with $a_1 = 507, a_n = 988, d = 13$

Now finding n ;

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow 988 = 507 + (n - 1) \times 13 \therefore n = 38$$

$$\text{Now } S_{38} = \frac{38}{2} [2 \times 507 + (38 - 1) \times 13] = 28\,405$$

- b) (i) Natural numbers divisible by 5 and not by 2

$$\Rightarrow 5, 15, 25, \dots, 955$$

These form an A.P. finding how many terms they are,

$$a_1 = 5, a_n = 955, d = 10, n = ?$$

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow 955 = 5 + (n - 1) \times 10 \therefore n = 100$$

$$\therefore S_{100} = \frac{100}{2} [2 \times 5 + (100 - 1) \times 10] = 50\,000$$

- ii) Natural numbers not divisible by 2 are;

$$\Rightarrow 1, 3, 5, 9, \dots, 999$$

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$999 = 1 + (n - 1) \times 2 \therefore n = 500$$

$$\text{Using } S_n = \frac{n}{2} [a_1 + a_n]$$

$$\therefore S_{500} = \frac{500}{2} [1 + 999] = 250\,000$$

Sum of natural numbers divisible by 5 is 50 000 as in above.

Natural numbers not divisible by 2 and not divisible by 5 is;

$$S_{500} - S_{100} = 250\,000 - 50\,000 = 200\,000$$

2. a) Find the sum of odd numbers between 100 and 200
- b) Find the sum of the even and odd numbers divisible by 3 lying between 400 and 500.

Solution

- a) 101, 103, 105, ..., 199 are odd numbers. Finding how many they are'

$$a_n = 199, a_1 = 101, d = 2, n = ?$$

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow 199 = 101 + (n - 1) \times 2 \therefore n = 50$$

$$\text{Now } S_{50} = \frac{50}{2} [101 + 199] = 7500$$

- b) Multiples of 3 after 400 and before 500 are:

$$402, 405, 408, \dots, 498$$

$$\left[\text{HINT: } \frac{400}{3} \approx 133, 134 \times 3 = 402; \frac{500}{3} \approx 167, 166 \times 3 = 498 \right]$$

Finding how many they are:

$$a_n = 498, a_1 = 402, d = 3, n = ?$$

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow 498 = 402 + (n - 1) \times 3 \therefore n = 33$$

$$\text{Now } S_{33} = \frac{33}{2} [402 + 498] = 14850$$

Also even multiples of 3 are:

$$402, 408, 414, \dots, 498$$

Finding how many they are:

$$a_n = 498, a_1 = 402, d = 6, n = ?$$

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow 498 = 402 + (n - 1) \times 6 \therefore n = 17$$

$$\text{Now } S_{17} = \frac{17}{2} [402 + 498] = 7650$$

Now sum of odd numbers divisible by 3 is;

$$S_{33} - S_{17} = 14850 - 7650 = 7200$$

Alternatively, odd numbers are;

$$405, 411, \dots, 495$$

Finding how many they are:

$$a_n = 495, a_1 = 405, d = 6, n = ?$$

$$\text{Using } a_n = a_1 + (n - 1)d$$

$$\Rightarrow 495 = 405 + (n - 1) \times 6 \therefore n = 16$$

$$\text{Now } S_{16} = \frac{16}{2} [405 + 495] = 7200$$

Task

1. Find the sum of all integers lying between 200 and 800 which are divisible by 9. **ANSWER: 32 967**
2. Find the sum of all numbers between 200 and 400 which are divisible by 7. **ANSWER: 8 729**
3. In an A.P consisting of 15 – terms, the middle term is 20. Find the sum of all the terms. **ANSWER: 300**
4. An A.P consists of 12 terms. The sum of the two middle terms is 24 and the sum of the first 3-terms is 9. Find the A.P. **ANSWER: 1, 3, 5, ..., 23**

PROOFS INVOLVING SUMMATION OF A.P FORMULAE,

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] \text{ or } S_n = \frac{n}{2}[a_1 + a_n]$$

1. a) The sum of the first m – terms of an A.P is n and the sum of the first n – terms is m . find the sum of the first $(m+n)$ – terms.
 b) The sum of the first m – terms of an A.P is the same as the sum of the first n – terms where $m \neq n$, show that the sum of the first $(m+n)$ – terms is zero.

Solution

a) Given: $S_m = n$, $S_n = m$

Required: $S_{(m+n)} = ?$

$$S_m = n = \frac{m}{2}[2a + (m-1)d]$$

$$\Rightarrow \frac{2n}{m} = 2a + (m-1)d \dots\dots\dots (1)$$

$$S_n = m = \frac{n}{2}[2a + (n-1)d]$$

$$\Rightarrow \frac{2m}{n} = 2a + (n-1)d \dots\dots\dots (2)$$

In the above equations a and d are unknowns. Calculating for d ,

Eqn. (1) – eqn. (2) gives

$$\Rightarrow \frac{2n}{m} - \frac{2m}{n} = (m-n)d$$

$$\Rightarrow \frac{2(n^2-m^2)}{mn} = (m-n)d$$

$$\Rightarrow -\frac{2(m^2-n^2)}{mn} = (m-n)d \text{ or } -\frac{2(m+n)(m-n)}{mn} = (m-n)d$$

$$\therefore d = -\frac{2(m+n)}{mn}$$

$$\text{Subst. } d = -\frac{2(m+n)}{mn} \text{ in to (1) or (2)}$$

From eqn. (1), $\frac{2n}{m} = 2a + (m-1)d$

$$\Rightarrow 2a = \frac{2n}{m} - (m-1)d$$

$$\Rightarrow 2a = \frac{2n}{m} - \frac{(m-1)[-2(m+n)]}{mn}$$

$$\begin{aligned} \Rightarrow 2a &= \frac{2n}{m} + \frac{2(m-1)(m+n)}{mn} \\ &= \frac{2n^2 + 2(m^2 + mn - m - n)}{mn} \end{aligned}$$

$$\text{Now } S_{(m+n)} = \frac{m+n}{2}[2a + (m+n-1)d]$$

$$\begin{aligned} S_{(m+n)} &= \frac{m+n}{2} \left[\frac{2n^2 + 2(m^2 + mn - m - n)}{mn} + (m+n-1) \times -\frac{2(m+n)}{mn} \right] \\ &= \frac{m+n}{mn} [n^2 + m^2 + mn - m - n - [(m+n)^2 - m - n]] \\ &= \frac{m+n}{mn} [n^2 + m^2 + mn - m - n - m^2 - 2mn - n^2 + m + n] \\ &= m + n \end{aligned}$$

b) Given: $S_m = S_n$; $m \neq n$

Required: $S_{(m+n)} = 0$

$$S_m = \frac{m}{2}[2a + (m-1)d] = \frac{n}{2}[2a + (n-1)d] = S_n$$

$$\Rightarrow m[2a + (m-1)d] = n[2a + (n-1)d]$$

$$\Rightarrow 2am + (m^2 - m)d = 2an + (n^2 - n)d$$

$$\begin{aligned} \Rightarrow 2a(m-n) &= (n^2 - n)d - (m^2 - m)d \\ &= [-(m^2 - n^2) + (m-n)]d \\ &= [-(m+n)(m-n) + (m-n)]d \end{aligned}$$

$$\Rightarrow 2a = [-(m+n) + 1]d$$

$$\therefore 2a = -(m+n-1)d$$

$$\text{But } S_{(m+n)} = \frac{m+n}{2} [2a + (m+n-1)d]$$

$$\Rightarrow S_{(m+n)} = \frac{m+n}{2} [2a - 2a] = 0, \text{ as required.}$$

2. a) If the first, second and the last term of an A.P are respectively a , b and x , show that the sum of the n - terms is $\frac{x+a}{2} + \frac{x^2-a^2}{2(b-a)}$.
- b) If the first and last terms of an A.P are a and L respectively and if S is the sum of all the n - terms, show that the common difference is $\frac{L^2-a^2}{2S-(L+a)}$.

Solution

- a) Given: $a_1 = a$, $a_2 = b$, $a_n = x$
 Required: $S_n = ?$
 $S_n = \frac{n}{2} [1^{st} + \text{last term}]$
 $S_n = \frac{n}{2} [a + x] \dots\dots\dots (1)$
 But $a_2 = a + d = b \dots\dots\dots (2)$
 Also $a_n = a + (n-1)d = x \dots\dots\dots (3)$
 In equations (2) and (3) d is the unknown.
 From (2), $d = b - a \dots\dots\dots (4)$
 Equation (4) in to (3)
 $\Rightarrow a + (n-1)d = x$, becomes
 $n = \frac{x-a}{d} + 1 \dots\dots\dots (5)$
 Equation (4) in to (5)
 $n = \frac{x-a}{d} + 1 = n = \frac{x-a+b-a}{b-a}$
 $\therefore n = \frac{x+b-2a}{b-a} \dots\dots\dots (6)$
 Now substitute equation (4) in to (1)
 $\Rightarrow S_n = \frac{n}{2} (a + x)$, becomes
 $\therefore S_n = \frac{\left[\frac{x-a}{b-a} + 1\right]}{2} \times (a + x)$
 $= \frac{(x-a)(x+a)}{2(b-a)} + \frac{a+x}{2}$
 $\therefore S_n = \frac{x+a}{2} + \frac{x^2-a^2}{2(b-a)}$, as required.
- b) Given: $a_1 = a$, $a_n = L$, $S_n = S$
 Required: $d = ?$
 From $a_n = a + (n-1)d = L \dots\dots\dots (1)$
 $S_n = \frac{n}{2} [2a + (n-1)d] = S \dots\dots\dots (2)$
 From equation (1) and (2), d is the unknown.
 From equation (1), $d = \frac{L-a}{n-1} \dots\dots\dots (3)$
 Now solving for n in equation (3)
 $\Rightarrow \frac{n}{2} [2a + (n-1)d] = S$, becomes
 $\therefore \frac{n}{2} \left[2a + (n-1) \times \frac{(L-a)}{n-1} \right] = S$
 $\Rightarrow n[2a + L - a] = 2S$
 $\therefore n = \frac{2S}{a+L} \dots\dots\dots (4)$
 Equation (4) in to (3)

$$\begin{aligned}\therefore d &= \frac{L-a}{\frac{2S}{a+L}-1} = \frac{L-a}{\frac{2S-(a+L)}{(a+L)}} \\ &= \frac{(L-a)(L+a)}{2S-(a+L)} = \frac{L^2-a^2}{2S-(a+L)}\end{aligned}$$

3. If p, q and r are sums $n, 2n$ and $3n$ terms of an A.P, prove that $r = 3(q - p)$

Solution

Given: $S_n = p, S_{2n} = q, S_{3n} = r$

From **R.H.S** $= 3(q - p)$

$$\Rightarrow p = \frac{n}{2}[2a + (n-1)d] \dots\dots\dots (2)$$

$$\Rightarrow q = \frac{2n}{2}[2a + (2n-1)d] \dots\dots\dots (3)$$

$$\Rightarrow r = \frac{3n}{2}[2a + (3n-1)d] \dots\dots\dots (4)$$

$$\text{Now } 3(q - p) = 3\left[\frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]\right]$$

$$\Rightarrow 3(q - p) = \frac{3n}{2}[2[2a + (2n-1)d] - [2a + (n-1)d]]$$

$$\Rightarrow 3(q - p) = \frac{3n}{2}[2a + (3n-1)d] = r, \text{ as required.}$$

Deducing last term of an A.P from summation of n - terms of an A.P

In general if S_n is the sum of n -terms of an A.P and S_{n-1} is the summation of $(n-1)$ - terms of an A.P, then;

Last term $= a_n = S_n - S_{n-1}$

Example

1. The sum of n -terms of a series is $n^2 + 2n$, prove that this series is an A.P, and find its first term and common difference.

Solution

Given: $S_n = n^2 + 2n$

$$\begin{aligned}\text{Also } S_{n-1} &= (n-1)^2 + 2(n-1) \\ &= n^2 - 2n + 1 + 2n - 2\end{aligned}$$

$$\therefore S_{n-1} = n^2 - 1$$

Now $a_n = S_n - S_{n-1}$

$$\Rightarrow a_n = n^2 + 2n - (n^2 - 1) = 2n + 1$$

$\therefore$ Last term, $a_n = 2n + 1$

Now for $n = 1, a_1 = 2 + 1 = 3$

$$n = 2, a_2 = 4 + 1 = 5$$

$$n = 3, a_3 = 3 + 1 = 7$$

The terms are: 3, 5, 7, ...; which form an A.P of first term 3 and common difference 2.

2. If the sum up to n -terms of an A.P is given by $\frac{n}{2}(3n + 5)$, show that the last term is $3n + 1$, and that the series is an A.P.

Solution

Given: $S_n = \frac{n}{2}(3n + 5)$

$$\text{Also } S_{n-1} = \frac{(n-1)}{2}[3(n-1) + 5] = \frac{(n-1)(3n+2)}{2}$$

Now $a_n = S_n - S_{n-1}$

$$\begin{aligned}\Rightarrow a_n &= \frac{n}{2}(3n + 5) - \frac{(n-1)(3n+2)}{2} \\ &= \frac{3n^2+5n}{2} - \frac{3n^2+2n-3n-2}{2}\end{aligned}$$

$$= \frac{1}{2}(6n + 2) = 3n + 1$$

$\therefore$ Last term, $a_n = 3n + 1$, as required.

By putting $n = 1, 2, 3, 4, \dots$, we get a series of;

4, 7, 10, $\dots$, which is an A.P.

Task

1. The sum of the first n -terms of a certain series is $n^2 + 5n$ for all integral values of n . find the first three terms and prove that the series is an A.P.

ANSWER: 6, 8, 10

2. The sum of the first n -terms of an A.P is $3n^2 + 2n$. Find the r^{th} - term. **ANSWER: $6r - 1$**

3. The second term of an A.P is -4 and the sixth term is -24 . Find the fifteenth term and sum of the first 15 - terms. **ANSWER: $-69, 540$**

4. The eighth term of an A.P is twice the third term, and the sum of the first 8-terms is 39. Find the first three terms of the progression, and show that the sum of n -terms is $\frac{3}{8}n(n + 5)$. **ANSWER: $\frac{9}{4}, 3, \frac{15}{4}$**

5. The sum of n -terms of a series 2, 5, 8, $\dots$ is 950. Find n . **ANSWER: 25**

6. The number of terms in an A.P is even; the sum of the odd terms is 24, of the even terms 30 and the last term exceeds the first by $10\frac{1}{2}$. Find the number of terms. **ANSWER: 8**

7. If p^{th} - term of an A.P is $\frac{1}{q}$ and the q^{th} - term is $\frac{1}{p}$, show that the sum of pq - terms is $\frac{1}{2}(pq + 1)$

8. Find the sum of all natural numbers from 100 up to 300 which are divisible by 4. **ANSWER: 10200**

9. In an A.P the sum of the first five terms is 30, and the third term is equal to the sum of the first two. Write down the first five terms of the progression. **ANSWER: 2, 4, 6, 8, 10**

10. The sum of the first three terms of an A.P is 3 and the sum of the first five terms is 20. Find the first five terms of the progression. **ANSWER: $-2, 1, 4, 4, 7, 10$**

11. The sum of the first six terms of an A.P is 21 and the seventh term is three times the sum of the third term and fourth term. Find the first term and the common difference. **ANSWER: $-9, 5$**

12. Sum of n -terms of three A.P's are S_1 , S_2 and S_3 . The first term of each of them is 1 and the common differences are 1, 2 and 3 respectively. Show that S_1 , S_2 , S_3 are in an A.P.

$$\left[\begin{array}{l} \text{HINT: } S_1 = \frac{n}{2}[2 + n - 1] = \frac{n(n+1)}{2}, S_2 = \frac{n}{2}[2 + (n - 1) \times 2 = n^2] \\ S_3 = \frac{n}{2}[2 + (3n - 1) \times 2 = \frac{n(3n-1)}{2}]; \text{ now show that } S_2 - S_1 = S_3 - S_2 \end{array} \right]$$

WORD PROBLEMS IN A.P'S

1. A firm produced 1000 sets of T.V during its first year. The total sum of the firm's production at the end of 10 years operation is 14 500 sets.
 - i) Estimate how many units production increased by each year if the increase each year is uniform.
 - ii) Fore cast based on the estimate of annual increments in production the level of output for the 15th year.

Solution

- i) Given: $a = 1000, S = 14\,500, n = 10$
 Required: $d = ?$
 Using $\Rightarrow S_n = \frac{n}{2}[2a + (n - 1)d]$
 $\Rightarrow 14\,500 = \frac{10}{2}[2 \times 1000 + (10 - 1)d]$
 $\Rightarrow 14\,500 = 5[2000 + 9d] \therefore d = 100$
 Hence 100 units is the increase per annum.
- ii) Required: $a_{15} = ?$
 $a_{15} = a + (15 - 1)d = a + 14d$
 $= 1000 + 14 \times 100 = 2\,400$
2. Two posts were offered to a man. In one the starting salary was £120 per month and the annual increment was £8.
 In the other post the salary commenced at £85 per month but the annual increment was £12. The man decided to accept the post which will give him more earnings in the first twenty years of the service. Which post was acceptable to him? Justify your answer.

Solution

The total earnings of the man in the first twenty years for the first job are $S_{20} = ?$:

Given: $a = £120, d = 8, n = 20$
 $\Rightarrow S_{20} = \frac{20}{2}[2 \times 120 + (20 - 1) \times 8] \times 12 = £47\,040$

His total earnings in the first twenty years for the second job are $S_{20} = ?$:

Given: $a = £85, d = 12, n = 20$
 $\Rightarrow S_{20} = \frac{20}{2}[2 \times 85 + (20 - 1) \times 12] \times 12 = £47\,760$

Total earnings for the second job are greater than the first job hence the second job was accepted by the man.

3. The annual salary of an employee is in an A.P over the first 7 – years, during which the annual increment is £200. Over the next 8 – years, the salary is in another A.P during which the annual increment is £300. After that the salary is constant.

If the starting salary is £14 000, calculate the employee's total earnings in the first 25 years of service.

Solution

Case I

Given: $a = 14\,000, d = 200, n = 7$

Required: $S_7 = ?$

$\Rightarrow S_7 = \frac{7}{2}[2 \times 14\,000 + (7 - 1) \times 200] = £102\,200$

Case II

At the end of the 7th – year, the annual salary is given from $a_n = a + (n - 1)d$

$$\Rightarrow a_7 = 14\,000 + (7 - 1) \times 200 = \text{£}15\,200$$

Now the next 8 – years , the sum of all his earnings is given by:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\Rightarrow S_8 = \frac{8}{2}[2 \times 15\,200 + (8 - 1) \times 300] = \text{£}130\,000$$

Now at the end of the 15 – years. i. e. 1st 7 – years + next 8 – years , the total earnings are given by;

$$\text{£}130\,000 + \text{£}102\,200 = \text{£}232\,200$$

Remaining years = 25 – 17 = 10 years

The amount in the 15th – year is given by $a_n = a + (n - 1)d$; $a = 15\,200$, $n = 8$, $d = 300$

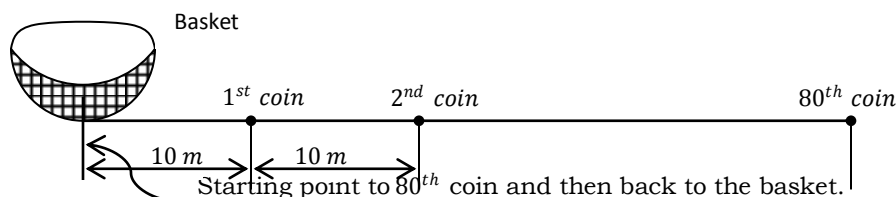
$$\Rightarrow a_{15} = 15\,200 + (8 - 1) \times 300 = \text{£}17\,300$$

Now for the remaining 10 years, total amount = $17\,300 \times 10 = \text{£}173\,000$

$\therefore$ Total earnings for 25 years is = $173\,000 + 232\,200 = \text{£}405\,200$

4. *Eighty coins are placed in a line on the ground. The distance between any two consecutive coins is 10 meters. How far must a person travel to bring them one by one to a basket placed 10 meters behind the first coin?*

Solution



Given: $a = 10$, $d = 10$ (distance between 2 successive coins), $n = 80$ (No. of coins)

$$\text{Required: } S_{80} = \frac{80}{2}[2 \times 10 + (80 - 1) \times 10] = 32\,400\text{ m}$$

Total distance = $32\,400$ (to 80th coin) + $32\,400$ (back to basket) = $64\,800\text{ m}$

5. *A man saved £16 500 in 10 years. In each year after the first he saved £100 more than he did in the preceding year. How much did he save in the first year?*

Solution

Given: $S_{10} = \text{£}16\,500$, $d = \text{£}100$, $n = 10$

Let a = Amount saved in 1st year

$$\text{From } S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\Rightarrow 16\,500 = \frac{10}{2}[2a + (10 - 1) \times 100]$$

$$\Rightarrow 3\,300 = 2a + 900$$

$$\therefore d = \text{£}1\,200$$

6. *Miss Sikiyomu Jane takes a loan of £2 000 from Anita and agrees to repay in a number of installment, each installment (beginning with the second) exceeding the previous by £10. If the first installment is £5, find how many installments will be necessary to wipe out the loan completely?*

Solution

Given: $S_{10} = \text{£}2\,000$, $d = \text{£}10$, $n = ?$, $a = \text{£}5$

$$\text{Using } S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\Rightarrow 2\,000 = \frac{n}{2}[2 \times 5 + (n - 1) \times 10]$$

$$\Rightarrow 2000 = 5n + 5n^2 - 5$$

$$\Rightarrow n^2 + n - 401 = 0$$

$$\Rightarrow n = \frac{-1 \pm \sqrt{1+4 \times 401}}{2}$$

$\therefore n \approx 20$, for a negative n , it is rejected

$\therefore$ Number of installments will be 20.

7. *The monthly salary of a person was £320 for each of the first three years. He next got annual increments of £40 per month for each of the following successive 12 years. His salary remained stationary till retirement when he found that his average monthly salary during the service period was £698. Find the period of his service.*

Solution

Let n = total no. of years of the person's service

Given: Average monthly salary = £698 = $\frac{\text{Amount for } n\text{-years}}{\text{No. of years}}$

$$\Rightarrow 698 \times 12 = \frac{\text{Amount for } n\text{-years}}{n}$$

$$\therefore \text{Amount for } n\text{-years} = 698 \times 12n$$

Total salary in the first 3 - years = £320 $\times$ 3 $\times$ 12 = £960 $\times$ 12

In the 4th year, his monthly salary = £(320 + 40) = £360

In the 5th year, his monthly salary = £(360 + 40) = £400

For the next 12 - years, his total salary is;

$$= £12 \times [360 + 400 + \dots \dots \dots \text{up to } 12^{\text{th}} - \text{term}]$$

$$= 12 \times \left[\frac{12}{2} [2 \times 360 + (12 - 1) \times 40] \right]$$

$$= £6960 \times 12$$

At the end of following 12 years, his monthly salary was;

$$= £[360 + (12 - 1) \times 40] = £800$$

Now total salary for all the 15 years so far is

$$= £6960 \times 12 + £960 \times 12 = £7920 \times 12$$

Amount received after 15 years .i.e. $(n - 15)$ years was constant for each year .i.e. amount in the 15th year = £800

$$\Rightarrow \text{Total amount} = £(n - 15) \times 800 \times 12$$

Total amount throughout his service

$$= (n - 15) \times 800 \times 12 + 7920 \times 12 = 698 \times 12n$$

$$\Rightarrow 800n - 12000 + 7920 = 698n$$

$$\therefore n = 40$$

8. *Mr. Peter arranges to pay off a debt of £9600 in 48 annual installments which form an arithmetical series. When 40 of these installments are paid, Mr. Peter becomes insolvent and his creditor finds that £2400 still remains unpaid. Find the value of each of the first three installments of Mr. Peter. Ignore the interest.*

Solution

Let $a, a + d, a + 2d, a + 3d, \dots \dots \dots$, be the annual installments

Given: $S_{48} = £9600$

$$\Rightarrow 9600 = \frac{48}{2} [2a + (48 - 1)d] \text{ or } 9600 = 24[2a + 47d]$$

$$\Rightarrow 2a + 47d = 400 \dots \dots \dots (1)$$

After 40 installments are paid, the balance is £2400 .i.e. the amount paid in 40 installments is;

$$= 9\,600 - 2\,400 = \text{£}7\,200$$

⇒ Sum of 40 terms of an A.P is £7 200

$$\Rightarrow 7\,200 = \frac{40}{2}[2a + (20 - 1)d] \text{ or } 7\,200 = 20[2a + 39d]$$

$$\Rightarrow 2a + 39d = 360 \dots\dots\dots (2)$$

Subtracting the equations

$$\Rightarrow 8d = 40 \therefore d = 5$$

$$\text{From (2) } 2a + 39d = 360, a = 82.5$$

Hence the first installment of Mr. Peter is £82.5 , the second installment is £(82.5 + 5) .i.e. £87.5 and the third installments is £(87.5 + 5) .i.e. £92.5

Task

1. Phionah secures an interest free loan of £14 500 from a friend and agrees to repay it in 10 installments. She pays £1 000 as first installment and then increases each installment by equal amount over the preceding installment. What will be her last installment?

$$\left[\begin{array}{l} \text{HINT: } s_{10} = 14\,500 = 5(2a + 9d), a = 1000 \therefore d = 100 \\ \text{Required: } a_{10} = a + (10 - 1)d = 1000 + 9 \times 100 = \text{£}1\,900 \end{array} \right]$$

2. An enterprise produced 600 units in the third year of its existence and 700 units in the seventh year. What was the initial production in the first year?

$$\text{ANSWER: } \text{£}550 \left[\begin{array}{l} \text{HINT: } a_3 = 600 + 2d, a_7 = a + 6d, \text{ solving the equations} \\ a = 550, d = 45 \therefore 1^{\text{st}} \text{ production is } \text{£}550 \end{array} \right]$$

3. A piece of equipment costs £600 000. It depreciates in value, 15% the first year, 13 ½ % the next year, 12% the third year and so on. What will be the value at the end of 10 years all percentages applying to the original cost? **ANSWER £ 105 000**

4. A money lender lends £1 000 and charges an overall interest of £140. He recovers the loan and interest by 12 monthly installments each less by £10 than the preceding one. Find the amount of the first installment. **ANSWER: £ 150**

5. A man saves £20 in the first month, £30 in the second month, £40 in the third month, and so on. How much has he saved at the end of 5 years? **ANSWER: £ 18 900**

6. A man saves £8 250 in 10 years. In each year after the first he saved £50 more than he did in previous year. How much did he save the first year? **ANSWER: £ 600**

7. A man borrows £5 000 and agrees to repay with a total interest as £1 000 in 12 installments, each installment being less than the preceding by £50. What should be the first installment? **ANSWER : £ 775**

8. A man borrows £4 500 and promises to repay back in 30 installments, each of value £10 more than the previous one. Find the first and last installment. **ANSWER: £ 5 , £ 295**

9. A man of 60 years old has 9 children born at equal intervals. The sum of the ages of father and the nine children is known to be 222 years. Calculate the age of the youngest son if the eldest one is 30 years. **ANSWER: 6 years**

THE GEOMETRIC PROGRESSION (G.P)

A series in which non-zero quantities $a_1, a_2, a_3, \dots, a_n, \dots$ each term of which is equal to the product of the preceding term and a constant number is a **Geometric Progression**. *i.e.* Each term in a G.P is obtained from the preceding one by multiplying or dividing by a constant number. *e.g.*

- $1, 2, 4, 8, 16, \dots$ is a G.P
- $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \dots$ is a G.P
- $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$ is a G.P
- $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \dots$ is a G.P
- $a, ar, ar^2, ar^3, ar^4, \dots$ is a G.P

The constant term is called a **common ratio**, r of a G.P

Consider the G.P below;

$$a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$$

This will be a G.P if:

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots = \frac{a_n}{a_{n-1}}, \text{ for all integers } n > 1$$

From the above examples of G.P series,

- $1, 2, 4, 8, 16, \dots$, has common ratio $r = \frac{2}{1} = \frac{4}{2} = 2$
- $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \dots$, has common ratio $r = -\frac{1}{3} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$
- $1, \sqrt{2}, 2, 2\sqrt{2}, \dots$, has common ratio $r = \frac{\sqrt{2}}{1} = \frac{2}{\sqrt{2}}$
- $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \dots$, has common ratio $r = \frac{\frac{1}{5}}{1} = \frac{\frac{1}{25}}{\frac{1}{5}} = \frac{1}{5}$
- $a, ar, ar^2, ar^3, ar^4, \dots$, has common ratio $r = \frac{ar}{a} = \frac{ar^2}{ar} = r$

To find the n^{th} - term of a G.P (General term of a G.P)

Let $a_1 + a_2 + a_3 + \dots$

Let $r = \text{common ratio}$

By definition,

$$r = \frac{a_2}{a_1} \Rightarrow a_2 = a_1 r$$

$$r = \frac{a_3}{a_2} \Rightarrow a_3 = a_2 r = a_1 r \cdot r = a_1 r^2$$

$$r = \frac{a_4}{a_3} \Rightarrow a_4 = a_3 r = a_1 r^2 \cdot r = a_1 r^3$$

Observation of terms

$$a_2 = a_1 r = a_1 r^{2-1}$$

$$a_3 = a_1 r^2 = a_1 r^{3-1}$$

$$a_4 = a_1 r^3 = a_1 r^{4-1}$$

⋮

$$a_n = a_1 r^{n-1}$$

Thus the n^{th} - term of a G.P is

$$a_n = ar^{n-1}, \text{ where } a = 1^{\text{st}} - \text{term}, n = \text{no. of terms}, r = \text{common ratio}$$

NOTE:

1. The n^{th} - term/ general term of a G.P is $a_n = ar^{n-1}$
It contains 4 - terms, is 3 - terms are given, the 4^{th} - term can be evaluated.
2. From the general term of a G.P, the standard form of a G.P series can be stated as;

$$a + ar + ar^2 + ar^3 + ar^4, \dots \dots \dots + ar^{n-1}$$

3. The most convenient terms of a G.P given their product are:

- a) If *three terms* are in a G.P, then use

$$\frac{a}{r}, a, ar$$

- b) If *four terms* are in a G.P, then use

$$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$$

NOTICE:

1. In both cases, when terms are multiplied, the common ratio vanishes.
2. In the case of *odd number terms*, the middle terms a and the common ratio is r .
3. In the case of *even number terms*, the middle terms are $\frac{a}{r}$, ar and the common ratio is r^2 .

Properties of a G.P

1. If every term of a G.P is multiplied or divided by a constant, the resulting series is a G.P.
2. If the terms of a G.P are raised to same power, the resulting series is a G.P.
3. If the reciprocals of terms in a G.P are got, the resulting series is a G.P.

Examples

1. Write down the terms indicated in each of the following series.

a) $2 + 4 + 8 + \dots \dots \dots 11^{\text{th}} - \text{term}$

b) $\frac{2}{3} + \frac{3}{4} + \frac{27}{32} + \dots \dots \dots 12^{\text{th}} - \text{term}$

c) $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, \dots \dots \dots 8^{\text{th}} - \text{term}$

Solution

a) $2 + 4 + 8 + \dots \dots \dots 11^{\text{th}} - \text{term}$

This is not an A.P because $4 - 2 \neq 8 - 4$

This is a G.P because $\frac{4}{2} = \frac{8}{4} = 2$

Using $a_n = ar^{n-1}$

Given: $a_n = ?, n = 11, r = 2, a = 2$

$$\therefore a_{11} = 11 \times (2)^{11-1} = 11 \times 2^{10} = 11264$$

b) $\frac{2}{3} + \frac{3}{4} + \frac{27}{32} + \dots \dots \dots 12^{\text{th}} - \text{term}$

Given: $a_n = ?$, $n = 12$, $r = \frac{\frac{3}{4}}{\frac{2}{3}} = \frac{9}{8}$, $a = \frac{2}{3}$

Required: $a_{12} = ?$

$\therefore a_{12} = \left(\frac{2}{3}\right) \times \left(\frac{9}{8}\right)^{12-1} = \frac{2}{3} \times \left(\frac{9}{8}\right)^{11} = 11264$

c) $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, \dots \dots \dots 8^{th} \text{ term}$

Given: $a_n = ?$, $n = 8$, $r = \frac{-2}{\frac{1}{\sqrt{2}}} = \frac{\frac{8}{\sqrt{2}}}{-2} = -\frac{4}{\sqrt{2}}$, $a = \frac{1}{\sqrt{2}}$

Required: $a_8 = ?$

$\therefore a_8 = \left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{4}{\sqrt{2}}\right)^{8-1} = \frac{1}{\sqrt{2}} \times \left(-\frac{4}{\sqrt{2}}\right)^7 = \left(\frac{1}{\sqrt{2}}\right)^8 \times (-4)^7 = -1024$

2. Find the number of terms in the G.P's below.

a) $0.03 + 0.06 + 0.12 + \dots \dots \dots + 1.92$

b) $81 + 27 + 9 + \dots \dots \dots + \frac{1}{27}$

c) $a + ar + ar^2 + \dots \dots \dots + ar^{n-1}$

Solution

a) $0.03 + 0.06 + 0.12 + \dots \dots \dots + 1.92$

Given: $a = 0.03$, $r = \frac{0.06}{0.03} = 2$, $a_n = 1.92$

Required: $n = ?$

Using $a_n = ar^{n-1}$

$\Rightarrow 1.92 = 0.03 \times (2)^{n-1}$

$\Rightarrow \frac{1.92}{0.03} = 2^{n-1} \text{ or } 2^6 = 2^{n-1}$

$\therefore n - 1 = 6 \text{ or } n = 7$

$\therefore \text{No. of terms is } 7$

b) $81 + 27 + 9 + \dots \dots \dots + \frac{1}{27}$

Given: $a = 81$, $r = \frac{27}{81} = \frac{1}{3}$, $a_n = \frac{1}{27}$

Required: $n = ?$

Using $a_n = ar^{n-1}$

$\Rightarrow \frac{1}{27} = 81 \times \left(\frac{1}{3}\right)^{n-1}$

$\Rightarrow \frac{1}{27 \times 81} = \frac{1}{3^{n-1}} \text{ or } 3^3 \times 3^4 = 3^{n-1}$

$\Rightarrow 7 = n - 1 \therefore n = 8$

c) $a + ar + ar^2 + \dots \dots \dots + ar^{n-1}$

Using $A_N = A.R^{N-1}$

Given: $A = a$, $R = \frac{ar}{a} = r$, $A_N = ar^{n-1}$

$\Rightarrow a.r^{n-1} = a.r^{N-1}$

$\Rightarrow n - 1 = N - 1 \therefore n = N$

$\therefore \text{No. of terms is } n$

3. The third term of a G.P is 10, and the sixth term is 80. Find the common ratio, the first term and the twelfth term.

Solution

Given: $a_3 = 10$, $a_6 = 80$

Using $a_n = ar^{n-1}$

$\Rightarrow 10 = a.r^{3-1} = a.r^2 \dots \dots \dots (1)$

$\Rightarrow 80 = a.r^{6-1} = a.r^5 \dots \dots \dots (2)$

eqn. (2) $\div$ eqn. (1)

$$\Rightarrow \frac{80}{10} = \frac{a \cdot r^5}{a \cdot r^2} = r^3$$

$$\Rightarrow 8 = r^3 \therefore r = 2$$

From eqn. (1)

$$a = \frac{10}{r^2} = \frac{10}{4} = \frac{5}{2}$$

$$\text{Now } 12^{\text{th}} - \text{term} = a_{12} = a \cdot r^{12-1} = a \cdot r^{11}$$

$$\therefore 12^{\text{th}} - \text{term} = \frac{5}{2} \times 2^{11} = 5 \times 2^{10}$$

$$\therefore 1^{\text{st}} - \text{term} = \frac{5}{2}, 12^{\text{th}} - \text{term} = 5(2)^{10}$$

4. The 3rd - term of a G.P is 2, and the 5th - term is 18. Find two possible values of common ratio, and the 2nd - term in each case.

Solution

$$\text{Given: } a_3 = 2, a_5 = 18$$

$$\text{Required: } r = ?, a_2 = ?$$

$$\text{Using } a_n = ar^{n-1}$$

$$\Rightarrow 2 = a \cdot r^{3-1} = a \cdot r^2 \dots\dots\dots (1)$$

$$\Rightarrow 18 = a \cdot r^{5-1} = a \cdot r^4 \dots\dots\dots (2)$$

eqn. (2) $\div$ eqn. (1)

$$\Rightarrow \frac{18}{2} = \frac{a \cdot r^4}{a \cdot r^2} = r^2$$

$$\Rightarrow 9 = r^2 \therefore r = \pm 3$$

Now from eqn. (1)

$$a = \frac{2}{r^2} = \frac{2}{(\pm 3)^2} = \frac{2}{9}$$

$$\text{Now } a_2 = ar^{2-1} = ar = \frac{2}{9} \times \pm 3 = \pm \frac{2}{3}$$

$\therefore$ For 1st G.P with $r = 3, a = 2$, the 2nd - term is $\frac{2}{3}$ and in the 2nd G.P with $r = -3, a = 2$, 2nd - term is $-\frac{2}{3}$

5. The ratio of the 4th to the 12th - term of a G.P with a positive common ratio is $\frac{1}{256}$. If the sum of the two terms is 61.68. find the 10th - term.

Solution

$$\text{Given: } \frac{a_4}{a_{12}} = \frac{1}{256} \Rightarrow \frac{a \cdot r^3}{a \cdot r^{11}} = \frac{1}{256} \text{ or } \frac{1}{r^8} = \frac{1}{256}$$

$$\Rightarrow r^8 = 256 \text{ . let } r^4 = x$$

$$\Rightarrow x^2 = 256 \therefore x = \pm 16$$

$$\text{Now } r^4 = \pm 16 \text{ or } r^4 = 16 \text{ (Taking the + ve value)}$$

$$\therefore r = \pm 2$$

Since r is required as +ve, then $r = 2$ is taken, neglecting the negative sign.

$$\text{Also given: } a_4 + a_{12} = 61.68$$

$$\Rightarrow a \cdot r^3 + a \cdot r^{11} = 61.68$$

$$\Rightarrow a(r^3 + r^{11}) = 61.68$$

$$\therefore a = \frac{61.68}{(r^3 + r^{11})} = \frac{61.68}{(2^3 + 2^{11})} = 0.03$$

$$\text{Now } a_{10} = ar^9 = 0.03 \times 2^9 = 15.36$$

$$\therefore \text{The } 10^{\text{th}} - \text{term is } 15.36$$

The Sum and Product of convenient terms of a G.P

Examples:

1. Find three numbers in a G.P such that
 - a) Their sum is 19 and their product is 216
 - b) Their sum is 21 and the sum of their squares is 651
 - c) The sum of their squares is 819 and their product is 729.

Solution

Let the numbers be $\frac{a}{r}, a, ar$

a) Given: $\frac{a}{r} + a + ar = 19$ or $a\left(\frac{1}{r} + r + 1\right) = 19$ (1)

Also $\frac{a}{r} \cdot a \cdot ar = 216$ or $a^3 = 216 \therefore a = \sqrt[3]{216} = 6$ (2)

Substituting eqn. (2) in to eqn. (1)

$$\Rightarrow \left(\frac{1}{r} + r + 1\right) \cdot 6 = 19$$

$$\Rightarrow \left(\frac{r^2+r+1}{r}\right) \times 6 = 19$$

$$\Rightarrow 6r^2 + 6r + 6 = 19r \text{ or } 6r^2 - 13r + 6 = 0$$

$$\Rightarrow 6r^2 - 9r - 4r + 6 = (3r - 2)(2r - 3) = 0$$

$$\therefore r = \frac{2}{3}, \frac{3}{2}$$

$$\therefore \text{The numbers are: } \frac{6}{\frac{2}{3}}, 6, 6 \times \frac{2}{3} \text{ for } r = \frac{2}{3} \text{ or } 9, 6, 4$$

For $r = \frac{3}{2}$, the numbers are in the reverse order of 9, 6, 4

b) Given: $\frac{a}{r} + a + ar = 21$ or $a\left(\frac{1}{r} + r + 1\right) = 21$ (1)

Also $\left(\frac{a}{r}\right)^2 + (a)^2 + (ar)^2 = 651$ or $a^2\left(\frac{1}{r^2} + r^2 + 1\right) = 651$ (2)

Squaring equation (1)

$$\Rightarrow a^2\left(\frac{1}{r} + r + 1\right)^2 = 441$$

$$\Rightarrow a^2\left[\left(\frac{1}{r} + r\right)^2 + 2\left(\frac{1}{r} + r\right) + 1\right] = 441$$

$$\Rightarrow a^2\left[\frac{1}{r^2} + r^2 + 2 + 2\left(\frac{1}{r} + r\right) + 1\right] = 441$$

$$\Rightarrow a^2\left[\left(\frac{1}{r^2} + r^2 + 1\right) + 2\left(\frac{1}{r} + r + 1\right)\right] = 441 \text{ (3)}$$

Dividing eqn. (2) by eqn. (3)

$$\Rightarrow \frac{a^2\left(\frac{1}{r^2} + r^2 + 1\right)}{a^2\left[\left(\frac{1}{r^2} + r^2 + 1\right) + 2\left(\frac{1}{r} + r + 1\right)\right]} = \frac{651}{441} \text{ or } 441\left(\frac{1}{r^2} + r^2 + 1\right) = 651\left(\frac{1}{r^2} + r^2 + 1\right) +$$

$$1302\left(\frac{1}{r} + r + 1\right)$$

$$\Rightarrow (651 - 441)\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right) = 0$$

$$\therefore 210\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right)$$

$$\text{Let } y = \frac{1}{r} + r \Rightarrow y^2 - 2 = \frac{1}{r^2} + r^2$$

$$\Rightarrow 210(y^2 - 2 + 1) + 1302(y + 1) = 0$$

$$\Rightarrow 210y^2 + 1302y + 1092 = 0$$

Solving the equation,

$$y = -1, -\frac{26}{5}$$

$$\text{Now for } y = -1, y = \frac{1}{r} + r$$

$$\Rightarrow \frac{1}{r} + r + 1 = 0 \text{ or } r^2 + r + 1 = 0$$

Solving the equation, no real values of r

$$\text{For } y = -\frac{26}{5}$$

$$\Rightarrow \frac{1}{r} + r + \frac{26}{5} = 0 \text{ or } 5r^2 + 26r + 5 = 0$$

$$\text{Solving the equation, } r = -5, -\frac{1}{5}$$

$$\text{For } r = -5 \text{ in eqn. (1) .i.e. } a\left(\frac{1}{r} + r + 1\right) = 21$$

$$\Rightarrow a\left(-\frac{1}{5} - 5 + 1\right) = 21 \therefore a = -5$$

$$\therefore \text{The numbers are: } -\frac{5}{5}, -5, -5 \times -\frac{1}{5} \text{ or } 25, -5, 1$$

$$\text{For } r = -\frac{1}{5}, \text{ the numbers are in the reverse order}$$

c) *Left as an exercise.*

$$\text{ANSWER: } r = \pm 3, r = \pm \frac{1}{3} : \text{No.s are: } 3, 9, 27 \text{ or } 27, 9, 3$$

2. Find five numbers in a G.P such that:

a) Their product is 32 and the product of the last two terms is 108

b) Their sum is $7\frac{3}{4}$, their product is unity.

Solution

$$\text{Let the numbers be: } \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$$

$$\text{a) Given: } \frac{a}{r^2} \cdot \frac{a}{r} \cdot a \cdot ar \cdot ar^2 = 32 \Rightarrow a^5 = 32 \therefore a = 2$$

$$\text{Also } ar \cdot ar^2 = 108 \Rightarrow r^3 = \frac{108}{a^2} = \frac{108}{4} = 27$$

$$\therefore r = 3$$

$$\Rightarrow \text{The numbers are: } \frac{2}{9}, \frac{2}{3}, 2, 6, 18$$

$$\text{b) Given: } \frac{a}{r^2} + \frac{a}{r} + a + ar + ar^2 = \frac{31}{4}$$

$$\Rightarrow a\left(\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2\right) = \frac{31}{4} \dots\dots\dots (1)$$

$$\text{Also } \frac{a}{r^2} \cdot \frac{a}{r} \cdot a \cdot ar \cdot ar^2 = 1 \Rightarrow a^5 = 1 \therefore a = 1 \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2 = \frac{31}{4} \text{ or } \frac{1}{r} + r + \frac{1}{r^2} + r^2 = \frac{27}{4}$$

$$\text{Let } y = \frac{1}{r} + r \Rightarrow y^2 - 2 = \frac{1}{r^2} + r^2$$

$$\Rightarrow y + y^2 - 2 = \frac{27}{4} \text{ or } y + y^2 = \frac{35}{4}$$

$$\Rightarrow 4y^2 + 4y - 35 = 0 \text{ or } (2y - 5)(2y + 7) = 0$$

$$\therefore y = \frac{5}{2}, y = -\frac{7}{2}$$

$$\text{For } r = \frac{5}{2}, \Rightarrow r + \frac{1}{r} = \frac{5}{2} \text{ or } 2r^2 - 5r + 2 = 0$$

$$\Rightarrow 2r^2 - 4r - r + 2 = 0$$

$$\Rightarrow (2r - 1)(r - 2) = 0 \therefore r = 2, \frac{1}{2}$$

$$\text{For } r = 2, \text{ the numbers are: } \frac{1}{4}, \frac{1}{2}, 1, 2, 4$$

$$\text{Also for } r = -\frac{7}{2} \Rightarrow r + \frac{1}{r} = -\frac{7}{2} \text{ or } 2r^2 + 7r + 2 = 0$$

$$r = \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 2}}{4} = \frac{-7 \pm \sqrt{33}}{4}$$

The values of r can be neglected since they are irrational, the numbers are:

$$\frac{1}{4}, \frac{1}{2}, 1, 2, 4$$

Task

1. The sum of three numbers in a G.P is 35 and their product is 1000. Find the numbers. **ANSWER:** For $r = 2$, nos. are 5, 10, 20; for $r = \frac{1}{2}$, nos. are 20, 10, 5
2. The sum of three numbers in a G.P is $\frac{124}{5}$ and their product is 64. Find the numbers. **ANSWER:** For $r = 5$ or $\frac{1}{5}$, nos. are $\frac{4}{5}$, 4, 20
3. Find three numbers in a G.P such that;
 - i) Their sum is $\frac{2}{5}$ and their product is -8. **ANSWER:** $\frac{2}{5}$, -2, 10
 - ii) Their sum is 28 and their product is 512. **ANSWER:** 4, 8, 16 or 16, 8, 4

TO FIND THE SUM OF n - terms OF A G.P

Let $a, ar, ar^2, \dots, ar^{n-1}$ be the n - terms in a G.P.

Let $S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$

..... (1)

S_n is the sum of n - terms

Multiplying eqn. (1) by r

$\Rightarrow rS_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n$

(2)

eqn. (1) - eqn. (2)

$\Rightarrow S_n - rS_n = a - ar^n$

$\Rightarrow S_n(1 - r) = a(1 - r^n)$ or $S_n = \frac{a(1-r^n)}{1-r}$

$S_n = \frac{a(1-r^n)}{1-r}, \text{ where } r \neq 1$

NOTE:

- 1) In case $r = 1$, $S_n = a + a + a + \dots + a(n - \text{times}) = na$
- 2) In case $|r| < 1$, then $S_n = \frac{a(1-r^n)}{1-r}$
- 3) In case $|r| > 1$, then $S_n = \frac{a(r^n-1)}{r-1}$

Examples

1. Find the sum of 11 terms of a G.P
 - a) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$
 - b) $3 + 33 + 333 + \dots$

Solution

- a) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$

Common ratio: $r = -\frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2}$

Given: $r = -\frac{1}{2}$, $a = 1$, $n = 11$

Using $S_n = \frac{a(1-r^n)}{1-r}$ since $|r| < 1$

$$\begin{aligned} \Rightarrow S_n &= \frac{1\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} = \frac{1+\frac{1}{2^{11}}}{1+\frac{1}{2}} \\ &= \frac{2^{11}+1}{2^{11} \times \frac{3}{2}} = \frac{2^{11}+1}{3 \times 2^{10}} = \frac{2050}{3072} = \frac{1025}{1536} \end{aligned}$$

- b) $3 + 33 + 333 + \dots$

$$\text{Common ratio: } r = \frac{33}{3} = \frac{333}{33} = 11$$

$$\text{Given: } r = 11, a = 3, n = 11$$

$$\text{Using } S_n = \frac{a(r^n - 1)}{r - 1} \text{ since } |r| > 1$$

$$\Rightarrow S_n = \frac{3(11^{11} - 1)}{11 - 1} = \frac{3}{10}(11^{11} - 1)$$

2. Find how many terms of the G.P $1 + 3 + 9 + \dots$ are required to make a total of more than 1 million.

Solution

$$1 + 3 + 9 + \dots, \text{ has } r = 3$$

$$\text{Using } S_n = \frac{a(r^n - 1)}{r - 1}, |r| > 1$$

$$\Rightarrow S_n = \frac{1(3^n - 1)}{3 - 1} = \frac{1}{2} \times (3^n - 1)$$

$$\Rightarrow 10^6 = \frac{1}{2} \times (3^n - 1)$$

$$\Rightarrow 2 \times 10^6 + 1 = 3^n$$

$$\Rightarrow 3^n = 2\,000\,001$$

$$\Rightarrow \log 3^n = \log 2\,000\,001$$

$$\therefore n = \frac{\log 2\,000\,001}{\log 3} = 13.206$$

Thus the number of terms required to make a total of more than 1 million will be 14.

3. What is the smallest number of terms of the G.P $5, 10, 20, \dots$ can give a sum greater than 500 000

Solution

$$5, 10, 20, \dots, \text{ has } r = 2$$

$$\text{Using } S_n = \frac{a(r^n - 1)}{r - 1}, |r| > 1$$

$$\Rightarrow S_n = \frac{5(2^n - 1)}{2 - 1} = 5 \times (2^n - 1)$$

$$\Rightarrow 500\,000 = 5 \times (2^n - 1)$$

$$\Rightarrow 100\,000 + 1 = 2^n$$

$$\Rightarrow 2^n = 100\,001$$

$$\Rightarrow \log 2^n = \log 100\,001$$

$$\therefore n = \frac{\log 100\,001}{\log 2} = 16.6097$$

Thus the number of terms required to make a total exceeding 500 000 is 17

4. The sum of the first eight terms of a G.P (of real terms) is five times the sum of the first four terms. Find the common ratio.

Solution

$$\text{Let the G.P be } a, ar, ar^2, ar^3, \dots$$

$$S_8 = \text{Sum of first 8 terms} = \frac{a(1 - r^8)}{1 - r}$$

$$S_4 = \text{Sum of first 4 terms} = \frac{a(1 - r^4)}{1 - r}$$

$$\text{Given: } S_8 = 5S_4$$

$$\Rightarrow \frac{a(1 - r^8)}{1 - r} = \frac{5a(1 - r^4)}{1 - r}$$

$$\Rightarrow 1 - r^8 = 5(1 - r^4)$$

$$\Rightarrow 1 - r^8 = 5 - 5r^4$$

$$\therefore r^8 - 5r^4 + 4 = 0$$

$$\text{Let } r^4 = x$$

$$\Rightarrow x^2 - 5x + 4 = 0 \text{ or } (x - 1)(x - 4) = 0$$

$$\therefore x = 1, 4$$

$$\Rightarrow r^4 - 1 = 0 \text{ or } (r^2 - 1)(r^2 + 1) = 0$$

$$\therefore r^2 - 1 = 0 \text{ or } r = \pm 1; r^2 + 1 = 0, \text{ has no real roots}$$

Now for $r = 1$, the series is $a + a + a + a + \dots$

But then, $S_8 = 8a$ and $S_4 = 4a$

So $S_8 \neq 4S_4$, hence $r \neq 1$

In case $r = -1$, we get $S_8 = 0$ and $S_4 = 0$

Thus $S_8 = 5S_4$ for $r = -1$

$$\text{Again } r^4 - 4 = 0 \text{ or } (r^2 - 2)(r^2 + 2) = 0$$

$$\therefore r^2 - 2 = 0 \text{ or } r = \pm\sqrt{2}$$

$$\therefore r = -1; \pm\sqrt{2}$$

5. The sum of the last three terms of a G.P having n - terms is 1024 times the sum of the first 3 - terms of the progression. If the third term is 5, find the last term.

Solution

Let the G.P be $a, ar, ar^2, ar^3, \dots, ar^{n-3}, ar^{n-2}, ar^{n-1}$

$$\text{Given: } ar^{n-3} + ar^{n-2} + ar^{n-1} = 1024 \times S_3$$

$$a_3 = ar^2 = 5$$

$$\text{Also } S_3 = \frac{a(1-r^3)}{1-r}$$

$$\Rightarrow ar^{n-3} + ar^{n-2} + ar^{n-1} = 1024 \times \frac{a(1-r^3)}{1-r}$$

$$\Rightarrow r^{n-3} + r^{n-2} + r^{n-1} = 1024 \times \frac{(1+r+r^2)(1-r)}{1-r}$$

$$\Rightarrow r^{n-3}[1+r+r^2] = 1024(1+r+r^2)$$

$$\Rightarrow r^{n-3} = 1024 \dots\dots\dots (1)$$

$$\text{Also } ar^2 = 5 \dots\dots\dots (2)$$

$$\text{Now last term, } a_n = ar^{n-1} \dots\dots\dots (3)$$

Eqn. (2) $\div$ eqn. (3) to eliminate a

$$\Rightarrow \frac{ar^2}{ar^{n-1}} = \frac{5}{a_n}$$

$$\therefore a_n = 5 \times r^{n-3} \dots\dots\dots (4)$$

Eqn. (1) in to eqn. (4)

$$\Rightarrow a_n = 5 \times r^{n-3} = 5 \times 1024 = 5120$$

$\therefore$ Last term is 5120

6. The second term and the third term of a G.P are 24 and $12(b + 1)$ respectively. Find b if the sum of the first 3 - terms of progression is 76.

Solution

$$\text{Given: } a_2 = 24, a_3 = 12(b + 1), S_3 = 76$$

$$a_2 = ar = 24 \dots\dots\dots (1)$$

$$a_3 = ar^2 = 12(b + 1) \dots\dots\dots (2)$$

$$S_3 = \frac{a(1-r^3)}{1-r} = 76 \dots\dots\dots (3)$$

Eqn. (1) $\div$ eqn. (2) to eliminate a

$$\Rightarrow \frac{ar}{ar^2} = \frac{24}{12(b+1)} \text{ or } \frac{1}{r} = \frac{2}{b+1}$$

$$\Rightarrow r = \frac{b+1}{2} \dots\dots\dots (4)$$

From eqn. (3)

$$\frac{a(1-r^3)}{1-r} = 76 \text{ or } \frac{a(1+r+r^2)(1-r)}{1-r} = 76$$

$$\Rightarrow a(1+r+r^2) = 76 \dots\dots\dots (5)$$

Eqn. (5) $\div$ eqn. (1) to eliminate a

$$\Rightarrow \frac{a(1+r+r^2)}{ar} = \frac{76}{24} = \frac{19}{6}$$

$$\Rightarrow 6 + 6r + 6r^2 = 19r$$

$$\Rightarrow 6r^2 - 13r + 6 = 0 \text{ or } (2r-3)(3r-2) = 0 \therefore r = \frac{2}{3}, \frac{3}{2}$$

From eqn. (4) $r = \frac{b+1}{2}$

For $r = \frac{b+1}{2}$, $r = \frac{2}{3}$

$$\Rightarrow \frac{b+1}{2} = \frac{2}{3} \text{ or } 4 = 3b + 3 \therefore b = \frac{1}{3}$$

For $r = \frac{b+1}{2}$, $r = \frac{3}{2}$

$$\Rightarrow \frac{b+1}{2} = \frac{3}{2} \text{ or } 1 = b + 1 \therefore b = 2$$

$$\therefore b = \frac{1}{3}, 2$$

7. The n^{th} - term of a G.P is a_n . Given that $a_1 = 1$, $a_2 = 1 + x$, $a_3 = 5 + x^2$, $a_4 = a + x^4$. Find a and x , hence determine the sum of the first 10 - terms of a G.P.

Solution

Given: $a_1 = 1$, $a_2 = 1 + x$, $a_3 = 5 + x^2$, $a_4 = a + x^4$

$$\Rightarrow r = \frac{1+x}{1} = \frac{5+x^2}{1+x} = \frac{a+x^4}{5+x^2}$$

$$\Rightarrow \frac{1+x}{1} = \frac{5+x^2}{1+x} \text{ or } (1+x)^2 = 5 + x^2$$

$$\Rightarrow 1 + 2x + x^2 = 5 + x^2$$

$$\Rightarrow 1 + 2x = 5 \therefore x = 2$$

$$\therefore r = 1 + x = 1 + 2 = 3$$

From $\frac{a+x^4}{5+x^2} = r$

$$\Rightarrow \frac{a+x^4}{5+x^2} = 3 \therefore \frac{a+2^4}{5+2^2} = 3$$

$$\Rightarrow a + 16 = 27 \therefore a = 11$$

$$\therefore a = 11, x = 2$$

$$S_{10} = \frac{1 \times (3^{10} - 1)}{3 - 1} = \frac{1}{2} \times (3^{10} - 1) = 29\,524$$

Task

1. The ratio of the 9^{th} - term of a G.P to the 6^{th} - term is -8 , and the 5^{th} - term is 16. Find the G.P.

ANSWER: 1, -2, 4, -8, 16

2. The 10^{th} - term of a G.P is double the 12^{th} - term. If the 3^{rd} - term is 6. Find the 5^{th} - term.

ANSWER: 3

3. The sum of the first three terms of a G.P is 7 and the fourth term exceeds the first by 7. Find the seventh term.

ANSWER: 64

PROOFS INVOLVING SUMMATION OF A G.P . i. e. $S_n = \frac{a(r^n-1)}{r-1}$ FORMULAE
AND $n^{\text{th}} - \text{TERM } a_n = ar^{n-1}$

Examples

1. The 4^{th} , 7^{th} and 10^{th} terms of a G.P are l, m and n respectively. Prove that $m^2 = ln$.

Solution

Given: $a_4 = l, a_7 = m, a_{10} = n$

$$\Rightarrow l = ar^3 \dots\dots\dots (1)$$

$$\Rightarrow m = ar^6 \dots\dots\dots (2)$$

$$\Rightarrow n = ar^9 \dots\dots\dots (3)$$

Eqn. (1) $\div$ eqn. (2) to eliminate a

$$\Rightarrow \frac{l}{m} = \frac{ar^3}{ar^6} = \frac{1}{r^3} \therefore r^3 = \frac{m}{l} \dots\dots\dots (4)$$

Eqn. (2) $\div$ eqn. (3) to eliminate a

$$\Rightarrow \frac{m}{n} = \frac{ar^6}{ar^9} = \frac{1}{r^3} \therefore r^3 = \frac{n}{m} \dots\dots\dots (5)$$

Eqn. (4) = eqn. (5)

$$\Rightarrow \frac{m}{l} = \frac{n}{m} \therefore m^2 = ln$$

2. If $p^{\text{th}}, q^{\text{th}}, r^{\text{th}}$ term of a G.P are respectively equal to a, b, c , then prove that $a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$

Solution

Let the $1^{\text{st}} - \text{term} = A$ and common ratio = R

Using $A_n = AR^{n-1}$

$$\text{Given: } A_p = a = AR^{p-1} \dots\dots\dots (1)$$

$$A_q = b = AR^{q-1} \dots\dots\dots (2)$$

$$A_r = c = AR^{r-1} \dots\dots\dots (3)$$

$$\text{Now } a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = [AR^{p-1}]^{q-r} \cdot [AR^{q-1}]^{r-p} \cdot [AR^{r-1}]^{p-q}$$

$$\begin{aligned} &= A^{(q-r)} \cdot R^{(p-1)(q-r)} \cdot A^{(r-p)} \cdot R^{(q-1)(r-p)} \cdot A^{(p-q)} \cdot R^{(r-1)(p-q)} \\ &= A^{(q-r+r-p+p-q)} \cdot R^{(p-1)(q-r)(q-1)(r-p)+(r-1)(p-q)} \\ &= A^0 \cdot R^{pq-pr-q+r+qr-pq-r+p+pr-qr-p+q} \\ &= A^0 \cdot R^0 = 1 \end{aligned}$$

3. If x, y and z are $n^{\text{th}}, (2n)^{\text{th}}$ and $(3n)^{\text{th}}$ terms of a G.P, show that $y^2 = xz$.

Solution

Given: $a_n = x, a_{2n} = y$ and $a_{3n} = z$

$$\Rightarrow x = ar^{n-1}, y = ar^{2n-1}, z = ar^{3n-1}$$

$$\Rightarrow \frac{x}{y} = \frac{ar^{n-1}}{ar^{2n-1}} = r^{n-1-2n+1} = r^{-n}$$

$$\therefore \frac{y}{x} = r^n \dots\dots\dots (1)$$

$$\frac{y}{z} = \frac{ar^{2n-1}}{ar^{3n-1}} = r^{2n-1-3n+1} = r^{-n}$$

$$\therefore \frac{z}{y} = r^n \dots\dots\dots (2)$$

eqn. (1) = eqn. (2)

$$\Rightarrow \frac{y}{x} = \frac{z}{y} \therefore y^2 = xz$$

4. The p^{th} term of a G.P is p and the q^{th} term is q . show that the n^{th} term is given as $p \cdot \left[\frac{p^{n-q}}{q^{n-p}} \right]^{\frac{1}{p-q}}$

Solution

$$\text{Given: } a_p = p = ar^{p-1} \dots \dots \dots (1)$$

$$a_q = q = ar^{q-1} \dots \dots \dots (2)$$

In these two equations, a and r are unknowns. Eliminating a from the equations,

$$\Rightarrow \frac{p}{q} = \frac{ar^{p-1}}{ar^{q-1}} = r^{p-q}$$

$$\therefore r = \left[\frac{p}{q} \right]^{\frac{1}{p-q}} \dots \dots \dots (3)$$

eqn. (3) in to eqn. (1) to find a ,

$$\text{From } p = ar^{p-1} \Rightarrow a = \frac{p}{r^{p-1}} = \frac{p}{\left(\left[\frac{p}{q} \right]^{\frac{1}{p-q}} \right)^{p-1}} = \frac{p}{\left[\frac{p}{q} \right]^{\frac{p-1}{p-q}}}$$

$$\therefore a = \frac{p}{\left[\frac{p}{q} \right]^{\frac{p-1}{p-q}}} \dots \dots \dots (4)$$

$$\text{Now } a_n = ar^{n-1}$$

$$\begin{aligned} \Rightarrow a_n &= \frac{p}{\left[\frac{p}{q} \right]^{\frac{p-1}{p-q}}} \times \left[\left[\frac{p}{q} \right]^{\frac{1}{p-q}} \right]^{n-1} \\ &= \frac{p}{\left[\frac{p}{q} \right]^{\frac{p-1}{p-q}}} \times \left[\frac{p}{q} \right]^{\frac{n-1}{p-q}} = p \times \left[\frac{p}{q} \right]^{\frac{n-1}{p-q} - \frac{p-1}{p-q}} = p \cdot \left[\frac{p}{q} \right]^{\frac{n-p}{p-q}} = p \times \left[\frac{p^{n-p}}{q^{n-p}} \right]^{\frac{1}{p-q}} \end{aligned}$$

Task:

Prove that the $(n+1)^{\text{th}}$ term of a G.P of which the first term is a and the third term is b , is equal to $(2n+1)^{\text{th}}$ term of a G.P of which the first term is a and the fifth term is b .

5. If S be the sum, P the product and R be sum of reciprocal of n - terms in a G.P. prove that $P^2 R^n = S^n$.

Solution

Let $a, ar, ar^2, \dots, ar^{n-1}$ be n - terms in a G.P with $|r| < 1$

$$\Rightarrow S = a + ar + ar^2 + \dots + ar^{n-1}$$

$$\therefore S = \frac{a(1-r^n)}{1-r} \dots \dots \dots (1)$$

$$\Rightarrow P = a \cdot ar \cdot ar^2 \cdot \dots \cdot ar^{n-1}$$

$$P = a^{1+1+1+\dots+n} \cdot r^{1+2+3+\dots+n-1}$$

$$= a^n \cdot r^{\frac{n(n-1)}{2}} \left[\begin{array}{l} \because 1+2+3+\dots+n-1 \text{ is an A.P with } a=1 \\ d=1, n=n-1 \therefore S_n = \frac{n-1}{2}(2+(n-1+1)) = \frac{n(n-1)}{2} \end{array} \right]$$

$$\therefore P = a^n \cdot r^{\frac{n(n-1)}{2}} \dots \dots \dots (2)$$

$$\Rightarrow R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots + \frac{1}{ar^{n-1}}$$

$$= \frac{1}{a} \left[\frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \dots + \frac{1}{r^{n-1}} \right]$$

$$= \frac{1}{a} \left[\frac{1 - \left(\frac{1}{r} \right)^n}{\frac{1}{r} - 1} \right] \left[\because \text{Since } |r| > 1, \text{ then using } S_n = \frac{a(r^n-1)}{r-1} \right]$$

$$\therefore R = \frac{1}{a} \left[\frac{(1-r^n)r}{(1-r)r^n} \right] \dots \dots \dots (3)$$

eqn. (1) in to eqn. (3)

$$\Rightarrow R = \frac{1}{a} \cdot \left[\frac{S}{a} \cdot \frac{r}{r^n} \right] = \frac{S \cdot r}{a^2 \cdot r^n} = \frac{S}{a^2 \cdot r^{n-1}}$$

$$\text{Now } R^n = \left[\frac{S}{a^2 \cdot r^{n-1}} \right]^n = \frac{S^n}{a^{2n} \cdot r^{n(n-1)}}$$

$$\Rightarrow a^{2n} \cdot r^{n(n-1)} R^n = S^n \dots\dots\dots (4)$$

But from eqn. (1)

$$\Rightarrow P = a^n \cdot r^{\frac{n(n-1)}{2}} \therefore \frac{P}{a^n} = r^{\frac{n(n-1)}{2}}$$

Squaring

$$\Rightarrow \frac{P^2}{a^{2n}} = r^{n(n-1)} \dots\dots\dots (5)$$

eqn. (5) in to eqn. (4)

$$\Rightarrow S^n = a^{2n} \cdot r^{n(n-1)} R^n = a^{2n} \times \frac{P^2}{a^{2n}} \times R^n$$

$$\therefore P^2 R^n = S^n$$

6. If S_1, S_2, S_3 be respectively the sum of $n, 2n, 3n$ terms of a G.P, prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$

Solution

$$\text{Given: } S_1 = S_n = \frac{a(1-r^n)}{1-r} \dots\dots\dots (1)$$

$$S_2 = S_{2n} = \frac{a(1-r^{2n})}{1-r} \dots\dots\dots (2)$$

$$S_3 = S_{3n} = \frac{a(1-r^{3n})}{1-r} \dots\dots\dots (3)$$

For all $|r| < 1$

$$\begin{aligned} \text{Now } S_3 - S_2 &= \frac{a(1-r^{3n})}{1-r} - \frac{a(1-r^{2n})}{1-r} \\ &= \frac{a}{1-r} \cdot (r^{2n} - r^{3n}) = \frac{ar^{2n}}{1-r} \cdot (1 - r^n) \dots\dots\dots (4) \end{aligned}$$

$$\text{Again } S_1(S_3 - S_2) = \frac{a(1-r^n)}{1-r} \times \frac{ar^{2n}}{1-r} \cdot (1 - r^n)$$

$$\Rightarrow S_1(S_3 - S_2) = \frac{ar^{2n}(1-r^n)^2}{(1-r)^2} \dots\dots\dots (5)$$

$$\begin{aligned} \mathbf{R.H.S} &= (S_2 - S_1)^2 = \left[\frac{a(1-r^{2n})}{1-r} - \frac{a(1-r^n)}{1-r} \right]^2 \\ &= a^2 \left[\frac{r^n - r^{2n}}{1-r} \right]^2 = a^2 \left[\frac{r^n(1-r^n)}{1-r} \right]^2 = \frac{ar^{2n}(1-r^n)^2}{(1-r)^2} \end{aligned}$$

$$\therefore \mathbf{R.H.S} = \mathbf{L.H.S}$$

7. If S_n represent sum of n – terms of a G.P whose first term and common ratio are a and r respectively. Prove that $S_1 + S_2 + S_3 + \dots + S_n = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

$$\frac{ar(1-r^n)}{(1-r)^2}$$

Solution

$$\text{Given: } S_n = \frac{a(1-r^n)}{1-r}$$

$$S_1 = \frac{a(1-r^1)}{1-r} = a; S_2 = \frac{a(1-r^2)}{1-r}; S_3 = \frac{a(1-r^3)}{1-r}$$

$$S_1 + S_2 + S_3 + \dots + S_n$$

$$= a + \frac{a(1-r^2)}{1-r} + \frac{a(1-r^3)}{1-r} + \dots + \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} [(1-r) + (1-r^2) + (1-r^3) + \dots + (1-r^n)]$$

$$= \frac{a}{1-r} [(1 + 1 + 1 + 1 + \dots + n \text{ – times}) - (r + r^2 + r^3 + \dots + r^n)]$$

$$= \frac{a}{1-r} \left[n - \frac{r(1-r^n)}{1-r} \right] = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$$

$$\therefore S_1 + S_2 + S_3 + \dots + S_n = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$$

8. The sum of $2n$ – terms of a G.P whose first term is a and the common ratio r is equal to the sum of n – terms of a G.P whose first term is b and common ratio r^2 . Prove that b is equal to the sum of the first two terms of the series.

Solution

Given: $S_{2n} = S_n$

1st series: $a, ar, ar^2, ar^3, \dots, ar^{2n-1}$

2nd series: $b, br^2, br^4, br^6, \dots, br^{2n-1}$

Now $S_{2n} = \frac{a(1-r^{2n})}{1-r}, S_n = \frac{b[1-(r^2)^n]}{1-r^2}$

From $S_{2n} = S_n$

$$\Rightarrow \frac{a(1-r^{2n})}{1-r} = \frac{b(1-r^{2n})}{1-r^2} \therefore \frac{a(1-r^{2n})}{1-r} = \frac{b(1-r^{2n})}{(1+r)(1-r)}$$

$$\Rightarrow a = \frac{b}{1+r} \therefore b = a + ar$$

Thus b is the sum of the first two terms of the series.

Task

- 1) If in a G.P, $(p+q)^{th}$ term is m and $(p-q)^{th}$ is n . show that p^{th} term is $\sqrt{mn}$ and the q^{th} term is $m^{\frac{2q-p}{2q}} \cdot n^{\frac{p}{2q}}$
- 2) There are n – terms in a G.P, show that the n^{th} root of their product is equal to the square root of the product of its first and last terms.

Other forms of proofs in G.Ps

1. if a, b, c are in a G.P and $a^x = b^y = c^z$, prove that $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$

Solution

a, b, c are in a G.P if

$$\frac{b}{a} = \frac{c}{b} \text{ or } b^2 = ac \dots\dots\dots (1)$$

Expressing ac in terms of b

$$a^x = b^y \Rightarrow a = b^{y/x} \dots\dots\dots (2)$$

$$b^y = c^z \Rightarrow c = b^{y/z} \dots\dots\dots (3)$$

eqn. (2), (3) in to (1)

$$\Rightarrow b^2 = b^{y/x} \cdot b^{y/z} = b^{\frac{y}{x} + \frac{y}{z}}$$

$$\therefore 2 = \frac{y}{z} + \frac{y}{x} \text{ or } \frac{1}{x} + \frac{1}{z} = \frac{2}{y}$$

2. if a, b, c, d are in a G.P, show that:
 - i) $(a+b), (b+c), (c+d)$ are in a G.P
 - ii) $(a-b)^2, (b-c)^2, (c-d)^2$ are in a G.P
 - iii) $a^2 - b^2, b^2 - c^2, c^2 - d^2$

Solution

- i) a, b, c, d are in a G.P if:

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} \text{ or } b = ar, c = br = ar^2, d = cr = ar^3$$

Now $(a+b), (b+c), (c+d)$ are in a G.P if:

$$\frac{b+c}{a+b} = \frac{ar+ar^2}{a+ar} = \frac{r(1+r)}{1+r} = r$$

$$\frac{c+d}{b+c} = \frac{ar^2+ar^3}{ar+ar^2} = \frac{r^2(1+r)}{r(1+r)} = r$$

- Since $\frac{b+c}{a+b} = \frac{c+d}{b+c} = r$, then $(a+b), (b+c), (c+d)$ are in a G.P
- ii) $(a-b)^2, (b-c)^2, (c-d)^2$ are in a G.P if:
- $$\frac{(b-c)^2}{(a-b)^2} = \frac{(c-d)^2}{(b-c)^2}$$
- Now $\frac{(b-c)^2}{(a-b)^2} = \frac{(ar-ar^2)^2}{(a-ar)^2} = \frac{a^2r^2(1-r)^2}{a^2(1-r)^2} = r^2$
- Also $\frac{(c-d)^2}{(b-c)^2} = \frac{(ar^2-ar^3)^2}{(ar-ar^2)^2} = \frac{a^2r^4(1-r)^2}{a^2r^2(1-r)^2} = r^2$
- Since $\frac{(b-c)^2}{(a-b)^2} = \frac{(c-d)^2}{(b-c)^2} = r^2$, then $(a-b)^2, (b-c)^2, (c-d)^2$ are in a G.P
- iii) $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in a G.P if:
- $$\frac{b^2-c^2}{a^2-b^2} = \frac{c^2-d^2}{b^2-c^2}$$
- Now $\frac{b^2-c^2}{a^2-b^2} = \frac{(ar)^2-(ar^2)^2}{a^2-(ar)^2} = \frac{a^2r^2(1-r^2)}{a^2(1-r^2)} = r^2$
- Also $\frac{c^2-d^2}{b^2-c^2} = \frac{(ar^2)^2-(ar^3)^2}{(ar)^2-(ar^2)^2} = \frac{a^2r^4(1-r^2)}{a^2r^2(1-r^2)} = r^2$
- Thus $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in a G.P since $\frac{b^2-c^2}{a^2-b^2} = \frac{c^2-d^2}{b^2-c^2}$
3. If a, b, c, d are in a G.P, show that:
- $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$
 - $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$

Solution

- i) a, b, c, d are in a G.P if:
- $$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} \text{ or } b = ar, c = br = ar^2, d = cr = ar^3$$
- Now $(b-c)^2 + (c-a)^2 + (d-b)^2 = (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar^2)^2$
- $$= a^2r^2(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$$
- $$= a^2[r^2(1-2r+r^2) + r^4 - 2r^2 + 1 + r^2(r^4 - 2r^2 + 1)]$$
- $$= a^2[r^2 - 2r^3 + r^4 + r^4 - 2r^2 + 1 + r^6 - 2r^4 + r^2]$$
- $$= a^2[r^6 - 2r^3 + 1] = a^2(r^3 - 1)^2$$
- Now **R.H.S** $= (a-d)^2 = (a-ar^3)^2 = a^2(1-r^3)^2 = a^2(r^3-1)^2$
- Thus **R.H.S** = **L.H.S**
- ii) $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$
- $$\frac{ab-cd}{b^2-c^2} = \frac{a \cdot ar - ar^2 \cdot ar^3}{a^2r^2 - (ar^2)^2} = \frac{1-r^4}{r^2-r^4} = \frac{1-r^4}{r(1-r^2)}$$
- $$= \frac{(1-r^2)(1+r^2)}{r(1-r^2)} = \frac{1+r^2}{r}$$
- Now $\frac{a+c}{b} = \frac{a+ar^2}{ar} = \frac{1+r^2}{r}$
- Thus **R.H.S** = **L.H.S**

Task

If a, b, c, d are in a G.P, show that

- $a^2b^2c^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) = a^3 + b^3 + c^3$
- $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{a+b+c}{a-b+c}$
- $(ab+bc+cd)^2 = (a^2+b^2+c^2)(b^2+c^2+d^2)$
- $a^2 + b^2, ab + bc, b^2 + c^2$, are in a G.P

4. If a, b, c form an A.P, b, c, a form a G.P, show that $\frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ form an A.P.

Solution

If a, b, c form an A.P, then

$$b - a = c - b \text{ or } a + c = 2b \dots\dots\dots (1)$$

b, c, a , form a G.P if;

$$\frac{c}{b} = \frac{a}{c} \text{ or } ab = c^2 \dots\dots\dots (2)$$

Now for $\frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ to be in an A.P, then

$$\frac{1}{a} - \frac{1}{c} = \frac{1}{b} - \frac{1}{a} \Rightarrow \frac{2}{a} = \frac{1}{b} + \frac{1}{c}$$

Generating $\frac{2}{a} = \frac{1}{b} + \frac{1}{c}$ from eqn. (1) and eqn. (2)

Eqn. (1) $\div$ eqn. (2)

$$\Rightarrow \frac{a+c}{ab} = \frac{2b}{c^2}$$

But $ab = c^2$,

$$\Rightarrow \frac{1}{b} + \frac{c}{c^2} = \frac{2b}{ab} \therefore \frac{2}{a} = \frac{1}{b} + \frac{1}{c}$$

5. If a, b, c are in A.P, a, x, b and b, y, c are in a G.P, show that x^2, b^2, y^2 are in an A.P.

Solution

a, b, c are in an A.P if;

$$b - a = c - b \text{ or } 2b = a + c \dots\dots\dots (1)$$

a, x, b are in a G.P if;

$$\frac{x}{a} = \frac{b}{x} \text{ or } x^2 = ab \dots\dots\dots (2)$$

b, y, c are in a G.P if;

$$\frac{y}{b} = \frac{c}{y} \text{ or } y^2 = bc \dots\dots\dots (3)$$

Now x^2, b^2, y^2 are in an A.P if;

$$b^2 - x^2 = y^2 - b^2 \text{ or } 2b^2 = x^2 + y^2 \dots\dots\dots (4)$$

Generating eqn. (4) from the other three equations

Now eqn. (2) + eqn. (3)

$$\Rightarrow x^2 + y^2 = ab + bc = b(a + c)$$

From eqn. (1), $2b = a + c$

$\therefore x^2 + y^2 = 2b^2$, which is eqn. (4). Hence x^2, b^2, y^2 are in an A.P.

Task

- If a, b, c are in G.P, prove that $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in an A.P.
- If b, a, c are in an A.P and c, b, a are in a G.P, show that $\frac{a+b}{b^2} = \frac{2}{c}$

Combination of A.P and G.P questions

- Three numbers are in an A.P and their sum is 21, if 1, 5, 15 be added to them respectively, they form a G.P. find the numbers.

Solution

Let the 3 numbers in an A.P be $a - d, a, a + d$

Given: $a - d + a + a + d = 21 \therefore a = 7$

Also: $(a - d + 1), (a + 5), (a + d + 15)$ are in a G.P

$$\Rightarrow \frac{a+5}{a-d+1} = \frac{a+d+15}{a+5}, \text{ but } a = 7$$

$$\Rightarrow \frac{7+5}{7-d+1} = \frac{7+d+15}{7+5}$$

$$\Rightarrow \frac{12}{8-d} = \frac{22+d}{12} \text{ or } 144 = (8-d)(22+d) = 176 + 8d - 22d - d^2$$

$$\Rightarrow d^2 + 14d - 32 = 0$$

$$\Rightarrow d^2 + 16d - 2d - 32 = 0 \text{ or } (d-2)(d+16) = 0$$

$$\therefore d = 2, -16$$

Thus for $a = 7, d = 2$, the numbers are: 6, 12, 24

For $a = 7, d = -16$, the numbers are: 6, 12, 24

- 2) The sequence of three numbers a, b, c is an A.P whose sum is 18. If a and b are each increased by 4 and c increased by 36, the new numbers form a G.P. find a, b, c .

Solution

a, b, c are in an A.P if;

$$b - a = c - b \text{ or } 2b = a + c \dots\dots\dots (1)$$

$$a + b + c = 18 \dots\dots\dots (2)$$

Given: $a + 4, b + 4, c + 36$ are in a G.P if;

$$\frac{b+4}{a+4} = \frac{c+36}{b+4} \Rightarrow (b+4)^2 = (a+4)(c+36) \dots\dots\dots (3)$$

Solving the equations simultaneously

Eqn. (1) in to eqn. (2)

$$\Rightarrow b + 2b = 18, b = 6$$

Put $b = 6$ in to eqn. (1) and (3)

$$\Rightarrow a + c = 12 \dots\dots\dots (4)$$

$$\Rightarrow 100 = (a+4)(c+36) \dots\dots\dots (5)$$

$$\text{From eqn. (4) } a = 12 - c \dots\dots\dots (6)$$

Eqn. (6) in to (5)

$$\Rightarrow 100 = (12 - c + 4)(c + 36) = (16 - c)(c + 36)$$

$$\Rightarrow 100 = -c^2 - 20c + 576$$

$$\therefore c^2 + 20c + 476 = 0 \therefore c = 14, -34$$

From eqn. (6), $a = 12 - c$

For $c = 14, a = 12 - 14 = -2$

For $c = -34, a = 12 + 34 = 46$

Hence numbers are: -2, 6, 14 or 46, 6, -34

- 3) The 1st, 10th and 28th terms of an A.P are three successive terms of a G.P. find the common ratio of a G.P. given that the sum of the first 28 terms of an A.P is 210, find its first term.

Solution

Given: $a_1 = a, a_{10} = a + 9d, a_{28} = a + 27d$

Now $a, a + 9d, a + 27d$ are in a G.P

$$\Rightarrow \frac{a+9d}{a} = \frac{a+27d}{a+9d} \dots\dots\dots (1)$$

Let the three consecutive terms of a G.P be: a, ar, ar^2

Note: the first term of an A.P is equal to the first term of a G.P

$$\Rightarrow ar = a + 9d \text{ or } ar - a = 9d \dots\dots\dots (2)$$

$$\Rightarrow ar^2 = a + 27d \text{ or } ar^2 - a = 27d \dots\dots\dots (3)$$

$$\text{Given: } S_{28} = \frac{28}{2}(2a + 27d) = 210$$

$$\Rightarrow 2a + 27d = 15 \dots\dots\dots (4)$$

eqn. (2) $\div$ eqn. (3)

$$\Rightarrow \frac{ar-a}{ar^2-a} = \frac{r-1}{r^2-1} = \frac{1}{3}$$

$$\Rightarrow \frac{r-1}{(r+1)(r-1)} = \frac{1}{3} \text{ or } 3 = r + 1 \therefore r = 2$$

Substitute $r = 2$ in to eqn. (2)

$$\Rightarrow 2a - a = 9d \therefore a = 9d \dots\dots\dots (5)$$

eqn. (5) in to eqn. (4)

$$\Rightarrow 2(9d) + 27d = 15 \therefore d = \frac{15}{45} = \frac{1}{3}$$

$$\therefore 1^{st} \text{ term} = 9d = 9 \times \frac{1}{3} = 3$$

- 4) *The sum of the first three terms of two series: one an A.P and the other a G.P is the same. If the first term of each of these is $\frac{2}{3}$ and the common difference of an A.P is equal to the common ratio of the G.P, find the sum of each series to 20 terms.*

Solution

$$\text{In an A.P, } S_3 = \frac{3}{2}(2a + 2d) = 3a + 3d$$

$$\text{In a G.P, } S_3 = \frac{a(1-r^3)}{1-r} = \frac{a(1-r)(1+r+r^2)}{1-r} = a(1+r+r^2)$$

$$\text{Given: } a = \frac{2}{3}, r = d$$

$$\Rightarrow 3a + 3d = a(1+r+r^2)$$

$$\Rightarrow 3 \times \frac{2}{3} + 3r = \frac{2}{3}(1+r+r^2) \text{ or } 6 + 9r = 2 + 2r + 2r^2$$

$$\Rightarrow 2r^2 - 7r - 4 = 0 \text{ or } (2r+1)(r-4) = 0$$

$$\therefore r = 4, -\frac{1}{2}$$

Now when $r = d = 4$, for an A.P,

$$S_{20} = \frac{20}{2} \left(2 \times \frac{2}{3} + 19 \times 4 \right) = 773 \frac{1}{3}$$

$$\text{For a G.P, } S_{20} = \frac{\frac{2}{3}(4^{20}-1)}{4-1} = \frac{2}{9}(4^{20}-1)$$

Also when $r = d = -\frac{1}{2}$, for an A.P

$$S_{20} = \frac{20}{2} \left(2 \times \frac{2}{3} + 19 \times -\frac{1}{2} \right) = -\frac{245}{3}$$

$$\text{For a G.P, } S_{20} = \frac{\frac{2}{3} \left(1 - \left(-\frac{1}{2} \right)^{20} \right)}{1 - -\frac{1}{2}} = \frac{1}{9} \frac{(2^{20}-1)}{2^{18}}$$

- 5) *Find three numbers a, b, c between 2 and 18 such that their sum is 25, the numbers 2, a, b are consecutive terms of an A.P and the numbers $b, c, 18$ are the consecutive terms of a G.P.*

Solution

$$\text{Given: } a + b + c = 18 \dots\dots\dots (1)$$

2, a, b form an A.P

$$\Rightarrow a - 2 = b - a \text{ or } 2a = b + 2 \dots\dots\dots (2)$$

Also $b, c, 18$ form a G.P

$$\Rightarrow \frac{c}{b} = \frac{18}{c} \text{ or } c^2 = 18b \dots\dots\dots (3)$$

Solving the equations simultaneously

Eqn. (3) in to eqn. (2)

$$\Rightarrow 2a = \frac{c^2}{18} + 2 \text{ or } 36a = c^2 + 36 \dots\dots\dots (4)$$

Eliminating b from eqn. (1) and eqn. (2)

$$\text{From eqn. (2), } b = 2a - 2$$

Substituting In equation (1)

$$\Rightarrow a + 2a - 2 + c = 25$$

$$\Rightarrow 3a + c = 27 \dots\dots\dots (5)$$

From eqn. (5), $c = 27 - 3a$

Substituting In equation (4)

$$\Rightarrow 36a = (27 - 3a)^2 + 36$$

$$\Rightarrow 36a = 9(9 - a)^2 + 36 \dots\dots\dots (\div 9)$$

$$\Rightarrow 4a = 81 - 18a + a^2 + 4 \text{ or } a^2 - 22a + 85 = 0$$

$$\Rightarrow (a - 5)(a - 17) = 0 \therefore a = 5, 17$$

From $c = 27 - 3a$

For $a = 5$, $c = 27 - 15 = 12$

For $a = 17$, $c = 27 - 51 = -24$ (*Rejected*)

Thus $a = 5$, $c = 12$ are used to find b.

From eqn. (3), $b = \frac{c^2}{18}$, for $c = 12$

$$\Rightarrow b = \frac{144}{18} = 8$$

$$\therefore a = 5, b = 8, c = 12$$

Hence the numbers are: 5, 8, 12

- 6) *Three integers form an increasing G.P. if the third is decreased by 16 an A.P is formed. If the second number is decreased by 2, a G.P is formed. Find the numbers.*

Solution

Let the integers be: a, ar, ar^2

Decreasing the 3rd number by 16 gives $a, ar, ar^2 - 16$ are in an A.P

$$\Rightarrow ar - a = ar^2 - 16 - ar$$

$$\Rightarrow 2ar - a = ar^2 - 16 \dots\dots\dots (1)$$

Decreasing the 2nd number by 2 from the new set of numbers give $a, ar - 2, ar^2 - 16$ are in a G.P

$$\Rightarrow \frac{ar-2}{a} = \frac{ar^2-16}{ar-2}$$

$$\Rightarrow (ar - 2)^2 = a(ar^2 - 16) \text{ or } a^2r^2 - 4ar + 4 = a^2r^2 - 16a$$

$$\Rightarrow -4ar + 4 = -16a \text{ or } 4 = a(4r - 16)$$

$$\therefore a = \frac{1}{r-4} \dots\dots\dots (2)$$

eqn. (2) in to eqn. (1)

$$\Rightarrow 2\left(\frac{1}{r-4}\right)r - \frac{1}{r-4} = \frac{1}{r-4} \times r^2 - 16$$

$$\Rightarrow \frac{2r-1}{r-4} = \frac{r^2-16r+64}{r-4} \text{ or } 2r - 1 = r^2 - 16r + 64$$

$$\Rightarrow r^2 - 18r + 64 = 0 \text{ or } (r - 5)(r - 13) = 0$$

$$\therefore r = 5, 13$$

From eqn. (2), $a = \frac{1}{r-4}$

For $r = 5$, $a = 1$

For $r = 13$, $a = \frac{1}{9}$

Thus the numbers are formed by $a = 1, r = 5$.i.e. 1, 5, 25

Task

1. The 2^{nd} , 4^{th} and 8^{th} terms of an A.P series are in a G.P series. The sum of the 3^{rd} and 5^{th} terms is 20. Find the sum of the first 20 terms of the progression.

ANSWER: 525

2. An arithmetic series and geometric series have r as the common difference and the common ratio respectively. The first term of the arithmetic series is 1 and the first term of the geometric series is 2. If the 4^{th} term of the arithmetic series is equal to the sum of the third and fourth terms of the geometric series, find three possible values of r when $|r| < 1$, find in the form $p + q\sqrt{2}$ the sum of the first 10 – terms of the arithmetic series. **ANSWER:** $\frac{-70+45\sqrt{2}}{2}$

3. A G.P and an A.P have the same first term. The sums of their first, second and third terms are 6, 10, 10.5 and 18 respectively. Calculate the sum of their fifth terms. (**Uneb 1995**)

HINT:
 For an A.P: $a = 1^{\text{st}}$ term, $a_2 = a + d$, $a_3 = a + 2d$, $a_5 = a + 4d$
 For a G.P: $a = 1^{\text{st}}$ term, $a_2 = ar$, $a_3 = ar^2$, $a_5 = ar^4$
 Now $a + a = 6 \therefore a = 3 \dots \dots \dots$ (1)
 $\Rightarrow a + d + ar = 10.5 \therefore 3 + d + 3r = 10.5 \dots \dots \dots$ (2)
 $\Rightarrow a + 2d + ar^2 = 18 \therefore 3 + 2d + 3r^2 = \dots \dots \dots$ (3)
 Solving eqn. (2) and eqn. (3) gives; $3 + 6r - 3r^2 = 3 \therefore 2r - r^2 = 0$
 $\therefore r = 0, r = 2$
 From eqn. (2) $d = 10.5 - 3 - 6 = 1.5 \therefore d = 1.5$
 Now $a_5 = a + 4d + ar^4 = \text{Sum of their fifth terms}$
 $\therefore 3 + 4 \times 1.5 + 3 \times 2^4 = 57$

4. The 2^{nd} term, 5^{th} term and 11^{th} term of an A.P are in a G.P. if the 11^{th} term is 4, find the difference between the sums of the 8 – terms of each progressions.

Solution

Given: $a_2 = a + d$, $a_5 = a + 4d$, $a_{11} = a + 10d = 4$ are in a G.P;

Let $a = 1^{\text{st}}$ term $d = \text{common difference}$

$$\Rightarrow \frac{a+4d}{a+d} = \frac{a+10d}{a+4d} = r = \text{common ratio}$$

$$\Rightarrow (a + 4d)^2 = (a + 10d)(a + d) \dots \dots \dots (1)$$

$$\text{Also } a + 10d = 4 \dots \dots \dots (2)$$

Solving eqn. (2) and eqn. (1)

$$\Rightarrow (4 - 10d + 4d)^2 = 4(4 - 10d) \text{ or } 4(2 - 3d)^2 = 4(4 - 10d)$$

$$\Rightarrow 4 - 12d + 9d^2 = 4 - 9d \therefore 9d^2 - 3d = 0$$

$$\therefore d = 0, \frac{1}{3}$$

$$\text{From eqn. (2) and for } d = \frac{1}{3}, a = 4 - \frac{10}{3} = \frac{2}{3}$$

$$\text{For an A.P, } S_8 = \frac{8}{2} \left[2 \times \frac{2}{3} + 7 \times \frac{1}{3} \right] = 4 \times \frac{11}{3} = \frac{44}{3}$$

$$\text{Now the common ratio, } r = \frac{a+4d}{a+d} = \frac{\frac{2}{3} + \frac{4}{3}}{\frac{2}{3} + \frac{1}{3}} = 2$$

Now the G.P is; $(a + d)$, $(a + 4d)$, $(a + 10d)$

$$1^{\text{st}} \text{ term} = a + d = \frac{2}{3} + \frac{1}{3} = 1$$

$$\text{Using } S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_8 = \frac{2^8 - 1}{2 - 1} = 255$$

$$\therefore S_{8(G.P)} - S_{8(A.P)} = 255 = \frac{44}{3} = \frac{721}{3}$$

5. If a, b, c are in a G.P, show that $\log a, \log b, \log c$ are in an A.P.

TO FIND THE SUM TO INFINITY OF A G.P WHOSE COMMON RATIO IS LESS THAN ONE

Let $a, ar, ar^2, ar^3 \dots \dots \dots ar^{n-1}$ be a G.P with $|r| < 1$

$$\text{Sum of } n - \text{terms is } S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

Since $|r| < 1$, then $r^2 < r, r^3 < r^2 \dots \dots \dots r^n < r^{n-1}$

As the power of r goes on increasing, the corresponding term in a G.P decrease in value. So assuming that n becomes indefinitely large, r^n becomes indefinitely small. i.e. $r^n \rightarrow 0$

$$\text{Thus from } S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

As $n \rightarrow \infty, r^n \rightarrow 0$

$$\therefore S_\infty = \frac{a}{1-r}, \text{ where } |r| < 1 \text{ or } -1 < r < 1$$

This is the sum to infinity. i.e. the limiting value to which the sum approaches/tends to.

$$\text{i.e. } a + ar + ar^2 + ar^3 + \dots \dots \dots = \frac{a}{1-r} = S_\infty$$

Examples

1. Express the recurring decimals to fraction form in lowest term.

a) $0.1\dot{7}$

b) $0.0\dot{7}$

c) $1.\overline{45}$

Solution

a) $0.1\dot{7} = 0.17777 \dots \dots \dots$

$$0.1\dot{7} = 0.1 + 0.07 + 0.007 + 0.0007 + \dots \dots \dots$$

$$= \frac{1}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \dots \dots \dots$$

$$= \frac{1}{10} + \left\{ \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \dots \dots \dots \right\}$$

The bracketed series is a G.P with first term as $\frac{7}{100}$ and common ratio as $\frac{1}{10}$

$$\Rightarrow \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \dots \dots \dots = S_\infty = \frac{\frac{7}{100}}{1 - \frac{1}{10}} = \frac{7}{90}$$

$$\therefore 0.1\dot{7} = \frac{1}{10} + \frac{7}{90} = \frac{8}{45}$$

b) $0.0\dot{7} = 0.07777 \dots \dots \dots$

$$= 0.07 + 0.007 + 0.0007 + \dots \dots \dots$$

$$= \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \dots \dots \dots = S_\infty = \frac{\frac{7}{100}}{1 - \frac{1}{10}} = \frac{7}{90}$$

$$\therefore 0.0\dot{7} = \frac{7}{90}$$

c) $1.\overline{45} = 1.45454545 \dots \dots \dots$

$$= 1 + 0.45 + 0.0045 + 0.00045 + \dots \dots \dots$$

$$= 1 + \left\{ \frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + \dots \dots \dots \right\}$$

$$= 1 + \frac{\frac{45}{100}}{1 - \frac{1}{100}} = \frac{45}{99} + 1 = \frac{16}{11}$$

$$\therefore 1.\overline{45} = \frac{16}{11}$$

2. a) Find the first term of a G.P whose sum to infinite terms is 8 and the second term is 2
 b) The sum of infinite terms of a G.P is 15 and the sum of their squares is 45. Find the series.

Solution

- a) Given: $S_{\infty} = 8 = \frac{a}{1-r}$, $ar = 2$

Dividing the equations

$$\frac{8}{2} = \frac{\frac{a}{1-r}}{ar} = \frac{1}{r-r^2} \text{ or } 4r - 4r^2 = 1 \therefore 4r^2 - 4r + 1 = 0$$

$$\therefore r = \frac{1}{2}$$

$$\therefore 1^{\text{st}} \text{ term} = a = \frac{2}{r} = \frac{2}{\frac{1}{2}} = 4$$

- b) Let $a, ar, ar^2, ar^3 \dots$ be the G.P with $|r| < 1$

$$\text{Given: } S_{\infty} = a + ar + ar^2 + ar^3 \dots = \frac{a}{1-r} = 15 \dots \dots \dots (1)$$

Sum to infinity of squares = 45

$$\Rightarrow S_{\infty} = a^2 + a^2r^2 + a^2r^4 + a^2r^6 \dots = \frac{a^2}{1-r^2} = 45 \dots \dots \dots (2)$$

From eqn. (1), $a = 15(1-r)$, substituting in eqn. (2)

$$\Rightarrow \frac{[15(1-r)]^2}{1-r^2} = 45 \text{ or } \frac{225(1-r)^2}{(1+r)(1-r)} = 45$$

$$\Rightarrow \frac{225(1-r)}{1+r} = 45 \therefore 5 - 5r = 1 + r \therefore r = \frac{2}{3}$$

$$\text{But } a = 15(1-r) \Rightarrow a = 15\left(1 - \frac{2}{3}\right) = 5$$

Thus the series is $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \dots$

3. The second term of a G.P is 24 and its sum to infinity is 100. Find two possible values of the common ratio and the corresponding first terms.

Solution

$$\text{Given: } ar = 24 \dots \dots \dots (1)$$

$$S_{\infty} = a + ar + ar^2 + ar^3 \dots = \frac{a}{1-r} = 100$$

$$\therefore a = 100 - 100r \dots \dots \dots (2)$$

Eqn. (2) in to eqn. (1) to eliminate a

$$(100 - 100r)r = 24 \text{ or } 25r - 25r^2 - 6 = 0$$

$$\therefore 25r^2 - 25r + 6 = 0$$

$$\Rightarrow r = \frac{25 \pm \sqrt{(25)^2 - 4 \times 25 \times 6}}{2 \times 25} = \frac{25 \pm 5}{50}$$

$$\therefore r = \frac{3}{5}, \frac{2}{5}$$

$$\text{From eqn. (1), for } r = \frac{3}{5}, a = \frac{24}{\frac{3}{5}} = 40$$

$$\text{For } r = \frac{2}{5}, a = \frac{24}{\frac{2}{5}} = 60$$

$$\therefore \frac{3}{5}, 40; \frac{2}{5}, 60$$

4. The sum of infinite terms of a G.P is 4 and the sum of their cubes is 192. Find the series.

Solution

Let the series be: $a, ar, ar^2, ar^3, \dots$

Cubing gives $a^3, a^3r^3, a^3r^6, a^3r^9, \dots$

$$\text{Given: } S_{\infty} = a^3 + a^3r^3 + a^3r^6 + a^3r^9 + \dots = \frac{a^3}{1-r^3} = 192$$

$$\frac{a^3}{1-r^3} = \frac{a^3}{(1-r)(1+r+r^2)} = 192 \quad (1)$$

$$\text{Also } S_{\infty} = a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r} = 4$$

$$\frac{a}{1-r} = 4 \text{ or } a = 4(1-r) \quad (2)$$

eqn. (2) in to eqn. (1)

$$\Rightarrow \frac{[4(1-r)]^3}{(1-r)(1+r+r^2)} = 192$$

$$\Rightarrow \frac{64(1-r)^3}{(1-r)(1+r+r^2)} = 192 \text{ or } \frac{(1-r)^2}{(1+r+r^2)} = 3$$

$$\Rightarrow 1 - 2r + r^2 = 3 + 3r + 3r^2$$

$$\therefore 2r^2 + 5r + 2 = 0 \text{ or } (2r+1)(r+1) = 0$$

$$\therefore r = -\frac{1}{2}, -1, \text{ rejecting } r = -1 \text{ since } |r| < 1$$

$$\text{For } r = -\frac{1}{2}, a = 4 - 4 \times -\frac{1}{2} = 6$$

$$\therefore \text{The series is } 6, -3, \frac{3}{2}, -\frac{3}{4}, \dots$$

Task

- Show that there are two possible G.Ps, in each of which the first term is 8, and the sum of three terms is 14. Find the sum to infinity in each.
- The sum infinity of a G.P with positive common ratio is 9 and the sum of the first two terms is 5. Find the first four terms of the G.P.

ANSWER: $3, 2, \frac{4}{3}, \frac{8}{9}$

- Show that there are two G.P's in which the second term is $-\frac{4}{3}$ and the sum of the first 3 - terms is $\frac{28}{9}$. Show that one of these progressions converges to a sum of 3.
- If $x = 1 + a + a^2 + \dots + \infty$, $y = 1 + b + b^2 + \dots + \infty$, prove that if $|a| < 1$ and $|b| < 1$ then $1 + ab + a^2b^2 + \dots = \frac{xy}{x+y-1}$
- Show that the sum of the series $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \frac{1}{2^7} + \dots$ to $\infty = \frac{19}{24}$
- a) If $S_1, S_2, S_3, \dots, S_n$ are sums of infinite G.P series whose first terms are $1, 2, 3, \dots, n$ and the common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+1}$ respectively. Show that $S_1 + S_2 + S_3 + \dots + S_n = \frac{n(n+3)}{2}$
- If S_1, S_2 and S are sums of n - terms, $2n$ - terms and to infinity of a G.P respectively, show that $S_1(S_1 - S) = S(S_1 - S_2)$

Solution

a) $S_{\infty} = \frac{a}{1-r}$

$$S_1 = \frac{1}{1-\frac{1}{2}} = 2, S_2 = \frac{2}{1-\frac{1}{3}} = 3, S_3 = \frac{3}{1-\frac{1}{4}} = 4, S_n = \frac{n}{1-\frac{1}{n+1}} = n+1$$

L.H.S: $S_1 + S_2 + S_3 + \dots + S_n = 2 + 3 + 4 + \dots + n + 1$

This is an A.P with 1^{st} term = 2, common difference = 1 and last term = $n + 1$

Using $S_N = \frac{N}{2}(1^{st} \text{ term} + \text{last term})$. Finding N, using

Using $a_N = A + (N - 1)D$ or $N = \frac{a_N - A}{D} + 1$

$$\Rightarrow N = \frac{n+1-2}{1} + 1 = n$$

$$\therefore S_n = \frac{n}{2}(2 + n + 1) = \frac{n}{2}(n + 3)$$

b) Given: $S_n = S_1, S_{2n} = S_2, S_{\infty} = S_3$; let 1^{st} term = a , common ratio = r

$$\text{Using } S_n = \frac{a(1-r^n)}{1-r} = S_1 \quad \dots\dots\dots (1)$$

$$S_{2n} = \frac{a(1-r^{2n})}{1-r} = S_2 \quad \dots\dots\dots (2)$$

$$S_{\infty} = \frac{a}{1-r} = S \quad \dots\dots\dots (3)$$

$$\begin{aligned} \text{Now } S_1(S_1 - S) &= \frac{a(1-r^n)}{1-r} \left[\frac{a(1-r^n)}{1-r} - \frac{a}{1-r} \right] \\ &= \frac{a^2(1-r^n)}{(1-r)^2} [1 - r^n - 1] = \frac{-a^2 r^n (1-r^n)}{(1-r)^2} \end{aligned}$$

$$\therefore S_1(S_1 - S) = \frac{a^2 r^n (r^n - 1)}{(1-r)^2}$$

$$\begin{aligned} \text{Also } S(S_1 - S_2) &= \frac{a}{1-r} \left[\frac{a(1-r^n)}{1-r} - \frac{a(1-r^{2n})}{1-r} \right] \\ &= \frac{a^2}{(1-r)^2} [1 - r^n - 1 + r^{2n}] \end{aligned}$$

$$\therefore S(S_1 - S_2) = \frac{a^2}{(1-r)^2} (r^{2n} - r^n) = \frac{a^2 r^n (r^n - 1)}{(1-r)^2}$$

$$\therefore S_1(S_1 - S) = S(S_1 - S_2) = \frac{a^2 r^n (r^n - 1)}{(1-r)^2}$$

5. Find the sum to infinity of the G.P $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ and show that is S denotes S_{∞} and S_n denotes sum of n - terms then $S - S_n < \frac{1}{1000}$ if $n \geq 11$.

Solution

$$S = \frac{1}{1-\frac{1}{2}} = 2, S_n = \frac{a(1-r^n)}{1-r} = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}$$

$$\text{Now } S - S_n < \frac{1}{1000} \Rightarrow 2 - 2 \left(1 - \frac{1}{2^n} \right) < \frac{1}{1000}$$

$$\Rightarrow \frac{2}{2^n} < \frac{1}{1000} \text{ or } 2^{n-1} < 10^{-3}$$

Introducing logarithms

$$\Rightarrow \log(2^{n-1}) < \log 10^{-3}$$

$$\Rightarrow (1 - n) \log 2 < -3 \log 10$$

$$\Rightarrow 1 - n < -\frac{3}{\log 2}$$

$$\therefore n > 1 + \frac{3}{\log 2} \text{ or } n > 10.96578425$$

$$\therefore n \geq 11$$

PROOF BY INDUCTION

Proof by induction is a mathematical technique that uses reasoning that if “proposition P_n ” is true for a particular value of n , say $n = 1$ then it must be true for the next value of n . i.e. $n = 2$, and now if it is true for $n = 2$, then it must be also true for $n = 3$ and hence true for all positive integral values of n .

Characteristics of positive integral values (Natural numbers)

Axiom of Induction:

If T is a set of natural numbers with properties;

- i) $1 \in T$ and
- ii) $k \in T$ implies $k + 1 \in T$

, then T is a set of Natural numbers N .

Note:

Both conditions must be satisfied for the Axiom of induction to hold.

Illustration

Consider $T = \{1, 2, 3\}$

Clearly $1 \in T$, but is not true that $k + 1 \in T$ every time $k \in T$ e.g. $3 \in T$ but $4 \notin T$. Thus $T \neq N$

Also let $T = \{10, 11, 12, \dots\}$

Clearly if $k \in T$, then $k + 1 \in T$ but $1 \notin T$. Thus again $T \neq N$

Now let $T = \{1, 2, 3, \dots\}$. Both $1, k, k + 1 \in T$, hence $T = N$.

Statement of Principle of Proof by Induction

For every $n \in N$, let P_n be a statement which is either true or false. If

- i) P_1 is true and
- ii) Whenever this statement is true for $n = k$, it is true for $n = k + 1$, then P_n is true for all $n \in N$

In general, to apply the statement of proof by induction, two things must be done.

- i) First verify that P_1 is true and
- ii) Verify that P_{k+1} is true whenever P_k is true.

Examples

Prove by induction that

- a) $n^3 + 2n$ is divisible by 3
- b) $n^3 - n$ is divisible by 6
- c) $3^{2n} - 1$ is a multiple of 8
- d) $9^n - 1$ is a multiple of 8
- e) $8^n - 7n + 6$ is divisible by 7

Solution

- a) Let $P_n = n^3 + 2n$

Verify that P_1 is true

$\Rightarrow P_1 = 1^3 + 2(1) = 3$, which is divisible by 3. i.e. $P_1 = 3(1)$

Verify that whenever P_n is true for a natural number, say $n = k$, then it is true for the next natural number $n = k + 1$. To this you must assume that P_k is true for some $k \in N$. This assumption is the *induction hypothesis*.

Assume P_k is true .i.e. $k^3 + 2k$ is divisible by 3.

Next show that P_k implies P_{k+1} .i.e. $(k + 1)^3 + 2(k + 1)$ is divisible by 3.

$$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= (k^3 + 2k) + 3k^2 + 3k + 3$$

$$= P(k) + 3(k^2 + k + 1)$$

Since $P(k)$ is divisible by 3 and $3(k^2 + k + 1)$ is also divisible by 3, $P(k + 1)$ is divisible by 3. Thus both conditions of principle of proof by induction are satisfied. You must conclude that $n^3 + 2n$ is divisible by 3 for $n \in N$.

NOTE:

The symbol $P(k)$ in the working stand for " $k^3 + 2k$ ". Don't confuse it with the symbol P_k which stands for the sentence " $k^3 + 2k$ is divisible by 3".

Similarly $P(k + 1)$ stands for " $(k + 1)^3 + 2(k + 1)$ "

- b) Let $P_n = n^3 - n$

For $n = 1, P_1 = 1^3 - 1 = 0 = 6(0)$. Hence it is divisible by 6

Now for $n = k$, let P_k be true .i.e. $k^3 - k$ is divisible by 6.

Now for $n = k + 1, P_{k+1}$ must also be true .i.e. $(k + 1)^3 - (k + 1)$ is divisible by 6

$$\text{But } (k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= (k^3 - k) + 3k^2 + 3k = P(k) + 3(k^2 + k)$$

Since $P(k)$ is divisible by 6 and $3(k^2 + k)$ is divisible by 6 for any $k \geq 1$ and $k \in N$ then $P(k + 1)$ is divisible by 6. Hence $n^3 - n$ is divisible by 6 for $n \in N$.

- c) Let $P_n = 3^{2n} - 1$

For $n = 1, P_1 = 3^{2(1)} - 1 = 8 = 8(1)$. Hence P_1 is true.

Now for $n = k$, let P_k be true .i.e. $3^{2k} - 1$ be a multiple of 8.

Now for $n = k + 1, P_{k+1}$ must also be true .i.e. $3^{2(k+1)} - 1$ a multiple of 8.

$$\text{But } 3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 3^2 - 1 = 9 \cdot 3^{2k} - 1$$

$$= 9 \left[3^{2k} - \frac{1}{9} \right] = 9 \left[3^{2k} - 1 + 1 - \frac{1}{9} \right]$$

$$= 9 \left[(3^{2k} - 1) + \frac{8}{9} \right] = 9(3^{2k} - 1) + 8 = 9.P(k) + 8$$

Since $9.P(k)$ is a multiple of 8 and 8 is a multiple of 8, then $P(k + 1)$ is a multiple of 8. Thus $3^{2n} - 1$ is a multiple of 8 for $n \in N$.

- d) Let $P_n = 9^n - 1$ be a multiple of 8.

For $n = 1, P_1 = 9 - 1 = 8 = 8(1)$, hence P_1 is true.

Now for $n = k$, let P_k be true .i.e. $9^k - 1$ be a multiple of 8.

Now for $n = k + 1, P_{k+1}$ must also be true .i.e. $9^{k+1} - 1$ a multiple of 8

$$\text{But } 9^{k+1} - 1 = 9^k \cdot 9 - 1 = 9 \cdot [9^k - 1 + 1] - 1$$

$$= 9(9^k - 1) + 9 - 1 = 9.P(k) + 8$$

Since $9.P(k)$ is a multiple of 8 and 8 is a multiple of 8, then $P(k + 1)$ is a multiple of 8. Thus $9^n - 1$ is a multiple of 8 for $n \in N$.

- e) Let $P_n = 8^n - 7n + 6$

For $n = 1, P_1 = 8^1 - 7 \cdot 1 + 6 = 7 = 7(1)$, hence P_1 is true.

Now for $n = k$, let P_k be true .i.e. $8^k - 7k + 6$ be divisible by 7.

Now for $n = k + 1$, P_{k+1} must also be true .i.e. $8^{k+1} - 7(k + 1) + 6$ be divisible by 7

$$\begin{aligned}\text{But } 8^{k+1} - 7(k + 1) + 6 &= 8^k \cdot 8 - 7k - 7 + 6 = 8^k \cdot 8 - 7k - 1 \\ &= 8[8^k - 7k + 6 + 7k - 6] - 7k - 1 \\ &= 8(8^k - 7k + 6) + 8(7k) - 48 - 1 = 8.P(k) + 7(k - 7)\end{aligned}$$

Since $8.P(k)$ is divisible by 7 and $7(k - 7)$ is divisible by 7, then $P(k + 1)$ is divisible by 7. Thus $8^n - 7n + 6$ is divisible by 7 for $n \in N$

Task

Prove by induction that for positive integer of n ,

- i) $n^3 + 3n^2 - 10n$ is divisible by 3
- ii) $7^n - 4^n + 1^n$ is divisible by 6
- iii) $10^n - 1$ is divisible by 9
- iv) $7^n - 3^n$ is a multiple of 4
- v) $10^{2n-1} + 1$ is divisible by 11
- vi) $3^{2n} + 7$ is a multiple of 8
- vii) $n(n + 1)(n + 2)$ is a multiple of 6
- viii) $2^{4n-1} - 1$ is a multiple of 15
- ix) $n^3 + 6n^2 + 8n$ is divisible by 3

Extension of Proof by Induction Principle

Suppose that for some proposition P_n where n is a positive integer and that P_1 is false/meaningless but true for $P_k \Rightarrow P_{k+1}$, clearly P_n cannot be proved for all positive integral values of n . however, provided that there is some positive integer m such that P_m is true, then the induction P_n holds for all integral values of n greater than or equal to m .

Examples

Prove by induction that $2^n + 3^{2n-3}$ for $n \geq 2$ is always divisible by 7.

Solution

Let $P_n = 2^n + 3^{2n-3}$

For $n = 2$, $P_2 = 2^2 + 3^{2 \times 2 - 3} = 7$, hence proof holds for $n = 2$

Now for $n = k$, let P_k be true i.e. $2^k + 3^{2k-3}$ be divisible by 7

Now for $n = k + 1$, P_{k+1} must be true i.e. $2^{k+1} + 3^{2(k+1)-3}$ is divisible by 7.

$$\begin{aligned}\text{But } 2^{k+1} + 3^{2(k+1)-3} &= 2^k \cdot 2 + 3^{2k-1} \\ &= 2^k \cdot 2 + 3^{2k} \cdot 3^{-1} = 2^k \cdot 2 + \frac{3^{2k}}{3} \\ &= 2[2^k + 3^{2k-3} - 3^{2k-3}] + \frac{3^{2k}}{3} = 2(2^k + 3^{2k-3}) - 2 \cdot 3^{2k-3} + \frac{3^{2k}}{3} \\ &= 2(2^k + 3^{2k-3}) - 2 \cdot 3^{2k} \cdot 3^{-3} + \frac{3^{2k}}{3} = 2(2^k + 3^{2k-3}) - 2 \cdot \frac{3^{2k}}{27} + \frac{3^{2k}}{3} \\ &= 2(2^k + 3^{2k-3}) + 3^{2k} \left(-\frac{2}{27} + \frac{1}{3} \right) \\ &= 2(2^k + 3^{2k-3}) + 3^{2k} \cdot \frac{7}{27} = 2(2^k + 3^{2k-3}) + 7 \cdot 3^{2k-3} \\ \therefore 2^{k+1} + 3^{2k-1} &= 2P(k) + 7(3^{2k-3})\end{aligned}$$

Since $2P(k)$ is divisible by 7 and $7(3^{2k-3})$ is also divisible by 7, then $2^n + 3^{2n-3}$ is divisible by 7 for $n \geq 2$.

Proof by Induction of a finite series

Here a formula of n – terms of a sum of finite series and a general term(n^{th} – term) of the series must be known to prove by induction.

Examples

1. Prove by induction that

$$\begin{aligned} \text{a)} \quad & 1 + 2 + 3 + \cdots \dots \dots + n = \frac{n(n+1)}{2} \\ \text{b)} \quad & \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots \dots \dots + \frac{1}{n(n+1)} = \frac{n}{n+1} \\ \text{c)} \quad & 1^2 + 2^2 + 3^2 + \cdots \dots \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \\ \text{d)} \quad & \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \dots \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} \end{aligned}$$

Solution

a) Let $P_n = 1 + 2 + 3 + \cdots \dots \dots + n = \frac{n(n+1)}{2}$

For $n = 1, P_1 = 1 = \frac{1(1+1)}{2} = 1$, thus $LHS = RHS$. Hence P_1 holds.

Now for $n = k$

Let P_k be true. i.e. $1 + 2 + 3 + \cdots \dots \dots + k = \frac{k(k+1)}{2}$

Now for $n = k + 1, P_{k+1}$ must be true. i.e

$$LHS = 1 + 2 + 3 + \cdots \dots \dots + k + (k + 1)$$

$$RHS = \frac{k(k+1)}{2} + k + 1$$

NOTE: On LHS of P_{k+1} , $(k + 1)$ has been added, hence this proof is done by adding the general term of the series after replacing n by $k + 1$ on either side of P_k .

$$\begin{aligned} \therefore RHS &= \frac{k(k+1)}{2} + k + 1 = (k + 1) \left[\frac{k}{2} + 1 \right] \\ &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)[(k+1)+1]}{2} \end{aligned}$$

Hence proof holds for $n = k + 1$. Since the proof holds for $n = 1, n = k$ and $n = k + 1$, then it is true for all integral values of n .

b) $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots \dots \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

$$\text{Let } P_n = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots \dots \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

For $n = 1, P_1 = \frac{1}{1 \times 2} = \frac{1}{1+1} = \frac{1}{2}$. Hence P_1 is true.

Now for $n = k$

Let P_k be true. i.e. $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots \dots \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Now for $n = k + 1, P_{k+1}$ must be true. i.e

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots \dots \dots + \frac{1}{k(k+1)} + \frac{1}{k(k+1)(k+1+1)} &= \frac{k}{k+1} + \frac{1}{k(k+1)(k+1+1)} \\ &= \frac{1}{k(k+1)(k+2)} \\ &= \frac{1}{(k+1)} \left[k + \frac{1}{k+2} \right] \\ &= \frac{1}{(k+1)} \left[\frac{k^2 + 2k + 1}{k+2} \right] = \frac{1}{(k+1)} \cdot \frac{(k+1)^2}{(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{[(k+1)+1]} \end{aligned}$$

Hence proof holds for $n = k + 1$

Note:

LHS and **RHS** for P_{k+1} has been increased by $\frac{1}{n(n+1)}$ for $n = k + 1$.

$$i.e. \frac{1}{k(k+1)(k+1+1)} = \frac{1}{k(k+1)(k+2)}$$

Since proof holds for $n = 1, k, k+1$, the proof holds for all integral values of n .

$$c) \text{ Let } P_n = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\text{For } n = 1, P_1 = 1^2 = 1 = \frac{1(1+1)(2 \times 1 + 1)}{6} = 1. \text{ Hence } P_1 \text{ is true.}$$

Now for $n = k$

$$\text{Let } P_k \text{ be true. i.e. } 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now for $n = k+1, P_{k+1}$ must be true. i.e

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)}{6} [k(2k+1) + 6(k+1)] \\ &= \frac{(k+1)}{6} [2k^2 + k + 6k + 6] \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(2k^2 + 3k + 4k + 6)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6} \\ &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}, \text{ hence proof holds.} \end{aligned}$$

Since the proof holds for $n = 1, n = k$ and $n = k+1$, then it is true for all integral values of n

$$d) \text{ Let } P_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$\text{For } n = 1, P_1 = \frac{1}{2} = \frac{2^1 - 1}{2^1} = \frac{1}{2}. \text{ Hence } P_1 \text{ is true.}$$

Now for $n = k$

$$\text{Let } P_k \text{ be true. i.e. } \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

Now for $n = k+1, P_{k+1}$ must be true. i.e

$$\begin{aligned} \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\ &= \frac{1}{2^{k+1}} \left[\frac{2^k - 1}{2^{-1}} + 1 \right] = \frac{1}{2^{k+1}} [2(2^k - 1) + 1] \\ &= \frac{1}{2^{k+1}} [2^{k+1} - 1] = \frac{2^{k+1} - 1}{2^{k+1}}, \text{ hence proof holds.} \end{aligned}$$

Since the proof holds for $n = 1, n = k$ and $n = k+1$, then it is true for all integral values of n

2. Prove by induction that

$$a) \sum_{r=2}^n \frac{1}{r^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$

$$b) \sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

$$c) \sum_{r=1}^n \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$$

$$d) \sum_{r=1}^n 3^{r-1} = \frac{3^n - 1}{2}$$

Solution

$$a) \sum_{r=2}^n \frac{1}{r^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}; r = 2, 3, 4, \dots, n$$

Generating the series;

$$\begin{aligned} &\Rightarrow \frac{1}{2^2-1} + \frac{1}{3^2-1} + \frac{1}{4^2-1} + \dots + \frac{1}{n^2-1} \\ &\Rightarrow \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \dots + \frac{1}{n^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)} \end{aligned}$$

$$\text{Let } P_n = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \dots + \frac{1}{n^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$

For $n = 2, P_2 = \frac{1}{3} = \frac{3}{4} - \frac{2 \cdot 2 + 1}{2 \cdot 2(2+1)} = \frac{3}{4} - \frac{5}{12} = \frac{1}{3}$. Hence P_2 is true.

For $n = k$, let P_k be true .i.e. $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \dots + \frac{1}{k^2 - 1} = \frac{3}{4} - \frac{2k+1}{2k(k+1)}$

Now for $n = k + 1, P_{k+1}$ must be true .i.e

$$\begin{aligned} \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \dots + \frac{1}{k^2 - 1} + \frac{1}{(k+1)^2 - 1} &= \frac{3}{4} - \frac{2k+1}{2k(k+1)} + \frac{1}{(k+1)^2 - 1} \\ &= \frac{3}{4} + \frac{1}{(k+1)^2 - 1} - \frac{2k+1}{2k(k+1)} = \frac{3}{4} + \frac{1}{k(k+2)} - \frac{2k+1}{2k(k+1)} \\ &= \frac{3}{4} + \frac{2(k+1) - (2k+1)(k+2)}{2k(k+1)(k+2)} \\ &= \frac{3}{4} + \frac{2k+2 - (2k^2 + 4k + k + 2)}{2k(k+1)(k+2)} = \frac{3}{4} - \frac{2k^2 + 3k}{2k(k+1)(k+2)} \\ &= \frac{3}{4} - \frac{k(2k+3)}{2k(k+1)(k+2)} = \frac{3}{4} - \frac{2(k+1)+1}{2k(k+1)[(k+1)+1]} \end{aligned}$$

Hence P_{k+1} is true for $n = k + 1$. Thus since proof holds for $n = 2, k, k + 1$, then it holds for all $n \geq 2$

b) $\sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2); r = 1, 2, 3, 4, \dots, n$

Generating the series;

$$\Rightarrow 1 \cdot (1+1) + 2 \cdot (2+1) + 3 \cdot (3+1) + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

$$\Rightarrow 2 + 6 + 12 + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

$$\text{Let } P_n = 2 + 6 + 12 + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

For $n = 1, P_1 = 2 = \frac{1}{3} \cdot 1 \cdot (1+1)(1+2) = 2$. Hence P_1 is true.

For $n = k$, let proof be true .i.e. $2 + 6 + 12 + \dots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$

Now for $n = k + 1, P_{k+1}$ must be true .i.e

$$\begin{aligned} 2 + 6 + 12 + \dots + k(k+1) + (k+1)(k+2) &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= (k+1)(k+2) \left[\frac{k}{3} + 1 \right] = \frac{(k+1)(k+2)(k+3)}{3} = \frac{(k+1)[(k+1)+1][(k+1)+2]}{3} \end{aligned}$$

Hence P_{k+1} is true for $n = k + 1$. Thus since proof holds for $n = 2, k, k + 1$, then it holds for all integral values of n .

c) $\sum_{r=1}^n \frac{r}{2^r} = 2 - \frac{n+2}{2^n}; r = 1, 2, 3, \dots, n$

Generating the series;

$$\Rightarrow \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

$$\text{Let } P_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

For $n = 1, P_1 = \frac{1}{2} = 2 - \frac{1+2}{2^1} = \frac{1}{2}$. Hence P_1 is true.

For $n = k$, let the proof be true .i.e. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$

Now for $n = k + 1, P_{k+1}$ must be true .i.e

$$\begin{aligned} \Rightarrow \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \\ &= 2 + \frac{k+1}{2^{k+1}} - \frac{k+2}{2^k} = 2 + \frac{1}{2^k} \left[\frac{k+1}{2} - k + 2 \right] \\ &= 2 + \frac{1}{2^{k+1}} [-k - 3] = 2 - \frac{k+3}{2^{k+1}} \\ &= 2 - \frac{[(k+1)+2]}{2^{k+1}} \end{aligned}$$

Hence P_{k+1} is true for $n = k + 1$. Thus since proof holds for $n = 2, k, k + 1$, then it holds for all integral values of n .

d) $\sum_{r=1}^n 3^{r-1} = \frac{3^n - 1}{2}; r = 1, 2, 3, \dots, n$

Generating the series;

$$\Rightarrow 3^{1-1} + 3^{2-1} + 3^{3-1} + \dots + 3^{n-1} = \frac{3^n - 1}{2}$$

$$\text{Let } P_n = 1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$$

$$\text{For } n = 1, P_1 = 1 = \frac{3^1 - 1}{2} = 1. \text{ Hence } P_1 \text{ is true.}$$

$$\text{For } n = k, \text{ let the proof be true. i.e. } 1 + 3 + 3^2 + \dots + 3^{k-1} = \frac{3^k - 1}{2}$$

Now for $n = k + 1, P_{k+1}$ must be true. i.e.

$$\begin{aligned} \Rightarrow 1 + 3 + 3^2 + \dots + 3^{k-1} + 3^{k+1-1} &= \frac{3^k - 1}{2} + 3^{k+1-1} \\ &= \frac{3^k - 1}{2} + 3^k = \frac{3^k + 2 \cdot 3^k - 1}{2} \\ &= \frac{3 \cdot 3^k - 1}{2} = \frac{3^{k+1} - 1}{2} \end{aligned}$$

Hence P_{k+1} is true for $n = k + 1$. Thus since proof holds for $n = 2, k, k + 1$, then it holds for all integral values of n .

Task

1. Prove by induction that

a) $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$

b) $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$

c) $\sum_{r=1}^n r(r+1)(r+4) = \frac{1}{4}n(n+1)(n+4)(n+5)$

d) $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \dots + \frac{1}{n(n+3)} = \frac{11}{18} - \frac{1}{3} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \right)$

e) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

f) $\sum_{r=1}^n r(r+3) = \frac{1}{3}n(n+1)(n+5)$

g) $\sum_{r=1}^n 2r(2r-1) = \frac{1}{3}n(n+1)(4n-1)$

2. Prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integral values of n .

3. Show that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ and prove by induction that

$$(1+x)^n = 1 + nx + \dots + \binom{n}{r}x^r + \dots + x^n$$

4. $\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

SUMMATION OF A FINITE SERIES

There are various standard results that can be used to find the sum of finite series.

The two formulae of sums of A.P's and G. P's are:

1) The sum of n -terms of the A.P with first term a and common difference d is:

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$

2) The sum of n - terms of a G.P with first term a and common ratio r is:

$$S_n = \frac{a(1-r^n)}{1-r} \text{ for } |r| < 1$$

The standard series are usually given as:

$$\begin{aligned} \text{i) } 1 + 2 + 3 + \dots + n &= \sum_{r=1}^n r = \frac{1}{2}n(n+1) \\ \text{ii) } 1^2 + 2^2 + 3^2 + \dots + n^2 &= \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1) \\ \text{iii) } 1^3 + 2^3 + 3^3 + \dots + n^3 &= \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2 = [\sum_{r=1}^n r]^2 \end{aligned}$$

The standard series above can be used in summation of finite number of terms.

Examples

1. Find the sum of the series:

- $1 + 2 + 3 + \dots + (2n+1)$
- $1^2 + 2^2 + 3^2 + \dots + (n-1)^2$
- $1^3 + 2^3 + 3^3 + \dots + (2n)^3$

Solution

NOTE: in all the above series, the last term is not a general term for $n \in N$. The series exercise the form of a standard series, hence the result of standard series will hold for the last term.

a) $1 + 2 + 3 + \dots + (2n+1)$

$$\text{Using } 1 + 2 + 3 + \dots + N = \frac{1}{2}N(N+1)$$

$$\Rightarrow N = 2n+1$$

$$\begin{aligned} \therefore 1 + 2 + 3 + \dots + (2n+1) &= \frac{1}{2}(2n+1)[(2n+1)+1] \\ &= (2n+1)(n+1) \end{aligned}$$

b) $1^2 + 2^2 + 3^2 + \dots + (n-1)^2$

$$\text{Using } 1^2 + 2^2 + 3^2 + \dots + N^2 = \frac{1}{6}N(N+1)(2N+1)$$

$$\text{Taking } N = (n-1)$$

$$\Rightarrow 1^2 + 2^2 + 3^2 + \dots + (n-1)^2 = \frac{(n-1)(n-1+1)[2(n-1)+1]}{6} = \frac{n(n-1)(2n-1)}{6}$$

c) $1^3 + 2^3 + 3^3 + \dots + (2n)^3$

$$\text{Using } 1^3 + 2^3 + 3^3 + \dots + N^3 = \frac{N^2(N+1)^2}{4}$$

$$\text{Taking } N = 2n$$

$$1^3 + 2^3 + 3^3 + \dots + (2n)^3 = \frac{(2n)^2((2n)+1)^2}{4} = n^2(2n+1)^2$$

2. Find the sum of the series;

- $1^3 + 3^3 + 5^3 + \dots + (2n+1)^3$
- $2^3 + 4^3 + 6^3 + \dots + (2n)^3$
- $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$

Solution

a) $1^3 + 3^3 + 5^3 + \dots + (2n+1)^3$

(Summation of the cubes of odd terms of natural numbers)

$$\text{Using } 1^3 + 2^3 + 3^3 + \dots + N^3 = \frac{N^2(N+1)^2}{4}$$

$$\text{Now } 1^3 + 2^3 + 3^3 + \dots + (2n)^3 + (2n+1)^3 = \frac{(2n+1)^2((2n+1)+1)^2}{4}$$

$$= (2n+1)^2(n+1)^2$$

Finding the sum of the series; $2^3 + 4^3 + 6^3 + \dots + (2n)^3$

$$2^3 + 4^3 + 6^3 + \dots + (2n)^3 = 2^3 [1^3 + 2^3 + 3^3 + \dots + n^3]$$

$$= 8 \left[\frac{n^2(n+1)^2}{4} \right] = 2n^2(n+1)^2$$

$$\therefore 1^3 + 3^3 + 5^3 + \dots + (2n+1)^3 = (2n+1)^2(n+1)^2 - 2n^2(n+1)^2$$

$$= (n+1)^2[(2n+1)^2 - 2n^2]$$

$$= (n+1)^2[4n^2 + 4n + 1 - 2n^2] = (n+1)^2(2n^2 + 4n + 1)$$

b) $2^3 + 4^3 + 6^3 + \dots + (2n)^3 = 2^3 [1^3 + 2^3 + 3^3 + \dots + n^3]$

$$= 8 \left[\frac{n^2(n+1)^2}{4} \right] = 2n^2(n+1)^2$$

c) $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$

First summing the natural numbers to the $(2n)^{th}$ term,

$$\Rightarrow 1^3 + 2^3 + 3^3 + \dots + (2n-1)^3 + (2n)^3 = \frac{(2n)^2(2n+1)^2}{4} = n^2(2n+1)^2$$

Now summing the series $2^3 + 4^3 + 6^3 + \dots + (2n)^3$

$$2^3 + 4^3 + 6^3 + \dots + (2n)^3 = 2n^2(n+1)^2 \text{ (See previous number above)}$$

Now subtracting the series,

$$\Rightarrow 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n+1)^2 - 2n^2(n+1)^2$$

$$= n^2[(2n+1)^2 - (n+1)^2]$$

$$= n^2[4n^2 + 4n + 1 - n^2 - 2n - 1]$$

$$= n^2(2n^2 - 1)$$

NOTE:

In this last example the series $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$ has the last term as $(2n-1)^3$ and is the general term of the series for all $n \in \mathbb{N}$

Alternative summation of the series

$$\begin{aligned} 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 &= \sum_{r=1}^n (2r-1)^3 \\ &= \sum_{r=1}^n (8r^3 - 12r^2 + 6r - 1) = 8 \sum_{r=1}^n r^3 - 12 \sum_{r=1}^n r^2 + 6 \sum_{r=1}^n r - \sum_{r=1}^n 1 \\ &= 8 \left[\frac{n^2(n+1)^2}{4} \right] - 12 \left[\frac{n(n+1)(2n+1)}{6} \right] + 6 \left[\frac{n(n+1)}{2} \right] - n \\ &= 2n^2(n+1)^2 - n(n+1)(2n+1) + 3n(n+1) - n \\ &= n(n+1)[2n(n+1) - 2(2n+1) + 3] - n \\ &= n(n+1)[2n^2 + 2n - 4n - 2 + 3] - n \\ &= n(n+1)(2n^2 - 2n + 1) - n \\ &= n[2n^3 - 2n^2 + n + 2n^2 - 2n + 1 - 1] \\ &= n[2n^3 - n] = n^2(2n^2 - 1) \end{aligned}$$

3. a) It can be proved by induction that for positive n ,

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

From this result deduce that

$$(n+1)^3 + (n+2)^3 + (n+3)^3 + \dots + (2n)^3 = \frac{1}{4}n^2(3n+1)(5n+3)$$

b) Show that the sum of the series

$$(n+1)^2 + (n+2)^2 + (n+3)^2 + \dots + (2n)^2 = \frac{1}{6}n(2n+1)(7n+1)$$

Solution

a) $1^3 + 2^3 + 3^3 + \dots + n^3 + (n+1)^3 + (n+2)^3 + (n+3)^3 + \dots + (2n)^3$

$$= \frac{(2n)^2(2n+1)^2}{4} = n^2(2n^2 + 1)$$

$$\text{Also } 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

Subtracting the series,

$$\begin{aligned} (n+1)^3 + (n+2)^3 + (n+3)^3 + \dots + (2n)^3 &= n^2(2n^2 + 1) - \frac{n^2(n+1)^2}{4} \\ &= \frac{1}{4}n^2[4(4n^2 - 4n + 1) - (n^2 + 2n + 1)] \\ &= \frac{1}{4}n^2[16n^2 + 16n + 4 - n^2 - 2n - 1] = \frac{1}{4}n^2[15n^2 + 14n + 3] \\ &= \frac{1}{4}n^2[15n^2 + 9n + 5n + 3] = \frac{1}{4}n^2[3n(5n + 3) + (5n + 3)] \\ &= \frac{1}{4}n^2(3n + 1)(5n + 3) \end{aligned}$$

$$\begin{aligned} \text{b) From } 1^2 + 2^2 + 3^2 + \dots + n^2 &= \frac{1}{6}n(n+1)(2n+1) \\ \Rightarrow 1^2 + 2^2 + 3^2 + \dots + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + \dots + (2n)^2 \\ &= \frac{1}{6}(2n)(2n+1)(2(2n)+1) = \frac{2n(2n+1)(4n+1)}{6} \end{aligned}$$

Subtracting the series,

$$\begin{aligned} (n+1)^2 + (n+2)^2 + (n+3)^2 + \dots + (2n)^2 \\ &= \frac{2n(2n+1)(4n+1)}{6} - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n(2n+1)(7n+1) \\ &= \frac{n(2n+1)}{6}[2(4n+1) - (n+1)] \\ &= \frac{n(2n+1)}{6}(8n+2-n-1) = \frac{1}{6}n(2n+1)(7n+1) \end{aligned}$$

4. Show that the sum of the series

$$3 + 6 + 11 + \dots + (n^2 + 2) = \frac{1}{6}n(2n^2 + 3n + 13)$$

Solution

For this series, $(n^2 + 2)$ is the general term of the series for all $n \in \mathbb{N}$

$$\begin{aligned} \therefore 3 + 6 + 11 + \dots + (n^2 + 2) &= \sum_{r=1}^n (r^2 + 2) = \sum_{r=1}^n r^2 + \sum_{r=1}^n 2 \\ &= \frac{1}{6}n(n+1)(2n+1) + 2n \\ &= \frac{n}{6}[(n+1)(2n+1) + 12] = \frac{n}{6}[2n^2 + 3n + 1 + 12] \\ &= \frac{1}{6}n(2n^2 + 3n + 13) \end{aligned}$$

Task

1. Show that the sum of the series;

$$\begin{aligned} \text{a) } 2 + 10 + 30 + \dots + (n^3 + n) &= \frac{n(n+1)(n^2+n+2)}{4} \\ \text{b) } 2 + 12 + 36 + \dots + (n^3 + n^2) &= \frac{n(n+1)(n+2)(3n+1)}{12} \\ \text{c) } 2 + 6 + \dots + (n^2 - n) &= \frac{n(n+1)(n-1)}{3} \\ \text{d) } 1.4 + 2.5 + 3.6 + \dots + n(n+3) &= \frac{n(n+1)(n+5)}{3} \\ \text{e) } 2^2 + 5^2 + 8^2 + \dots + (3n-1)^2 &= \frac{n(6n^2+3n-1)}{6} \\ \text{f) } 1.2 + 3.4 + 5.6 + \dots + 2n(2n-1) &= \frac{n(n+1)(4n-1)}{3} \\ \text{g) } 1.2.5 + 2.3.6 + 3.4.7 + \dots + n(n+1)(n+4) &= \frac{n(n+1)(n+2)(3n+12)}{12} \end{aligned}$$

2. Find the sum of the series

$$\begin{aligned} \text{a) } 21^3 + 22^3 + 23^3 + \dots + 35^3 \\ \text{b) } 1^3 + 3^3 + 5^3 + \dots + 25^3 \\ \text{c) } 1^3 - 2^3 + 3^3 - 4^3 + \dots + (2n-1)^3 - (2n)^3 \end{aligned}$$

$$\text{ANSWER: a) 352 800 b) 56 953 c) } -n(4n+3)$$

INTEREST AND ANNUITIES

Simple and Compound Interest

When person A lends money to person B he makes a charge for the use of his money. The amount of money lent is called *Principal* (P) and the charge made for lending is called *Interest* (I). The sum of interest and principal is the *Amount* or *Total amount* (A), which is the amount B will pay back to A after any interval of time. i.e. ($A = P + I$)

The *Rate per cent per annum* ($R\%$) is the interest payable on principal for one year.

Types of Interest

- Simple interest
- Compound interest

Simple Interest (S.I)

Simple Interest is the interest that is paid on the principal only.

S.I is paid on a given principal, and is the same after equal intervals of time.

The principal is always constant.

To find the Interest and Amount of a given sum in a given time at Simple Interest

When money is put out at S.I, the interest is payable for each year, but is not added to the principal.

Let P be the given principal and n the number of years for which the principal is lent, R the rate of interest per annum.

The interest of P for 1-year = $P \cdot \frac{R}{100}$ and therefore for n -years is $P \cdot \frac{R}{100} \cdot n$

$$\therefore \text{Amount at the end of } 1\text{-year} = A_1 = P + P \cdot \frac{R}{100} = P \left(1 + \frac{R}{100} \right)$$

Since the principal is lent at S.I, the interest $P \cdot \frac{R}{100}$ will be added to the amount at the end of each year.

Thus the amount at the end of n -years will be;

$$\begin{aligned} A_n &= P \left(1 + \frac{R}{100} \right) + \frac{PR}{100} + \frac{PR}{100} + \frac{PR}{100} + \cdots \cdots \cdots + \frac{PR}{100} \\ &= P + \underbrace{\frac{PR}{100} + \frac{PR}{100} + \frac{PR}{100} + \cdots \cdots \cdots + \frac{PR}{100}}_{n\text{-times}} \end{aligned}$$

$$A_n = P + \frac{PRn}{100} = P \left(1 + \frac{nR}{100} \right), \text{ which is the amount at the end of } n\text{-years.}$$

$$\therefore \text{Amount for } n\text{-years, } A_n = P \left(1 + \frac{nR}{100} \right)$$

, which is the formula for amount at the end of n -years, where;

P = Principal

n = Number of years principal is lent

R = Rate per cent per annum

Note:

1. The amount at the end of each year constitute a series in an A.P with a common difference $\frac{PR}{100}$
2. Simple interest can be paid at different intervals .i.e. monthly, quarterly, half yearly or yearly.

3. In $A = P \left(1 + \frac{nR}{100} \right)$, formulae contain four quantities .i.e. P, A, R, n and if three are given, the fourth can be evaluated.
4. When working out the interest due on a sum of money for a period between two specific dates, always take a year as containing 365 days.
5. For purposes of interest, the day a deposit is first made does not count. However the day on which money is withdrawn does count. Thus if someone deposits money on an account on 15th March and withdraws it on 17th April, he is not paid interest for 15th March but he is paid interest for 17th April.

Examples

1. Find the simple interest on and the amount at simple interest of;
 - a) Sh. 8000 for $2\frac{1}{4}$ years at $8\frac{1}{5}\%$ p.a.
 - b) Sh. 480 for 2 months at $3\frac{3}{4}\%$ p.a.

Solution

a) Simple Interest $= \frac{P.R.n}{100}$; $P = 8000, R = 8.2\%, n = 2.25 \text{ years}$

$$= \frac{8000 \times 8.2 \times 2.25}{100} = 1476$$

$$\text{Amount} = P + I = 8000 + 1476 = 9476 / =$$

b) $P = 480, R = 3.75\%, n = \frac{20}{12} \text{ years}$

$$\therefore \text{Simple Interest} = \frac{480 \times 3.75 \times \frac{20}{12}}{100} = 30$$

$$\text{Amount} = 30 + 480 = 510 / =$$

2. Calculate to the nearest shilling, the simple interest on;
 - a) Sh. 550 from March 6 to May 6 at 11% p.a.
 - b) Sh. 990 from August 20 to December 13 at $9\frac{1}{2}\%$ p.a.

Solution

a) Simple Interest $= \frac{P.R.n}{100}$, $P = 550, R = 11\%$,

$$n = (31 - 6)\text{days} + (30 + 6)\text{days} = 61\text{days} = \frac{61}{365} \text{ years}$$

[April = 30 days, May 6 = 6 days, March 6 = (31 - 6)days]

$$\therefore \text{Simple Interest} = \frac{550 \times 11 \times \frac{61}{365}}{100} = 10 / =$$

b) $P = 990, R = 9\frac{1}{2}\%$,

$$n = (31 - 20)\text{days} + (30 + 31 + 30 + 13)\text{days} = 115\text{days} = \frac{115}{365} \text{ years}$$

$$\therefore \text{Simple Interest} = \frac{990 \times 9\frac{1}{2} \times \frac{115}{365}}{100} = 30 / =$$

3. a) In what time will Sh. 3 200 amount to Sh. 4 188 at $9\frac{1}{2}\%$ p.a. simple interest?
- b) A sum of money invested at 9% p.a. simple interest amounts after 3 years to sh. 7 620. Find the sum invested.

Solution

a) Given: $P = 3200, A = 4188, R = 9\frac{1}{2}\%$, $n = ?$

$$\text{From Simple Interest} = \frac{P.R.n}{100} \Rightarrow n = \frac{100I}{P.R}$$

$$\text{But Interest} = \text{Amount} - \text{Principal}$$

$$\therefore I = 4188 - 3200 = 988$$

$$\therefore n = \frac{100 \times 988}{3200 \times 9\frac{1}{2}} = 3\frac{1}{4} \text{ years}$$

- b) Given: Amount = 7 620 = Principal + Interest

$$\text{Interest} = \frac{P.R.n}{100} = \frac{P \times 9 \times 3}{100}$$

$$\text{Now } 7\,620 = P + \frac{27P}{100} = \frac{127P}{100} \therefore P = 6\,000/=$$

Task

1. Find the un known quantities in the table below

Principal	Interest	Amount	Time	Rate % p.a.
sh. 650	sh. 156		3 years	
.....	sh. 1 942.25		$1\frac{1}{4}$ years	$9\frac{1}{4}$
sh. 8 450			9 months	
.....	sh. 170.40		16 months	$11\frac{1}{4}$
sh. 1 818		sh. 1 908.90		$8\frac{1}{3}$

ANSWER:

- a) sh. 806 ; 8% b) sh. 16 800 ; sh. 18 742.50 c) sh. 549.25 ; $8\frac{2}{3}\%$ d) sh. 1 136 ; sh. 1 306.40
 e) sh. 90.90 ; 219 days
 2. Two men both invest the same sum, one at $9\frac{3}{4}\%$ p.a. if the difference in the interest they receive is sh. 100 , what sum each man invested is.

ANSWER: sh. 20 000

3. Two men both invest sh. 800 at 10% and $8\frac{1}{2}\%$ p.a respectively. What is the difference in the interest they earn after 9 months?

ANSWER: sh. 9

4. A club of 320 members has to meet a bill of sh. 9 500 in 5 months time. Each member subscribes sh. 30 and the money accumulates at $8\frac{1}{4}\%$ p.a simple interest. What balance remains when the bill has been paid?

ANSWER: sh. 430

$$\left[\begin{array}{l} \text{Hint: Total principal} = 30 \times 320 = 9\frac{600}{=}, n = \frac{5}{12}, I = \frac{9600 \times 8\frac{1}{4} \times \frac{5}{12}}{100} = 330 \\ \text{Now New amount} = 330 + 9600 = \text{Sh. } 9\,930 \therefore \text{Balance} = 9930 - 9500 = 430/= \end{array} \right]$$

5. A sum of money amounts at simple interest to sh. 819 in 2 years at $8\frac{1}{2}\%$ p.a. to what would it amount in 1 year 9 months at 8% p.a? ANSWER: sh. 798

$$\left[\begin{array}{l} \text{Hint: Let sum} = P = \text{Principal}, A = P + I, I = \frac{P \times 8\frac{1}{2} \times 2}{100} = 0.17P \\ \text{Now } 819 = P + 0.17P, P = 700 \\ \text{Now at } 8\% \text{ p.a, amount, } A = 700 + \frac{700 \times 8\frac{1}{2} \times 1\frac{9}{12}}{100} = 798/= \end{array} \right]$$

6. Diana invests $\frac{2}{5}$ of her capital at 15% p.a and the remainder at 10% p.a simple interest. After 25 cents in the shillings has been deducted for tax, she receives sh. 720 interest each year. What is the size of her capital?

ANSWER: sh. 8000

7. A man borrows sh. 5000 at 10 $\frac{1}{2}$ % p.a payable yearly. After one year he pays back sh. 2500. How much will he need to clear his date completely at the end of another one year? **ANSWER: sh. 3 343**

$$\left[\begin{array}{l} \text{Hint: } P = 5000, R = 10.5\%, \text{ Amount after 1 year} = 5000 + \frac{10.5}{100} \times 5000 = 5525 \\ \text{Balance after paying off 2500} = 5525 - 2500 = 3025 \\ \text{Now amount after another 1 year} = 3025 + 3025 \times \frac{10.5}{100} = 3343 /= \end{array} \right]$$

8. Godfrey borrows sh. 20 000 from a finance corporation and pays 10% interest p.a on an amount still owing at the beginning of that year. After one year he pays sh. 5 000 back, sh. 6 000 a year later and sh. 7 000 at the end of the third year. How much does he still need to clear the debt completely? **ANSWER: sh. 6 970**

$$\left[\begin{array}{l} \text{Hint: } P = 20\,000, R = 10\% \text{ p.a. Amount after 1 year} = 20000 + \frac{10}{100} \times 20000 = 22\,000; \\ \text{Amount remaining after paying 5000} = 22\,000 - 5\,000 = 17\,000 \\ \text{Expected amount after another year} = 17000 + \frac{10}{100} \times 17000 = 18\,700; \\ \text{Balance after paying sh. 6000} = 18\,700 - 6000 = 12\,700/= \\ \text{Expected amount after another 3}^{\text{rd}} \text{ year} = 12\,700 + 12\,700 \times \frac{10}{100} = 13\,970/= \\ \text{Now balannce after clearing sh. 7000} = 13\,970 - 7000 = 6\,970/= \\ \text{NOTE: Principal is not constant every year but interest is the same.} \end{array} \right]$$

COMPOUND INTEREST (C.I)

Compound interest is the interest calculated on the principal and the accrued interest.

The interest unpaid is added to the principal and for the next period the interest is computed. *i.e.* Compound interest is the interest on the growing principal.

NOTE

- Both principal and compound interest change from period to period since the principal changes.
- Interest earned on the principal is not withdrawn.
- Compound interest can be computed annually, half yearly, quarterly or monthly.
- When a given amount is due after a specified period of time, its present value (principal for the next period) is that sum together with the interest for that period produces the given amount.

To find Interest and Amount of a given sum at Compound Interest

If the money is lent at compound interest the interest is added each year to the principal and for the following year the interest is calculated on their sum.

Let P be the principal and n the number or years for which the principal is lent at $r\%$ per annum compound interest.

$$\text{The amount at the end of one year will be} = P + \frac{Pr}{100} = P \left(1 + \frac{r}{100} \right) = PR$$

$$\text{The amount at the end of two years will be} = PR + PR \cdot \frac{r}{100} = PR \left(1 + \frac{r}{100} \right) = PR^2$$

The amount at the end of three years will be $= PR^2 + PR^2 \cdot \frac{r}{100} = PR^2 \left(1 + \frac{r}{100}\right) = PR^3$

Then arguing in this way, the amount at the end of n years will be $= PR^n$

Thus the amount, A at the end of n years for principal P lent at $r\%$ per annum is

$$A = P \left(1 + \frac{r}{100}\right)^n \text{ or } A = PR^n$$

, where $A = \text{Amount}$

$P = \text{Principal}$

$r = \text{rate per cent per annum}$

$$R = \left(1 + \frac{r}{100}\right)$$

Deduction

From $A = PR^n$, $P = \frac{A}{R^n}$ this enables us to find the sum which will

produce amount A in n years.

Also $R = \sqrt[n]{\frac{A}{P}}$, from which interest can be calculated.

Again from $R^n = \frac{A}{P}$, $\log(R^n) = \log\left(\frac{A}{P}\right)$

$\Rightarrow n \log R = \log A - \log P$ $\therefore n = \frac{\log A - \log P}{\log R}$, from which time taken

for P to amount to A can be evaluated.

Note

The amount in each year constitute terms in a G.P with first term PR and common ratio $\left(1 + \frac{r}{100}\right)$

Examples

1. Find the compound interest on;
 - a) Sh. 10 000 for 2 years at 10% p.a paid annually
 - b) Sh. 6 000 for 3 years at 10% p.a paid annually

Solution

- a) $P = 10\,000, n = 2, r = 10\%$

Interest, $I = \text{Amount} - \text{Principal}$

$$I = P \left(1 + \frac{r}{100}\right)^n - P = 10\,000 \left(1 + \frac{10}{100}\right)^2 - 10\,000 = \text{Sh. } 2\,100$$

- b) $P = 6\,000, n = 3, r = 10\%$

$$A = 6000 \left(1 + \frac{10}{100}\right)^3 = 7\,986$$

$$\therefore \text{Interest} = 7\,986 - 6\,000 = \text{sh. } 1\,986$$

2. Prove that in order that a sum of money may double itself in 10 years by investment at compound interest payable annually, the interest should be approximately 7.2 per cent per annum.

Solution

Let $P = \text{Amount invested for}$

$n = 10$ years, which amounts to

$A = 2P$ at a rate of

$r = r\%$ per annum

Using $A = P \left(1 + \frac{r}{100}\right)^n$,

$$\Rightarrow 2P = P(1 + 0.01r)^{10} \text{ or } 2 = (1 + 0.01r)^{10}$$

Introducing logarithms

$$\Rightarrow \log 2 = \log(1 + 0.01r)^{10} = 10 \log(1 + 0.01r)$$

$$\Rightarrow \frac{\log 2}{10} = \log(1 + 0.01r)$$

$$\Rightarrow 2^{0.1} = 1 + 0.01r \therefore r = \frac{2^{0.1} - 1}{0.01} \approx 7.2\%$$

3. How many years (correct to the nearest whole year) does it take a sum of money to double in value if compound interest is paid at a rate of 9% p.a?

Solution

Let the sum be P be invested for n – years to amount to $2P$ at a rate of 9% p.a

Using $A = P \left(1 + \frac{r}{100}\right)^n$,

$$\Rightarrow 2P = P \left(1 + \frac{9}{100}\right)^n \text{ or } 2 = 1.09^n$$

Introducing logarithms

$$\Rightarrow \log 2 = \log 1.09^n = n \log 1.09$$

$$\Rightarrow \therefore n = \frac{\log 2}{\log 1.09} \approx 8 \text{ years}$$

To find Amount of a given sum for $\left(n + \frac{1}{m}\right)$ years at Compound Interest

In business transactions when the time contains a fraction of a year, allow Simple Interest for the fraction of a year, and Compound Interest for the whole years.

Let P be invested for $\left(n + \frac{1}{m}\right)$ years at a rate per cent per annum ($r\%$)

Compound Interest.

Amount, A = Amount after n – years + Interest for $\frac{1}{m}$ years

after $\left(n + \frac{1}{m}\right)$ years (principal) due to C.I

on principal (Amount due to C.I)

Amount after n – years = $P(1 + r)^n = PR^n$

Now principal for $\frac{1}{m}$ years = PR^n

Interest for $\frac{1}{m}$ years = $PR^n \times r \times \frac{1}{m} = PR^n \times \frac{r}{m}$

Now Total Amount = $PR^n + PR^n \times \frac{r}{m} = PR^n \left(1 + \frac{r}{m}\right)$

$$A = PR^n \left(1 + \frac{r}{m}\right)$$

Example

1. Find Compound Interest on;

a) Sh. 6 000 for $2\frac{1}{2}$ years at 10% p.a paid annually

b) Sh. 8 000 for $1\frac{3}{4}$ years at 7.5% p.a paid annually

Solution

- a) $P = 6\,000$, $n = 2$, $\frac{1}{m} = \frac{1}{2}$, $r = 10\%$ p.a.

Using $A = PR^n \left(1 + \frac{r}{m}\right)$, $R = 1 + r = 1 + \frac{10}{100} = 1.1$

Amount after $2\frac{1}{2}$ years = $A = 6\,000 \times (1.1)^2 \times \left(1 + \frac{10}{100} \times \frac{1}{2}\right) = 7\,623$

Now *Compound Interest* = *Amount* – *Principal*
 $= 7\,623 - 6\,000 = \text{sh. } 1\,623$

- b) $P = 8\,000, n = 1, \frac{1}{m} = \frac{3}{4}, r = 7.5\% \text{ p. a.}$

Amount after $1\frac{3}{4}$ years = $A = 8\,000 \times \left(1 + \frac{7.5}{100}\right)^1 \times \left(1 + \frac{7.5}{100} \times \frac{3}{4}\right) = \text{Sh. } 9\,083.75$

$\therefore \text{Compound Interest} = 9\,083.75 - 8\,000 = \text{Sh. } 1\,083.75$

2. A person owes a merchant Sh. 5 702.27 to be paid at the end of $2\frac{1}{4}$ years from now. How much the merchant ought to accept now in order to clear off the accounts taking the money to be worth 6% p. a compound interest?

Solution

Let P = Sum to be invested, $A = \text{Sh. } 5\,702.27, r = 6\% \text{ p. a.}, n = 2, \frac{1}{m} = \frac{1}{4}$

Using $A = PR^n \left(1 + \frac{r}{m}\right)^n, R = 1 + r = 1 + \frac{6}{100}$

$$\Rightarrow 5\,702.27 = P \times \left(1 + \frac{6}{100}\right)^2 \times \left(1 + \frac{6}{100} \times \frac{1}{4}\right)$$

$$\Rightarrow 5\,702.27 = P(1.06)^2 \times (1.015)$$

$$\therefore P = \frac{5\,702.27}{(1.06)^2 \times (1.015)} = \text{Sh. } 5\,000$$

$\therefore$ The merchant has to accept now Sh. 5 000 in order to get Sh. 5 702.27 after $2\frac{1}{4}$ years at a rate of 6% p. a C. I

Note

- The rate $r\%$ must be in per annum
- If the investment is for only n years, then $\frac{1}{m} = 0$. Hence the formula

$$A = P(1 + r)^n$$

3. A sum of money is put out at compound interest amounts in 2 years to R. s 2 809 and in 3 years to R. s 2 977.54. Find the rate of interest and the original sum.

Solution

Amount for 2 years = R. s 2 809

Amount for 3 years = R. s 2 977.54

$$\Rightarrow A = P(1 + r)^2 = 2\,809 \dots\dots\dots (1)$$

$$\Rightarrow A = P(1 + r)^3 = 2\,977.54 \dots\dots\dots (2)$$

Dividing the equations

$$\Rightarrow \frac{P(1+r)^2}{P(1+r)^3} = \frac{2\,809}{2\,977.54}$$

$$\therefore 1 + r = 1.06 \text{ or } r = 0.06$$

From eqn. (1)

$$P(1 + r)^2 = 2\,809 \Rightarrow P(1 + 0.06)^2 = 2\,809$$

$$\therefore P = \frac{2\,809}{(1.06)^2} = \text{R. s } 2\,500$$

4. A person wishes to divide Rs. 65 050 between his two sons who are 13 and 15 years old respectively in such a way that their shares if invested at 4% compound interest should produce the same amount when they become 18 years of age. Find the share of each.

Solution

Let the share of the young son = Rs. x

$\therefore$ The share of the older son = $65\,050 - x$

The young son has to go for 5 years and the elder son for 3 years to become 18 years old.

Case I: younger son

$$P = x, n = 5, r = 4\%, A = ?$$

$$A = x(1 + 0.04)^5 = x \cdot (1.04)^5$$

Case II: Elder son

$$P = 65\,050 - x, n = 3, r = 4\%, A = ?$$

$$A = (65\,050 - x)(1 + 0.04)^3 = (65\,050 - x) \cdot (1.04)^3$$

Now the amounts must be the same

$$\Rightarrow x \cdot (1.04)^5 = (65\,050 - x) \cdot (1.04)^3$$

$$\Rightarrow x \cdot (1.04)^2 = 65\,050 - x$$

$$\therefore 1.0816x = 65\,050 \Rightarrow \text{Rs. } 31\,250$$

$\therefore$ The share of the younger son is Rs. 31 250 and the share of the elder son is Rs. 65 050 - 31 250 = Rs. 33 800

Find the sum if the difference between the simple interest and compound interest on the sum is £30 at 9% p.a for 3 years.

Solution

Let $P = \text{sum}$, $r = 9\% \text{ p.a}$, $n = 3 \text{ years}$

$$\text{At Simple Interest, simple interest} = \frac{P \times 9 \times 3}{100} = 0.27P$$

At Compound Interest, compound interest = Amount - Principal

$$\begin{aligned} \text{compound interest} &= P \left(1 + \frac{9}{100}\right)^3 - P = P(1.09)^3 - P \\ &= 0.295029P \end{aligned}$$

$$\text{Now } 0.295029P - 0.27P = 30 \text{ or } 0.025029P = 30$$

$$\therefore P = \text{£}1198.6$$

Task

1. Find the compound interest on;
 - a) Sh. 12 000 for $2\frac{1}{2}$ years at 10% p.a.
 - b) Sh. 10 000 for $1\frac{1}{4}$ years at 5% p.a.

ANSWER: a) Sh. 3 246 b) Sh. 631.25

2. Find the compound interest on the sums, giving your answers to 3 significant figures
 - a) Sh. 500 for 4 years at 10% p.a.
 - b) Sh. 18 765 for 4 years at 9% p.a.
 - c) Sh. 15 000 for 3 years at 20% p.a.

ANSWER: a) Sh. 232 b) Sh. 7 690 c) Sh. 10 920

3. On a certain sum, the compound interest for 2 years is Sh. 91.35 and the simple interest is Sh. 90. Find the sum and the rate of interest per annum.

ANSWER: Sh. 1 500, 3%

4. A sum of money invested at compound interest, payable yearly, amounts to Sh. 2 704 at the end of the second year and to Sh. 2 812.16 at the end of the third year. Find the rate of interest and the sum.

ANSWER: Sh. 25 000, 4%

5. A person deposited in a bank £ 1000 at 4% p.a interest being added annually. After 5 years the rate of interest was raised to 5% and after four more years,

the rate was further increased to 6%. The money was withdrawn at the end of 12 years. Find the amount.

ANSWER: Sh. 1 761. 40

6. A person has two sons aged 12 and 15 years respectively. He wants to divide Sh. 51 783 between them and deposit the share of each separately in the bank so that each gets the same amount when he is 18 years old. If the bank pays compound interest at the rate of 5% p. a, what should be the share of each?

ANSWER: Sh. 24 000 for young son; Sh. 27 783 for elder son

To find Amount on a sum (P) in given n-years, interest payable q-times a year at a rate r% per annum

Suppose you deposit \$ 1 000 in a savings and a loan that pays 8% compounded semi annually, how much will the savings and the loan owe you at the end of 2 years?

Solution

Compounded semi annually means “Interest is paid to your account at the end of each 6 month period, and the interest will in turn earn interest”

Annual rate $r = 8\%$

Semi annual rate $= \frac{8}{2} = 4\%$

Now amount after 6 months $= 1000 + \frac{4}{100} \times 1000 = 1000(1 + 0.04)$

Principal for the next 6 months $= 1000(1 + 0.04)$

Now amount for 1 year $= 1000(1 + 0.04) + 1000(1 + 0.04) \times \frac{4}{100}$
 $= 1000(1 + 0.04)^2$

Again amount for another 6 months

$= 1000(1 + 0.04)^2 + 1000(1 + 0.04)^2 \times \frac{4}{100} = 1000(1 + 0.04)^3$

This amount for 2 years $= 1000(1 + 0.04)^3 + 1000(1 + 0.04)^3 \times \frac{4}{100}$
 $= 1000(1 + 0.04)^4$

Now the amount that would be expected at the end of 6 years will be

$= 1000(1 + 0.04)^{12}$

In general,

“If the interest is payable q – times a year and r is the nominal rate, the amount of P in n – years or $q.n$ intervals is;

$$A = P \left(1 + \frac{r}{q} \right)^{n.q}$$

NOTE

- Since the principal P represents the initial amount in the account and A represents the amount in the account n -years later, P is called *present value* of the account and A is called the *future value* of the account.
- $r\%$ p. a is the annual rate/nominal rate and $\frac{r}{q}\%$ per $\frac{1}{q^{th}}$ annum is the rate for compounding period/interval.

Example

- Find how much Peter will have on the account at the end of 2 years if he invests a sum of \$ 100 in an account paying 8% p. a interest compounded;
 - Annually

- b) *Semi annually*
- c) *Quarterly*
- d) *Weekly*
- e) *Daily*
- f) *Hourly*

Solution

Using $A = P \left(1 + \frac{r}{q}\right)^{n \cdot q}$		
Compounding frequency	q	$A = 100 \left(1 + \frac{0.08}{q}\right)^{2q}$
Annually	1	\$116.64
Semi annually	2	\$116.9859
Quarterly	4	\$117.1659
Weekly	52	\$117.3367
Daily	365	\$117.3490
Hourly	8 760	\$117.3501

NOTE:

- 1 year = 12 months = 365 days = 52 weeks = 8760 hours
- Semi annually = $\frac{1}{2} \times 12 = 6$ months; Now in 1 year, $q = 2$
- Quarterly = $\frac{1}{4} \times 12 = 3$ months; now in 1 year, $q = 4$
- Weekly = $\frac{1}{7} \times 365 \approx 52$ weeks $\therefore q = 52$

2. Find compound interest on;

- a) Sh. 5 000 for $1\frac{1}{2}$ years at 5% p.a compound interest is paid half yearly
- b) Sh. 1 000 for 1 year at 10% p.a compound interest is paid quarterly
- c) Sh. 10 000 for $2\frac{1}{2}$ years at 15% p.a compound interest is paid weekly

Solution

Using $A = P \left(1 + \frac{r}{q}\right)^{n \cdot q}$, $A = P + I$

- a) $P = 5000$, $q = 2$, $n = 1\frac{1}{2}$ years = $\frac{3}{2}$ years, $r = 5\% = 0.05$

$$A = 5000 \left(1 + \frac{0.05}{2}\right)^{\frac{3}{2} \times 2} = 5000(1.025)^3 = \text{Sh. } 5\,384.45$$

$$\therefore \text{Compound Interest} = A - P = 5\,384.45 - 5\,000 = \text{Sh. } 384.45$$

- b) $P = 1000$, $q = 4$, $n = 1$, $r = 10\% = 0.1$

$$A = 1000 \left(1 + \frac{0.1}{4}\right)^{4 \times 1} = \text{Sh. } 1\,103.83$$

$$\therefore \text{Compound Interest} = 1\,103.83 - 1\,000 = \text{Sh. } 103.83$$

- c) $P = 10\,000$, $q = 52$, $n = 2$, $r = 15\% = 0.15$

$$A = 10000 \left(1 + \frac{0.15}{52}\right)^{52 \times 2} = \text{Sh. } 13\,492.76$$

$$\therefore \text{Compound Interest} = 13\,492.76 - 10\,000 = \text{Sh. } 3\,492.76$$

Task

1. Find the amount the principal Sh. 10 000 will accumulate at a rate of 5% p.a compound interest for 2 years if compound interest is paid;
 - i) Annually **ANSWER: Sh. 1 025**
 - ii) Half yearly **ANSWER: Sh. 1 038.13**
 - iii) Quarterly **ANSWER: Sh. 1 044.86**

2. How many years to the nearest year will it take a sum of money to double if it is invested at 15% compounded annually.

ANSWER: 5 years. [Use $A = P \left(1 + \frac{r}{q}\right)^{n \cdot q}$]

3. How many years to the nearest year will it take money to quadruple if it is invested at 20% compounded annually. **ANSWER: 8 years.** [Use $A =$

$$P \left(1 + \frac{r}{q}\right)^{n \cdot q}$$

4. If you deposit \$5 000 in an account paying 9% compounded daily, how much will you have in an account in 5 years. Compute your answer to the nearest cent. **ANSWER: \$7 841.12**

Other problems related to compound interest

Problems relating

- Depreciation
- Change of population growth
- Annuity calculations

, are majorly on the principle of compound interest.

DEPRECIATION

This happens when the value of a new item decreases with passage of time. Depreciation can be expressed in terms of an *annual percentage rate*, so as to calculate the decrease in value after a given period of time.

Let P = price of a new commodity, which depreciates at $r\%$ p.a for n – years. The price after n -years will be calculated as below.

$$\text{Value after 1 year} = P \times \left(\frac{100-r}{100}\right) = P \left(1 - \frac{r}{100}\right)$$

$$\text{Value after 2 years} = P \left(1 - \frac{r}{100}\right) \times \left(\frac{100-r}{100}\right) = P \left(1 - \frac{r}{100}\right)^2$$

$$\text{Value after 3 years} = P \left(1 - \frac{r}{100}\right)^2 \times \left(\frac{100-r}{100}\right) = P \left(1 - \frac{r}{100}\right)^3$$

$$\text{Value after } n \text{ years, } A = P \left(1 - \frac{r}{100}\right)^n$$

Example

1. A machine costing £15 000 is depreciated at 10% p.a. how long will it take to reduce the value to £1 400?

Solution

The cost of machine, $P = £15\,000$

Rate of depreciation, $r = 10\% = 0.1$

Value of the machine after n years, $A = £1\,400$

But value of the machine after n years is $A = P \left(1 - \frac{r}{100}\right)^n$

$$\Rightarrow 1400 = 15000(1 - 0.1)^n$$

$$\Rightarrow \frac{1400}{15000} = (0.9)^n$$

$$\Rightarrow \log\left(\frac{14}{150}\right) = \log(0.9)^n = n \log 0.9$$

$$\therefore n = \frac{\log\left(\frac{14}{150}\right)}{\log 0.9} = \frac{0.8451 - 1.8751}{-1.9542} = 22.5 \text{ years}$$

2. A machine costing £30 000 has an estimated life time of 15 years. If the depreciation is at a rate of 20% p.a., find the scrap value of the machine.

Solution

The cost of the machine = $P = £30\,000$

Life of the machine = $n = 15$ years

Rate of depreciation = $r = 20\% = 0.2$

$$\begin{aligned}\therefore \text{The value of the machine at the end of } 15^{\text{th}} \text{ year} &= P \left(1 - \frac{r}{100}\right)^n \\ &= 30\,000(1 - 0.2)^{15} = 30\,000 \times (0.8)^{15} = £1\,055\end{aligned}$$

3. A machine was bought at Rs. 25 000 on Jan. 1, 1981. It was depreciated at 10% p.a. It was sold on 31.12.1983 at Rs. 20 047.50. Find out the percentage of profit on the reduced cost.

Solution

The cost of the machine = $P = \text{Rs. } 25\,000$

Life of the machine = $n = 3$ years (i.e. Period from Jan. 1, 1981 to 31.12.1983)

Rate of depreciation = $r = 10\%$

$$\begin{aligned}\text{Expected value after } n = 3 \text{ years} &= P \left(1 - \frac{r}{100}\right)^n \\ &= 25\,000(1 - 0.1)^3 = 25\,000(0.9)^3 \\ &= \text{Rs. } 18\,225\end{aligned}$$

Selling price = Rs. 20 047.50

Profit realized = $20\,047.50 - 18\,225 = \text{Rs. } 1\,822.5$

$$\% \text{ profit} = \frac{1822.5}{18225} \times 100 = 10\%$$

Task

- A machine depreciates in value each year at a rate of 10% of its value at the beginning of a year. The machinery was purchased for Rs. 10 000. Obtain to the nearest rupee its value at the end of the tenth year. **Answer: Rs. 3 483**
- A machine depreciates at the rate of 7% of its value at the beginning of the year. If the machine was purchased for Rs. 8 500, what is the minimum number of complete years at the end of which the worth of the machine will be less than or equal to half of its original price?

ANSWER: 10 years

- John bought a new car for Sh. 650 000. The car depreciated in value by 25%, 15%, 10% and 5% of its value at the beginning of each year for four successive years. Find its value at the end of the fourth year. If after the fourth year it depreciated at a uniform annual rate of 4%, find its value at the end of 10 years (Give your answers to 3 significant figures).

ANSWER: Sh. 354 000, Sh. 277 000

- A woman opened a hair dressing saloon but went out of business three years later. She sold off various pieces of equipment for Sh. 12 000. This represented a uniform rate of depreciation of 15% p.a. estimate the cost price of the equipment giving your answers to 3 s.f. **ANSWER: Sh. 19 500**
- A man bought a sewing machine for Sh. 10 800. After 12 years he sold it for Sh. 4 000. At what annual rate did the machine depreciate? (Answer to the nearest integer)

What was the value of the machine when it was 6 years old? (Answer to the 3 s.f.)

ANSWER: 8 %, Sh. 65 500

POPULATION GROWTH

When the population P increases at a uniform rate $r\%$ p.a for n – years, at the end of the n^{th} year, the population can be calculated as below.

Initial (present) population = P

Rate on increase = $r\%$

$$\text{Population after 1 year} = P + P \times \frac{r}{100} = P \left(1 + \frac{r}{100}\right)$$

$$\begin{aligned} \text{Population after 2 years} &= P \left(1 + \frac{r}{100}\right) + P \left(1 + \frac{r}{100}\right) \times \frac{r}{100} \\ &= P \left(1 + \frac{r}{100}\right) \left(1 + \frac{r}{100}\right) = P \left(1 + \frac{r}{100}\right)^2 \end{aligned}$$

$$\begin{aligned} \text{Population after 3 years} &= P \left(1 + \frac{r}{100}\right)^2 + P \left(1 + \frac{r}{100}\right)^2 \times \frac{r}{100} \\ &= P \left(1 + \frac{r}{100}\right)^2 \left(1 + \frac{r}{100}\right) = P \left(1 + \frac{r}{100}\right)^3 \end{aligned}$$

$$\therefore \text{Population after } n \text{ years} = P \left(1 + \frac{r}{100}\right)^n$$

Note

If the population is decreasing at a rate $r\%$ with passage of time, then the population after n years can be evaluated as $P \left(1 - \frac{r}{100}\right)^n$

Example

1. The population of a town increases yearly at a rate of 25 per 1000. It is now 378 530, what it be 3 years hence?

Solution

Present population, $P = 378\,530$

Time, $n = 3$ years

$$\text{Rate of growth, } r = \frac{25}{1000} = 2.5\%$$

$$\begin{aligned} \therefore \text{Population after 3 years} &= P \left(1 + \frac{r}{100}\right)^n \\ &= 378\,530(1 + 0.025)^3 \\ &= 378\,530(1.025)^3 = 407\,635 \end{aligned}$$

2. In a certain town, population raised at 20 per 1000. Find the number of years in which the population will be doubled assuming there is neither immigration nor migration.

Solution s

Let present population = P

$$\text{Rate of growth} = \frac{20}{1000} = 2\% = 0.02$$

Time of growth = n – years

Population after n -years = $2P$

$$\text{But population after } n \text{ years} = P(1 + 0.02)^n = P(1.02)^n$$

$$\Rightarrow 2P = P(1.02)^n \text{ or } 2 = (1.02)^n$$

$$\log 2 = \log(1.02)^n = n \log 1.02$$

$$\therefore n = \frac{\log 2}{\log 1.02} = 35 \text{ years}$$

3. In a Military Academy, 150 trainees were recruited in 2000. The recruiting team planned to increase the number of trainees each subsequent year at a rate of 20% of the number for the previous year. Determine the number expected to be in 2010.

Solution

In 2000 there were 150

In 2001 there were $150 + 0.2(150) = 150(1 + 0.2) = 150(1.2)$

In 2002 there were $150(1.2) + (150)(1.2)(0.2) = 150(1.2)(1 + 0.2) = 150(1.2)^2$

In 2003 the number will be $150(1.2)^2 + 150(1.2)^2(0.2)$

$$= 150(1.2)^2(1 + 0.2) = 150(1.2)^3$$

$\therefore$ In 2010, after 10 years, the number is $150(1.2)^{10} \approx 929$ trainees.

Task

1. *The population of a country increases every year by 2.4% of the population at the beginning of that year. In what time will the population double itself?*

ANSWER: 29 years

2. *In Jinja district, 6500 pupils registered for PLE 1996. In 1997 the number of students increased by 1250 and in each subsequent year the number of pupils increased by $\frac{1}{5}$ more than the previous year. Find the number of pupils that will be registered in 2006. **ANSWER: 40 000 to 3 s. f***

3. *If Kenya has a population of 23 million people and a doubling time of 19 years, and growth continues at the same rate, find the population in;*

a) 10 years.

ANSWER: 33 million

b) 30 years.

ANSWER: 69 million

4. *If the population in a particular third world country is growing at 4% compounded annually how long will it take the population to double?*

ANSWER: 18 years

ANNUITY CALCULATIONS

FUTURE VALUE OF ANNUITY

Bankers call a series of equal annual payment made at equal intervals of time an *Annuity*.

Terms associated with Annuities

i) **Immediate/Ordinary Annuity**

If each of the annuity payment is made at the end of each period (time interval), is called *immediate annuity*.

ii) **Future value of an annuity**

Is the sum of all the annuity payments plus any accumulated interest.

To find the Amount of Annuity left unpaid for a given number of years, allowing Compound Interest.

Let R be the periodic deposit, i the rate per period and n the number of periods in the given years.

Case I: Payment at the end of each period.

At the end of the 1st year, R is due and the amount of this sum in the remaining $(n - 1)$ year is $R(1 + i)^{n-1}$

At the end of 2nd year, another R is due and the amount of this sum in the remaining $(n - 2)$ years is $R(1 + i)^{n-2}$

$$\begin{aligned}\therefore \text{Future value} &= R(1 + i)^{n-1} + R(1 + i)^{n-2} + \cdots \cdots \cdots + R(1 + i)^2 + R(1 + i)^1 + R \\ &= R[1 + (1 + i) + (1 + i)^2 + \cdots \cdots \cdots + (1 + i)^{n-2} + (1 + i)^{n-1}] \\ &= R[1 + (1 + i) + (1 + i)^2 + \cdots \cdots \cdots \text{to } n - \text{terms}] \\ &= \frac{R[(1+i)^n - 1]}{(1+i) - 1} = \frac{R[(1+i)^n - 1]}{i}\end{aligned}$$

In formula form,

Future value of Immediate Annuity is:

$$FV = PMT \cdot \frac{(1+i)^n - 1}{i}$$

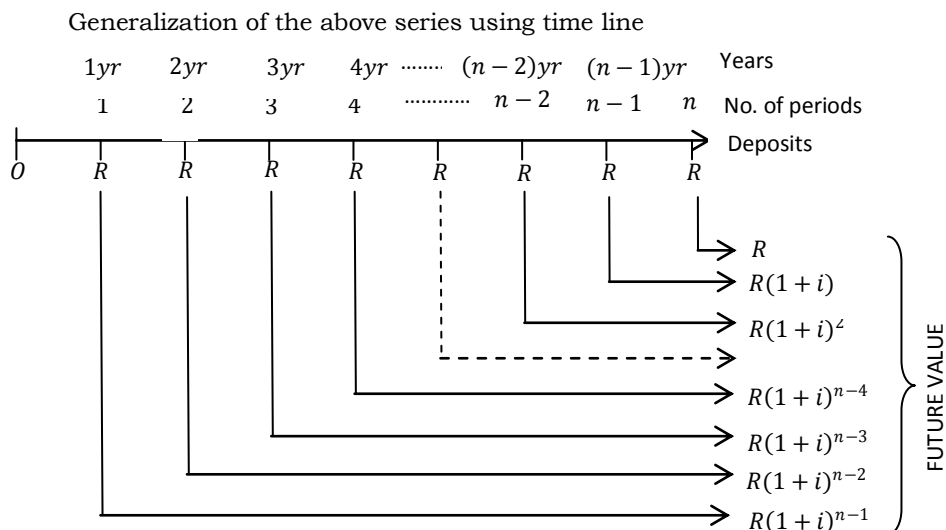
PMT = Periodic payment

i = Rate per period

n = Number of payments (Periods)

FV = Future Value (Amount)

[Payments are made at the end of each period]



Hence the sum of all these annuities plus any interest is;
 $= R + R(1+i) + R(1+i)^2 + \dots + R(1+i)^{n-4} + R(1+i)^{n-3} + R(1+i)^{n-2} + R(1+i)^{n-1}$

NOTE:

1. The future value of annuity is considered when payments are made at the end of each period (Immediate Annuity)
2. Banks usually state interest rates as *Annual Percentage Rates*(APRS)

Case II: Payment at the beginning of every period (payment for every period)

At the beginning of 1st year, R is due and amount of this sum for n years is $R(1+i)^n$

At the beginning of 2nd year, another R is due and the amount of the remaining (n - 1) year is $R(1+i)^{n-1}$

$$\begin{aligned}
 \therefore \text{Future value} &= R(1+i)^n + R(1+i)^{n-1} + \dots + R(1+i)^2 + R(1+i) \\
 &= R[(1+i) + (1+i)^2 + \dots + (1+i)^{n-1} + (1+i)^n] \\
 &= R(1+i)[1 + (1+i) + (1+i)^2 + \dots + (1+i)^{n-2} + (1+i)^{n-1}] \\
 &= R(1+i)[1 + (1+i) + (1+i)^2 + \dots \text{to } n - \text{terms}] \\
 &= \frac{R(1+i)[(1+i)^n - 1]}{i}
 \end{aligned}$$

In formula form,

Future value of an Annuity due is:

$$FV = PMT \cdot \frac{(1+i)}{i} \cdot [(1+i)^n - 1]$$

PMT = Periodic payment

i = Rate per period

n = Number of payments (Periods)

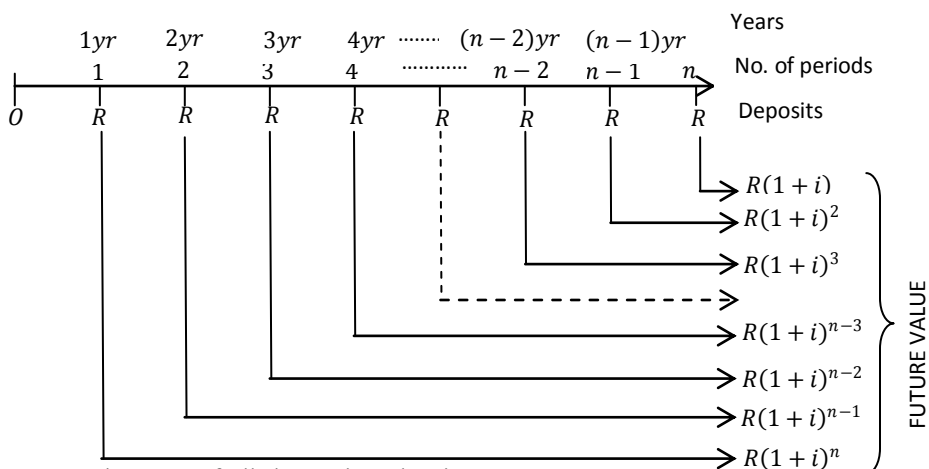
FV = Future Value (Amount)

[Payments are made at the beginning of each period/payments are made for every period]

Case III: Annuity due

Is an annuity where the first payment falls due in the beginning of the first period or interval, hence Case II.

Generation of Case II series above using time line.



Hence the sum of all the series plus interest

$$\begin{aligned}\therefore \text{Future value} &= R(1+i)^n + R(1+i)^{n-1} + \cdots + R(1+i)^2 + R(1+i) \\ &= R(1+i)[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-2} + (1+i)^{n-1}]\end{aligned}$$

General examples:

(Immediate Annuity)

1. A person wishes to deposit \$ 1000 at the end of each year in a bank which pays 10% p.a Compound Interest. If installments are allowed to accumulate, what will be the total accumulation at the end of 10 years?

Solution

At the end of 1yr, 1st installment of \$ 1000 amounts in 9 years to:

$$1000(1+0.1)^9 = 1000(1.1)^9$$

The 2nd installment of \$ 1000 amounts in 8 years to: $1000(1+0.1)^8 = 1000(1.1)^8$

The last installment of \$ 1000 amounts to: $1000(1+0.1)^0 = 1000$

So the total accumulated at the end of 10 years is:

Total = $1000 + 1000(1.1) + 1000(1.1)^2 + \cdots + 1000(1.1)^8 + 1000(1.1)^9$, which is a G.P of 10 terms.

$$= \frac{1000[(1.1)^{10}-1]}{1.1-1} = \frac{1000[(1.1)^{10}-1]}{0.1} = \$ 15\,937$$

Alternatively, using the formula,

$$P = \$ 1000, i = 0.1, n = 10, A = ?$$

$$A = \frac{P}{i}[(1+i)^n - 1]$$

$$= \frac{1000}{0.1}[(1+0.1)^{10} - 1] = \$ 15\,937$$

(Immediate Annuity)

2. Jessica deposits \$ 100 every 6 months in an account that pays 6% Compound Interest compounded semi annually. If she made 6 deposits one at the end of each interest payment period over 3 years, how much money will be on the account after the last deposit is made?

Solution

% rate p. a = 6, % rate per 6 – months = $\frac{6}{2} = 3\%$. i. e. $i = 0.03$, no. of periods = 6

At the end of the 1st 6 months, \$100 is due, and this amounts to:

$$\$100(1 + 0.03)^5$$

The 2nd installment of \$ 100 amounts to: $\$100(1 + 0.03)^4 = \$100(1.03)^4$

The 3rd installment of \$ 100 amounts to: $\$100(1 + 0.03)^3 = \$100(1.03)^3$

The last installment of \$ 100 amounts to: $\$100(1 + 0.03)^0 = \$100(1.03)^0 = \$100$

Thus the total accumulated at the end of the 6th deposit (at the end of 3 years) is:

$$= 100 + 100(1.03) + 100(1.03)^2 + 100(1.03)^3 + \dots + 100(1.03)^5$$

$$= 100[1 + 1.03 + (1.03)^2 + \dots + (1.03)^5]$$

$$= \frac{100[(1.03)^6 - 1]}{1.03 - 1} = \$646.84$$

Alternatively, using the formula

$$P = \$100, i = \frac{6}{2} = 3\% = 0.03, n = 3 \text{ yrs} = 6 \text{ periods}, A = ?$$

$$A = 100 \left[\frac{(1.03)^6 - 1}{0.03} \right] = \$646.84$$

(Immediate Annuity)

3. When Jim started his first job after he finished college, he opened an Individual Retirement Account (IRA). He plans to contribute \$ 2 500 per year for 38 years until he reaches age 62. He hopes to earn an average annual percentage rate (APR) of 8% over the 38 year period.

a) If Mr. Jim contributes to his IRA at the rate he plans, how much will his account be worth when he is 62 years old?

b) How much interest will be earned on the account?

Solution

Given: $P = \$2\,500, i = 8\% = 0.08, n = 38 \text{ yrs}$

- a) At the end of 1st year \$ 2 500 is due, and this amounts to:
 $\$2\,500(1 + 0.08)^{37} = \$2\,500(1.08)^{37}$ for the remaining years.

The 2nd installment of \$ 2 500 amounts to:

$$\$2\,500(1 + 0.08)^{36} = \$2\,500(1.08)^{36}$$

The 2nd installment of \$ 2 500 amounts to:

$$\$2\,500(1 + 0.08)^{35} = \$2\,500(1.08)^{35}$$

$\therefore$ the last deposit amounts to:

$$\$2\,500(1 + 0.08)^0 = \$2\,500(1.08)^0 = \$2\,500$$

$\therefore$ the accumulated amount after 38 years is:

$$= 2500 + 2500(1.08) + 2500(1.08)^2 + \dots + 2500(1.08)^{37}$$

$$= 2500[1 + 1.08 + (1.08)^2 + \dots + (1.08)^{37}]$$

$$= \frac{2500[(1.08)^{38} - 1]}{1.08 - 1} = \$550\,789.86$$

- b) Mr. Jim will pay $\$2\,500 \times 38 = \$95\,000$ in to his IRA, so the account will earn:
 $\$ (550\,789.86 - 95\,000) = \$455\,789.86$

4. John, procrastinating, does not make his first \$ 2000 deposit in to an IRA account until he is 32, but then he continues to deposit \$ 2000 on every birthday until he is 65. If his account also earns 9% compounded annually, how much (to the nearest dollar) will he have on his account when he makes the last deposit on his 65th birth day?

Solution

The 1st deposit is at when John is 32 years.

The last deposit is at when John is 65 years.

∴ number of deposits in all = 34, at the end of every year $i = 9\%$ p.a, $P = \$2000$

The 1st deposit (at 32 years) grows to $2000(1 + 0.09)^{33}$

The 2nd deposit (at 33 years) grows to $2000(1 + 0.09)^{32}$

∴ the last deposit (at 65 years) grows to $2000(1 + 0.09)^0$

∴ the total amount accumulated at the end of the 65th birthday is;

$$= 2000 + 2000(1 + 0.09) + 2000(1 + 0.09)^2 + \dots + 2000(1 + 0.09)^{33}$$

$$= 2000[1 + 1.09 + (1.09)^2 + \dots + (1.09)^{33}]$$

$$= \frac{2000[(1.09)^{34} - 1]}{1.09 - 1} = \$393\,965$$

5. A man puts Rs. 10 at the end of every year in the savings Bank at $2\frac{1}{2}\%$ compound interest. How much will his savings amount to in 15 years?

Solution

At the end of 1st year, Rs. 10 is due, and amounts to $10(1 + 0.025)^{14}$

At the end of the 2nd year, Rs. 10 is again due, and amounts to $10(1 + 0.025)^{13}$

The last deposit grows to $10(1 + 0.025)^0$

∴ total amount = $10 + 10(1.025) + 10(1.025)^2 + \dots + 10(1.025)^{14}$

$$= 10[1 + 1.025 + (1.025)^2 + \dots + (1.025)^{14}]$$

$$= \frac{10[(1.025)^{15} - 1]}{1.025 - 1} = \text{Rs. } 179.32$$

6. A man deposits \$ 100 per month in to an account earning 9% compounded monthly for 5 years. How much interest does he receive?

Solution

At the end of the 1st month, 1st deposit grows to $\$100\left(1 + \frac{0.09}{12}\right)^{59}$ in the remaining 59 months.

At the end of the 2nd month, 2nd deposit grows to $\$100\left(1 + \frac{0.09}{12}\right)^{58}$ in the remaining 58 months.

∴ at the end of the last month of deposit, deposit grows to $\$100\left(1 + \frac{0.09}{12}\right)^0 = \$100$

∴ total accumulation at the end of 5 years (60 months) is:

$$= 100 + 100\left(1 + \frac{0.09}{12}\right)^1 + 100\left(1 + \frac{0.09}{12}\right)^2 + \dots + 100\left(1 + \frac{0.09}{12}\right)^{59}$$

$$= 100\left[1 + \left(1 + \frac{0.09}{12}\right) + \left(1 + \frac{0.09}{12}\right)^2 + \dots + \left(1 + \frac{0.09}{12}\right)^{59}\right]$$

$$= \frac{100\left[\left(1 + \frac{0.09}{12}\right)^{60} - 1\right]}{\left(1 + \frac{0.09}{12}\right) - 1} = \frac{100\left[\left(1 + \frac{0.09}{12}\right)^{60} - 1\right]}{0.0075} = \$7\,542.41$$

Now total amount deposited = $60 \times 100 = 6\,000$

Interest = Value – Deposits

interest = $7\,542.41 - 6\,000 = \$1\,542.41$

Note:

Since 9% is compounded monthly, then $i = \frac{0.09}{12} = 0.0075$

7. Jane deposits \$ 2000 a year in to an IRA account earning 9% compounded annually. She makes her first deposit on her 24th birthday and her last deposit on her 31st birthday (8 deposits in all). Making no additional deposits, she leaves the accumulated amount from the 8 deposits in an account, earning interest at 9% compounded annually until her 65th birthday. How much to the nearest dollar will be in her account on her 65th birthday?

Solution

Given: $P = \$ 2000, i = 9\% = 0.09, n = 8$

At the end of the 1st year, \$ 2000 is due and amounts to $2000(1 + 0.09)^7$ in the remaining 7 years.

At the end of the 2nd year, \$ 2000 is again due and amounts to $2000(1 + 0.09)^8$ in the remaining 6 years.

$\therefore$ the last deposit of \$ 2000 grows to $2000(1 + 0.09)^0 = 2000$

$\therefore$ the total amount for the 8 years of depositing is:

$$\begin{aligned} &= 2000 + 2000(1.09) + 2000(1.09)^2 + \dots + 2000(1.09)^7 \\ &= 2000[1 + 1.09 + (1.09)^2 + \dots + (1.09)^7] \\ &= \frac{2000[(1.09)^8 - 1]}{1.09 - 1} = \$ 22\,056.94759 \end{aligned}$$

Now since no other deposit after the 8 – deposits, the money grows at compound interest at a rate of 9% p.a for 34 years after the 31st birthday.

Using $A = P(1 + i)^n$

$$A = 22\,056.94759(1 + 0.09)^{34} = \$ 413\,092$$

(Annuity due)

8. A person invests £2000 every year with a company which pays interest 10% p.a. he allows his deposit to accumulate at compound interest. Find the amount to the credit of the person at the end of the 5th year.

Solution

At the beginning of the 1st year, £2000 is due and his sum amounts in 5 years to

$$£2000(1 + 0.1)^5$$

At the beginning of the 2nd year, another £2000 is due and his sum amounts in 4 years to

$$£2000(1 + 0.1)^4$$

The last deposit (5th year) grows to $£2000(1 + 0.1)^1$

$\therefore$ total amount accumulated

$$\begin{aligned} &= 2000(1 + 0.1) + 2000(1 + 0.1)^2 + \dots + 2000(1 + 0.1)^5 \\ &= 2000(1.1)[1 + 1.1 + (1.1)^2 + \dots + (1.1)^4] \\ &= \frac{2000(1.1)[(1.1)^5 - 1]}{1.1 - 1} = \frac{2000(1.1)[(1.1)^5 - 1]}{0.1} = £13\,431.22 \end{aligned}$$

Alternatively by formula method

$$P = £2000, n = 5, i = 0.1, A = \frac{P(1+i)}{i} [(1+i)^n - 1]$$

$$A = £2000 \times \frac{1.1}{0.1} \times [(1.1)^5 - 1] = £13\,431.22$$

(Annuity due)

9. Mr. John deposits £2000 per year in the bank for 10 years at 5% p.a. find how much will he receive at the end of 10 years if the money accumulates at compound interest.

Solution

At the end of the 10th deposit, the last deposit grows to £2000(1 + 0.05)

At the end of the 9th deposit, the amount grows to £2000(1 + 0.05)²

∴ the 1st deposit grows to £2000(1 + 0.05)¹⁰

∴ total amount accumulated = 2000(1.05) + 2000(1.05)² + ... + 2000(1.05)¹⁰

$$\begin{aligned}
 &= 2000(1.05)[1 + 1.05 + (1.05)^2 + \dots + (1.05)^9] \\
 &= \frac{2000(1.05)[(1.05)^{10}-1]}{1.05-1} = \frac{2000(1.05)[(1.05)^{10}-1]}{0.05} \\
 &= £26\,413.57
 \end{aligned}$$

10. Jack operates an account with a certain bank which pays a compound interest rate of 13.5% p.a. he opened the account at the beginning of the year with Sh. 500 000 and deposits the same amount of money at the beginning of every year. Calculate how much will he receive at the end of 9 years. After how long will the money have accumulated to Sh. 3.32 million.

Solution

At the beginning of the 1st year, Sh. 500 000 is due, and the amount in 9 years is

$$500\,000(1 + 0.135)^9 = 500\,000(1.135)^9$$

At the beginning of the 2nd year, another Sh. 500 000 is due and amounts in 8 years to

$$500\,000(1 + 0.135)^8 = 500\,000(1.135)^8$$

∴ the last deposit in the 9th year, amounts to 500 000(1 + 0.135)¹ = 500 000(1.135)

∴ the total amount = 500000(1.135) + 500000(1.135)² + ... + 500000(1.135)⁹

$$\begin{aligned}
 &= 500000(1.135)[1 + 1.135 + (1.135)^2 + \dots + (1.135)^8] \\
 &= \frac{500000(1.135)[(1.135)^9-1]}{1.135-1} = \frac{500000(1.135)[(1.135)^9-1]}{0.135} \\
 &= \text{Sh. } 8\,936\,281
 \end{aligned}$$

Let the money accumulate in n – years to 3.32 million

The 1st deposit grows to 500 000(1 + 0.135) ^{n} = 500 000(1.135) ^{n} in n – years.

The 2nd deposit grows to 500 000(1 + 0.135) ^{$n-1$} = 500 000(1.135) ^{n} in ($n-1$) – years.

∴ the n^{th} deposit grows to 500 000(1.135)

∴ the total amount after n – years

$$\begin{aligned}
 &= 500\,000(1.135) + 500\,000(1.135)^2 + \dots + 500\,000(1.135)^n \\
 &= 500\,000(1.135)[1 + 1.135 + (1.135)^2 + \dots + (1.135)^{n-1}] \\
 &= \frac{500\,000(1.135)[(1.135)^n-1]}{1.135-1} = \frac{500\,000(1.135)[(1.135)^n-1]}{0.135}
 \end{aligned}$$

$$\therefore 3.32 \times 10^6 = \frac{500\,000(1.135)[(1.135)^n-1]}{0.135}$$

$$0.789779735 = 1.135^n - 1 \text{ or } 1.135^n = 1.789779735$$

$$\ln(1.135)^n = \ln 1.789779735$$

$$\therefore n = \frac{\ln 1.789779735}{\ln 1.135} = 4.6 \text{ years}$$

11. Five million shillings are invested each year at a rate of 20% interest. In how many years will it accumulate to have more than 50 million shillings?

Solution

Since deposits are made each/every year, this is a case of annuity due.

For the 1st year, Sh. 5 000 000 is due, and amounts in n years to $Sh. 5\,000\,000(1 + 0.2)^n$

For the 2nd year, another Sh. 5 000 000 is due, and amounts in $n - 1$ years to $Sh. 5\,000\,000(1 + 0.2)^{n-1}$

$\therefore$ the last deposit in the n^{th} year amounts to $Sh. 5\,000\,000(1 + 0.2)$

$\therefore$ total accumulation after n years is:

$$\begin{aligned} &= 5\,000\,000(1.2) + 5\,000\,000(1.2)^2 + \cdots \cdots \cdots + 5\,000\,000(1.2)^n \\ &= 5\,000\,000(1.2)[1 + 1.2 + (1.2)^2 + \cdots \cdots \cdots + (1.2)^{n-1}] \\ &= \frac{5\,000\,000(1.2)[(1.2)^n - 1]}{1.2 - 1} = \frac{5\,000\,000(1.2)[(1.2)^n - 1]}{0.2} \end{aligned}$$

Given the total accumulation is Sh. 50 000 000

$$\Rightarrow 50\,000\,000 = \frac{5\,000\,000(1.2)[(1.2)^n - 1]}{0.2}$$

$$\Rightarrow \frac{5}{3} = 1.2^n - 1 \text{ or } 1.2^n = \frac{8}{3}$$

$$\Rightarrow \ln 1.2^n = \ln \left(\frac{8}{3}\right) \therefore n = \frac{\ln \left(\frac{8}{3}\right)}{\ln 1.2} = 5.379666948$$

$\therefore$ in 6 years the money would have accumulated more than 50 million.

Task

1. Find the amount of annuity due of £2000 per year payable in the beginning of every year for 10 years at 5% p.a. **ANSWER: £26 413.60**
2. Five million shillings are invested each year at a rate of 15% interest. In how many years will it accumulate to more than Sh. 50 million?

ANSWER: 6 years

3. A man puts Rs. 10 at the end of every year in the savings Bank at 2 ½ % compound interest. How much will his savings amount to in 15 years?

ANSWER: Rs. 179

4. A company sets aside a sum of £20 000 annually to enable it pay off a debenture issue of £230 000 at the end of 10 years. Assuming that the sum accumulates at 4% p.a compound interest, find the surplus after paying off a debenture.

HINT:

£20 000 will amount to (at the end of 10 years) $20\,000(1.04)^{10}$

The 2nd sum amounts to $20\,000(1.04)^9$

$\therefore$ total money at the end of the 10th year will be:

$$\begin{aligned} &20\,000(1.04)[1 + 1.04 + (1.04)^2 + \cdots \cdots \cdots + (1.04)^9] \\ &= \frac{20\,000(1.04)[(1.04)^{10} - 1]}{0.04} = £249\,080 \end{aligned}$$

Thus the surplus = $249\,080 - 230\,000 = £19\,080$

5. Phionah deposited \$1000 every 6 months in to an account earning 8% compounded semi annually. How much will the \$1000 amount to at the end of 10 years? How much of this value is interest?

HINT:

1st deposit grows to $1000\left(1 + \frac{0.08}{2}\right)^{19}$ in 19 months

2nd deposit grows to $1000\left(1 + \frac{0.08}{2}\right)^{18}$ in 19 months

$\therefore$ last deposit grows to $1000\left(1 + \frac{0.08}{2}\right)^0$

$$\therefore \text{Total value} = 1000[1 + 1.04 + (1.04)^2 + \dots + (1.04)^{19}]$$

$$= \$ 29\,778.4$$

$$\therefore \text{Interest} = 29\,778.4 - 20 \times 1000 = \$ 9\,778.4$$

6. Beginning in January, a person plans to deposit \$ 100 at the end of each month in to an account earning 9% compounded monthly. Each year taxes must be paid on the interest earned during that year. Find the interest earned during each year for the first 3 – years.

HINT:

$$i = \frac{0.09}{12} = 0.0075 \text{ per month, } P = \$ 100, n = 1 \text{ year} = 12 \text{ months.}$$

$$\text{Value after 1st year} = \frac{100[(1+0.0075)^{12}-1]}{0.0075} = \$ 1\,250.76$$

$$\text{Interest for 1st year} = 1\,250.76 - 12 \times 100 = \$ 50.76$$

$$\text{Value after 2nd year} = \frac{100[(1+0.0075)^{24}-1]}{0.0075} = \$ 2\,618.85$$

$$\text{Interest after 2 years} = 2\,618.85 - 24 \times 100 = \$ 218.85$$

$$\therefore \text{Interest for 2nd year} = 218.85 - 50.76 = \$ 168.09$$

$$\text{Value after 3rd year} = \frac{100[(1+0.0075)^{36}-1]}{0.0075} = \$ 4\,115.27$$

$$\therefore \text{Interest after 3 years} = 4\,115.27 - 36 \times 100 = \$ 512.27$$

Now interest for 3rd year

= Interest for 3 years minus interest for the 1st two years

$$= 512.27 - 218.85 = \$ 296.42$$

7. What is the annuity at the end of 5 years if \$ 100 per month is deposited in to an account earning 9% compounded monthly? How much of this value is interest? **Answer: \$ 1 542.41**
8. A man pays a premium of £ 100 at the beginning of every year to an insurance company on the understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? (Give your answer correct to 3 s. f)

HINT:

At the beginning of 1st year, £ 100 is due and amounts to $\pounds 100(1 + 0.05)^{15}$ for remaining 15 years.

At the beginning of 2nd year, another £ 100 is due and amounts to $\pounds 100(1 + 0.05)^{14}$ for the remaining 14 years.

The last deposit grows to $\pounds 100(1 + 0.05)$

$$\therefore \text{Total} = 100(1.05) + 100(1.05)^2 + \dots + 100(1.05)^{15}$$

$$= \frac{100(1.05)[(1.05)^{15}-1]}{0.05} = 2\,265.749177$$

$$\approx 22700$$

9. A man earned in a certain year £ 2000 from a certain source and his annual earnings from this time continued to increase at a rate of 5%. Find to the nearest £ the whole amount he received from this source in this year and the next seven years. Give your answer correct to 3 s. f.

HINT:

Amount in the 1st year = $2000(1 + 0.05) = £2100$

Now amount in 2nd year grows to $2000(1 + 0.05)^7$ in the remaining 7 years or grows to $2000(1 + 0.05)^6$ in the remaining 6 years, taking £2100 as the beginning amount.

Amount in 3rd year grows to $2000(1 + 0.05)^6$ in the remaining 6 years or grows to $2000(1 + 0.05)^5$ in the remaining 5 years.

∴ Amount in the last year grows to $2000(1 + 0.05)^0$

∴ Total amount accumulated in the next 7 years is:

$$= 2000 + 2000(1.05) + 2000(1.05)^2 + \dots + 2000(1.05)^7$$

$$= 2000[1 + 1.05 + (1.05)^2 + \dots + (1.05)^7]$$

$$= \frac{2000[(1.05)^8 - 1]}{0.05} = £19\,098.21775$$

$$\approx £19\,100$$

10. A credit society gives out a compound interest of 4.5% p.a. Mugaga deposits Sh. 300 000 at the beginning of each year. How much money will he have at the end of 4 years if there are no withdraws during this period. **(Uneb 2006) Answer: Sh. 1 341 213**
11. A man invests £100 at the end of 1971, £200 at the end of 1972, £300 at the end of 1973, £400 at the end of 1974, £500 at the end of 1975. If all interest accumulated at 5% p.a, what do his investments amount to at the end of 1976?

HINT:

£100 is invested for 5 years, £200 for 4 years, £300 for 3 years, £400 for 2 years
£500 for 1 year

Total value at the end of 1976 is;

$$A = 100(1.05)^5 + 100(1.05)^4 + 100(1.05)^3 + 100(1.05)^2 + 100(1.05) = £1\,684.2$$

PERMUTATIONS AND COMBINATIONS

FACTORIALS:

Are used in the computations of permutations and combinations

Definition:

The continued product of the first n – *natural numbers* i.e. the continued product of n -consecutive integers beginning with 1 and ending with n , is factorial n , denoted by the symbol $n!$

$$\begin{aligned}\therefore n! &= 1 \times 2 \times 3 \times \dots \times n \\ &= n(n-1)(n-2)(n-3) \dots \times 2 \times 1\end{aligned}$$

Examples

1. How many different 3-digit number can be constructed from numerals 5,7 and 8 taking each once?

Solution

3-choices arise for the 1st numeral and for each choice; there are a further of 2-choices for the 2nd numeral. The third numeral is the one that is left
 $\therefore 3 \times 2 \times 1 = 6$ choices for the 1st and 2nd numerals combined exist.
 i.e. $3! = 3 \times 2 \times 1 = 6$

2. How many 4-digit numbers can be constructed using numerals 1,2,3 and 4 once each?

Solution

4-choices arise for the 1st numeral; each choice leaves 3-ways to select 2nd numeral
 $\Rightarrow 4 \times 3 = 12$ ways of selecting the first 2-numerals exist.
 Each combination of the first two numerals leaves 2-ways to select the third numeral.
 $\Rightarrow$ There are $4 \times 3 \times 2 \times 1 = 24$ ways of selecting first 3-numerals. The last numeral is the one left.
 $\therefore 4 \times 3 \times 2 \times 1 = 4! = 24$ are the number of ways.

3. Evaluate

- a) $6!$
- b) $\frac{8!}{3!}$
- c) $(7-2)!$
- d) $\frac{3!}{0!}$
- e) $\frac{5!}{(5-2)!2!}$

Solution

- a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
- b) $\frac{8!}{3!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 6720$
- c) $(7-2)! = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- d) $\frac{3!}{0!} = \frac{3 \times 2 \times 1}{1} = 6$ [$\because 0! = 1$]
- e) $\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = 10$

Note

- 1) From the definition of $n!$
 $n! = n(n-1)(n-2)(n-3) \dots \times 2 \times 1$

$$= n \times (n-1)!$$

$$\therefore n = \frac{n!}{(n-1)!}$$

$$2) \quad 2n! = 2 \times n!; (2n)! = (2n)(2n-1)(2n-2) \dots \times 2 \times 1$$

Example

1. Simplify

$$a) \frac{(2n+1)!}{(2n-1)!}$$

$$b) \frac{n!}{(n+1)!}$$

$$c) \frac{(n+1)!}{(n-1)!}$$

$$d) \frac{(2n)!}{(2n+2)!}$$

Solution

$$a) \frac{(2n+1)!}{(2n-1)!} = \frac{(2n+1)(2n+1-1)(2n+1-2) \times \dots \times 2 \times 1}{(2n-1)!}$$

$$= \frac{(2n+1)(2n)(2n-1)!}{(2n-1)!} = 2n(2n+1)$$

$$b) \frac{n!}{(n+1)!} = \frac{n!}{(n+1)(n+1-1)!} = \frac{n!}{(n+1) \times n!} = \frac{1}{n+1}$$

$$c) \frac{(n+1)!}{(n-1)!} = \frac{(n+1)(n+1-1)(n+1-2) \dots \times 2 \times 1}{(n-1)!}$$

$$= \frac{(n+1) \times n \times (n-1)!}{(n-1)!} = n(n+1)$$

$$d) \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+2-1)(2n+2-2) \dots \times 2 \times 1}$$

$$e) = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{2(n+1)(2n+1)}$$

2. Write each of the following in factorial notation

$$a) \quad 4 \times 3 \times 2 \times 1$$

$$b) \quad 6 \times 5 \times 4$$

$$c) \quad \frac{(7 \times 6) \times (3 \times 2 \times 1)}{2}$$

Solution

$$a) \quad 4 \times 3 \times 2 \times 1 = 4!$$

$$b) \quad 6 \times 5 \times 4 = \frac{(6 \times 5 \times 4) \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{6!}{3!}$$

$$c) \quad \frac{(7 \times 6) \times (3 \times 2 \times 1)}{2} = \frac{[(7 \times 6) \times 5 \times 4 \times 3 \times 2 \times 1] \times (3 \times 2 \times 1)}{(2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)} = \frac{7!3!}{2!5!}$$

Note

Some other terms have been introduced but without changing the meaning of the question so as to apply $n!$ definition.

PERMUTATIONS

The knowledge of permutations helps in finding *the number of ways a set of objects may be grouped/arranged under certain conditions.*

Definition

Each arrangement/*ordered* set of objects which can be obtained from a given set of objects by taking some or all at a time is called *permutation of the given set of objects.*

Notation

If the number of objects in the given set is n and the number of objects in each permutation is r (i.e. $n \geq r$), the total number of permutation is denoted as ${}^n p_r$.

Consider the example below

Qn. Assuming in the department of math, there are four teachers A, B, C and D; a committee consisting of two members is to be formed for drafting the math syllabus, list the possible groups.

ANSWER: AB, AC, AD, BC, BD, CD

Hence there is 6-groups altogether i.e. 6 combinations

Qn. Suppose one is to be selected as a president of the syllabus committee and the other as a secretary, list the possible groups.

ANSWER:

Let the 1st one denote the president and the 2nd as the secretary.

The possibilities are;

$A_{\text{president}} B_{\text{secretary}}$, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC

Hence there is now 12-possibilities altogether.

⇒ This problem is not only one of *grouping* but also *ordering* them.

Thus each of the 12-arrangements made out of 4-letters A, B, C, D by taking 2-letters at a time viz: AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC is a *permutation of the 4-letters taken 2-at a time*. i.e. ${}^4 p_2 = 12$

NOTE

1. Permutations involve *ordering* and *arranging*. i.e. AB and BA are different; AB means A is the president, B is the secretary, while BA means B is the president and A is the secretary. However for purpose of *grouping*, AB and BA mean one and same group.
2. Grouping is more in *combinations*

IMPORTANT RESULTS OF PERMUTATIONS

- a) *Permutations of n-different objects*

Permutations of n-different objects taken r-at a time $r < n$ is

$${}^n p_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \dots \dots \dots (n-r+1)$$

- b) *Permutations in which objects are repeated*

The number of permutations of n-different objects taken r-at a time in which each object is repeated r-times in any arrangement is n^r

- c) *Permutations of objects not all different*

The number of permutations of n-objects of which p-objects are alike of one kind, q-objects are alike of second kind, r-objects are alike of a third kind, and all the rest are all different is

- d) *Circular permutations*

$$\frac{n!}{p!q!r!}$$

In circular permutations, fix the position of one object and then arrange the remaining $(n - 1)$ out of n -objects in all the possible ways. This can be done in only

$$(n - 1)! \text{ ways}$$

e) *Restricted permutations*

- i) The number of permutations of n -different objects taken r -at a time in which p -particular objects do not occur is

$${}^{n-p}p_r$$

- ii) The number of permutations of n -different objects taken r -at a time in which p -particular objects are present is

$${}^{n-p}p_{r-p} \times {}^r p_p$$

Examples

1. Find the value of n if:

- a) ${}^n p_1 = 3$
 b) ${}^n p_2 = 2$
 c) ${}^{2n} p_2 = 6$
 d) ${}^{3n} p_2 = 30$

Solution

- a) ${}^n p_1 = 3 \Rightarrow \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = 3 \therefore n = 3$
 b) ${}^n p_2 = 2 \Rightarrow \frac{n!}{(n-2)!} = 2$ or $\frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = 2$
 $\Rightarrow n^2 - n - 2 = 0 \therefore n = 2, -1$ Required $n = 2$
 c) ${}^{2n} p_2 = 6 \Rightarrow {}^{2n} p_2 = \frac{(2n)!}{(2n-2)!} = \frac{(2n)(2n-1)(2n-2)!}{(2n-2)!} \therefore 2n(2n-1) = 6$ or $4n^2 - 2n - 6 = 0$
 solving $n = 1.5, -1$. Since n is a counting number, then $n = 2$
 d) ${}^{3n} p_2 = 30 \Rightarrow {}^{3n} p_2 = \frac{(3n)!}{(3n-2)!} = \frac{(3n)(3n-1)(3n-2)!}{(3n-2)!}$
 $\therefore 3n(3n-1) = 30$ or $9n^2 - 3n - 30 = 0$. Solving $n = 2, -\frac{5}{3} \therefore$
 Required $n = 2$

2. Find n if

- i) ${}^n p_5 : {}^n p_3 = 2 : 1$
 ii) ${}^{2n+1} p_{n-1} : {}^{2n-1} p_n = 3 : 5$
 iii) ${}^{56} p_{n+6} : {}^{54} p_{n+3} = 30800 : 1$

Solution

- i) ${}^n p_5 : {}^n p_3 = 2 : 1 \Rightarrow {}^n p_5 / {}^n p_3 = \frac{2}{1}$ or $\frac{\frac{n!}{(n-5)!}}{\frac{n!}{(n-3)!}} = \frac{n!}{(n-5)!} \times \frac{(n-3)!}{n!} = 2$
 $\Rightarrow \frac{(n-3)!}{(n-5)!} = 2$ or $\frac{(n-3)(n-4)(n-5)!}{(n-5)!} \therefore (n-3)(n-4) = 2$
 $\Rightarrow n^2 - 7n + 10 = 0 \therefore n = 2$ or 5
 Since $n > r$, then required $n = 5$ (Reject $n = 2$)
 ii) ${}^{2n+1} p_{n-1} : {}^{2n-1} p_n = 3 : 5 \Rightarrow {}^{2n+1} p_{n-1} / {}^{2n-1} p_n = 3 / 5$
 $\therefore \frac{(2n+1)!}{[(2n+1)-(n-1)]!} \div \frac{(2n-1)!}{[(2n-1)-n]!} = \frac{3}{5}$

$$\begin{aligned}
 \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} &= \frac{3}{5} \Rightarrow \frac{(2n+1)!}{(2n-1)!} \times \frac{(n-1)!}{(n+2)!} \\
 \frac{(2n+1)(2n+1-1)(2n+1-2)!}{(2n-1)!} \times \frac{(n-1)!}{(n+2)(n+2-1)(n+2-2)(n+2-3)!} &= \frac{3}{5} \\
 \frac{(2n+1)(2n)(2n-1)}{(2n-1)!} \times \frac{(n-1)!}{(n+2)(n+1).n.(n-1)!} &= \frac{3}{5} \\
 \Rightarrow \frac{2n(2n+1)}{n(n+1)(n+2)} = \frac{3}{5} \text{ or } \frac{2(2n+1)}{(n+1)(n+2)} = \frac{3}{5} \\
 \therefore 10(2n+1) &= 3(n^2 + 3n + 2) \text{ or } 3n^2 - 11n - 4 = 0 \\
 \Rightarrow (n-4)(3n+1) &= 0 \therefore n = 1, -\frac{1}{3} \text{ Required } n = 1 \\
 \text{and non integer } n &= -\frac{1}{3} \text{ rejected}
 \end{aligned}$$

iii) ${}^{56}p_{n+6} : {}^{54}p_{n+3} = 30800 : 1$

$$\begin{aligned}
 {}^{56}p_{n+6} / {}^{54}p_{n+3} &= 30800/1 \Rightarrow \frac{56!}{[56-(n+6)]!} \div \frac{54!}{[54-(n+3)]!} = \frac{30800}{1} \\
 \Rightarrow \frac{56!}{[50-n]!} \div \frac{54!}{[51-n]!} &= \frac{30800}{1} \Rightarrow \frac{56!}{54!} \times \frac{[51-n]!}{[50-n]!} = 30800 \\
 \Rightarrow \frac{56 \times 55 \times 54!}{54!} \times \frac{(51-n)(51-n-1)!}{(50-n)!} &= 30800 \\
 \therefore 3080(51-n) &= 30800 \text{ or } 51-n = 10 \therefore n = 41
 \end{aligned}$$

Task

1) If ${}^{10}p_{n-1} : {}^{11}p_{n-2} = 30 : 11$, find n

ANSWER: $n = 7$

2) Find n if

a) $5 \times {}^{56}p_{n+6} = 12 \times {}^{n-1}p_3$

b) ${}^{n+1}p_5 : {}^{n-1}p_5 = 9 : 2$

c) ${}^{2n-1}p_n : {}^{2n+1}p_{n-1} = 5 : 3$

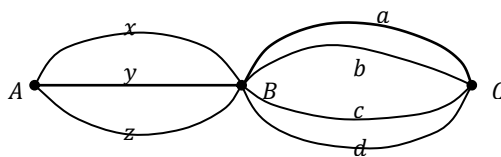
THE FUNDAMENTAL PRINCIPLE OF COUNTING

Consider the figure below

A, B and C are 3-stations.

x, y and z are 3-routes from A to B

a, b, c and d are 4-routes from B to C



There are 3-independent ways of going from A to B , and corresponding to each way of going from A to B , there are 4-ways of going from B to C .

Hence there $3 \times 4 = 12$ ways of going from A to C .

i.e. $xa, xb, xc, xd, ya, yb, yc, yd, za, zb, zc, zd$

With this background the fundamental rules are;

1. Multiplication principle

If one operation can be performed independently in m different ways, and another operation can be performed independently in n different ways, the total number of ways in which both operations can be performed simultaneously is $(m \times n)$ ways

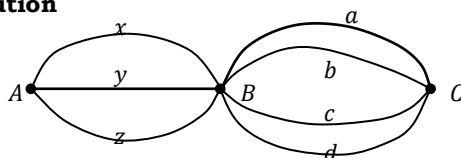
2. Addition principle

If one operation can be performed in **m** ways, and another operation can be performed in **n** ways, and only one operation can be done at a time, then either of the two operations can be performed in **(m + n) ways**

Examples

1. There are four bus lines between A and B and three bus lines between B and C.
 - a) In how many ways can a man travel by bus from A to C by way of B?
 - b) In how many ways can a man travel round trip by bus from A to C by way of B, if he does not want to use a bus line more than once?

Solution



- a) $4 \times 3 = 12$ ways
- b) Travelling from A to C, no. of ways = $4 \times 3 = 12$ ways. Since the man does not want to use the bus line more than once, the number of bus lines from C to B are 2 and from B to A are 3.
 $\therefore$ no. of ways in which he can travel from C to A by way of B is $2 \times 3 = 6$ ways
 $\therefore$ the total number of ways in which a man can travel round trip by bus from A to C by way of B not using bus line more than once is $12 \times 6 = 72$ ways

General examples on permutations

- 1) Find the number of words that can be formed by considering all possible permutations of the letters of the word "FATHER". How many of these words begin with A and end with R?

Solution

Since all the letters are different,

$n = 6, r = 6$ (Taking all the letters for a word)

$$\Rightarrow {}^n p_r = {}^6 p_6 = 6! = \frac{6!}{0!} = 720 \text{ ways}$$

Since A and R are fixed, permutations are to be done on FTHE

$$\therefore n = 4, r = 4 \Rightarrow {}^4 p_4 = 24 \text{ ways}$$

- 2) How many different words containing all the letters of the word TRIANGLE can be formed?

How many of them

- a) Begin with T?
- b) Begin with E?
- c) Begin with T and end with E?

Solution

All the letters are different

$$n = 8, r = 8 \therefore {}^8 p_8 = 8!$$

- a) Since T is fixed for all words, permutations are to be made on R, I, A, N, G, L, E

$$\therefore n = 7, r = 7 \Rightarrow {}^7p_7 = 7!$$

- b) Since E is now fixed, permutations are to be made on the remaining letters

$$\therefore n = 7, r = 7 \Rightarrow {}^7p_7 = 7!$$

- c) Now T and E are fixed, hence permutations are to be made on the remaining 6-letters

$$\therefore n = 6, r = 6 \Rightarrow {}^6p_6 = 6!$$

- 3) i) In how many ways can the letters of the word "MOBILE" be arranged so that the consonants always occupy the odd places?
 ii) How many words can be formed out of the letters "ARTICLE" so that the vowels occupy the even places?

Solution

- i) There are 6-letters of which 3 (M, B, L) are consonants and 3 (I, O, E) are vowels.

The positions to be filled up with consonants are as below

1	2	3	4	5	6
(M)	—	(B)	—	(L)	—

The total no. of ways consonants can be filled $= {}^3p_3 = 3! = 6$ ways

The total no. of ways 3-vowels can be arranged $= {}^3p_3 = 3! = 6$ ways

By fundamental principle,

Total arrangements $= 6 \times 6 = 36$ ways

- ii) There are 7-letters of which 3 (I, O, E) are vowels

1	2	3	4	5	6	7
—	(A)	—	(I)	—	(E)	—

Arrangements of vowels $= {}^3p_3 = 3! = 6$ ways

Remaining 4-letters can be arranged in 4 places in ${}^4p_4 = 4!$ ways

By fundamental theorem, total no. of arrangements $= 3! \times 4! = 144$ ways

- 4) In how many ways can the letters of the word "STRANGE" be arranged so that the vowels may appear in the odd places?

Solution

There are 7 letters with 2 (A, E) vowels.

1	2	3	4	5	6	7
(A)	—	(E)	—	—	—	—

The 2 vowels occupy only in the 2 out of 4 positions, in which this can be done in

$${}^4p_2 = \frac{4!}{2!} = 12 \text{ ways}$$

The rest of the letters occupy empty spaces. This can be done in

$${}^5p_5 = 5! = 120 \text{ ways}$$

$$\therefore \text{total no. of arrangements are } 120 \times 12 = 1440 \text{ ways}$$

- 5) How many words can be formed from the letters of the word "FATEPUR" when

- The 3-letters PUR occur together?
- Vowels occur at even places?

Solution

- i. There are 8 letters. Since the 3 (PUR) letters occur together, we may regard them as forming one letter, so there are now 6 letters for permutation.

$\Rightarrow$ The 6 letters can be arranged in ${}^6p_6 = 6!$ ways

$\Rightarrow$ The 3 (PUR) letters can be arranged in ${}^3p_3 = 3!$ ways

By fundamental principle, total arrangement = $6! \times 3!$ ways

- ii.

1	2	3	4	5	6	7	8
–	(A)	–	(E)	–	(U)	–	(–)

3-vowels occupy 3 out of 4 positions in

$${}^4p_3 = \frac{4!}{1!} = 24 \text{ ways}$$

Remaining consonants occupy 5-empty spaces (1, 3, 5, 7, 8) in

$${}^5p_5 = 5! = 120 \text{ ways}$$

$$\therefore \text{total no. of arrangements} = 120 \times 24 = 2880 \text{ ways}$$

- 6) How many different words containing all the letters of the word “TRIANGLE” can be formed? How many of them
- When consonants are never together?
 - When no two vowels are never together?
 - When consonants and vowels are both always together?
 - Vowels occupy odd places?
 - Relative positions of vowels and consonants remain unaltered?

Solution

TRIANGLE contain 8- different letters, which can be arranged in ${}^8p_8 = 8!$ ways

Consonants are (T, R, N, G, L)

Vowels are (I, A, E)

- i) When consonants are never together

$$= \left(\begin{matrix} \text{Total arrangements} \\ \text{of 8 – letters} \end{matrix} \right) - \left(\begin{matrix} \text{consonants when} \\ \text{altogether} \end{matrix} \right)$$

Permutations for consonants when together = ${}^5p_5 = 5!$

Permutations for the word with consonants together .i.e. (TRNGL), I, A, E .i.e.

4-letters can be arranged in ${}^4p_4 = 4!$ ways

$\Rightarrow$ The total arrangements when consonants are together = $5! \times 4!$

$\therefore$ When consonants are never together = $8! - 5! \times 4!$ ways can arise.

- ii) When no two vowels are together = $8!$ – permutation for vowels when together. Different arrangement for 2 vowels out of 3

$$= {}^3p_2 = \frac{3!}{1!} = 3! \text{ ways}$$

Arrangement for vowels (two) occurring together = ${}^2p_2 = 2!$ ways arise

Arrangement for the remaining consonants = ${}^5p_5 = 5!$ ways arise

$\therefore$ Total arrangement for vowels = $2! \times 3! \times 5!$

$\therefore$ When no two vowels occur together = $(8! - 2! \times 3! \times 5!)$ ways arise

- iii) When consonants and vowels are both always together .i.e. (TRNGL), (IAE) give two letters that can be arranged in

$${}^2p_2 = 2! \text{ ways}$$

Consonants are arranged in ${}^5p_5 = 5!$

Vowels are arranged in ${}^3p_3 = 3!$

$\therefore$ Total arrangements = $2! \times 3! \times 5!$

iv) Vowels occupy odd places

1	2	3	4	5	6	7	8
(A)	–	(I)	–	(E)	–	()	–

Vowels occupy 3 out of 4 positions in ${}^4P_3 = 4! \text{ ways}$

Remaining consonants occupy 5 places in ${}^5P_5 = 5! \text{ ways}$

$\therefore$ Total arrangements $4! \times 5!$

v) T R I A N G L E

$\Rightarrow$ For consonants, ${}^5P_5 = 5! \text{ ways occur}$

$\Rightarrow$ For vowels, ${}^3P_3 = 3! \text{ ways occur}$

$\therefore$ Relative positions of vowels and consonants remain unaltered in the word is $5! \times 3! \text{ ways}$

7) *How many permutations can be made out of the letters of the word “MISSISSIPPI” taken altogether?*

Solution

MISSISSIPPI contains repeated letters . i.e. “S” occurs 4-times, “I” occurs 4-times and “P” occurs 2-times.

Using permutation of objects not all different . i.e. $\frac{n!}{p!q!r!}$, $\Rightarrow$ number of permutations for 11-letters will be = $\frac{11!}{4!4!2!}$

8) *In how many ways can the letters of the word “CONSTITUTION” be arranged? How many of these will have the letter “N” both at the beginning and at the end?*

Solution

Word “CONSTITUTION” consists of 12-letters having O, N, I two times each, T- 3 times and U, C, S once each.

Required number of different ways is = $\frac{12!}{2!2!2!3!} = 9979200 \text{ ways}$

For having both N at the beginning and at the end, then N is fixed at the beginning and at the end, there by leaving 10-letters having the above characteristics.

$\therefore$ Required number of ways is = $\frac{10!}{2!2!3!} = 151200$

9) *In how many ways can the letters of the word “ARRANGE” be arranged? How many of these arrangements are there in which*

- a) The two R’s come together?
- b) The two R’s do not come together?
- c) The two R’s and the two A’s come together?

Solution

ARRANGE has 7-letters out of which 2 are A, 2 are R, 1 is N, 1 is G and 1 is E

$\therefore$ Required no. of ways = $\frac{7!}{2!2!} = 1260 \text{ ways}$

a) Two R’s come together give the word A (RR) A N G E consisting of now 6-letters.

Now word has 2 A’s, N, G, E are all one.

$$\Rightarrow \text{Required arrangements} = \frac{6!}{2!} = 360 \text{ ways}$$

$$\left[\text{For } 2R's, \text{ arrangement} = \frac{2!}{2!} = 1 \right]$$

- b) Required no. of arrangements = $1260 - 360 = 900$ ways
 c) Treating both R and A as one individual letter the word becomes (AA)(RR)NGE consisting of 5-letters that are different
 $\therefore$ No. of ways = $5! = 120$ ways
- 10) a) How many different arrangements can be made by using all the letters in the word "MATHEMATICS"? How many of them begin with
 i) C? ii) T?
 b) How many permutations can be made out of the letters of the word "INDEPENDENCE"? How many of them have vowels occurring together?

Solution

- a) MATHEMATICS has 11-letters of which 'M' occurs twice, 'A' occurs twice and 'T' also occurs twice. Remaining letters .i.e. $11 - 6 = 5$ letters H, E, I, C, S are all different from one another.

$\therefore$ No. of permutations of these 11- letters taken all at a time is

$$\frac{11!}{2!2!2!} = 4989600 \text{ ways}$$

- i) Fixing C in first position, then the permutation shall be done on the remaining letters .i.e. 10 letters with the above stated characteristics

$$\text{No. of ways} = \frac{10!}{2!2!2!} = 453600 \text{ ways}$$

- ii) Fixing T in first position, permutation is to be done on the remaining letters .i.e. T, M, A, T, H, E, M, A, I, C, S .i.e. in these 10-letters, M and A both occur twice, T-no longer occurs twice for one of them occupied the first position.

$$\therefore \text{No. of ways} = \frac{10!}{2!2!} = 907200 \text{ ways}$$

- b) INDEPENDENCE has 12-letters in which 'N' occurs thrice, 'D' occurs twice, 'E' occurs 4-times and the remaining 3-letters I, P, C are all different from each other.

$$\therefore \text{No. of permutations of 12-letters taking all at a time} = \frac{8!}{3!2!4!}$$

INDEPENDENCE, consists of 5-vowels .i.e. I, E, E, E, E (4 - letters are similar)

Taking the 5-vowels as one letter, then (IEEEE)NDPNDNC is the word of 8-letters consisting of 3 N's and 2 D's

$$\therefore \text{Arrangement} = \frac{8!}{3!2!}$$

$$\text{Permutation of the tied/bracketed vowels} = \frac{5!}{4!}$$

$$\therefore \text{Total no. of arrangements} = \frac{8!}{3!2!} \times \frac{5!}{4!}$$

- 11) a) How many numbers greater than 1 million can be formed with digits 4, 5, 5, 0, 5, 4, 3?

Solution

There are 7 digits in all, of which there are 2-fours, 3-fives and the rest are different.

$$\therefore \text{Total no. of arrangements} = \frac{7!}{2!3!} = 420$$

Of these numbers, some begin with 0 and are less than 1-million, so they must be rejected.

Fixing 0 at the beginning, possible no. of permutations are $= \frac{6!}{2!3!} = 60$

$\therefore$ Required number $= 420 - 60 = 360$

b) (i) Find all the six digit numbers formed from the digits 0, 1, 2, 3, 4, 5.

(ii) How many numbers of 4-digits can be formed from out of the digits 3, 4, 5, 6, 7, 8, 0 if no digit is to be repeated?

Solution

i) The digits can be arranged in ${}^6P_6 = 6!$ ways

Out of these 6! Ways, there are numbers that begin with "0"

$\therefore$ Number of such numbers $= {}^5P_5 = 5!$

6-digit number $= 6! - 5! = 600$

ii) From the 7-digits, 4-digit number is to be formed.

Number of ways for this, ${}^7P_4 = \frac{7!}{3!}$

Out of these numbers, there are numbers starting with 0-digit hence reducing to 3- digit number.

Number of ways for this $= {}^6P_3 = \frac{6!}{3!}$

Required numbers $= \frac{7!}{3!} - \frac{6!}{3!} = 720$

c) How many odd numbers of 5-digits can be formed with digits 3, 2, 7, 4, 0 when no digit is repeated?

Solution

Odd numbers have 3 or 7 digit in the last place.

Odd numbers starting with 0 and have 3 or 7 in the last place exist.

Let us find all 5-digit odd numbers with 3 in last place $= {}^4P_4 = 4! = 24$ ways

Now all 5-digit odd numbers with 7 in last place $= {}^4P_4 = 4! = 24$ ways

Now finding all odd numbers with 0 in first place but ending with 3

.i.e. 4 - digit number

$= {}^3P_3 = 3! = 6$ ways

Also finding numbers with 0 in first place and 7 in the last place. *i.e. 4 - digit number*

$= {}^3P_3 = 3! = 6$ ways

Now total number of odd numbers with 0 in first place $= 6 + 6 = 12$ numbers

Total number of odd numbers $= 24 + 24 = 48$

$\therefore$ Number of odd numbers with 5-digits $= 48 - 12 = 36$

d) (i) How many numbers of 4-digits can be formed with digits 3, 5, 7, 8, 9, 0 digit being repeated? How many of these will be greater 8000?

ii) A number of 4-different digits is to formed using digits 1, 2, 3, 4, 5, 6, 7 in all possible ways. Find how many of such numbers can be formed. Hence how many such numbers are greater than 3400?

Solution

i) 4-digit numbers $= {}^5P_4 = 5!$

For 4-digit number to be greater than 8000, 8 or 9 must occupy the first place.

Permutations of the remaining digits *.i.e* 3 out of 4 after the first being occupied by 8 or 9

$$= {}^4p_3 = 4! \text{ ways}$$

Since numbers are of 2 types .i.e 8 or 9 starting digits, then

$$\text{Total number of numbers} = 4! \times 2 = 48$$

Alternatively

Number greater than 8000 = $5! - (\text{Numbers less than 8000})$

Finding 4 – digit numbers < 8000

Numbers < 8000 begin with 3, 5 or 7

Since first place is being occupied in 3-ways, then

4 – digit numbers < 8000 = $3 \times$ Permutation of the 3-digits after the first one being occupied by 3 or 5 or 7

$$\therefore \text{Numbers} < 8000 = 3 \times {}^4p_3 = 3 \times 4!$$

$$\text{Required numbers} > 8000 = 5! - 3 \times 4! = 48$$

ii) 4-digit numbers formed out of 7-digits in ${}^7p_4 = \frac{7!}{3!}$ ways

For a number to be greater than 3400,

Case I

The first digit on LHS must be 4, 5, 6, 7 and the 3-digits occupied in ${}^6p_3 = \frac{6!}{3!}$

The first place in 4-ways

Case II

The first digit on LHS is 3, the second digit must be 5, 6, 7 and the other two places are occupied by any of the two remaining 5-digits

Case III

The first digit on LHS is occupied by 3, second digit by 4 and the other two places are occupied by two of the remaining digits

For case I: Total arrangement = $4 \times {}^6p_3$

For case II: Total arrangement = $3 \times {}^5p_2$

For case III: Total arrangement = ${}^5p_2$

$$\text{Total} = 4 \times {}^6p_3 + 3 \times {}^5p_2 + {}^5p_2 = 560 \text{ ways}$$

e) How many numbers greater than 7000 can be formed with digits 3, 5, 7, 8, 9 no digit being repeated?

Solution

Cases in the question

Case I: All numbers with 5-digits are > 7000

This is done in ${}^5p_5 = 5!$ ways

Case II: There will be 4-digit numbers which are > 7000. These start with 8, 9 or 7 but never with 3 or 5

A. 4-digit number > 7000, starting with 8 or 9 are:

$$2 \times {}^4p_3 \left[\begin{array}{l} \because \text{the remaining 3 – digits} \\ \text{are permuted out of 4} \end{array} \right]$$

B. If the number is to start with 7, the 3-digits must be filled up from the 4-remaining digits in ${}^4p_3 = 4!$ ways

$$\therefore \text{Total arrangement} = {}^4p_3 + 2 \times {}^4p_3 + 5! = 192$$

Task

1. Uneb 2004

How many arrangements can be made from the letters of the name "MISSISSIPPI"

- When all letters are taken at a time?
- If the two letters P begin every word?

ANSWER: a. $\frac{11!}{4!4!2!}$ **b.**

$\frac{9!}{4!4!}$ [HINT: MISSISSI(PP)I has 10 letters of which 9] are to be permuted

- How many different words can be formed with the letter of the word "CAPTAIN"? In how many of these C and T are never together?

ANSWER: $\frac{7!}{2!}$, taking C and T as one unit $\Rightarrow \frac{6!}{2!}$, (CT) occur in ${}^2P_2 = 2!$ $\therefore$ total = $\frac{6!}{3!} \times 2!$

- In how many ways the letters of the word "PETROL" can be arranged? How many of these do not begin with P? How many begin with P but do not end with L? Also find the number of words which can be formed if O and L are never together?

ANSWER: i) 720 **ii)** 720 – 120 **iii)** 120 – 24 **iv)** 720 – 2 $\times$ 120

- How many permutations can be made out of letters all taken together

- EXAMINATION
- PERMUTATION
- ACCOUNTANT
- MATAAMA

ANSWER: a) 4 989 600 **b)** $\frac{11!}{2!}$ **c)** $\frac{10!}{2!2!2!}$ **d)** $\frac{7!}{2!4!}$

Circular permutation

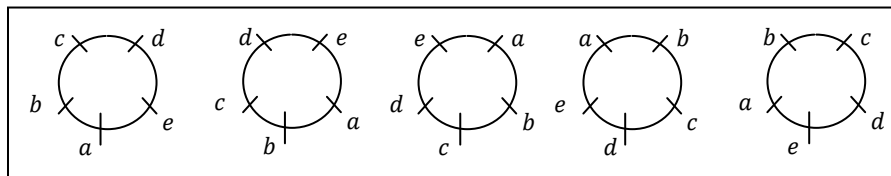
- In how many ways can 5 persons be seated on a round table conference?

Solution

Let the 5 persons be denoted by letters a, b, c, d and e

NOTE

If these persons are to be seated in a row the five linear arrangements are; abcde, bcdea, cdeab, deabc, eabcd, are all different from one another. But if are to be seated at a round table in the same sense as in the figure below,



All the arrangements are the same, for starting with different letters and reading them in clockwise direction. i.e. abcde, bcdea, cdeab, deabc, eabcd, which are the same as the linear arrangements.

$\therefore$ A single circular arrangement of the n letters is equivalent to n different linear arrangements.

i.e. $n \times \text{a circular arrangement} = n! \times (\text{linear arrangement of } n - \text{letters})$

$\Rightarrow \text{No. of circular arrangements} = (n - 1)!$

Let the circular arrangement = x

$\Rightarrow 5 \times x = 5!$ (linear arrangements of 5 – letters)

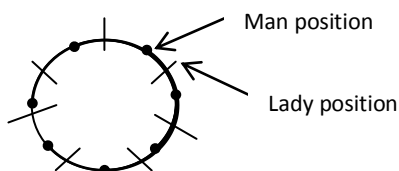
$\therefore x = \frac{1}{5} \times 5! = 4!$

$\therefore \text{No. of circular arrangements of 5 – persons} = (5 - 1)! = 4!$

2. a) In how many ways can 7 gentlemen and 7 ladies sit at a round table so that no two ladies are together?
- b) In how many ways can 5 gentlemen and 3 ladies sit at a round table so that no two ladies are together?

Solution

a)

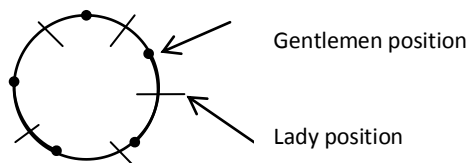


No. of ways in which 7 gentlemen can sit on round table = $(7 - 1)! = 6!$

Since no two ladies can be sitting together, ladies can be put in between the men .i.e. there are exactly 7 places, 7 ladies can use. This can be done in 7! Ways.

$\therefore \text{Total no. of ways} = 6! \times 7! = 3\,628\,800$

b)



5 gentlemen sit on a round table in $(5 - 1)! = 4!$ ways

There are 5 positions 3 ladies can occupy in ${}^5P_3 = \frac{5!}{2!}$ ways

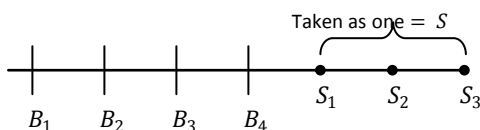
$\therefore \text{Total nu. of ways required} = 4! \times \frac{5!}{2!} = 1440 \text{ ways}$

Task

1. In how many ways can 7 people be arranged at a round table so that two particular persons be together?
2. A family of 4 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated if
 - a) All sisters sit together?
 - b) No two sisters sit together?

HINT:

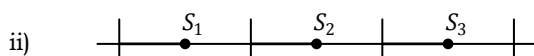
- a) Total no. of arrangements in which all sisters sit together = $5! \times 3! = 720$



$\therefore$ Permutation of $B_1 B_2 B_3 B_4 S = 5!$

Now permutation of $S_1 S_2 S_3 = 3!$

Total permutation = $5! \times 3!$



Permut B_1 in all B_2, B_3, B_4 sons = B_3 B_4

Permutations of sisters seated together

$$= 5! \times 3!$$

$\therefore$ Total = $7! - 5! \times 3! = 4320$

3. There are 6 books of Economics, 3 of Math and 2 of Accountancy. In how many ways can these be placed on a shelf if the books on the same subject are to be together?

HINT: 6E's, 3M's and 2A's are different $\Rightarrow 6! \times 3! \times 2!$

4. A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of 3 books and a single copy of 8 books. In how many ways can all the books be arranged on the shelf?

HINT: total number of books of same kind = $1 \times 5 + 4 \times 2 + 6 \times 3 + 8 =$

39 books of which same types are 5, 4, 4, 6, 6, and 6

$\therefore$ Total number of arrangements = $\frac{39!}{5!4!4!6!6!6!} = \frac{39!}{5!(4!)^2(6!)^3}$

COMBINATIONS

Definition:

The different groups/collections/selections that can be formed out of a given set of objects by taking some or all of them at a time (without regard to the order of arrangements) are called

Combinations

NOTATION

The number of combinations of n -different objects taken r -at a time is denoted as

$${}^n C_r = \frac{{}^n P_r}{r!} = \frac{n!}{r!(n-r)!}, \text{ called combinatorial coefficient}$$

IMPORTANT DEFINITIONS IN COMBINATIONS

- a) **Complementary combinations**

- i) ${}^n C_r = {}^n C_{n-r}$.i.e. The number of combinations of n different

objects taken r at a time is equal to the number of combinations of n different objects taken $(n-r)$ at a time.

Proof

$${}^n C_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)![n-n+r]} = \frac{n!}{(n-r)![n-(n-r)]} = {}^n C_{n-r}$$

- ii) ${}^n C_n = {}^n C_0 = 1$

Proof

$${}^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$$

$$\text{Also } {}^nC_0 = \frac{n!}{n!(n-0)!} = \frac{n!}{n!0!} = 1$$

$$\text{iii) } \boxed{{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1} \quad \text{OR} \quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r}$$

i.e. The number of combination of n objects taken r at a time plus the number of combinations of n taken (r+1) or (r-1) in the above formulae respectively at a time is equal to the number of combinations (n+1) objects taken (r+1) or respectively at a time.

Proof

$${}^nC_n + {}^nC_{r+1} = {}^{n+1}C_{r+1}$$

$$\begin{aligned} {}^nC_n + {}^nC_{r+1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-r-1]!} \\ &= \frac{n!}{(n-r-1)!r!} \left[\frac{1}{n-r} + \frac{1}{r+1} \right] \text{ taking out common factor} \\ &= \frac{n!}{(n-r-1)!r!} \left[\frac{r+1+n-r}{(n-r)(r+1)} \right] \text{ adding two fractions} = \frac{n!}{(n-r-1)!r!} \cdot \frac{(n+1)}{(n-r)(r+1)} \\ &= \frac{(n+1)!}{(n-r)!(r+1)!} \left[\begin{array}{l} \because (n-r-1)!(n-r) = (n-r)! \\ r!(r+1) = (r+1)! \\ n!(n+1) = (n+1)! \end{array} \right] \\ &= \frac{(n+1)!}{[(n+1)-(r+1)]!(r+1)!} [\because (n+1)-(r+1) = n-r] \\ &= {}^{n+1}C_{r+1} \end{aligned}$$

b) Restricted combinations

- i. The number of combinations on n objects taken r at a time in which p particular objects are always included is

$$\boxed{{}^{n-p}C_{r-p}}$$

- ii. The number of combinations of n objects taken r at a time in which p particular objects never occur is

$$\boxed{{}^{n-p}C_r}$$

c) Total number of combinations of n-objects all different

The total number of combinations of n-different objects taken some or all at a time is

$$\boxed{{}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n - 1}$$

d) Group division

The number of ways in which (m + n)objects m and n being unequal, can be divided in to two groups containing m and n objects respectively is

$$\boxed{\frac{(m+n)!}{m!n!}}$$

181

Solution

4 men can be selected from 10 men in $^{10}C_4$

3 women are selected from 8 women in $^8C_3$

Total number of ways of selecting committee $= {}^{10}C_4 \times {}^8C_3 = 11\,760$ ways

7. *In how many ways can a committee of 6 men and 2 women be formed out of 10 men and 5 women?*

Solution

6 men selected from 10 men in $^{10}C_6$ ways

2 women selected from 5 women in $^5C_2$

Total number of ways of selecting a committee $= {}^{10}C_6 \times {}^5C_2 = 2\,100$

8. *A group of nine has to be selected from 10 boys and 8 girls. It can consist of either 5 boys and 4 girls or 4 boys and 5 girls. How many different groups can be chosen?(Uneb 2006)*

Solution

Case I

5 boys selected from 10 boys in $^{10}C_5$

4 girls selected from 8 girls in $^8C_4$

Total number of selection $= {}^{10}C_5 \times {}^8C_4$

Case II

4 boys selected from 10 boys in $^{10}C_4$

5 girls selected from 8 girls in $^8C_5$

Total number of selection $= {}^{10}C_4 \times {}^8C_5$

$\therefore$ Overall total number of selection $= {}^{10}C_5 \times {}^8C_4 + {}^{10}C_4 \times {}^8C_5 = 29\,400$

9. *In a multinational company, 3 branches in a particular country are managed by 4, 5 and 8 persons. In how many ways can 20 persons be allotted to the different branches?*

Solution

Let the branches be A, B and C

For branch A

Let the 1st 4 persons be selected out 20 persons.

Number of ways $= {}^{20}C_4$

Having selected 4 persons for branch A, 5 persons can be selected out of the remaining 16 for **branch B** in $^{16}C_5$ ways

After this 8 persons for **branch C** out of the remaining 11 in $^{11}C_8$ ways

By fundamental theorem, selection of persons can be made in

${}^{20}C_4 \times {}^{16}C_5 \times {}^{11}C_8$ ways

10. *At an election 3 wards of a town are canvassed by 4, 5 and 8 respectively. If 20 men volunteer, in how many ways can they be allotted to the different wards?*

Solution

Let 3 wards be A, B and C

Let A be canvassed by the 1st 4 out of 20 in ${}^{20}C_4$ ways

B will be canvassed by 5 out of remaining 16 in ${}^{16}C_5$ ways

C will be canvassed by 8 out of the remaining 11 in ${}^{11}C_8$ ways

$\therefore$ Total number of ways $= {}^{20}C_4 \times {}^{16}C_5 \times {}^{11}C_8$

11. From 6 boys and 4 girls, 5 are to be selected for admission for a particular course. In how many ways can this be done if there must be exactly 2 girls?

Solution

<u>Type of selection</u>	<u>Boys (6)</u>	<u>Girls (4)</u>
A	3	2

$\Rightarrow$ A can be done in ${}^6C_3 \times {}^4C_2 = 120$ ways

12. From 5 gentlemen and 6 ladies, a committee of 5 is to be formed. In how many ways can this be done if

- The committee is to include at least a lady
- There is no restriction about its formation
- Not more than 3 gentlemen?

Solution

a)	<u>Lady (6)</u>	<u>Gentlemen (5)</u>	<u>No. ways of formation</u>
1		4	${}^6C_1 \times {}^5C_4$
2		3	${}^6C_2 \times {}^5C_3$
3		2	${}^6C_3 \times {}^5C_2$
4		1	${}^6C_4 \times {}^5C_1$
5		0	${}^6C_5 \times {}^5C_0$

$\therefore$ Number of ways in which a committee includes at least a lady is

$$= {}^6C_1 \times {}^5C_4 + {}^6C_2 \times {}^5C_3 + {}^6C_3 \times {}^5C_2 + {}^6C_4 \times {}^5C_1 + {}^6C_5 \times {}^5C_0$$

$$= 30 + 50 + 200 + 75 + 6 = 461 \text{ ways}$$

- b) When there is no restriction about formation 5 are selected out of 11 in
 ${}^{11}C_5 = 231$ ways

c)	<u>Gentlemen (5)</u>	<u>Lady (6)</u>	<u>Type of committee</u>	<u>No. of ways</u>
	0	5	A	${}^5C_0 \times {}^6C_5$
	1	4	B	${}^5C_1 \times {}^6C_4$
	2	3	C	${}^5C_2 \times {}^6C_3$
	3	2	D	${}^5C_3 \times {}^6C_2$

$\therefore$ Number of ways in which a committee consisting of not more than 3 gentlemen is formed in

$${}^5C_0 \times {}^6C_5 + {}^5C_1 \times {}^6C_4 + {}^5C_2 \times {}^6C_3 + {}^5C_3 \times {}^6C_2 = 6 + 75 + 200 + 150 = 431 \text{ ways}$$

13. The question paper on math and statistics contains 10 questions divided in to two groups of 5 questions each. In how many ways can an examinee select 6 questions taking at least 2 questions from each group?

Solution

<u>Group (A) [5 questions]</u>	<u>Group (B) [5 questions]</u>	<u>No. of ways</u>
2	4	${}^5C_2 \times {}^5C_4$
3	3	${}^5C_3 \times {}^5C_3$
4	2	${}^5C_4 \times {}^5C_2$
5	1	
6	0	

Not considered since at least two questions from each group are to be considered.

$$\begin{aligned}\therefore \text{Total number of selection} &= {}^5C_2 \times {}^5C_4 + {}^5C_3 \times {}^5C_3 + {}^5C_4 \times {}^5C_2 \\ &= 50 + 100 + 50 = 200 \text{ ways}\end{aligned}$$

14. A council consists of 11 members of which 5 are permanent and the remaining are temporary. A solution is declared to be passed by the council if it receives at least 7 votes including all the permanent members. Find the number of ways in which the resolution can be passed. What will be the number of ways if;

- For passing a resolution, only a simple majority was selected
- Any 7 or more votes could secure the passing of the resolution?

Solution

<u>Permanent vote (5)</u>	<u>Temporary vote (6)</u>	<u>No. of selection</u>
5	2	${}^5C_5 \times {}^6C_2 = 15$
5	3	${}^5C_5 \times {}^6C_3 = 20$
5	4	${}^5C_5 \times {}^6C_4 = 15$
5	5	${}^5C_5 \times {}^6C_5 = 6$
5	6	${}^5C_5 \times {}^6C_6 = 1$

$$\therefore \text{Total number of ways} = 15 + 20 + 15 + 6 + 1 = 57$$

- For simple majority either 6 or 7 or 8 or 9 or 10 or 11 votes out of 11 are required. This is done in

$${}^{11}C_6 + {}^{11}C_7 + {}^{11}C_8 + {}^{11}C_9 + {}^{11}C_{10} + {}^{11}C_{11} = 1024 \text{ ways}$$

- Number of ways $= {}^{11}C_7 + {}^{11}C_8 + {}^{11}C_9 + {}^{11}C_{10} + {}^{11}C_{11} = 562 \text{ ways}$

15. A council consists of 10 members, 6 belonging to the party A and 4 to the party B. in how many ways can a committee of 5 be selected so that the members of the party A are in the majority?

Solution

<u>A (6)</u>	<u>B (4)</u>	<u>No. of ways</u>
5	0	${}^6C_5 \times {}^4C_0 = 6$
4	1	${}^6C_4 \times {}^4C_1 = 60$
3	2	${}^6C_3 \times {}^4C_2 = 120$
2	3	
1	4	

, cannot be taken since party (A) must be the majority.

$$\therefore \text{Total number of selection} = 6 + 60 + 120 = 186$$

16. Among 20 members of a cricket club there are 2 wicket keepers and 5 bowlers. In how many ways can 11 be chosen so as to include only one of the wicket keepers and at least 3 bowlers?

Solution

<u>Type</u>	<u>Wicket players [2 max]</u>	<u>Bowler [5max]</u>	<u>No. of players other than Wicket of Bowlers [13 max]</u>	<u>Number of ways</u>
A	1	3	7	${}^2C_1 \times {}^5C_3 \times {}^{13}C_7$
B	1	4	6	${}^2C_1 \times {}^5C_4 \times {}^{13}C_6$
C	1	5	5 (Exhausted)	${}^2C_1 \times {}^5C_5 \times {}^{13}C_5$

$$\therefore \text{Total number of ways} = {}^2c_1 \times {}^5c_3 \times {}^{13}c_7 + {}^2c_1 \times {}^5c_4 \times {}^{13}c_6 + {}^2c_1 \times {}^5c_5 \times {}^{13}c_5 = 54\,054 \text{ ways}$$

16. A cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket keepers. In how many ways can a team be formed so that the team consists of at least 3 bowlers and at least one wicket keeper?

Solution

Wicket keeper (2 max) bowlers (4 max) Neither wicket nor keeper (10 max)

1	3	7
2	4	5
1	4	6
2	3	6



Further two possibilities

$$\therefore \text{Total selection} = {}^2c_1 \times {}^4c_3 \times {}^{10}c_7 + {}^2c_1 \times {}^4c_4 \times {}^{10}c_6 + {}^2c_2 \times {}^4c_3 \times {}^{10}c_6 + {}^2c_2 \times {}^4c_4 \times {}^{10}c_5 = 2\,472 \text{ ways}$$

3. a) A bag contains a mixture of 8 Rupee, 6 Fifty paise and 4 Twenty paise coins. In how many ways selection of 3 coins can be made so that

- all the three are Rupee coins
- one is of each denomination
- none is a Rupee coin

Solution

- For all to be Rupee coins, the 3 coins are selected from 8 Rupee in ${}^8c_3 \times {}^6c_0 \times {}^4c_0 = 56 \text{ ways}$
- One of each denomination is to be chosen; 1 Rupee can be chosen out of 8 in ${}^8c_1$, 1 Fifty paise is taken out of 6 in ${}^6c_1$ and 1 Twenty paise in ${}^4c_1$ ways.
 $\therefore \text{Total number of choices} = {}^8c_1 \times {}^6c_1 \times {}^4c_1 = 192 \text{ ways}$
- None is a Rupee coin if the 3 coins are to be selected from Fifty paise and Twenty paise .i.e. a total of $6 + 4 = 10$
 $\therefore \text{Total number of ways} = {}^{10}c_3 = 120 \text{ ways}$

- b) A box contains 7 red, 6 white and 4 blue balls. How many selection of 3 balls can be made so that

- All three are red balls
- None is a red ball
- There is one ball of each color?

Solution

- All 3 balls are red and are selected from only 7 red balls in ${}^7c_3 = 35 \text{ ways}$
 - No red ball, the 3 balls are selected from the other colors .i.e. out of $6 + 4 = 10$ balls
 $\therefore {}^{10}c_3 = 120 \text{ ways of selection}$
 - One ball of each color is selected in ${}^7c_1 \times {}^6c_1 \times {}^4c_1 = 168 \text{ ways}$
- c) A committee of 4 persons is to be appointed from 3 officers of the production department, 3 officers of the sales department, 2 officers of the purchase department and 1 cost accountant. In how many ways should a committee be formed so that

- i) *There must be one from each category*
- ii) *It should have at least one from purchase department?*

Solution

- i) Number of ways of forming a committee from each category = ${}^3c_1 \times {}^3c_1 \times {}^2c_1 \times {}^1c_1 = 18$ ways
- ii) A committee with at least 1 from purchase department
 = No. of ways of (1 – from purchase department +3 from other depts.)
 +(2 – from purchase dept.+ 2 from other depts.)
 = ${}^2c_1 \times {}^7c_3 + {}^2c_2 \times {}^7c_2 = 92$ ways

d) *An urn contains 8 white and 3 red balls. In how many ways should 2 balls be drawn so that*

- i) *Both are white*
- ii) *Both are red*
- iii) *One is of each color?*

Solution

- i) 2 white balls are drawn from 8 in ${}^8c_2 = 28$ ways
- ii) 2 red balls are drawn from 3 in ${}^3c_2 = 3$ ways
- iii) No. of ways in which 1 white and 1 red can be drawn out of 8 white and 3 red = ${}^8c_1 \times {}^3c_1 = 24$ ways

e) *A subcommittee of 16 members is to be formed out of a group of 7 men and 4 ladies. In how many ways can the subcommittee be formed in order to consist of*

- i) *Exactly two ladies*
- ii) *At least 3 ladies*

Solution

- i) 2 ladies out of 4 and 4 men out of 7 can be selected in
 ${}^4c_2 \times {}^7c_4 = 210$ ways
- ii) The subcommittee will consist of at least 2 ladies if there are 2, 3 or 4 ladies in it in each case the number of men would be 4, 3 and 2 respectively.
 No. of ways = ${}^4c_2 \times {}^7c_4 + {}^4c_3 \times {}^7c_3 + {}^4c_4 \times {}^7c_2 = 371$ ways

f) *A bag contains 6 white and 9 black balls. Two drawings of 4 balls are made such that*

- i) *The balls are replaced before the second trial*
- ii) *The balls are not replaced before the second trial*
In how many ways should the first drawing give 4 white and the second 4 black balls in each case? What if no restriction of which is the first or second drawn, find the number of ways this can be done in each case.

Solution

- i) No. of ways in which 4 white balls are drawn = ${}^6c_4$ ways
 No. of ways in which 4 black balls are drawn = ${}^9c_4$ ways
- ii) At the first trial number of ways of drawing 4 white = ${}^6c_4$ ways. When 4 white balls have been removed the bag contains 2 white and 9 black balls
 $\therefore$ No. of ways = ${}^9c_4$ ways
 If no restriction; (i) ${}^{15}c_4 \times {}^{15}c_4$ (ii) ${}^{15}c_4 \times {}^{11}c_4$

Task

1. A student has to answer 8 out of 10 questions in an examination.
 - a) How many choices have he?
 - b) How many if he must answer the first three questions
 - c) How many if he must answer at least 4 of the first five questions?

Solution

- a) 8 out of 10 can be selected in ${}^{10}C_8 = 45$ ways
- b) Since the first three questions are compulsory only 5 out of the remaining 7 questions must be selected. This can be done in ${}^7C_5 = 21$ ways
- c) **Case I:** 4 from the first 5 questions and 4 from the remaining 5 questions

Case II: 5 from the first 5 questions and 3 from the 3 remaining questions

$$\therefore \text{No. of ways} = {}^5C_4 \times {}^5C_4 + {}^5C_5 \times {}^5C_3 = 35$$

2. How many combinations can be formed of 8 counters marked 1, 2, 3, 4, 5, 6, 7, 8 taking them 4 at a time, there being at least one odd and one even counter in each combinations?

Solution

$${}^4C_3 \times {}^4C_1 + {}^4C_2 \times {}^4C_2 + {}^4C_1 \times {}^4C_3 = 68$$

3. Find the number of combinations of the words "ACCOMMODATION", "COLLEGE" taken 4 at a time. **ANSWER:** For ACCOMMODATION: **167**, COLLEGE, **${}^5C_4 + {}^2C_1 \times {}^4C_2 + {}^2C_2 = 18$**

4. i) ${}^nC_{10} = {}^nC_{12}$ ii) ${}^nC_7 = {}^nC_3$ iii) ${}^nC_5 + {}^nC_3 = 6$ **ANSWER:** (i) 22 (ii) 10

5. Find the value of r if

$$\text{a) } {}^np_r = 30240 \text{ and } {}^nc_r = 252$$

$$\text{ANSWER: } r = 5$$

$$\text{b) } {}^nc_r = {}^nc_{2r-5}$$

A committee of 4 men and 3 women is to be formed from 10 men and 8 women. In how many ways can the committee be formed? (Uneb 2011)

$$\text{ANSWER: } {}^{10}C_4 \times {}^8C_3 = 11\,760$$

POLYNOMIAL FUNCTIONS AND EQUATIONS

Polynomial functions of one variable

A Polynomial in the symbol x over the real numbers $\mathbb{R}$, is an algebraic expression that can be written:

$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$, where n is a non-negative integer, the *coefficient* a_i in $\mathbb{R}$, and $\mathbb{R} a_n \neq 0$

The number n is said to be the *degree* of a polynomial and a_n is called the *leading coefficient*.

A *polynomial function* $f(x)$ is an algebraic function that can be written as:

$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$, where x is a variable whose domain is the set of real number $\mathbb{R}$

Examples of polynomial functions

i) $f(x) = x^5 + 2x^3 + 1$, degree = 5

ii) $f(x) = 2x^3 + 5x + 1$, degree = 3

Examples of non-polynomial functions

a) $f(x) = \sqrt{x} + 2x^3 + 1$; $x^{\frac{1}{2}}$ non permissible

b) $r(x) = \frac{3}{2}ix^2 + 2$; $\frac{3}{2}i \notin \mathbb{R}$

Division of a polynomial by another polynomial

Two methods may be applied

- Long division
- Synthetic division

The polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$ can be divided by a DIVISOR to either get a QUOTIENT only or *quotient* + REMAINDER. The polynomial being divided is called a DIVIDEND.

LONG DIVISION

Steps followed are:

- 1) Arrange the dividend in descending powers of the variable
- 2) Insert with 0 coefficients any missing terms of less than degree n
- 3) Divide the 1st term of the divisor in to the 1st term of the dividend
- 4) Multiply the divisor by the result from step (3) above, line up like terms
- 5) Subtract as arithmetic and bring down now the remaining terms
- 6) Repeat the process until the degree of the remainder is less than that of the divisor.

NOTE:

Division of a polynomial by a divisor will be possible if the degree of polynomial is greater than that of divisor.

Examples

Divide

a) $f(x) = 5 + 4x^3 - 3x$ by $2x - 3$

b) $f(x) = x^6 - 2x^4 + 2$ by $x^2 - x - 2$

c) $f(x) = x^5 + 2x^3 + 2$ by $x^3 + x + 1$

Solution

a)

$2x - 3$	$2x^2 + 3x + 3$	}	$Quotient$
$2x - 3$	$4x^3 + 0x^2 - 3x + 5$	}	$Dividend$
	$4x^3 - 6x^2$		
	$6x^2 - 3x + 5$		
	$6x^2 - 9x$		
	$6x + 5$		
	$6x - 9$		
	14	}	$Remainder$

$$\therefore \frac{f(x)}{2x-3} = 2x^2 + 3x + 3 + \frac{14}{2x-3} \text{ or } f(x) = (2x^2 + 3x + 3)(2x - 3) + 14$$

In general, $Dividend = (Divisor)(Quotient) + Remainder$

NOTE:

- i) Divisor $2x - 3$ is *linear*
- ii) Dividend $4x^3 - 3x + 5$ is of degree $n = 3$
- iii) Quotient $2x^2 + 3x + 3$ is of degree $n - 1 = 2$
- iv) Remainder 14 is a *constant*

b)

$x^2 - x - 2$	$x^4 + x^3 + x^2 + 3x + 6$
$x^2 - x - 2$	$x^6 + 0x^5 - 2x^4 + 0x^3 + x^2 + 0x - 2$
	$x^6 - x^5 - 2x^4$
	$x^5 + 0x^4 + 0x^3 + x^2 + 0x$
	$x^5 - x^4 - 2x^3$
	$x^4 + 2x^3 + x^2 + 0x - 2$
	$x^4 - x^3 - 2x^2$
	$3x^3 + 3x^2 + 0x - 2$
	$3x^3 - 3x^2 - 6x$
	$6x^2 + 6x - 2$
	$6x^2 + 6x - 12$
	$12x + 10$

$$\therefore \frac{f(x)}{x^2-x-2} = x^4 + x^3 + x^2 + 3x + \frac{12x+10}{x^2-x-2}$$

NOTE:

- i) Divisor $x^2 - x - 2$ is a *Quadratic*
- ii) Dividend $x^6 - 2x^4 + 2$ by $x^2 - x - 2$ is of degree $n = 6$
- iii) Quotient $x^4 + x^3 + x^2 + 3x + 6$ is of degree $n - 2 = 4$
- iv) Remainder $12x + 10$ is *linear*

$$\begin{array}{r}
 x^4 + x^3 + x^2 + 3x + 6 \\
 \hline
 x^3 + 0x^2 - x - 2 \quad \begin{array}{l} x^5 + 0x^4 + 2x^3 + 0x^2 + 0x + 2 \\ x^5 + 0x^4 + x^3 + x^2 \\ \hline x^3 - x^2 + 0x + 2 \\ x^3 + 0x^2 + x + 1 \\ \hline -x^2 - x + 1 \end{array}
 \end{array}$$

c)

$$\therefore \frac{f(x)}{x^3+x+1} = x^2 + 1 + \frac{-x^2-x+1}{x^3+x+1}$$

NOTE:

- Divisor $x^3 + x + 1$ is a *Cubic*
- Dividend $x^5 + 2x^3 + 2$ is of degree $n = 5$
- Quotient $x^2 + 1$ is of degree $n - 3 = 2$
- Remainder $-x^2 - x + 1$ is *Quadratic*

Summary of the results:

Degree of Polynomial	Divisor	Degree of Quotient	Remainder
n	$(ax + b)$ <i>Linear</i>	$n - 1$	R <i>Constant</i>
	$(ax + b)^2$ or $(ax + b)(px + q)$ or $ax^2 + bx + c$ <i>Quadratic</i>	$n - 2$	$Ax + B$ <i>Linear</i>
	$(ax + b)^3$ or $(ax + b)(px + q)(mx + n)$ or $ax^3 + bx^2 + cx + d$ <i>Cubic</i>	$n - 3$	$Ax^2 + Bx + C$

The above table is important in finding the remainder basing on the type of the divisor. It is so much applied in remand theorem.

SYNTHETIC DIVISION

Key steps in synthetic division process

To divide $f(x) = \text{Polynomial}$ by $x - a$

- Arrange the coefficients of $f(x)$ in descending powers of x . write zero as the coefficient for each missing power
- After writing the divisor in the form $(x - a)$, use a to generate the 2^{nd} and 3^{rd} rows as follows: bring down the 1^{st} coefficient of the dividend and multiplying it by a ; then add the product to the 2^{nd} coefficient of the dividend. Multiplying this sum by a and add the product to the 3^{rd} coefficient of the dividend.
Repeat the procedure until a product is added to the constant term of the $f(x)$.
- The last number to the right in the 3^{rd} row of numbers is the remainder. The other numbers in the 3^{rd} row are the coefficients of the quotient which is of degree $< f(x)$.

Examples

Use synthetic division to find the quotient and remainder resulting from dividing;

a) $f(x) = 4x^5 - 30x^3 - 50x - 2$ by $x + 3$

b) $f(x) = 2x^4 + 3x^3 - x - 5$ by $x + 2$

Solution

a) Divisor = $x + 3 = x - (-3) \therefore a = -3$

Dividend = $4x^5 + 0x^4 - 30x^3 + 0x^2 - 50x - 2$

$$\begin{array}{r}
 1^{st} \text{ row: } 4 \quad 0 \quad -30 \quad 0 \quad -50 \quad -2 \\
 2^{nd} \text{ row: } \quad -12 \quad 36 \quad -18 \quad 54 \quad -12 \\
 \hline
 -3 \overline{) 4 \quad -12 \quad 6 \quad -18 \quad 4 \quad -14}
 \end{array}$$

$\therefore$ The quotient is: $4x^4 - 12x^3 + 6x^2 - 18x + 4$

The remainder is: -14

$\therefore \frac{f(x)}{x+3} = 4x^4 - 12x^3 + 6x^2 - 18x + 4 - \frac{14}{x+3}$

b) Divisor = $x + 2 = x - (-2) \therefore a = -2$

Dividend = $2x^4 + 3x^3 + 0x^2 - x - 5$

$$\begin{array}{r}
 2 \quad 3 \quad 0 \quad -1 \quad - \\
 \quad -4 \quad 2 \quad -4 \quad 10 \\
 \hline
 -2 \overline{) 2 \quad -1 \quad 2 \quad -} \quad 10
 \end{array}$$

Quotient Coefficient

Remainder

$\therefore$ The quotient is: $2x^3 - x^2 + 2x - 5$

The remainder is: 5

$\therefore \frac{f(x)}{x+2} = 2x^3 - x^2 + 2x - 5 - \frac{5}{x+2}$

THE REMAINDER AND FACTOR THEOREM
Remainder Theorem

If $f(x)$ is a polynomial of degree n , $n > 0$ and if c is a number, then the remainder in division of $f(x)$ by $(x - c)$ is $f(c)$. In symbols,

$$f(x) = (x - c).Q(x) + f(c), \text{ where } Q(x) \text{ is a quotient of degree } n - 1$$

Proof:

Dividing $f(x)$ by $(x - c)$, you obtain a quotient and a remainder r (a real number)

$$\Rightarrow \frac{f(x)}{(x-c)} = Q(x) + \frac{r}{(x-c)}$$

$$\therefore f(x) = Q(x).(x - c) + r$$

This is an identity, true for all values of x .

If $x = c$, then $f(xc) = Q(c).(c - c) + r$

$$\therefore r = f(c)$$

NOTE:

The theorem is stated in terms of $(x - c)$, the *difference* between x and the number c .

The factor Theorem

A number c is a zero of polynomial function f of degree $n, n > 0$, if and only if $x - c$ is a factor of $f(x)$.

Proof:

From the remainder theorem $f(x) = Q(x) \cdot (x - c) + r$, if $f(c) = 0$

(Remainder $f(c) = 0$), then $f(x) = (x - c) \cdot Q(x)$.

Consequently $(x - c)$ is a factor of $f(x)$.

Now from $f(x) = (x - c) \cdot Q(x)$, for $x = c$ then $f(c) = Q(c) \cdot (c - c) = 0$. Thus for a polynomial function $f(x)$, if $x - c$ is its factor, then $f(c) = 0$. i.e. remainder is zero.

To test whether or not a linear polynomial $mx + b, m \neq 0$ is a factor of $f(x)$ you write;

$$mx + b = m \left(x + \frac{b}{m} \right) = m \left[x - \left(-\frac{b}{m} \right) \right]$$

, and see whether $f \left(-\frac{b}{m} \right) = 0$

By factor theorem, $mx + b$ is a factor of $f(x)$ if and only if $f \left(-\frac{b}{m} \right) = 0$

Examples

1. Find the remainder when $x^5 + x - 9$ is divided by;

a) $x + 2$

b) $x - 3$

Solution

Let $f(x) = x^5 + x - 9$

a) $x + 2 = x - (-2)$

$$\therefore f(-2) = (-2)^5 + (-2) - 9 = -43$$

b) $x - 3$

$$\therefore f(3) = (3)^5 + (3) - 9 = 239$$

2. Find the remainder when $4x^3 - 5x + 4$ when divided by;

a) $2x - 1$

b) $2x + 3$

Solution

Let $f(x) = 4x^3 - 5x + 4$

a) $2x - 1 = 2 \left(x - \frac{1}{2} \right)$

$$\therefore f \left(\frac{1}{2} \right) = 4 \left(\frac{1}{2} \right)^3 - 5 \left(\frac{1}{2} \right) + 4 = 2$$

$$\therefore f \left(-\frac{3}{2} \right) = 4 \left(-\frac{3}{2} \right)^3 - 5 \left(-\frac{3}{2} \right) + 4 = -2$$

3. Find the value of constant a if;

i) $x^3 - 3x^2 + ax + 5$ has a remainder 17 when divided by $x - 3$

ii) $x^5 + 4x^4 - 6x^2 + ax + 2$ has a remainder 6 when divided by $x + 2$

iii) $4x^3 - ax + 5$ has a remainder $2 \frac{1}{2}$ when divided by $2x - 1$

iv) $3x^3 + ax^2 - 6x + 3$ has a remainder 5 when divided by $3x + 1$

Solution

i) Let $f(x) = x^3 - 3x^2 + ax + 5$, Remainder = 17 when divided by $x - 3$

$$\therefore \text{Remainder} = f(3) = 17$$

$$\Rightarrow (3)^3 - 3(3)^2 + a(3) + 5 = 17$$

- $\therefore 3a + 5 = 17 \therefore a = 4$
- ii) Let $f(x) = x^5 + 4x^4 - 6x^2 + ax + 2$, Remainder = 6 when divided by $x + 2$;
 Divisor = $x + 2 = x - (-2)$
 Remainder = $f(-2) = 6$
 $\Rightarrow (-2)^5 + 4(-2)^4 - 6(-2)^2 + a(-2) + 2 = 6$
 $\therefore -2a + 10 = 6 \therefore a = 2$
- iii) Let $f(x) = 4x^3 - ax + 5$, Remainder = $2\frac{1}{2}$ when divided by $2x - 1$
 Divisor = $2x - 1 = 2\left(x - \frac{1}{2}\right)$
 Remainder = $f\left(\frac{1}{2}\right) = 2\frac{1}{2}$
 $\Rightarrow 4\left(\frac{1}{2}\right)^3 - a\left(\frac{1}{2}\right) + 5 = 2\frac{1}{2}$
 $\therefore \frac{a}{2} + \frac{11}{2} = \frac{5}{2} \therefore a = -6$
- iv) Let $f(x) = 3x^3 + ax^2 - 6x + 3$, Remainder = 5 when divided by $3x + 1$
 Divisor = $3x + 1 = 3\left(x + \frac{1}{3}\right) = 3\left[x - \left(-\frac{1}{3}\right)\right]$
 Remainder = $f\left(-\frac{1}{3}\right) = 5$
 $\Rightarrow 3\left(-\frac{1}{3}\right)^3 + a\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 3 = 5$
 $\therefore \frac{a}{9} + \frac{46}{9} = 5 \therefore a = -1$
4. Show that;
- $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$
 - $2x^3 + x^2 - 8x - 4$ is divisible by $2x + 1$
 - $x + 2$ and $x + 1$ are factors of $2x^3 + 5x^2 + x - 2$

Solution

- i) Let $f(x) = 12x^3 + 16x^2 - 5x - 3$
 $f(x)$ is divisible by $2x - 1$ if Remainder = 0
 Remainder = $f\left(\frac{1}{2}\right) = 0 \left[\because 2x - 1 = 2\left(x - \frac{1}{2}\right)\right]$
 $\Rightarrow f\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^3 + 16\left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) - 3 = 0$
 $\therefore 2x - 1$ is a factor of $12x^3 + 16x^2 - 5x - 3$ or $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$
- ii) Let $f(x) = 2x^3 + x^2 - 8x - 4$
 $2x + 1 = 2\left[x - \left(-\frac{1}{2}\right)\right]$ is a factor of $f(x)$ if Remainder = 0
 Remainder = $f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 - 8\left(-\frac{1}{2}\right) - 4 = 0$
 $\therefore 2x + 1$ is a factor of $2x^3 + x^2 - 8x - 4$
- iii) Let $f(x) = 2x^3 + 5x^2 + x - 2$
 $x + 2 = x - (-2)$ is a factor of $f(x)$ if Remainder = $f(-2) = 0$
 $\Rightarrow f(-2) = 2(-2)^3 + 5(-2)^2 + (-2) - 2 = 0$. Hence $x + 2$ is a factor.
 Also $x + 1 = x - (-1)$ is a factor of $f(x)$ if Remainder = $f(-1) = 0$
 $\Rightarrow f(-1) = 2(-1)^3 + 5(-1)^2 + (-1) - 2 = 0$. Hence $x + 1$ is a factor.

NOTE:

For all the above examples the factors and divisors used are all *linear*. Now let us look at examples of non-linear divisors.

Examples

1. Find the remainder when;

- $x^4 - 5x^3 + 6x^2 - 7$ is divided by $(x - 1)(x - 3)$
- $x^4 + x^2 - 7$ is divided by $x^2 - 4$
- $x^5 - 7x^3 + 4x - 2$ is divided by $(x + 1)(x - 1)(x - 3)$
- $x^5 + x^2 - 4x - 3$ is divided by $(x^2 - 4)(x + 1)$

Solution

- Let $f(x) = x^4 - 5x^3 + 6x^2 - 7$
 Divisor = $(x - 1)(x - 3)$. i. e. Quadratic divisor.
 Let Remainder = $Ax + B$. i. e. Linear remainder.
 $\Rightarrow f(x) = (x - 1)(x - 3).Q(x) + Ax + B$
 For $x = 1$
 $\Rightarrow f(1) = (1 - 1)(1 - 3).Q(1) + A(1) + B = A + B \therefore f(1) = A + B$
 Also from $f(x) = x^4 - 5x^3 + 6x^2 - 7 \Rightarrow f(1) = (1)^4 - 5(1)^3 + 6(1)^2 - 7 = -5$
 $\therefore f(1) = -5$
 $\therefore A + B = -5$ (1)
 Also for $x = 3$
 $\Rightarrow f(3) = (3 - 1)(3 - 3).Q(3) + A(3) + B = 3A + B \therefore f(3) = 3A + B$
 Now from $f(x) = x^4 - 5x^3 + 6x^2 - 7 \Rightarrow f(3) = (3)^4 - 5(3)^3 + 6(3)^2 - 7 = -7$
 $\therefore f(3) = -7$
 $\therefore 3A + B = -7$ (2)
 Solving (1) and (2) simultaneously, eqn. (1) - eqn. (2)
 $\Rightarrow -2A = 2 \therefore A = -1$
 From eqn. (1), $B = -5 - A = -5 + 1 = -4$
 $\therefore$ Remainder = $-x - 4$
- Let $f(x) = x^4 + x^2 - 7$
 Divisor = $(x^2 - 4) = (x + 2)(x - 2)$. i. e. Quadratic divisor.
 Let Remainder = $Ax + B$. i. e. Linear remainder.
 $\Rightarrow f(x) = (x + 2)(x - 2).Q(x) + Ax + B$ or $x^4 + x^2 - 7 = (x + 2)(x - 2).Q(x) + Ax + B$
 For $x = 2$
 $f(2) = 2A + B = (2)^4 + (2)^2 - 7 = 13$
 $\therefore 2A + B = 13$ (1)
 For $x = -2$
 $f(-2) = -2A + B = (-2)^4 + (-2)^2 - 7 = 13$
 $\therefore -2A + B = 13$ (2)
 Solving (1) and (2) simultaneously, eqn. (1) + eqn. (2)
 $\Rightarrow 2B = 26 \therefore B = 13$
 From eqn. (1), $B = \frac{1}{2}(13 - B) = \frac{1}{2}(13 - 13) = 0$
 $\therefore$ Remainder = 13
- Let $f(x) = x^5 - 7x^3 + 4x - 2$
 Divisor = $(x + 1)(x - 1)(x - 3)$. i. e. Cubic divisor.
 Let Remainder = $Ax^2 + Bx + C$. i. e. Quadratic remainder.
 $\Rightarrow f(x) = (x + 1)(x - 1)(x - 3).Q(x) + Ax^2 + Bx + C = x^5 - 7x^3 + 4x - 2$
 For $x = 1$
 $f(1) = A + B + C = (1)^5 - 7(1)^3 + 4(1) - 2 = -4$
 $\therefore A + B + C = -4$ (1)

For $x = -1$

$$f(-1) = A - B + C = (-1)^5 - 7(-1)^3 + 4(-1) - 2 = 0$$

$$\therefore A - B + C = 0 \dots\dots\dots (2)$$

For $x = 3$

$$f(3) = 9A + 3B + C = (3)^5 - 7(3)^3 + 4(3) - 2 = 64$$

$$\therefore 9A + 3B + C = 64 \dots\dots\dots (3)$$

Solving the equations simultaneously

Eqn. (1) – eqn. (2) to eliminate C

$$\begin{array}{r} A + B + C = -4 \\ - \quad A - B + C = 0 \\ \hline 2B = -4 \quad \therefore B = -2 \end{array}$$

Eqn. (1) – eqn. (3) to eliminate C

$$\begin{array}{r} A - B + C = 0 \\ 9A + 3B + C = 64 \\ \hline -8A - 4B = -64 \end{array}$$

From $B = -2$, $-8A = -64 + 4B = -64 - 8 = -72$

$$\therefore A = 9$$

From eqn. (1), $A + B + C = -4$

$$\Rightarrow 9 - 2 + C = -4 \therefore C = -11$$

$$\therefore \text{Remainder} = 9x^2 - 2x - 11$$

d) Let $f(x) = x^5 + x^2 - 4x - 3$

Divisor $= (x + 1)(x^2 - 4) = (x + 1)(x - 2)(x + 2)$. i. e. *Cubic divisor*.

Let Remainder $= Ax^2 + Bx + C$. i. e. *Quadratic remainder*.

$$\Rightarrow f(x) = (x + 2)(x - 2)(x + 1). Q(x) + Ax^2 + Bx + C = x^5 + x^2 - 4x - 3$$

For $x = 2$

$$f(2) = 4A + 2B + C = (2)^5 + (2)^2 - 4(2) - 3 = 25$$

$$\therefore 4A + 2B + C = 25 \dots\dots\dots (1)$$

For $x = -2$

$$f(-2) = 4A - 2B + C = (-2)^5 + (-2)^2 - 4(-2) - 3 = -23$$

$$\therefore 4A - 2B + C = -23 \dots\dots\dots (2)$$

For $x = -1$

$$f(-1) = 4A - 2B + C = (-1)^5 + (-1)^2 - 4(-1) - 3 = 1$$

$$\therefore A - B + C = 1 \dots\dots\dots (3)$$

Solving the equations simultaneously

Eqn. (1) – eqn. (2) to eliminate C

$$\begin{array}{r} 4A + 2B + C = 25 \\ - \quad 4A - 2B + C = -23 \\ \hline 4B = 48 \end{array}$$

$$\therefore B = 12$$

Eqn. (2) – eqn. (3) to eliminate C

$$\begin{array}{r} 4A + 2B + C = 25 \\ - \quad A - B + C = 1 \\ \hline 3A - B = -24 \end{array}$$

$$\text{From } B = 12, 3A = B - 24 = 12 - 24 = -12$$

$$\therefore A = -4$$

$$\text{From eqn. (3), } A - B + C = 1$$

$$\Rightarrow -4 - 12 + C = 1 \therefore C = 17$$

$$\therefore \text{Remainder} = -4x^2 + 12x + 17$$

NOTE:

If the divisor is not factorisable to linear factors, you can use long division to find the remainder.

2. Find the quotient when;

a) $2x^3 - 3x^2 + 2x - 4$ is divided by $x - 2$

b) $x^4 - x^3 + x - 5$ is divided by $x + 1$

Solution

a) Let $f(x) = 2x^3 - 3x^2 + 2x - 4$

$$\Rightarrow f(x) = (x - 2).Q(x) + R; R = f(2) = \text{constant}$$

$$\text{Now } f(2) = 2(2)^3 - 3(2)^2 + 2(2) - 4 = 4$$

$$\therefore f(x) = (x - 2).Q(x) + 4$$

$$\Rightarrow \frac{f(x)-4}{x-2} = Q(x) \text{ . Here } Q(x) \text{ has no remainder.}$$

$$\therefore Q(x) = \frac{2x^3-3x^2+2x-4-4}{x-2} = \frac{2x^3-3x^2+2x-8}{x-2}$$

Using long division,

$$\begin{array}{r} \overline{2x^2 + x + 4} \\ x-2 \overline{2x^3 - 3x^2 + 2x - 8} \\ \underline{2x^3 - 4x^2} \\ x^2 + 2x - 8 \\ \underline{x^2 - 2x} \\ 4x - 8 \\ \underline{4x - 8} \\ 0 \end{array} \quad \therefore Q(x) = 2x^2 + x + 4$$

Alternatively

$$\text{Let } f(x) = 2x^3 - 3x^2 + 2x - 4, \text{ Degree: } n = 3$$

$$\text{Divisor} = x - 2$$

$$\therefore f(x) = (x - 2).Q(x) + \text{Remainder}$$

$$\text{But Remainder} = f(2) = 2(2)^3 - 3(2)^2 + 2(2) - 4 = 4$$

$$\therefore f(x) = (x - 2).Q(x) + 4$$

Since the quotient has degree $n - 1 = 3 - 1 = 2$, then the quotient must be a quadratic. .i.e. $Ax^2 + Bx + C$

$$\Rightarrow f(x) = (x - 2).(Ax^2 + Bx + C) + 4$$

$$\therefore 2x^3 - 3x^2 + 2x - 4 \equiv (x - 2).(Ax^2 + Bx + C) + 4$$

$$\equiv Ax^3 + Bx^2 + Cx - 2Ax^2 - 2Bx - C + 4$$

$$\equiv Ax^3 + x^2(B - 2A) + x(C - 2B) + (4 - 2C)$$

Comparing coefficients;

$$\text{For } x^3 : 2 = A$$

$$\text{For } x^2 : -3 = B - 2A; B = -3 + 2A = -3 + 4 = 1$$

$$\text{For } x : 2 = C - 2B \Rightarrow C = 2 + 2B = 2 + 2 = 4$$

$$\text{For constants: } -4 = 4 - 2C \Rightarrow 2C = 8 \therefore C = 4$$

$$\therefore Q(x) = 2x^2 + x + 4$$

- b) Let $f(x) = x^4 - x^3 + x - 5$; $Divisor = x + 1$
 $\Rightarrow f(x) = (x + 1).Q(x) + R$; $R = f(-1) = \text{constant}$
 Now $f(-1) = (-1)^4 - (-1)^3 + (-1) - 5 = -4$
 $\therefore f(x) = (x + 1).Q(x) - 4$
 $\Rightarrow \frac{f(x)+4}{x+1} = Q(x)$. Here $Q(x)$ has no remainder.
 $\therefore Q(x) = \frac{x^4 - x^3 + x - 5 + 4}{x+1} = \frac{x^4 - x^3 + x - 1}{x+1} = \frac{x^4 - x^3 + 0x^2 + x - 1}{x+1}$

By long division.

$$\begin{array}{r}
 \overline{x^3 - 2x^2 + 2x - 1} \\
 x+1 \overline{x^4 - x^3 + 0x^2 + x - 1} \\
 \underline{x^4 + x^3} \\
 -2x^3 + 0x^2 + x - 1 \\
 \underline{-2x^3 - 2x^2} \\
 2x^2 + x - 1 \\
 \underline{2x^2 + 2x} \\
 -x - 1 \\
 \underline{-x - 1} \\
 0
 \end{array}
 \quad \therefore Q(x) = x^3 - 2x^2 + 2x - 1$$

Alternatively

Let $f(x) = x^4 - x^3 + x - 5$, Degree: $n = 4$

Divisor $= x + 1$

$\therefore f(x) = (x + 1).Q(x) + \text{Remainder}$

But $\text{Remainder} = f(-1) = (-1)^4 - (-1)^3 + (-1) - 5 = -4$

$\therefore f(x) = (x + 1).Q(x) - 4$

Since the quotient has degree $n - 1 = 4 - 1 = 3$, then the quotient must be a cubic. .i.e. $Ax^3 + Bx^2 + Cx + D$

$\Rightarrow f(x) = (x + 1).(Ax^3 + Bx^2 + Cx + D) - 4$

$\therefore x^4 - x^3 + x - 5 \equiv (x + 1).(Ax^2 + Bx + C) - 4$

$$\equiv Ax^4 + Ax^3 + Bx^3 + Bx^2 + Cx^2 + Cx + Dx + D - 4$$

$$\equiv Ax^4 + x^3(A + B) + x^2(B + C) + x(C + D) + D - 4$$

Comparing coefficients;

For x^4 : $1 = A$

For x^3 : $-1 = A + B \Rightarrow B = -1 - A = -1 - 1 = -2$

For x^2 : $0 = B + C \Rightarrow C = -B = -(-2) = 2$

For x : $1 = C + D \Rightarrow D = 1 - C = 1 - 2 = -1$

For constants: $-5 = D - 4 \Rightarrow D = -1$

$\therefore Q(x) = x^3 - 2x^2 + 2x - 1$

NOTE:

To determine the quotient using method II, given the polynomial function and divisor, first establish the degree of the quotient basing on that of the divisor, then determine the coefficients as presented as above.

3. Determine the quotient when;

a) $x^4 + x^2 - 7$ is divided by $x^2 - 4$

b) $x^5 - 7x^3 + 4x - 2$ is divided by $(x^2 - 1)(x - 3)$

Solution

- a) Let $f(x) = x^4 + x^2 - 7$, Degree: $n = 4$

$$\text{Divisor} = x^2 - 4$$

$$\therefore f(x) = (x^2 - 4).Q(x) + \text{Remainder}$$

$$\text{Let the remainder} = Ax + B$$

$$\Rightarrow f(x) = (x^2 - 4).Q(x) + Ax + B$$

$$\therefore f(x) = (x + 2)(x - 2).Q(x) + Ax + B$$

$$\text{Or } x^4 + x^2 - 7 \equiv (x + 2)(x - 2).Q(x) + Ax + B$$

First obtaining the remainder,

$$\text{For } x = 2$$

$$\Rightarrow (2)^4 + (2)^2 - 7 \equiv (2 + 2)(2 - 2).Q(2) + A(2) + B$$

$$\therefore 2A + B = 13 \dots\dots\dots (1)$$

$$\text{For } x = -2$$

$$\Rightarrow (-2)^4 + (-2)^2 - 7 \equiv (-2 + 2)(2 - 2).Q(-2) + A(-2) + B$$

$$\therefore -2A + B = 13 \dots\dots\dots (1)$$

Solving eqn. (1) and eqn. (2) simultaneously,

eqn. (1) + eqn. (2) to eliminate A

$$2B = 26 \therefore B = 13$$

$$\text{From eqn. (1), } A = \frac{1}{2}(13 - B) = \frac{1}{2}(0) = 0$$

$$\text{Remainder} = 13$$

$$\text{Now } f(x) = (x + 2)(x - 2).Q(x) + 13$$

Since the degree of $f(x)$ is $n = 4$ and the divisor is of degree $n - 2 = 2$, then the quotient is a quadratic .i.e. $mx^2 + nx + p$

$$\text{Let the quotient } Q(x) = mx^2 + nx + p$$

$$\Rightarrow f(x) = (x + 2)(x - 2).(mx^2 + nx + p) + 13$$

$$\begin{aligned} \therefore x^4 + 0x^3 + x^2 - 7 &\equiv (x + 2)(x - 2).(mx^2 + nx + p) + 13 \\ &\equiv (x^2 - 4)(mx^2 + nx + p) + 13 \\ &\equiv mx^4 + nx^3 + px^2 - 4mx^2 - 4nx - 4p + 13 \\ &\equiv mx^4 + nx^3 + x^2(p - 4m) - 4nx + (-4p + 13) \end{aligned}$$

Comparing coefficients;

$$\text{For } x^4 : 1 = m$$

$$\text{For } x^3 : 0 = n$$

$$\text{For } x^2 : 1 = p - 4m \Rightarrow p = 1 + 4m = 1 + 4 = 5$$

$$\text{For } x : 0 = -4n \therefore n = 0$$

$$\text{For constants: } -7 = 13 - 4p \therefore p = 5$$

$$\therefore Q(x) = x^2 + 5$$

b) Let $f(x) = x^5 - 7x^3 + 4x - 2$, Degree: $n = 5$

$$\text{Divisor} = (x^2 - 1)(x - 3)$$

$$\therefore f(x) = (x^2 - 1)(x - 3).Q(x) + \text{Remainder}$$

$$\text{Let the remainder} = px^2 + qx + r$$

$$\therefore x^5 - 7x^3 + 4x - 2 \equiv (x^2 - 1)(x - 3).Q(x) + px^2 + qx + r$$

$$\text{Or } x^5 - 7x^3 + 4x - 2 \equiv (x + 1)(x - 1)(x - 3).Q(x) + px^2 + qx + r$$

First obtaining the remainder,

$$\text{For } x = 1$$

$$(1)^5 - 7(1)^3 + 4(1) - 2 \equiv (1 + 1)(1 - 1)(1 - 3).Q(1) + p(1)^2 + q(1) + r$$

$$\therefore p + q + r = -4 \dots\dots\dots (1)$$

$$\text{For } x = -1$$

$$\begin{aligned} (-1)^5 - 7(-1)^3 + 4(-1) - 2 &\equiv (-1 + 1)(-1 - 1)(-1 - 3).Q(-1) + p(-1)^2 + \\ &+ q(-1) + r \end{aligned}$$

$$\therefore p - q + r = 0 \dots\dots\dots (2)$$

For $x = 3$

$$(3)^5 - 7(3)^3 + 4(3) - 2 \equiv (3 + 1)(3 - 1)(3 - 3). Q(3) + p(3)^2 + q(3) + r$$

$$\therefore 9p + 3q + r = 64 \dots\dots\dots (3)$$

Solving eqn. (1), eqn. (2) and eqn. (3) simultaneously,

eqn. (1) - eqn. (2) to eliminate r

$$\Rightarrow 2q = -4 \therefore q = -2$$

eqn. (2) - eqn. (3) to eliminate r

$$\Rightarrow -8p - 4q = -64, q = -2$$

$$\therefore -8p + 8 = -64, p = 9$$

$$\text{From eqn. (2), } p - q + r = 0 \Rightarrow r = q - p = -2 - 9 = -11$$

$$\therefore \text{Remainder} = 9x^2 - 2x - 11$$

$$\text{Now } \therefore f(x) = (x^2 - 1)(x - 3).Q(x) + 9x^2 - 2x - 11$$

Since the degree of $f(x)$ is $n = 5$ and the divisor is of degree $n - 3 = 2$, then the quotient is a quadratic . i.e. $Ax^2 + Bx + C$

Let the quotient $Q(x) = Ax^2 + Bx + C$

$$\Rightarrow f(x) = (x^2 - 1)(x - 3).(Ax^2 + Bx + C) + 9x^2 - 2x - 11$$

$$\therefore f(x) \equiv (x^2 - 1)(x - 3).(Ax^2 + Bx + C) + 9x^2 - 2x - 11$$

$$\equiv (x^3 - 3x^2 - x + 3)(Ax^2 + Bx + C) + 9x^2 - 2x - 11$$

$$\equiv Ax^5 + Bx^4 + Cx^3 - 3Ax^4 - 3Bx^3 - 3Cx^2 - Ax^3 - Bx^2 - Cx + 3Ax^2 + 3Bx + 3C + 9x^2 - 2x - 11$$

$$\therefore f(x) \equiv Ax^5 + x^4(B - 3) + x^3(C - 3B - A) + x^2(-3C - B + 3A + 9) + x(-C + 3B - 2) + 3C - 11$$

$$\text{Given } f(x) = x^5 + 0x^4 - 7x^3 + 0x^2 + 4x - 2$$

Comparing coefficients,

$$\text{For } x^5 : 1 = A$$

$$\text{For } x^4 : 0 = B - 3 \Rightarrow B = 3$$

$$\text{For } x^3 : -7 = C - 3B - A \Rightarrow C = -7 + 9 + 1 = 3$$

$$\text{For } x^2 : 0 = -3C - B + 3A + 9 \Rightarrow C = \frac{-3+3+9}{3} = 3 \text{ As before}$$

$$\text{For } x : 4 = -C + 3B - 2 \Rightarrow C = 9 - 2 - 4 = 3 \text{ As before}$$

$$\text{For constants: } -2 = 3C - 11 \Rightarrow C = 3 \text{ As before}$$

$$\therefore Q(x) = x^2 + 3x + 3$$

NOTE:

The above examples on determination of remainder and quotient when a polynomial is divided by a divisor using the above methods can still be done using long division.

4. Find the constants p, q, r if;

- $x^3 - px + q$ is divided by $x^2 - 3x + 2$ leaves a remainder $4x - 1$
- $px^4 + qx^3 + 3x^2 - 2x + 3$ has a remainder $x + 1$ when divided by $x^2 - 3x + 2$
- $(x - 1)$ and $(x + 1)$ are factors of the expression $x^3 + px^2 + qx + r$ and it leaves a remainder of 12 when divided by $x - 2$

Solution

a) Let $f(x) = x^3 - px + q$

$$\text{Divisor} = x^2 - 3x + 2 = (x - 1)(x - 2)$$

$$\text{Remainder} = 4x - 1$$

$$\Rightarrow f(x) \equiv Q(x) \cdot (x-1)(x-2) + 4x - 1$$

$$\therefore x^3 - px + q \equiv Q(x) \cdot (x-1)(x-2) + 4x - 1$$

$$\text{For } x = 1, (1)^3 - p(1) + q \equiv Q(1) \cdot (1-1)(1-2) + 4(1) - 1$$

$$\Rightarrow 1 - p + q = 3 \therefore q - p = 2 \dots\dots\dots (1)$$

$$\text{For } x = 2, (2)^3 - p(2) + q \equiv Q(2) \cdot (2-1)(2-2) + 4(2) - 1$$

$$\Rightarrow 8 - 2p + q = 7 \therefore q - 2p = -1 \dots\dots\dots (2)$$

Solving eqn. (1) and eqn. (2) simultaneously

eqn. (1) - eqn. (2), eliminate q

$$\Rightarrow p = 3$$

$$\text{From eqn. (1), } q - p = 2 \Rightarrow q = 2 + 3 = 5$$

$$\therefore p = 2, q = 5$$

Alternatively

By long division

$$\begin{array}{r} x^2 - 3x + 2 \overline{) x^3 + 0x^2 - px + q} \\ \underline{x^3 - 3x^2 + 2x} \\ -3x^2 + (-p-2)x + q \\ \underline{3x^2 - 9x + 6} \\ (-p+7)x + q - 6 \end{array}$$

Comparing the remainder,

$$(-p+7)x + q - 6 \equiv 4x - 1$$

$$\Rightarrow -p+7 = 4 \therefore p = 3$$

$$\Rightarrow q - 6 = -1 \therefore q = 5$$

b) Let $f(x) = px^4 + qx^3 + 3x^2 - 2x + 3$

$$\text{Divisor} = x^2 - 3x + 2 = (x-1)(x-2)$$

$$\text{Remainder} = x + 1$$

$$\Rightarrow f(x) \equiv Q(x) \cdot (x-1)(x-2) + x + 1$$

$$\therefore px^4 + qx^3 + 3x^2 - 2x + 3 \equiv Q(x) \cdot (x-1)(x-2) + x + 1$$

$$\text{For } x = 1, p(1)^4 + q(1)^3 + 3(1)^2 - 2(1) + 3 \equiv Q(1) \cdot (1-1)(1-2) + 1 + 1$$

$$\Rightarrow p + q + 3 - 2 + 3 = 2 \therefore p + q = -2 \dots\dots\dots (1)$$

$$\text{For } x = 2, p(2)^4 + q(2)^3 + 3(2)^2 - 2(2) + 3 \equiv Q(2) \cdot (2-1)(2-2) + 2 + 1$$

$$\Rightarrow 16p + 8q + 12 - 4 + 3 = 3$$

$$\therefore 16p + 8q = -8 \text{ or } 2p + q = -1 \dots\dots\dots (2)$$

Solving eqn. (1) and eqn. (2) simultaneously

eqn. (1) - eqn. (2), eliminate q

$$\Rightarrow -p = -1 \therefore p = 1$$

$$\text{From eqn. (1), } p + q = -2 \Rightarrow q = -2 - 1 = -3$$

$$\therefore p = 1, q = -3$$

c) Let $f(x) = x^3 + px^2 + qx + r$

$$\text{Divisor} = (x-1), \text{Remainder} = 0$$

$$\text{Divisor} = (x+1), \text{Remainder} = 0$$

$$\text{Divisor} = (x-2), \text{Remainder} = 12$$

$$\Rightarrow f(x) \equiv Q(x) \cdot (x-1)(x+1)$$

$$\therefore x^3 + px^2 + qx + r \equiv Q(x) \cdot (x-1)(x+1)$$

$$\text{For } x = 1, (1)^3 + p(1)^2 + q(1) + r \equiv Q(1) \cdot (1-1)(1+1)$$

$$\Rightarrow p + q + r = -1 \dots\dots\dots (1)$$

$$\text{For } x = -1, (-1)^3 + p(-1)^2 + q(-1) + r \equiv Q(-1) \cdot (-1-1)(-1+1)$$

$$\Rightarrow p - q + r = 1 \dots\dots\dots (2)$$

$$\Rightarrow f(x) \equiv H(x) \cdot (x-2) + 12$$

$$\therefore x^3 + px^2 + qx + r \equiv H(x).(x - 2) + 12$$

$$\text{For } x = 2, (2)^3 + p(2)^2 + q(2) + r \equiv H(2).(2 - 2) + 12$$

$$\Rightarrow 4p + 2q + r = 4 \dots\dots\dots (3)$$

Eliminating r from eqn. (1) and eqn. (2), eqn. (1) - eqn. (2)

$$2q = -2 \therefore q = -1$$

Eliminating r from eqn. (2) and eqn. (3), eqn. (2) - eqn. (3)

$$-3p - 3q = -3 \text{ or } p + q = 1$$

$$\text{But } q = -1 \therefore p = 1 - q = 1 + 1 = 2$$

$$\text{From (1) } r = -1 - p - q = -1 - 2 + 1 = -2$$

$$\therefore p = 2, q = -1, r = -2$$

5. Given the polynomial $f(x) = Q(x).g(x) + R(x)$ where $Q(x)$ is the quotient, $g(x) = (x - \alpha)(x - \beta)$ and $R(x)$ is the remainder, show that

$$R(x) = \frac{(x-\beta)f(\alpha) + (\alpha-x)f(\beta)}{\alpha-\beta} \text{ when } f(x) \text{ is divided by } g(x).$$

Hence find the remainder when $f(x)$ is divided by $x^2 - 9$ given that $f(x)$ divided by $x - 3$ and when divided by $x + 3$, the remainders are respectively 2 and -3.

Solution

$$f(x) = Q(x).g(x) + R(x)$$

$$\text{Divisor} = g(x) = (x - \alpha)(x - \beta)$$

$$\text{Remainder} = R(x) = Ax + B$$

$$\Rightarrow f(x) = Q(x).(x - \alpha)(x - \beta) + Ax + B$$

Now for $x = \alpha$

$$f(\alpha) = A\alpha + B \dots\dots\dots (1)$$

Now for $x = \beta$

$$f(\beta) = A\beta + B \dots\dots\dots (2)$$

Since $f(x)$ is known, $f(\alpha)$ and $f(\beta)$ will be known. Now finding the unknowns A and B,

Eliminating B from eqn. (1) and eqn. (2), eqn. (1) - eqn. (2)

$$\Rightarrow f(\alpha) - f(\beta) = A(\alpha - \beta)$$

$$\therefore A = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \dots\dots\dots (3)$$

From eqn. (1), $f(\alpha) = A\alpha + B \Rightarrow B = f(\alpha) - A\alpha$

$$\Rightarrow B = f(\alpha) - \alpha \left[\frac{f(\alpha) - f(\beta)}{\alpha - \beta} \right] = \frac{\alpha f(\alpha) - \beta f(\alpha) - \alpha f(\alpha) + \alpha f(\beta)}{\alpha - \beta}$$

$$\therefore B = \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta} \dots\dots\dots (4)$$

$\therefore R(x) = Ax + B$ becomes

$$R(x) = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} x + \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta} = \frac{xf(\alpha) - xf(\beta) + \alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta}$$

$$\therefore R(x) = \frac{(x-\beta)f(\alpha) + (\alpha-x)f(\beta)}{\alpha-\beta}, \text{ as required.}$$

From $f(x) = Q(x).(x - \alpha)(x - \beta) + Ax + B$

$$\text{Divisor} = g(x) = (x - \alpha)(x - \beta) \equiv x^2 - 9 = (x - 3)(x + 3)$$

By comparison, $\alpha = 3, \beta = -3$

$$\text{Given: } f(x) = G(x).(x - 3) + 2 \Rightarrow f(3) = 2 \equiv f(\alpha)$$

$$f(x) = H(x).(x + 3) - 3 \Rightarrow f(-3) = -3 \equiv f(\beta)$$

, where $G(x)$ and $H(x)$ are the new quotients due to different divisors.

$$\text{Remainder} = Ax + B \text{ is given by } \frac{(x-\beta)f(\alpha) + (\alpha-x)f(\beta)}{\alpha-\beta}$$

$$\Rightarrow R(x) = \frac{(x-3)(2)+(3-x)(-3)}{3-(-3)} = \frac{5x-3}{6}$$

$\therefore$ Remainder when $f(x)$ is divided by $x^2 - 9$ is $\frac{5x-3}{6}$

6. A polynomial $P(x)$ leaves a remainder 6 when divided by $x^2 - 4$ and a remainder $x + a$ when divided by $x^2 - x - 2$. Find;

- a) The value of the constant a
- b) The remainder when $P(x)$ is divided by;
 - i) $x^2 + 3x + 2$
 - ii) $x^3 + x^2 - 4x - 4$

Solution

Case I:

$$\begin{aligned} P(x) &= Q(x).(x^2 - 4) + 6 \\ &= Q(x).(x + 2)(x - 2) + 6 \end{aligned}$$

$$\text{For } x = 2, P(2) = 6$$

$$\text{For } x = -2, P(-2) = 6$$

Case II:

$$\begin{aligned} P(x) &= H(x).(x^2 - x - 2) + x + a \\ &= H(x).(x - 2)(x + 1) + x + a \end{aligned}$$

$$\text{For } x = 2, P(2) = 2 + a$$

$$\text{For } x = -1, P(-1) = -1 + a$$

a) But $P(2) = 6 = 2 + a \therefore a = 4$

b) (i) $P(x) = G(x).(x^2 + 3x + 2) + Ax + B$

[$\because$ Remainder is $Ax + B$ (Linear) since Divisor is Quadratic]

$$P(x) = G(x).(x + 1)(x + 2) + Ax + B$$

$$\text{For } x = -1, P(-1) = -A + B$$

$$\text{For } x = -2, P(-2) = -2A + B$$

$$\text{But } P(-1) = -1 + a = -A + B; -1 + a = -1 + 4 = 3$$

$$\Rightarrow -A + B = 3 \dots\dots\dots (1)$$

$$\text{Also } P(-2) = -2A + B = 6 \dots\dots\dots (2)$$

Eliminating B from eqn. (1) and eqn. (2)

$$\text{eqn. (1)} - \text{eqn. (2)}$$

$$\therefore A = -3$$

$$\text{From eqn. (1)} -A + B = 3$$

$$\Rightarrow B = A + 3 = -3 + 3 = 0 \therefore B = 0$$

Remainder when $P(x)$ is divided by $x^2 + 3x + 2$ is $-3x$

$$\begin{aligned} \text{(ii) } P(x) &= F(x).(x^3 + x^2 - 4x - 4) + mx^2 + nx + l \\ &= F(x).(x + 1)(x + 2)(x - 2) + mx^2 + nx + l \end{aligned}$$

$$\text{For } x = 2, P(2) = 4m + 2n + l$$

$$\text{For } x = -2, P(-2) = 4m - 2n + l$$

$$\text{For } x = -1, P(-1) = m - n + l$$

$$\text{But } P(2) = 6, P(-2) = 6, P(-1) = 3$$

$$\Rightarrow 4m + 2n + l = 6 \dots\dots\dots (i)$$

$$\Rightarrow 4m - 2n + l = 6 \dots\dots\dots (ii)$$

$$\Rightarrow m - n + l = 3 \dots\dots\dots (iii)$$

Eliminating l from eqn. (i) and eqn. (ii)

$$\text{eqn. (i)} - \text{eqn. (ii)}$$

$$4n = 0 \therefore n = 0$$

Eliminating l from eqn. (ii) and eqn. (iii)

eqn. (ii) – eqn. (iii)

$$3m - n = 3, n = 0$$

$$\therefore 3m = 3 \Rightarrow m = 1$$

$$\text{From eqn. (iii) } m - n + l = 3 \Rightarrow l = 3 - m + n$$

$$\Rightarrow l = 3 - 1 + 0 = 2$$

$$\therefore \text{Remainder is } x^2 + 2$$

Task

- Find the value of the constant k so that $f(x) = x^4 + kx^3 - 2kx^2 + 3x - 5$ has $(x - 1)$ as a factor. **Answer: $k = -1$**

- Find the value of k so that $f(x) = x^3 - x^2 + kx - 12$ has a factor $(x - 3)$
Answer: $k = -2$

- Find the value of a if;

- $x^3 - 3x^2 + ax + 5$ has a remainder 17 when divided $x - 3$.

Answer: 4

- $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$

Answer: 2

- $x - a$ will divide $f(x) = x^2 + 4x + 2$ with a remainder of -1

Answer: $a = -3, -1$

- Find the values of a and b if;

- When $x^5 + 4x^2 + ax + b$ is divided by $x^2 - 1$ the remainder is $2x + 3$

Answer: $a = 1, b = -1$

- When $ax^4 + bx^3 - x^2 + 2x + 3$ has a remainder $3x + 5$ when divided by $x^2 - x - 2$

Answer: $a = 1, b = -1$

- $x - 3$ and $x + 7$ are factors of the quadratic $ax^2 + 12x + b$

Answer: $a = 3, b = -63$

- a) If $f(x) = x^2 + px + q$ is exactly divisible by $x - a$ and $x - b$, show that $p = -a - b$ and

$$q = ab$$

- Show that;

- $x - a$ is a factor of $x^n - a^n$ when n is even

- $x + a$ is a factor of $x^n + a^n$ when n is odd

[HINT:
 $f(-a) = (-a)^n - a^n$. If n is odd, $(-a)^n = -a^n$ and $f(-a) = 0$. Thus $x + a$ is a factor of
 $x^n + a^n$. $Q(x) = x^{n-1} - ax^{n-2} + ax^{n-3} - \dots \dots \dots a^{n-1}$
]

- Show that the remainder when the polynomial $f(x)$ is divided by $x - a$ is $f(a)$. Show further that if $f(x)$ is divided by $(x - a)(x - b)$ where $a \neq b$, then the remainder is;

$$\left[\frac{f(a) - f(b)}{a - b} \right] x + \left[\frac{af(b) - bf(a)}{a - b} \right]$$

- A polynomial $f(x)$ is divided by $x^2 - a^2$ where $a \neq 0$ and the remainder is $px + q$ prove that $p = \frac{1}{2a} [f(a) - f(-a)]$, $q = \frac{1}{2a} [f(a) + f(-a)]$

- When a polynomial $p(x)$ is divided by $2x + 1$, the quotient is $x^2 - x + 4$ and the remainder is 3. Find $p(x)$ **[HINT: $p(x) = (2x + 1) \cdot (x^2 - x + 4) + 3$]**

- When a polynomial $p(x)$ is divided by $3x - 4$, the quotient is $x^3 + 2x + 2$ and the remainder is -1 . Find $p(x)$. **[HINT: $p(x) = (3x - 4) \cdot (x^3 + 2x + 2) - 1$]**

8. a) When the quadratic expression $ap^2 + bp + c$ is divided by $p - 1$, $p - 2$ and $p + 1$, the remainders are 1, 2 and 25 respectively. Determine the factors of the expression.
[HINT: Solve eqns. $a + b + c = 1$, $4a - 12b + c = 2$ and $a - b + c = 25$. Then $f(p) = 4p^2 - 12p + 9$. Now factoring $f(p) = (2p - 3)(2p - 3)$]
- b) Express $2x^3 + 5x^2 - 4x - 3$ in the form $(x^2 + x - 2) \cdot Q(x) + Ax + B$ where $Q(x)$ a polynomial in x and A and B are constants. Determine the values of constants A and B and the expression $Q(x)$. **Answer: $Q(x) = 2x + 3$, $A = -3$, $B = 3$ (Uneb 1993)**
9. if $4x^3 + kx^2 + px + 2$ is divisible by $x^2 + \lambda^2$, prove that $k\lambda = 8$.
[HINT: Let $f(x) = 4x^3 + kx^2 + px + 2 \equiv (x^2 + \lambda^2)(Ax + B)$, since the quotient must be of degree 1 when the divisor is of degree 2 for $f(x)$ is of degree 3]
10. A polynomial expression $P(x)$ when divided by $x - 1$ leaves a remainder of 3 and when divided by $x - 2$ leaves a remainder of 1. Show that when divided by $(x - 1)(x - 2)$ it leaves a remainder of $-2x + 5$.
**[HINT: $P(x) = (x - 1) \cdot Q(x) + 3$ or $P(1) = 3$
 $P(x) = (x - 2) \cdot H(x) + 1$ or $P(2) = 1$
Now $P(x) = (x - 1)(x - 2) \cdot G(x) + Ax + B$
 $P(1) = A + B = 3$; $P(2) = 2A + B = 1$
Solving the eqns. $A = -2$, $B = 5$
 $\therefore$ Remainder $= -2x + 5$]**
11. Find the quotient and the remainder when $f(x)$ is divided by a linear factor
i) $f(x) = 2x^3 - 3x^2 + 2x - 8$, $x - 2$
ii) $f(x) = x^7 - 31x^2$, $x - 2$
ANSWER: (i) $2x^2 + x + 4$; 0 (ii) $x^6 + 2x^5 + 4x^4 + 8x^3 + 16x^2 + x + 2$; 2
12. Find the constant p such that $x^2 + 2$ is a factor of $x^4 - 6x^2 + p$. Hence factorize it. **ANSWER: $p = -16$, $(x^2 + 2)(x - 2\sqrt{2})(x + 2\sqrt{2})$**

REPEATED ROOTS

(Equal Roots)

From factor theorem which states that "If for a polynomial function $f(x)$ if $f(a) = 0$ then $(x - a)$ is a factor of $f(x)$ "

If $f(x)$ has a repeated factor $(x - a)$. i.e. $f(x) = (x - a)^2 \cdot g(x)$, then;

$$f'(x) = \frac{d}{dx}[(x - a)^2 \cdot g(x)] = g(x) \cdot 2(x - a) + (x - a)^2 \cdot g'(x) \\ = (x - a)[(x - a) \cdot g'(x) + 2g(x)]$$

i.e. If $f(x)$ has a repeated factor $(x - a)$, then $f'(x)$ has a repeated factor $(x - a)$. This is true if and only if $f(a) = 0$, $f'(a) = 0$.

Examples

1. Without using long division find the remainder when $x^3 - 5x^2 + 6x - 2$ is divided by $(x - 2)^2$.

Solution

$$\text{Let } f(x) = x^3 - 5x^2 + 6x - 2$$

$$\Rightarrow f(x) = (x - 2)^2 \cdot Q(x) + Ax + B \dots\dots\dots (*)$$

$$\text{For } x = 2$$

$$f(2) = 2A + B = (2)^3 - 5(2)^2 + 6(2) - 2$$

$$\therefore 2A + B = -2 \dots\dots\dots (1)$$

No other condition linking A and B can be obtained from eqn. (*) without involving $Q(x)$.

But from $f(x) = x^3 - 5x^2 + 6x - 2$, $f'(x) = 3x^2 - 10x + 6$

$$\begin{aligned} f'(x) &= \frac{d}{dx} [(x-2)^2 \cdot Q(x) + Ax + B] = Q(x) \cdot 2(x-2) + (x-2)^2 \cdot Q'(x) + A \\ \Rightarrow 3x^2 - 10x + 6 &= Q(x) \cdot 2(x-2) + (x-2)^2 \cdot Q'(x) + A \\ &= (x-2)[2Q(x) + (x-2) \cdot Q'(x)] + A \end{aligned}$$

For $x = 2$

$$f'(2) = 3(2)^2 - 10(2) + 6 = A \therefore A = -2$$

$$\text{From eqn. (1) } 2A + B = -2 \Rightarrow B = -2 + 4 = 2$$

$$\therefore \text{Remainder} = -2x + 2$$

2. Find the constants m and n such that when $x^4 - mx^2 + n$ is divided by $(x+1)^2$ the remainder is $5x - 2$.

Solution

$$\text{Let } P(x) = x^4 - mx^2 + n = (x+1)^2 \cdot Q(x) + 5x - 2$$

For $x = -1$

$$P(-1) = (-1)^4 - m(-1)^2 + n = 5(-1) - 2$$

$$\therefore n - m = -8 \dots\dots\dots (1)$$

Since the divisor is a repeated factor, then

$$\begin{aligned} P'(x) &= 4x^3 - 2mx = Q(x) \cdot 2(x+1) + (x+1)^2 \cdot Q'(x) + 5 \\ &= (x+1)[2Q(x) + (x+1) \cdot Q'(x)] + 5 \end{aligned}$$

For $x = -1$

$$P'(-1) = 4(-1)^3 - 2m(-1) = 5$$

$$\Rightarrow 2m - 4 = 5 \therefore m = \frac{9}{2}$$

$$\text{From eqn. (1), } n = m - 8 = \frac{9}{2} - 8 = -\frac{7}{2}$$

$$m = \frac{9}{2}, n = -\frac{7}{2}$$

3. If the equation $3x^4 + 2x^3 - 6x^2 - 6x + p = 0$ has two equal roots, find the possible values of p .

Solution

$$\text{Let } f(x) = 3x^4 + 2x^3 - 6x^2 - 6x + p$$

The equation $f(x) = 0$ has two equal roots if $f(x)$ has two equal factors. i.e. *Repeated factors*.

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$.

$$\begin{aligned} f'(x) &\equiv 12x^3 + 6x^2 - 12x - 6 \\ &\equiv 6(x-1)(x+1)(2x+1) \end{aligned}$$

For $(x-1)$ to be a repeated factor of $f(x)$, then $f(1) = 0$

$$\Rightarrow f(1) = 3(1)^4 + 2(1)^3 - 6(1)^2 - 6(1) + p = 0 \therefore p = 7$$

Similarly $(x+1)$ is a repeated factor of $f(x)$, if $f(-1) = 0$

$$\Rightarrow f(-1) = 3(-1)^4 + 2(-1)^3 - 6(-1)^2 - 6(-1) + p = 0 \therefore p = -1$$

For $(2x+1)$ to be a repeated factor of $f(x)$, then $f\left(-\frac{1}{2}\right) = 0$

$$\Rightarrow f\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^4 + 2\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) + p = 0$$

$$\therefore p = -\frac{23}{16}$$

4. Determine whether $f(x) \equiv 3x^4 - 8x^3 - 6x^2 + 24x - 13$ has any repeated factors, and if so find them.

Solution

$$\begin{aligned}f'(x) &\equiv 12x^3 - 24x^2 - 12x + 24 \\&\equiv 12(x^3 - 2x^2 - x + 2) \\&\equiv 12((x-1)(x^2 - x - 2)) \equiv 12(x-1)(x+1)(x-2)\end{aligned}$$

Now $f'(x) = 0$ if $x = 1, -1, 2$

Checking the value of $f(x)$ for these values of x

$$\Rightarrow f(1) = 0, f(-1) \neq 0, f(2) \neq 0$$

So $(x+1)$ and $(x-2)$ are not factors of $f(x)$. Hence $(x-1)$ is the only repeated factor of $f(x)$

5. a) Find the values of p and q which make $x^4 + 6x^3 + 13x^2 + px + q$ a perfect square.
b) The polynomial $p(x) \equiv x^4 + 4x^3 + bx^2 + cx + d$ is a perfect square of degree two. Show that $c + 8 = 2b$, $16d = c^2$. Given that $P(x)$ leaves a remainder of 4 when divided by $x + 1$, determine the possible values of the constants b , c and d .

Solution

- a) Since $p(x) \equiv x^4 + 4x^3 + bx^2 + cx + d$ is of degree four (4^0), then its perfect square must be a quadratic.

$$\begin{aligned}\text{Let } x^4 + 6x^3 + 13x^2 + px + q &\equiv (Ax^2 + Bx + C)^2 \equiv [(Ax^2 + Bx) + C]^2 \\&\equiv (Ax^2 + Bx)^2 + 2C(Ax^2 + Bx) + C^2 \\&\equiv A^2x^4 + 2ABx^3 + B^2x^2 + 2ACx^2 + 2BCx + C^2 \\&\equiv A^2x^4 + (2AB)x^3 + (B^2 + 2AC)x^2 + (2BC)x + C^2\end{aligned}$$

Comparing coefficients

$$\text{For } x^4: 1 = A^2 \therefore A = \pm 1$$

$$\text{For } x^3: 6 = 2AB \therefore B = \frac{3}{A} = \pm 3$$

$$\text{For } x^2: 13 = B^2 + 2AC \therefore C = \frac{13-B^2}{2A} = \frac{13-9}{2(\pm 1)} = \pm 2$$

$$\text{For } x: p = 2BC \therefore p = 2(\pm 3)(\pm 2) = \pm 12$$

$$\text{For Constants: } q = C^2 \therefore q = (\pm 2)^2 = 4$$

$$\therefore p = 12, q = 4$$

- b) Let $P(x) = x^4 + 4x^3 + bx^2 + cx + d \equiv (Ax^2 + Bx + D)^2$
- $$\begin{aligned}&\equiv [(Ax^2 + Bx) + D]^2 \\&\equiv (Ax^2 + Bx)^2 + 2D(Ax^2 + Bx) + D^2 \\&\equiv A^2x^4 + 2ABx^3 + B^2x^2 + 2ADx^2 + 2BDx + D^2 \\&\equiv A^2x^4 + (2AB)x^3 + (B^2 + 2AD)x^2 + (2BD)x + D^2\end{aligned}$$

Comparing coefficients

$$\text{For } x^4: 1 = A^2 \therefore A = \pm 1$$

$$\text{For } x^3: 4 = 2AB \therefore B = \frac{2}{A} = \pm 2$$

$$\text{For } x^2: b = B^2 + 2AD \therefore b = 4 \pm 2D$$

$$\text{For } x: c = 2BD \therefore c = \pm 4D$$

$$\text{For Constants: } d = D^2 \therefore q = (\pm 2)^2 = 4$$

From $b = 4 \pm 2D$ and $c = \pm 4D$ Taking the + sign for both, eliminating D from the equations,

$$2b = 8 + 4D, \text{ but } c = 4D$$

$$\Rightarrow 2b = 8 + c \therefore c + 8 = 2b$$

From $d = D^2$ and $c = 4D$ or $D = \frac{c}{4}$, eliminating D from the equations,

$$\Rightarrow d = \left(\frac{c}{4}\right)^2 \therefore 16d = c^2$$

Given that $P(x) = Q(x) \cdot (x + 1) + 4$

For $x = -1, P(-1) = 4$

From $P(x) = x^4 + 4x^3 + bx^2 + cx + d$

$$\Rightarrow P(-1) = (-1)^4 + 4(-1)^3 + b(-1)^2 + c(-1) + d$$

$$\Rightarrow 4 = 1 - 4 + b - c + d$$

$$\therefore b - c + d = 7 \dots\dots\dots (1)$$

$$\text{From } 16d = c^2 \text{ or } d = \frac{c^2}{16}$$

$$\therefore b - c + \frac{c^2}{16} = 7 \dots\dots\dots (2)$$

$$\text{From } c + 8 = 2b \text{ or } \frac{c+8}{2} = b$$

Eqn. (2) becomes,

$$\frac{c+8}{2} - c + \frac{c^2}{16} = 7 \dots\dots\dots (\times 16)$$

$$\Rightarrow 64 + 8c - 16c + c^2 = 112 \text{ or } c^2 - 8c - 48 = 0$$

Solving the equation, $c = 12, -4$

$$\therefore b = \frac{c+8}{2} = \frac{12+8}{2} \text{ or } \frac{-4+8}{2}$$

$$\therefore b = 10, c = 12; b = 2, c = -4$$

$$\text{Now from } d = \frac{c^2}{16}$$

$$\text{For } c = -4, d = \frac{16}{16} = 1$$

$$\text{For } c = 12, d = \frac{144}{16} = 9$$

$$\therefore b = 10, c = 12, d = 9; b = 2, c = -4, d = 9$$

Task

- 1) Find the remainder when $x^3 - 5x^2 + 7$ is divided by $(x - 1)^2$. **ANSWER: $10 - 7x$**
- 2) If the equation $2x^3 - 9x^2 + 12x + p = 0$ has two equal roots, find the possible values of p . **ANSWER: $-5, -4$**
- 3) Find the value of a for which the function $2x^3 - ax^2 - 12x - 7$ has a repeated factor. **ANSWER: 3**
- 4) If $(x + 1)^2$ is a factor of $2x^4 + 7x^3 + 6x^2 + Ax + B$. Find the values of A and B . **ANSWER: $A = 7, B = 0$**
- 5) Given that $x^2 + c^2$ is a factor of $5x^4 + px^3 + qx^2 + rx + 4$, show that $5r^2 + 4p^2 - 4pr = 0$
- 6) Show that if $x^3 + 3px^2 + qx + r$ is a perfect cube for all real values of x then $q^3 = 27r^2$
- 7) Show that $x^2 + 3$ is a factor of $x^3 - x^2 + 3x - 3$
- 8) Find the constant m for which $x^2 + 1$ is a factor of $mx^4 + x^2 - 1$. **ANSWER: 2**
- 9) Determine whether the given functions have any repeated factors, and if any, find them.
 - a) $x^4 - 16$ **ANSWER: No**
 - b) $x^4 - 18x^2 + 81$ **ANSWER: $(x - 3)(x + 3)$**
 - c) $2x^3 - 3x^2 + 1$ **ANSWER: $x - 1$**
 - d) $x(x^4 - 4)$ **ANSWER: No**
- 10) Find the constants p and q such that $(x - 1)$ is a common factor of $x^4 - 2px^2 + 2$ and $x^4 + x^2 - q$. **ANSWER: $p = \frac{3}{2}, q = 2$**

SOLVING POLYNOMIAL EQUATIONS BY FACTORING

Important Theorems used:

- a) **The Factor Theorem**
 "For a polynomial $P(x)$, $x - a$ is a factor if and only if $P(a) = 0$ "
- b) **The Rational Theorem**
 "Let $P(x)$ be a polynomial of degree n with integral coefficients a_i and non zero constant term, $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_i \in I$.
 If one of the roots of the equation $P(x) = 0$ is $x = \frac{p}{q}$, $\frac{p}{q} \neq 0$, $q > 0$ and $\frac{p}{q}$ is expressed in lowest terms, then p must be a factor of a_0 and q must be a factor of a_n
- c) **The Fundamental Theorem of Algebra**
 If the complex system consisting of all real and imaginary numbers, if $P(x)$ is a polynomial of degree n , ($n > 0$) with complex coefficients, then the equation $P(x) = 0$ has exactly n - roots (provided a double root is counted as two roots, a triple is counted as 3 - roots)

Example

- $2ix^3 + \sqrt{5}x^2 + (3 + 2i)x + 7 - 0$ has 3 - roots
- $3x^4 - 11x^3 + 19x^2 + 25x - 36 = 0$ has 4 - roots

Other Theorems

- "If $P(x)$ is a polynomial of odd degree with real coefficients, then the equations $P(x) = 0$ has at least one real root"
- The relationship between the roots of a polynomial equation and coefficients of the polynomial is;
 For the equation $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ with $a_n \neq 0$, then

$$\text{Sum of roots} = -\frac{a_{n-1}}{a_n}$$

$$\text{Product of roots} = \begin{cases} \frac{a_0}{a_n} & \text{if } n \text{ is even} \\ -\frac{a_0}{a_n} & \text{if } n \text{ is odd} \end{cases}$$

Example

1. a) If one of the roots of the equation $P(x) = 3x^4 + 13x^3 + 15x^2 - 4 = 0$ is in the form $x = \frac{p}{q}$, state the possible rational roots of the equation $P(x) = 0$.
 b) Determine whether any of the possible rational roots really are roots. Then find all the other roots, real/imaginary.

Solution

- a) $x = \frac{p}{q}$ is a possible rational root if p divides -4 and q divides 3
 $\Rightarrow p$ could be equal to: $\pm 1, \pm 2$ or ± 4 and
 q could be equal to: $\pm 1, \pm 3$
 $\therefore \frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$
- b) Using Factor Theorem
 From $P(x) = 3x^4 + 13x^3 + 15x^2 - 4$
 Let $x = 1, P(1) \neq 0 \therefore x - 1$ is not a factor

$$x = -1, P(-1) \neq 0 \therefore x + 1 \text{ is not a factor}$$

$$x = 2, P(2) \neq 0 \therefore x - 2 \text{ is not a factor}$$

$$x = -2, P(-2) = 0 \therefore x + 2 \text{ is a factor}$$

Now by long division.

$$\begin{array}{r}
 \overline{3x^3 + 7x^2 + x - 2} \\
 x+2 \overline{3x^4 + 13x^3 + 15x^2 + 0x - 4} \\
 \underline{3x^4 + 6x^3} \\
 7x^3 + 15x^2 + 0x - 4 \\
 \underline{7x^3 + 14x^2} \\
 x^2 + 0x - 4 \\
 \underline{x^2 + 2x} \\
 -2x - 1 \\
 \underline{-2x - 1} \\
 0
 \end{array}
 \quad \therefore P(x) = (x+2)(3x^3 + 7x^2 + x - 2) = 0$$

Now finding roots of the depressed equation $3x^3 + 7x^2 + x - 2 = 0$

If $x = \frac{p}{q}$ is the root of the equation, p divides -2 and q divides 3

p could be equal to: $\pm 1, \pm 2$ and

q could be equal to: $\pm 1, \pm 3$

$$\therefore \frac{p}{q} = \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$$

$$\text{Let } Q(x) = 3x^3 + 7x^2 + x - 2 = 0$$

$$\text{For } x = 1, Q(1) \neq 0 \therefore x - 1 \text{ is not a factor}$$

$$x = -1, Q(-1) \neq 0 \therefore x + 1 \text{ is not a factor}$$

$$x = 2, Q(2) \neq 0 \therefore x - 2 \text{ is not a factor}$$

$$x = -2, Q(-2) = 0 \therefore x + 2 \text{ is a factor}$$

By long division,

$$\begin{array}{r}
 \overline{3x^2 + x - 1} \\
 x+2 \overline{3x^3 + 7x^2 + x - 2} \\
 \underline{3x^3 + 6x^2} \\
 x^2 + x - 2 \\
 \underline{x^2 + 2x} \\
 -x - 2 \\
 \underline{-x - 2} \\
 0
 \end{array}
 \quad \therefore Q(x) = (x+2)(3x^2 + x - 1) = 0$$

$$\therefore P(x) = (x+2)^2(3x^2 + x - 1) = 0$$

Now solving $3x^2 + x - 1 = 0$

$$x = \frac{-1 \pm \sqrt{(1)^2 + 4(3)(1)}}{2(3)} = \frac{-1 \pm \sqrt{13}}{6}$$

$$\therefore \text{Roots are: } x = \frac{-1 \pm \sqrt{13}}{6}, x = -2, x = -2$$

2. Find all the zeros and if possible, all the other real zeros of;

a. $f(x) = x^3 - x^2 - 14x + 24$

b. $f(x) = x^3 - 8x^2 + 5x + 14$

c. $f(x) = 3x^3 - x^2 - 6x + 2$

Solution

For rational zeros, $f(x) = 0$ is an equation to be solved.

a. $f(x) = x^3 - x^2 - 14x + 24 = 0$

Let $x = \frac{p}{q}$, p divides 24, q divides 1

p could be equal to: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 12, \pm 24$ and

q could be equal to: ± 1

$\therefore$ Possible values of $x = \frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 12, \pm 24$

By factor theorem,

For $x = 1, f(1) \neq 0 \therefore x - 1$ is not a factor

$x = -1, f(-1) \neq 0 \therefore x + 1$ is not a factor

$x = -2, f(-2) \neq 0 \therefore x + 2$ is not a factor

$x = 2, f(2) = 0 \therefore x - 2$ is a factor

By long division,

$$\begin{array}{r}
 x^2 + x - 12 \\
 x-2 \overline{) x^3 - x^2 - 14x + 24} \\
 \underline{x^3 - 2x^2} \\
 x^2 - 14x + 24 \\
 \underline{-x^2 + 2x} \\
 -12x + 24 \\
 \underline{-12x + 24} \\
 0
 \end{array}$$

$$\therefore f(x) = (x-2)(x^2 + x - 12) = 0$$

$$\text{Solving } x^2 + x - 12 = 0$$

$$\Rightarrow (x+4)(x-3) = 0$$

$$\therefore x = -4, 3$$

$\therefore$ The rational zeros are:

$$\therefore x = 2, -4, 3$$

b. $f(x) = x^3 - 8x^2 + 5x + 14 = 0$

Let $x = \frac{p}{q}$, p divides 14, q divides 1

Possible values of $p = \pm 1, \pm 2, \pm 7, \pm 14$

Possible values of $q = \pm 1$

Possible values of $x = \pm 1, \pm 2, \pm 7, \pm 14$

By factor theorem,

For $x = 1, f(1) \neq 0 \therefore x - 1$ is not a factor

$x = -1, f(-1) \neq 0 \therefore x + 1$ is a factor

$x = 2, f(2) = 0 \therefore x - 2$ is a factor

$x = 7, f(7) = 0 \therefore x - 7$ is a factor

$$\therefore f(x) = (x+1)(x-2)(x-7) = 0$$

The zeros of $f(x)$ are: $-1, 2, 7$

c. $f(x) = 3x^3 - x^2 - 6x + 2 = 0$

Let $x = \frac{p}{q}$, p divides 2, q divides 3

Possible values of $p = \pm 1, \pm 2$

Possible values of $q = \pm 1, \pm 3$

Possible values of $x = \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

By factor theorem,

For $x = 1, f(1) \neq 0 \therefore x - 1$ is not a factor

$x = -1, f(-1) \neq 0 \therefore x + 1$ is not a factor

$x = 2, f(2) \neq 0 \therefore x - 2$ is not a factor

$x = -\frac{1}{3}, f\left(-\frac{1}{3}\right) \neq 0 \therefore x + \frac{1}{3}$ is not a factor

$x = \frac{1}{3}, f\left(\frac{1}{3}\right) \neq 0 \therefore x - \frac{1}{3}$ is a factor

By long division

$$\begin{array}{r} x - \frac{1}{3} \overline{) 3x^2 - 6} \\ \underline{3x^3 - x^2 - 6x + 2} \\ 3x^3 - x^2 \\ \underline{-6x + 2} \end{array} \quad \therefore f(x) = (3x^2 - 6)\left(x - \frac{1}{3}\right) = 0$$

$$\therefore f(x) = (3x^2 - 6)\left(x - \frac{1}{3}\right) = 0 \Rightarrow 3x^2 - 6 = 0 \therefore x = \pm\sqrt{2}$$

The zeros of $f(x)$ are: $\pm\sqrt{2}, \frac{1}{3}$

3. Factorize the polynomial $P(x)$, hence find the zeros of $P(x)$.

a) $P(x) = x^3 + 2x^2 - 9x - 18$

b) $P(x) = x^4 - 11x^2 + 28$

c) $P(x) = 2x^4 - 5x^3 - 11x^2 + 20x + 12$

Solution

a) $P(x) = x^3 + 2x^2 - 9x - 18$

Let $x = \frac{p}{q}$, p divides -18 , q divides 1

Possible values of $p = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

Possible values of $q = \pm 1$

Possible values of $x = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

By factor theorem,

For $x = 1, P(1) \neq 0 \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \therefore x + 1$ is not a factor

$x = -2, P(-2) = 0 \therefore x + 2$ is a factor

$x = 3, P(3) = 0 \therefore x - 3$ is a factor

$x = -3, P(-3) \neq 0 \therefore x + 3$ is a factor

$$\therefore P(x) = (x + 3)(x - 3)(x + 2)$$

The zeros of $P(x)$ are: $\pm 3, -2$

b) $P(x) = x^4 - 11x^2 + 28$

Let $x = \frac{p}{q}$, p divides 28 , q divides 1

Possible values of $p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 7, \pm 14, \pm 28$

Possible values of $q = \pm 1$

Possible values of $x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 7, \pm 14, \pm 28$

By factor theorem,

For $x = 1, P(1) \neq 0 \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \therefore x + 1$ is not a factor

$x = 2, P(2) = 0 \therefore x - 2$ is a factor

$x = -2, P(-2) = 0 \therefore x + 2$ is a factor

$$\therefore P(x) = (x + 2)(x - 2).M(x) = (x^2 - 4).M(x)$$

Since $x + 2, x - 2$ are factors of $P(x)$, then $(x + 2)(x - 2) = x^2 - 4$ must also be a factor.

By long division to obtain $M(x)$

$$\text{Divisor} = x^2 - 4$$

$$\begin{array}{r} x^2 - 7 \\ x^2 - 4 \overline{) x^4 + 0x^3 - 11x^2 + 0x + 28} \\ \underline{x^4 + 0x^3 - 4x^2} \\ -7x^2 + 0x + 28 \\ \underline{7x^2 + 0x + 28} \\ 0 \end{array} \quad \therefore M(x) = x^2 - 7$$

$$\therefore P(x) = (x+2)(x-2)(x^2-7) = (x+2)(x-2)(x \pm \sqrt{7})$$

The zeros of $P(x)$ are: $\pm 2, \pm \sqrt{7}$

c) $P(x) = 2x^4 - 5x^3 - 11x^2 + 20x + 12$

Let $x = \frac{p}{q}$, p divides 12, q divides 2

Possible values of $p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Possible values of $q = \pm 1, \pm 2$

Possible values of $x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}$

By factor theorem,

For $x = 1, P(1) \neq 0 \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \therefore x + 1$ is not a factor

$x = 2, P(2) = 0 \therefore x - 2$ is a factor

$x = -2, P(-2) = 0 \therefore x + 2$ is a factor

$x = 3, P(3) = 0 \therefore x - 3$ is a factor

$$\therefore P(x) = (x+2)(x-2)(x-3).M(x) = (x^2-4)(x-3).M(x)$$

Since $x+2, x-2, x-3$ are factors of $P(x)$, then $(x+2)(x-2)(x-3)$ must also be a factor. By long division to obtain $M(x)$

$$\text{Divisor} = (x^2-4)(x-3) = x^3-3x^2-4x+12$$

$$\begin{array}{r} 2x+1 \\ x^3-3x^2-4x+12 \overline{) 2x^4-5x^3-11x^2+20x+12} \\ \underline{2x^4-6x^3-8x^2+24x} \\ x^3-7x^2+0x+28 \\ \underline{-x^3-7x^2+0x+28} \\ 0 \end{array}$$

$$\therefore P(x) = (x+2)(x-2)(x-3)(2x+1)$$

The zeros of $P(x)$ are: $\pm 2, -\frac{1}{2}, -3$

Task

1) Factorize $f(x)$, hence state the zeros of $f(x)$

a) $f(x) = x^3 + 4x^2 + x - 6$

ANSWER: $(x+2)(x+3)(x-1)$; **Zeros:** $-2, -3, 1$

b) $f(x) = 6x^3 - 5x^2 - 2x + 1$

ANSWER: $(3x-1)(2x+1)(x-1)$; **Zeros:** $-\frac{1}{2}, \frac{1}{3}, 1$

c) $f(x) = 4x^3 - 13x - 6$

ANSWER: $(x-2)(2x+1)(2x+3)$; **Zeros:** $-\frac{1}{2}, -\frac{3}{2}, 2$

2) Solve the equation giving real and imaginary roots if any;

a) $2x^3 - 7x + 2 = 0$ **ANSWER:** $\frac{2 \pm \sqrt{2}}{2}, -2$

b) $3x^3 - 4x^2 - 5x + 2 = 0$ **ANSWER:** $-1, 2, \frac{1}{3}$

c) $x^4 - 10x^2 + 9 = 0$ **ANSWER:** $\pm 1, \pm 3$

THE BINOMIAL THEOREM

Pascal's triangle

This is used to give the coefficients of the emerging terms during the expansion.

The coefficients are obtained as below.

Consider an array of triangular combinatorial coefficients.

		COLUMN (r)					
ROW (n)	0	0	1	2	3	4	5
	0	${}^0C_0$					
	1	${}^1C_0$	${}^1C_1$				
	2	${}^2C_0$	${}^2C_1$	${}^2C_2$			
	3	${}^3C_0$	${}^3C_1$	${}^3C_2$	${}^3C_3$		
	4	${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$	

Superscript on the left of each of the coefficients denotes *row number* and the subscript on the right denote *column number*.

Note:

- Superscript *must* be greater than or equal to subscript. *i.e.* for ${}^nC_r, n \geq r$
- The above array is generated using
Simplifying the table

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$$

		COLUMN (r)					
ROW (n)	0	0	1	2	3	4	5
	0	1					
	1	1	${}^1C_1$				
	2	1	2	1			
	3	1	3	3	1		
	4	1	4	6	4	1	

From the above table, from the coefficients of each expansion, the coefficient of the next expansion can be obtained using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

$$\begin{array}{ccccccc}
 {}^1C_0 & {}^1C_1 & & & & & \\
 \downarrow & & & & & & \\
 {}^1C_0 & [{}^1C_0 + {}^1C_1 = {}^2C_1] & {}^1C_1 & & & & \\
 \downarrow & & {}^1C_0 & {}^2C_1 & {}^1C_1 & & \\
 {}^1C_0 & [{}^2C_1 + {}^1C_0 = {}^3C_1] & [{}^1C_1 + {}^2C_1 = {}^3C_2] & & & & \\
 \downarrow & & {}^1C_0 & {}^3C_1 & {}^3C_2 & {}^1C_1 & \\
 {}^1C_0 & [{}^3C_1 + {}^1C_0 = {}^4C_1] & [{}^3C_2 + {}^3C_1 = {}^4C_2] & [{}^1C_1 + {}^3C_2 = {}^4C_3] & & & \\
 & & & \downarrow & & &
 \end{array}$$

add

add

add

	1	1	1			
		1	1	1	} add	
$(a + b)^2$ coeff:	1	2	1			
		1	2	1	} add	
$(a + b)^3$ coeff:	1	3	3	1		
		1	3	3	1	} add
$(a + b)^4$ coeff:	1	4	6	4	1	
		1	4	6	4	1} add
$(a + b)^5$ coeff:	1	5	10	10	5	1

$$\begin{array}{cccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
 1 & & 1 & & 4 & & 6 & & 4 & & 1 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
 \end{array}$$

1 6 15 20 15 6 1

$$\Rightarrow (x+6)^6 = 1.x^6 + 6.x^5(6)^1 + 15.x^4(6)^2 + 20.x^3(6)^3 + 15.x^2(6)^4 + 6.x^1(6)^5 + 1.(6)^6$$

$$= x^6 + 36x^5 + 540x^4 + 4320x^3 + 19440x^2 + 46656x + 46656$$

ii. $\left(2x - \frac{1}{4}\right)^4 = ?$

There are 5 terms. i. e.

$$(2x)^4 \quad (2x)^3\left(-\frac{1}{4}\right)^1 \quad (2x)^2\left(-\frac{1}{4}\right)^2 \quad (2x)^1\left(-\frac{1}{4}\right)^3 \quad \left(-\frac{1}{4}\right)^4$$

Coefficients from Pascal's triangle

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$\Rightarrow \left(2x - \frac{1}{4}\right)^4 = 1.(2x)^4 + 4.(2x)^3\left(-\frac{1}{4}\right)^1 + 6.(2x)^2\left(-\frac{1}{4}\right)^2 + 4.(2x)^1\left(-\frac{1}{4}\right)^3 + 1.\left(-\frac{1}{4}\right)^4$$

$$= 16x^4 - 8x^3 + \frac{3}{2}x^2 - \frac{1}{8}x + \frac{1}{256}$$

iii. $(2x+1)^3 = ?$

There are 4 terms. i. e.

$$(2x)^3 \quad (2x)^2(1)^1 \quad (2x)^1(1)^2 \quad (1)^3$$

Coefficients from Pascal's triangle

$$1 \quad 3 \quad 3 \quad 1$$

$$\Rightarrow (2x+1)^3 = 1.(2x)^3 + 3.(2x)^2(1)^1 + 3.(2x)^1(1)^2 + 1.(1)^3$$

$$= 8x^3 + 12x^2 + 6x + 1$$

iv. $\left(x - \frac{1}{x}\right)^5 = ?$

There are 6 terms. i. e.

$$x^5 \quad x^4\left(-\frac{1}{x}\right)^1 \quad x^3\left(-\frac{1}{x}\right)^2 \quad x^2\left(-\frac{1}{x}\right)^3 \quad x\left(-\frac{1}{x}\right)^4 \quad \left(-\frac{1}{x}\right)^5$$

Coefficients from Pascal's triangle

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^5 = 1.x^5 + 5.x^4\left(-\frac{1}{x}\right)^1 + 10.x^3\left(-\frac{1}{x}\right)^2 + 10.x^2\left(-\frac{1}{x}\right)^3 + 5.x\left(-\frac{1}{x}\right)^4 + 1.\left(-\frac{1}{x}\right)^5$$

$$= x^5 - 5x^2 + \frac{10}{x} - \frac{10}{x^4} + \frac{5}{x^7} - \frac{1}{x^{10}}$$

Finding approximations by Pascal's triangle expansions

- 2) Expand $\left(3 + \frac{1}{8}x\right)^4$ in ascending powers of x . by putting $x = 0.2$ in the first three terms of the expansion, find the approximate value of $(3.025)^4$ to 3d.p.

Solution

$\left(3 + \frac{1}{8}x\right)^4$ has 5 - terms. i. e.

$$(3)^4 \quad (3)^3\left(\frac{x}{8}\right) \quad (3)^2\left(\frac{x}{8}\right)^2 \quad (3)\left(\frac{x}{8}\right)^3 \quad \left(\frac{x}{8}\right)^4$$

Coefficients: $1 \quad 4 \quad 6 \quad 4 \quad 1$

$$\therefore \left(3 + \frac{1}{8}x\right)^4 = 1.(3)^4 + 4.(3)^3\left(\frac{x}{8}\right) + 6.(3)^2\left(\frac{x}{8}\right)^2 + 4.(3)\left(\frac{x}{8}\right)^3 + 1.\left(\frac{x}{8}\right)^4$$

$$= 81 + \frac{108}{8}x + \frac{54}{64}x^2 + \frac{12}{512}x^3 + \frac{x^4}{4096}$$

$$= 81 + \frac{27}{2}x + \frac{27}{32}x^2 + \frac{3}{128}x^3 + \frac{x^4}{4096}$$

Taking the first 3 -terms

$$\left(3 + \frac{1}{8}x\right)^4 = 81 + \frac{27}{2}x + \frac{27}{32}x^2$$

For $x = 0.2$

$$\Rightarrow \left(3 + \frac{1}{8} \times 0.2\right)^4 = 81 + \frac{27}{2} \times 0.2 + \frac{27}{32} \times (0.2)^2$$

$$\therefore (3.025)^4 \approx 83.734$$

- 3) Expand $\left(1 + \frac{1}{x}\right)^3$ as far as $\frac{1}{x^3}$. Hence find the value of $(1.025)^3$ giving your answer to 4 s.f.

Solution

$\left(1 + \frac{1}{x}\right)^3$ has 4 – terms. i. e.

$$(1)^3 \quad (1)^2 \left(\frac{1}{x}\right)^1 \quad (1)^1 \left(\frac{1}{x}\right)^2 \quad \left(\frac{1}{x}\right)^3$$

Coefficients from Pascal's triangle: **1 3 3 1**

$$\begin{aligned} \therefore \left(1 + \frac{1}{x}\right)^3 &= 1.(1)^3 + 3.(1)^2 \left(\frac{1}{x}\right)^1 + 3.(1)^1 \left(\frac{1}{x}\right)^2 + 1.\left(\frac{1}{x}\right)^3 \\ &= 1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3} \end{aligned}$$

Now finding a suitable value of x for substitution

$$\Rightarrow \left(1 + \frac{1}{x}\right)^3 = (1.025)^3 \quad \therefore 1 + \frac{1}{x} = 1.025 \text{ or } x = 40$$

$$\therefore \left(1 + \frac{1}{40}\right)^3 = (1.025)^3 = 1 + \frac{3}{40} + \frac{3}{40^2} + \frac{1}{40^3} = 1.0779$$

$$\therefore (1.025)^3 \approx 1.077$$

- 4) Write down the expansion of $\left(1 + \frac{x}{4}\right)^4$ as far as x^3 . Hence find the value of $(1.025)^4$ correct to 3d.ps.

Solution

$\left(1 + \frac{x}{4}\right)^4$ has 5 – terms. i. e.

$$(1)^4 \quad (1)^3 \left(\frac{x}{4}\right)^1 \quad (1)^2 \left(\frac{x}{4}\right)^2 \quad (1)^1 \left(\frac{x}{4}\right)^3 \quad \left(\frac{x}{4}\right)^4$$

Coefficients from Pascal's triangle: **1 4 6 4 1**

$$\begin{aligned} \therefore \left(1 + \frac{x}{4}\right)^4 &= 1.(1)^4 + 4.(1)^3 \left(\frac{x}{4}\right)^1 + 6.(1)^2 \left(\frac{x}{4}\right)^2 + 4.(1)^1 \left(\frac{x}{4}\right)^3 + 1.\left(\frac{x}{4}\right)^4 \\ &\approx 1 + x + \frac{3}{8}x^2 + \frac{1}{16}x^3; \left(\frac{x}{4}\right)^4 \text{ is ignored} \end{aligned}$$

Now finding a suitable value of x for substitution

$$\Rightarrow \left(1 + \frac{x}{4}\right)^4 = (1.025)^4 \quad \therefore 1 + \frac{x}{4} = 1.025 \text{ or } x = 0.1$$

$$\therefore \left(1 + \frac{0.1}{4}\right)^4 = (1.025)^4 = 1 + 0.1 + \frac{3}{8}(0.1)^2 + \frac{1}{16}(0.1)^3 = 1.104$$

$$\therefore (1.025)^4 \approx 1.104$$

- 5) Using Pascal's triangle, find the exact value of

a) $(2.95)^3$

b) $(2.998)^4$

Solution

- a) $(2.95)^3 = (3 - 0.05)^3 = (2 + 0.95)^3$ [Use any of binomials of your choice]

Using $(2.95)^3 = (3 - 0.05)^3$, has 4 – terms. i. e.

$$(3)^3 \quad (3)^2(-0.05)^1 \quad (3)^1(-0.05)^2 \quad (-0.05)^3$$

Coefficients from Pascal's triangle: **1 3 3 1**

$$\begin{aligned} \Rightarrow (2.95)^3 &= (3 - 0.05)^3 = 1.(3)^3 + 3.(3)^2(-0.05)^1 + 3.(3)^1(-0.05)^2 + 1.(-0.05)^3 \\ &= 27 - 1.35 + 0.0225 - 0.000125 \\ &= 25.672375 \end{aligned}$$

- b) $(2.998)^4 = (3 - 0.002)^4 = (2 + 0.998)^4$

Using $(2.998)^4 = (3 - 0.002)^4$, has 5 – terms. i. e.

$$(3)^4 \quad (3)^3(-0.002)^1 \quad (3)^2(-0.002)^2 \quad (3)^1(-0.002)^3 \quad (-0.002)^4$$

Coefficients from Pascal's triangle: **1 4 6 4 1**

$$\begin{aligned}
 \Rightarrow (2.998)^4 &= (3 - 0.002)^4 \\
 &= 1 \cdot (3)^4 + 4 \cdot (3)^3(-0.002)^1 + 6 \cdot (3)^2(-0.002)^2 + 4 \cdot (3)^1(-0.002)^3 + \\
 &1 \cdot (-0.002)^4 \\
 &= 81 - 0.216 + 0.000216 \\
 &= 80.784
 \end{aligned}$$

Task

- 1) Expand $(2 - x)^6$ in ascending powers of x . Taking the first 3-terms, find the value of $(1.998)^6$ as accurately as possible. **Answer: 63.617**
- 2) Write down the first three terms of $(3 - x)^6$. Hence find the value of $(2.998)^6$ as accurately as possible. **Answer: 80.784**

Note:

- 1) Pascal's triangle is applicable to $(a + b)^n$ where n is relatively small. i. e. $n \leq 6$ and is a natural number.
- 2) If n is large or negative or fractional, Binomial theorem can be used.

THE BINOMIAL THEOREM

This states that if n is a particular number then

$$\begin{aligned}
 (a + b)^n &= {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^{n-n} b^n \\
 &= a^n + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + b^n
 \end{aligned}$$

However if few terms of an expansion are required the theorem is used in the form

$$(a + b)^n = a^n + n a^{n-1} b^1 + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \dots + b^n$$

Note:

$${}^nC_1 = \frac{n!}{(n-1)!} = n, {}^nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2!}, \text{ and } {}^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!} \text{ e.t.c}$$

Putting all these results in the first stated Binomial theorem yields the second stated theorem above.

Example

Expand by Binomial theorem.

- a) $\left(x + \frac{1}{x}\right)^5$ as far as term in $\frac{1}{x^5}$
- b) $\left(3 + \frac{x}{5}\right)^4$ as far as term in x^3
- c) $\left(4 + \frac{x}{8}\right)^8$ as far as term in x^4

Solution

$$\text{a) } \left(x + \frac{1}{x}\right)^5 = ?$$

$$\text{Using } (a + b)^n = a^n + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + b^n$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^5 = x^5 + {}^5C_1 x^{5-1} \left(\frac{1}{x}\right)^1 + {}^5C_2 x^{5-2} \left(\frac{1}{x}\right)^2 + {}^5C_3 x^{5-3} \left(\frac{1}{x}\right)^3 + {}^5C_4 x^{5-4} \left(\frac{1}{x}\right)^4 + {}^5C_5 x^{5-5} \left(\frac{1}{x}\right)^5$$

$${}^5C_5 x^{5-5} \left(\frac{1}{x}\right)^5$$

$$= x^5 + \frac{5!}{4!1!} \cdot \frac{x^4}{x} + \frac{5!}{3!2!} \cdot \frac{x^3}{x^2} + \frac{5!}{2!3!} \cdot \frac{x^2}{x^3} + \frac{5!}{1!4!} \cdot \frac{x}{x^4}$$

$$\therefore \left(x + \frac{1}{x}\right)^5 = x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

$$\text{b) } \left(3 + \frac{x}{5}\right)^4 = ?$$

$$\begin{aligned} \left(3 + \frac{x}{5}\right)^4 &= (3)^4 + {}^4C_1(3)^{4-1}\left(\frac{x}{5}\right)^1 + {}^4C_2(3)^{4-2}\left(\frac{x}{5}\right)^2 + {}^4C_3(3)^{4-3}\left(\frac{x}{5}\right)^3 + {}^4C_4\left(\frac{x}{5}\right)^4 \\ &= 81 + \frac{4!}{3!1!}(3)^3\left(\frac{x}{5}\right) + \frac{4!}{2!2!}(3)^2\left(\frac{x}{5}\right)^2 + \frac{4!}{1!3!}(3)\left(\frac{x}{5}\right)^3 + \frac{x^4}{625} \end{aligned}$$

$$\therefore \left(3 + \frac{x}{5}\right)^4 = 81 + \frac{108}{5}x + \frac{54}{25}x^2 + \frac{12}{125}x^3$$

$$\text{c) } \left(4 + \frac{x}{8}\right)^8 = (4)^8 + {}^8C_1(4)^7\left(\frac{x}{8}\right)^1 + {}^8C_2(4)^6\left(\frac{x}{8}\right)^2 + {}^8C_3(4)^5\left(\frac{x}{8}\right)^3 + {}^8C_4(4)^4\left(\frac{x}{8}\right)^4$$

$$\therefore \left(4 + \frac{x}{8}\right)^8 = 65536 + 16384x + 1792x^2 + 112x^3 + \frac{35}{8}x^4$$

The general term of Binomial expansion

Recall: $(a + b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \dots + {}^nC_na^{n-n}b^n$

Each term looks like ${}^nC_ra^{n-r}b^r$ where $0 \leq r \leq n$.

Thus the expression ${}^nC_ra^{n-r}b^r$ is typical of each term in the expansion of $(a + b)^n$. Hence it is called **general term**.

Any specific term can be derived from the general term.

Example:

If $r = 2$, the general term becomes;

${}^nC_2a^{n-2}b^2 = \frac{n!}{(n-2)!2!}a^{n-2}b^2$, which is the 3^{rd} term of the expansion of $(a + b)^n$.

$\therefore$ The 3^{rd} term is obtained by letting $r = 2$.

General deduction

${}^nC_ra^{n-r}b^r$ represents $(r + 1)^{th}$ term in the expansion of $(a + b)^n$ for $0 \leq r \leq n$

Example

- State the required term in the expansions written in ascending powers of x .

- a) 10^{th} term in $(1 + x)^{15}$

- b) 8^{th} term in $\left(2 - \frac{x}{3}\right)^{12}$

- c) 4^{th} term in $\left(2 - \frac{3}{x}\right)^8$

Solution

Using $(a + b)^n$ expansion with general term ${}^nC_ra^{n-r}b^r$ for $(r + 1)^{th}$ term

- a) $(1 + x)^{15}$ gives: $a = 1, b = x, n = 15, r + 1 = 10 \Rightarrow r = 9$

$$\begin{aligned} \therefore 10^{th} \text{ term} &= {}^{15}C_9(1)^{15-9}x^9 = \frac{15!}{(15-9)!9!}x^9 \\ &= \frac{15!}{6!9!}x^9 = 5005x^9 \text{ [obtained from calculator]} \end{aligned}$$

- b) $\left(2 - \frac{x}{3}\right)^{12}$ gives: $a = 2, b = -\frac{x}{3}, n = 12, r + 1 = 8 \Rightarrow r = 7$

$$\begin{aligned} \therefore 8^{th} \text{ term} &= {}^{12}C_7(2)^{12-7}\left(-\frac{x}{3}\right)^7 = \frac{12!}{5!7!}(2)^5 \cdot \frac{x^7}{(-3)^7} \\ &= \frac{792 \times 32}{-2187}x^7 = -\frac{2816}{243}x^7 \end{aligned}$$

- c) $\left(2 - \frac{3}{x}\right)^8$ gives: $a = 2, b = -\frac{3}{x}, n = 8, r + 1 = 4 \Rightarrow r = 3$

$$\therefore 4^{\text{th}} \text{ term} = {}^8C_3(2)^{8-3}\left(-\frac{3}{x}\right)^3 = \frac{8!}{5!3!}(2)^5 \cdot \frac{(-3)^3}{x^3} = -\frac{48384}{x^3}$$

Note:

Sometimes a coefficient of term can be required. The general term can be used to state the required coefficient.

The required coefficient of term is derived by finding a suitable r .

2. Find the coefficient of x^{10} in the expansion of $(2x - 3)^{14}$.

Solution

From the general term $U_{r+1} = {}^nC_r a^{n-r} b^r$ of expansion of $(a + b)^n$

For $(2x - 3)^{14}$, $a = 2x$, $b = -3$, $n = 14$, $r + 1 = ? \Rightarrow r = ?$

$$\Rightarrow U_{r+1} = {}^{14}C_r (2x)^{14-r} (-3)^r = {}^{14}C_r \cdot (2)^{14-r} \cdot (x)^{14-r} \cdot (-3)^r$$

Since term in x^{10} is required, then equating $x^{10} = (x)^{14-r}$ gives the value of r which is the key determinant of required coefficient.

$$\Rightarrow 10 = 14 - r \therefore r = 4$$

Thus the required coefficient is in the 5th term of the expansion of $(2x - 3)^{14}$

$$\therefore U_5 = {}^{14}C_4 \cdot (2)^{14-4} \cdot (x)^{14-4} \cdot (-3)^4 = \frac{14!}{10!4!} \cdot 2^{10} \cdot (-3)^4 \cdot x^{10}$$

$$\text{Required coefficient} = 1001 \times 2^{10} \times (-3)^4 = 83026944$$

3. Find the coefficient of x^{-4} in the expansion of $\left(2 - \frac{3}{x}\right)^8$.

Solution

For $\left(2 - \frac{3}{x}\right)^8$, $a = 2$, $b = -\frac{3}{x}$, $n = 8$, $r + 1 = ? \Rightarrow r = ?$

$$\Rightarrow U_{r+1} = {}^8C_r (2)^{8-r} \left(-\frac{3}{x}\right)^r = {}^8C_r \cdot (2)^{8-r} \cdot (x)^{-r} \cdot (-3)^r$$

Finding a suitable value of r ;

Since term in x^{-4} is required, then equating $x^{-4} = x^{-r}$ gives the value of r
i.e. $r = 4$

Thus the required coefficient is in the 5th term of the expansion of $\left(2 - \frac{3}{x}\right)^8$

$$\therefore U_5 = {}^8C_4 \cdot (2)^{8-4} \cdot (x)^{-4} \cdot (-3)^4 = \frac{8!}{4!4!} \cdot 2^4 \cdot (-3)^4 \cdot x^{-4}$$

$$\text{Required coefficient} = \frac{8!}{4!4!} \cdot 2^4 \cdot (-3)^4 = 90\,720$$

4. Find the coefficient of $a^6 x^{14}$ in $(a + x)^{20}$.

Solution

For $(a + x)^{20}$, $A = a$, $B = x$, $N = 20$, $R + 1 = ? \Rightarrow R = ?$

$$\Rightarrow U_{R+1} = {}^{20}C_R a^{20-R} x^R$$

Finding a suitable value of R ;

Since term in $a^6 x^{14}$ is required, then equating either $x^{14} = x^R$ giving the value of $R = 14$ or $a^6 = a^{20-R} \Rightarrow R = 14$ as before

Thus the required coefficient is in the 15th term of the expansion of $(a + x)^{20}$

$$\therefore U_{15} = {}^{20}C_{14} a^{20-14} x^{14} = {}^{20}C_{14} \cdot a^6 x^{14}$$

$$\therefore \text{coefficient} = {}^{20}C_{14} = 38\,760$$

5. Write down the coefficient of the term in $x^4 y^5$ in the expansion of $(2x + 3y)^9$

Solution

For $(2x + 3y)^9$, $a = 2x$, $b = 3y$, $n = 9$, $r + 1 = ? \Rightarrow r = ?$

$$\Rightarrow U_{r+1} = {}^9C_r (2x)^{9-r} (3y)^r = {}^9C_r \cdot (2)^{9-r} \cdot (x)^{9-r} \cdot (3)^r \cdot (y)^r$$

Finding a suitable value of r ;

Since term in x^4y^5 is required, then equating either $x^4 = x^{9-r}$ giving the value of $r = 5$ or $y^5 = y^r \Rightarrow r = 5$ as before

Thus the required coefficient is in the 6th term of the expansion of $(2x + 3y)^9$

$$\therefore U_6 = {}^9C_5 \cdot (2)^{9-5} \cdot (x)^{9-5} \cdot (3)^5 (y)^5 = {}^9C_5 \cdot (2)^{9-5} \cdot (3)^5 \cdot x^4 y^5$$

$$\therefore \text{coefficient} = {}^9C_5 \cdot (2)^{9-5} \cdot (3)^5 = 489\,988$$

6. Expand $\left(1 + \frac{10}{x}\right)^{10}$ in terms of descending powers of x as far as term in $\frac{1}{x^3}$.

Find: a) The 8th term

b) The coefficient of x^{-8}

Solution

The expansion is left as an exercise.

- a) Using $U_{r+1} = {}^nC_r a^{n-r} b^r$ for $(r+1)^{\text{th}}$ term

$$\text{For } \left(1 + \frac{10}{x}\right)^{10}, a = 1, b = \frac{10}{x}, n = 10$$

$$\text{Given: } r + 1 = 8 \therefore r = 7$$

$$\therefore U_8 = {}^{10}C_7 (1)^{10-7} \left(\frac{10}{x}\right)^7 = {}^{10}C_7 \cdot \frac{10^7}{x^7} = \frac{10!}{7!3!} \cdot \frac{10^7}{x^7}$$

$$\therefore 8^{\text{th}} \text{ term} = \frac{1.2 \times 10^9}{x^7}$$

- b) From $U_{r+1} = {}^nC_r a^{n-r} b^r$, $a = 1, b = \frac{10}{x}, n = 10, r + 1 = ? \Rightarrow r = ?$

$$U_{r+1} = {}^{10}C_r (1)^{10-r} \left(\frac{10}{x}\right)^r = {}^{10}C_r \cdot 10^r \cdot x^{-r} \quad [\because (1)^{10-r} = 1]$$

Finding a suitable value of r ;

$$\text{Equating } x^{-r} = x^{-8} \therefore r = 8$$

$$\Rightarrow U_9 = {}^{10}C_8 \cdot 10^8 \cdot x^{-8}$$

$$\therefore \text{coefficient} = {}^{10}C_8 \cdot 10^8 = \frac{10!}{2!8!} \times 10^8 = 4.5 \times 10^8$$

7. Write down the middle terms of the expansions

a) $(a + b)^{14}$

b) $(a - b)^{16}$

c) $(1 + x)^9$

Solution

- a) $(a + b)^{14}$, has 15 – terms. Required term is the $\left(\frac{15+1}{2}\right)^{\text{th}}$ value = 8th term

$$\left[\because \text{For odd terms, the } n^{\text{th}} \text{ term is located using } \left(\frac{n+1}{2}\right)^{\text{th}} \text{ value}\right]$$

Using $U_{r+1} = {}^nC_r a^{n-r} b^r$ for $(r+1)^{\text{th}}$ term

$$\Rightarrow 8 = r + 1 \therefore r = 7$$

$$\therefore \text{Middle term} = 8^{\text{th}} \text{ term} = {}^{14}C_7 a^{14-7} b^7 \\ = {}^{14}C_7 a^7 b^7 = 3432 a^7 b^7$$

- b) $(a - b)^{16}$, has 17 – terms(odd). Required term is the $\left(\frac{17+1}{2}\right)^{\text{th}}$ value = 9th term

$$\text{The } 9^{\text{th}} \text{ term} = r + 1 \therefore r = 8$$

$$\Rightarrow U_9 = {}^{16}C_8 a^{16-8} (-b)^8 = {}^{16}C_8 a^8 b^8 = 12\,870 a^8 b^8$$

- c) $(1 + x)^9$, has 10 – terms(even). Hence there are 2 – middle terms.

$$\left[\because \text{For even terms, the } \frac{n^{\text{th}}}{2} \text{ and } \left(\frac{n^{\text{th}}}{2} + 1\right) \text{ are the middle terms}\right]$$

$$\text{Required terms are } \left(\frac{10}{2}\right)^{\text{th}} \text{ and } \left(\frac{10}{2} + 1\right)^{\text{th}} \text{ terms}$$

$$\text{For } 5^{\text{th}} \text{ term, } r + 1 = 5 \therefore r = 4$$

$$\Rightarrow 5^{\text{th}} \text{ term} = {}^9C_4 (1)^{9-4} (x)^4 = {}^9C_4 \cdot x^4 = 126 x^4$$

For 6^{th} term, $r + 1 = 6 \therefore r = 5$
 $\Rightarrow 6^{th} \text{ term} = {}^9C_5(1)^{9-5}(x)^5 = {}^9C_5 \cdot x^5 = 126x^5$

8. Find the last three terms of $(2x + y)^7$

Solution

$(2x + y)^7$ has 8 - terms

Required terms are: 8^{th} , 7^{th} and 6^{th} term each corresponding to r - values: 7, 6 and 5 respectively.

$$\therefore 8^{th} - \text{term} = {}^7C_7(2x)^{7-7}y^7 = y^7$$

$$\therefore 7^{th} - \text{term} = {}^7C_6(2x)^{7-6}y^6 = 7 \times 2xy^6 = 14xy^6$$

$$6^{th} - \text{term} = {}^7C_5(2x)^{7-5}y^5 = 21 \times (2x)^2y^5 = 84x^2y^5$$

9. Write down the terms involving:

a) $x^4 \left(\frac{1}{x}\right)^2$

b) $x^3 \left(\frac{1}{x}\right)^3$, in the expansion of $\left(x + \frac{1}{x}\right)^6$

Solution

For $\left(x + \frac{1}{x}\right)^6$, $a = x$, $b = \frac{1}{x}$, $n = 6$

$$\Rightarrow U_{r+1} = {}^6C_r \cdot x^{6-r} \left(\frac{1}{x}\right)^r = {}^6C_r \cdot x^{6-r} x^{-r} \cdot (1)^r = {}^6C_r \cdot x^{6-2r} \cdot (1)^r$$

- a) Now term involving $x^4 \left(\frac{1}{x}\right)^2 = x^4 \cdot x^{-2} \cdot 1^2 = x^2$ is equated to x^{6-2r}

$$\Rightarrow x^2 = x^{6-2r} \therefore 2 = 6 - 2r \text{ From which } r = 2$$

$$\therefore 3^{rd} - \text{term} = {}^6C_2 \cdot x^{6-2 \times 2} \cdot (1)^2 = 15x^2$$

- b) For term involving $x^3 \left(\frac{1}{x}\right)^3 = x^3 \cdot x^{-3} \cdot (1)^3 = x^0$

$$\text{Equating } x^0 = x^{6-2r} \Rightarrow 0 = 6 - 2r \text{ or } r = 3$$

$$\therefore 4^{th} - \text{term} = {}^6C_3 \cdot x^{6-2 \times 3} \cdot (1)^3 = 20$$

10. Write down the constant terms in the expansion of:

a. $\left(x - \frac{1}{x}\right)^8$

b. $\left(2x^2 - \frac{1}{x}\right)^6$

Solution

A constant term does not contain x. finding the suitable value for r;

a. $\left(x - \frac{1}{x}\right)^8 = {}^8C_r \cdot x^{8-r} \cdot \left(\frac{1}{x}\right)^r = {}^8C_r \cdot x^{8-r} \cdot x^{-r} \cdot (1)^r = {}^8C_r \cdot x^{8-2r} \cdot (1)^r$

$$\text{For a constant term, } x^{8-2r} = x^0 \Rightarrow 8 - 2r = 0 \therefore r = 4$$

$$\text{The constant term} = 5^{th} \text{ term} = {}^8C_4 \cdot x^{8-2 \times 4} \cdot (1)^4 = {}^8C_4 = 70$$

b. $\left(2x^2 - \frac{1}{x}\right)^6 = {}^6C_r \cdot (2x^2)^{6-r} \cdot \left(-\frac{1}{x}\right)^r = {}^6C_r \cdot 2^{6-r} \cdot x^{12-2r} \cdot x^{-r} \cdot (-1)^r$
 $= {}^6C_r \cdot 2^{6-r} \cdot x^{12-3r} \cdot (-1)^r$

$$\text{For a constant term, } x^{12-3r} = x^0 \Rightarrow 12 - 3r = 0 \therefore r = 4$$

$$\text{The constant term} = 5^{th} \text{ term} = {}^6C_4 \cdot 2^{6-4} \cdot x^{12-3 \times 4} \cdot (-1)^4$$

 $= {}^6C_4 \times 4 = 60$

11. Find the coefficients of the terms indicated in the expansions below.

a) $\left(x + \frac{1}{x}\right)^6$, term in x^4

b) $\left(2x + \frac{1}{x}\right)^7$, term in $\frac{1}{x^5}$

c) $\left(x - \frac{2}{x}\right)^8$, term in x^6

d) $\left(2 + \frac{x}{2}\right)^{10}$, term in x^7

Solution

Using $U_{r+1} = {}^nC_r a^{n-r} b^r$ for $(r+1)^{th}$ term for $(a+b)^n$

a) $\left(x + \frac{1}{x}\right)^6 = {}^6C_r \cdot x^{6-r} \cdot \left(\frac{1}{x}\right)^r = {}^6C_r \cdot x^{6-r} \cdot x^{-r} \cdot (1)^r$
 $= {}^6C_r \cdot x^{6-2r} \cdot (1)^r$

For term in x^4 a suitable value of r is;

$$x^4 = x^{6-2r} \Rightarrow 4 = 6 - 2r \therefore r = 1$$

$$\therefore \text{term in } x^4 = {}^6C_1 \cdot x^{6-2 \times 1} \cdot (1)^1 = 6x^4$$

$$\therefore \text{coefficient in } x^4 = 6$$

b) $\left(2x + \frac{1}{x}\right)^7 = {}^7C_r \cdot (2x)^{7-r} \cdot \left(\frac{1}{x}\right)^r = {}^7C_r \cdot 2^{7-r} \cdot x^{7-r} \cdot x^{-r} \cdot (1)^r$
 $= {}^7C_r \cdot 2^{7-r} \cdot x^{7-2r} \cdot (1)^r$

For term in $\frac{1}{x^5}$ a suitable value of r is;

$$x^{-5} = x^{7-2r} \Rightarrow -5 = 7 - 2r \therefore r = 6$$

$$\therefore \text{term in } \frac{1}{x^5} = {}^7C_6 \cdot x^{7-2 \times 6} \cdot (1)^6 = 14x^{-5}$$

$$\therefore \text{coefficient in } \frac{1}{x^5} = 14$$

c) $\left(x - \frac{2}{x}\right)^8 = {}^8C_r \cdot x^{8-r} \cdot \left(-\frac{2}{x}\right)^r = {}^8C_r \cdot x^{8-r} \cdot (-2)^r \cdot x^{-r}$
 $= {}^8C_r \cdot (-2)^r \cdot x^{8-2r}$

For term in x^6 a suitable value of r is;

$$x^6 = x^{8-2r} \Rightarrow 6 = 8 - 2r \therefore r = 1$$

$$\therefore \text{term in } x^6 = {}^8C_1 \cdot x^{8-2 \times 1} \cdot (-2)^1 = -16x^6$$

$$\therefore \text{coefficient in } x^6 = -16$$

d) $\left(2 + \frac{x}{2}\right)^{10} = {}^{10}C_r \cdot 2^{10-r} \cdot \left(\frac{x}{2}\right)^r = {}^{10}C_r \cdot x^r \cdot 2^{10-r} \cdot \left(\frac{1}{2}\right)^r$
 $= {}^{10}C_r \cdot 2^{10-2r} \cdot (1)^r \cdot x^r$

For term in x^7 a suitable value of r is;

$$x^7 = x^r \therefore r = 7$$

$$\therefore \text{term in } x^7 = {}^{10}C_7 \cdot x^7 \cdot 2^{10-2 \times 7} = {}^{10}C_7 \cdot 2^{-4} \cdot x^7 = \frac{15}{2} x^7$$

$$\therefore \text{coefficient in } x^7 = \frac{15}{2}$$

The binomial expansion of $(1+x)^n$

From $(a+b)^n = a^n + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + b^n$, if $a = 1, b = x$, then;

$$(1+x)^n = 1^n + {}^nC_1 (1)^{n-1} x^1 + {}^nC_2 (1)^{n-2} x^2 + \dots + x^n$$

$$= 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + x^n$$

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + x^n \quad (1)$$

$$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots + x^n \quad (2)$$

NOTE:

- 1) The expansion of $(1+x)^n$ has $(n+1)$ - terms.
- 2) The term in x^2 is the 3^{rd} - term, the term in x^3 is the 4^{th} - term and the term in x^r is the $(r+1)^{th}$ - term.

$$\therefore r^{\text{th}} \text{ term} = \binom{n}{r-1} \cdot x^{r-1} = {}^nC_{r-1} \cdot x^{r-1}$$

- 3) The coefficient of $x^r = {}^nC_r = \binom{n}{r}$
- 4) The form of the expansion given in (1) is useful when the coefficient of a larger power of x is required, or when general term is required.
The form of the expansion given in (2) is useful when the first few terms of an expansion are required.
- 5) The expansion of $(a+x)^n$ where n is a positive integer can be done as;

$$(a+b)^n = \left[a \left(1 + \frac{b}{a} \right) \right]^n = a^n \left(1 + \frac{b}{a} \right)^n$$

Replacing x by $\frac{x}{a}$ in the binomial expansion series gives;

$$\begin{aligned} (a+x)^n &= a^n \left[1 + \binom{n}{1} \left(\frac{x}{a} \right) + \binom{n}{2} \left(\frac{x}{a} \right)^2 + \dots + \binom{n}{r} \left(\frac{x}{a} \right)^r + \dots + \binom{n}{n} \left(\frac{x}{a} \right)^n x^n \right] \\ &= a^n + \binom{n}{1} a^{n-1} x + \binom{n}{2} a^{n-2} x^2 + \dots + \binom{n}{r} a^{n-r} x^r + \dots + \binom{n}{n} a^n x^n \end{aligned}$$

$$\begin{aligned} (a+x)^n &= a^n + \binom{n}{1} a^{n-1} x + \binom{n}{2} a^{n-2} x^2 + \dots + \binom{n}{r} a^{n-r} x^r + \dots + \binom{n}{n} a^n x^n \\ (a+x)^n &= a^n + na^{n-1} x + \frac{n(n-1)}{2!} a^{n-2} x^2 + \dots + x^n \end{aligned}$$

Example:

- 1) Write down the first 3 – terms in the expansion in ascending powers of x

a) $\left(1 - \frac{x}{2} \right)^{10}$

b) $(3 - 2x)^8$

Solution

Using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots + x^n$

Replacing x by $-\frac{x}{2}$

a) $\left(1 - \frac{x}{2} \right)^{10} = 1 + 10 \left(-\frac{x}{2} \right) + \frac{10 \times 9}{2!} \left(-\frac{x}{2} \right)^2 + \dots$
 $= 1 - 5x + \frac{45}{4} x^2 + \dots$

b) Using $(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!} a^{n-2}x^2 + \dots + x^n$

Replacing x by $-2x$

$$\begin{aligned} \Rightarrow (3-2x)^8 &= 3^8 + 8(3)^7(-2x) + \frac{8(7)}{2!} (3)^6(-2x)^2 + \dots \\ &= 3^8 - 16(3)^7 x + 112(3)^6 x^2 + \dots \end{aligned}$$

- 2) Write down the binomial expansion in ascending powers of x as far as and including the term in x^2 for each of the following;

a) $(1+x)(1-x)^9$

b) $(1-x)(1+2x)^{10}$

c) $(2+x) \left(1 - \frac{x}{20} \right)^{20}$

d) $(1+x)^2(1-5x)^{14}$

e) $(1+x-2x^2)^8$

f) $(1-x+2x^2)^5$

Solution

a) $(1+x)(1-x)^9 = ?$

Expanding $(1-x)^9$ as far as x^2

$$(1-x)^9 = [1+(-x)]^9 = 1 + 9(-x) + \frac{9(8)}{2!}(-x)^2 + \dots$$

$$= 1 - 9x + 36x^2 + \dots$$

$$\therefore (1+x)(1-x)^9 = (1+x)(1-9x+36x^2)$$

$$= 1(1-9x+36x^2) + x(1-9x+36x^2)$$

$$= 1-9x+36x^2 + x-9x^2 + \dots$$

$$\therefore (1+x)(1-x)^9 = 1-8x+27x^2 \quad [\text{terms in } x^3 \text{ are ignored}]$$

b) $(1-x)(1+2x)^{10}$

Expanding $(1+2x)^{10}$ as far as x^2

$$(1+2x)^{10} = 1 + 10(2x) + \frac{10(9)}{2!}(2x)^2 + \dots$$

$$= 1 + 20x + 180x^2 + \dots$$

$$\therefore (1-x)(1+2x)^{10} = (1-x)(1+20x+180x^2)$$

$$= 1(1+20x+180x^2) - x(1+20x+180x^2)$$

$$= 1+20x+180x^2 - x-20x^2 + \dots$$

$$\therefore (1+x)(1-x)^9 = 1+19x+160x^2 \quad [\text{terms in } x^3 \text{ are ignored}]$$

c) $(2+x)\left(1-\frac{x}{20}\right)^{20}$

Expanding $\left(1-\frac{x}{20}\right)^{20}$ as far as x^2

$$\left(1-\frac{x}{20}\right)^{20} = \left[1+\left(-\frac{x}{20}\right)\right]^{20} = 1 + 20\left(-\frac{x}{20}\right) + \frac{20(19)}{2!}\left(-\frac{x}{20}\right)^2 + \dots$$

$$= 1 - x + \frac{19}{40}x^2 + \dots$$

$$\therefore (2+x)\left(1-\frac{x}{20}\right)^{20} = (2+x)\left(1-x+\frac{19}{40}x^2\right)$$

$$= 2\left(1-x+\frac{19}{40}x^2\right) - x\left(1-x+\frac{19}{40}x^2\right)$$

$$= 2-2x+\frac{38}{40}x^2 + x-x^2 + \dots$$

$$\therefore (2+x)\left(1-\frac{x}{20}\right)^{20} = 2-x-\frac{1}{20}x^2$$

d) $(1+x)^2(1-5x)^{14}$

Expanding $(1+x)^2$ as far as x^2

$$(1+x)^2 = 1 + 2x + \frac{2(1)}{2!}x^2 = 1 + 2x + x^2$$

Expanding $(1-5x)^{14}$ as far as x^2

$$(1-5x)^{14} = [1+(-5x)]^{14} = 1 + 14(-5x) + \frac{14(13)}{2!}(-5x)^2$$

$$= 1 - 70x + 2275x^2$$

$$\therefore (1+x)^2(1-5x)^{14} = (1+2x+x^2)(1-70x+2275x^2)$$

$$= 1(1-70x+2275x^2) + 2x(1-70x+2275x^2) + x^2(1-70x+2275x^2)$$

$$= 1-70x+2275x^2 + 2x-140x^2 + x^2[\text{terms in } x^3 \text{ and higher power terms ignored}]$$

$$\therefore (1+x)^2(1-5x)^{14} = 1-68x+2136x^2$$

e) $(1+x-2x^2)^8$

$$(1+x-2x^2)^8 = [(1+x)-2x^2]^8 = [(1+x)+(-2x^2)]^8$$

$$\text{Using } (a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \dots + \binom{n}{r}a^{n-r}x^r + \dots + \binom{n}{n}a^n x^n$$

Replacing a by $(1+x)$ and x by $(-2x^2)$

$$[(1+x)+(-2x^2)]^8 = (1+x)^8 + \binom{8}{1}(1+x)^7(-2x^2) + \binom{8}{2}(1+x)^6(-2x^2)^2 + \dots$$

Now expanding $(1+x)^8$ and $(1+x)^7$ as far as term in x^2

$$(1+x)^8 = 1 + 8x + \frac{8(7)}{2!}x^2 = 1 + 8x + 28x^2$$

$$(1+x)^7 = 1 + 7x + \frac{7(5)}{2!}x^2 = 1 + 7x + 21x^2$$

The 3rd – term in the expansion of $(1+x-2x^2)^8$ yield higher power terms greater than 2

$$\begin{aligned}\therefore (1+x-2x^2)^8 &= (1+8x+28x^2) + \binom{8}{1}(1+7x+21x^2)(-2x^2) \\ &= 1+8x+28x^2 + \binom{8}{1}(-2x^2) \quad [x^3 \text{ and higher power terms neglected}] \\ &= 1+8x+28x^2 - 16x^2\end{aligned}$$

$$\therefore (1+x-2x^2)^8 = 1+8x+12x^2$$

f) $(1-x+2x^2)^5$

$$(1-x+2x^2)^5 = [(1-x)+2x^2]^5 = (1-x)^5 + \binom{5}{1}(1-x)^4(2x^2) + \dots\dots\dots$$

[other terms neglected because yield unwanted terms]

Now expanding $(1-x)^5$ and $(1-x)^4$ as far as term in x^2

$$(1-x)^5 = 1 - 5x + \frac{5(4)}{2!}x^2 = 1 - 5x + 10x^2$$

$$(1-x)^4 = 1 - 4x + \frac{4(3)}{2!}x^2 = 1 - 4x + 6x^2$$

$$\begin{aligned}\therefore (1-x+2x^2)^5 &= (1-5x+10x^2) + \binom{5}{1}(1-4x+6x^2)(2x^2) \\ &= 1-5x+10x^2 + 10x^2(1-4x+6x^2) \\ &= 1-5x+10x^2 + 10x^2\end{aligned}$$

[x^3 and higher power terms neglected]

$$\therefore (1-x+2x^2)^5 = 1-5x+20x^2$$

Task

Expand the following as far as term in x^3

- $(1-x-x^2)^4$
- $(3+x+x^3)^4$
- $(3-2x+2x^2)^4$
- $(1-x+x^2)^8$

ANSWER:

- $1-4x+2x^2+8x^3$
- $81+108x+54x^2+120x^3$
- $81-216x+324x^2-312x^3$
- $1-8x+36x^2-112x^3$

USING BINOMIAL EXPANSION TO FIND APPROXIMATIONS

- 1) If x is so small that x^2 and higher power terms can be neglected, show that

- $(1-x)^5 \left(2+\frac{x}{2}\right)^{10} \cong 2^9(2-5x)$
- $(2x+3)(1-2x)^{10} \cong 3-58x+500x^2$
- $\frac{x+3}{(1+x)^2} \cong 3-5x+7x^2-9x^3$
- $\frac{1+x}{1-x} \cong 1+2x+2x^2+2x^3$

Solution

- a) $(1-x)^5 \left(2+\frac{x}{2}\right)^{10} \cong 2^9(2-5x)$

Since the **R.H.S** contains terms with highest power as one i.e. x^1 , then, ignoring terms in x^2 and higher power terms

$$(1-x)^5 = 1-5x$$

$$\begin{aligned}\left(2 + \frac{x}{2}\right)^{10} &= 2^{10} + \binom{10}{1} 2^9 \cdot \left(\frac{x}{2}\right) = 2^{10} + 5 \cdot 2^9 x \\ \therefore (1-x)^5 \left(2 + \frac{x}{2}\right)^{10} &\cong (1-5x)(2^{10} + 5 \cdot 2^9 x) \\ &\cong 2^9 [(1-5x)(2+5x)] \quad \because [2^9 \text{ is factorised out}] \\ &\cong 2^9 [2+5x-10x] \quad [x^2 - \text{term ignored}]\end{aligned}$$

$$\therefore (1-x)^5 \left(2 + \frac{x}{2}\right)^{10} \cong 2^9 (2-5x)$$

b) $(2x+3)(1-2x)^{10} \cong 3 - 58x + 500x^2$

Since the **R.H.S** contains terms with highest power as *two* i.e. x^2 , then, ignoring terms in x^3 and higher power terms

$$\begin{aligned}(1-2x)^{10} &= 1 + 10(-2x) + \frac{10(9)}{2!}(-2x)^2 \\ &= 1 - 20x + 180x^2 \\ \therefore (2x+3)(1-2x)^{10} &\cong (2x+3)(1-20x+180x^2) \\ &\cong 2x - 40x^2 + 3 - 60x + 540x^2 \\ \therefore (2x+3)(1-2x)^{10} &\cong 3 - 58x + 500x^2\end{aligned}$$

c) $\frac{x+3}{(1+x)^2} \cong 3 - 5x + 7x^2 - 9x^3$

$$\frac{x+3}{(1+x)^2} = (x+3)(1+x)^{-2}$$

Expanding $(1+x)^{-2}$ as far as term in x^3 neglecting higher power terms

$$\begin{aligned}\Rightarrow (1+x)^{-2} &= 1 - 2x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 \\ &= 1 - 2x + 3x^2 - 4x^3 \\ \therefore \frac{x+3}{(1+x)^2} &\cong (x+3)(1-2x+3x^2-4x^3) \cong x - 2x^2 + 3x^3 + 3 - 6x + 9x^2 - 12x^3 \\ \therefore \frac{x+3}{(1+x)^2} &\cong 3 - 5x + 7x^2 - 9x^3\end{aligned}$$

d. $\frac{1+x}{1-x} \cong 1 + 2x + 2x^2 + 2x^3$

$$\frac{1+x}{1-x} = (1+x)(1-x)^{-1} \approx 1 + 2x + 2x^2 + 2x^3$$

Expanding $(1-x)^{-1}$ as far as term in x^3 neglecting higher power terms

$$\begin{aligned}\Rightarrow (1-x)^{-1} &= 1 + x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 = 1 + x + x^2 + x^3 \\ \therefore \frac{1+x}{1-x} &\cong (1+x)(1+x+x^2+x^3) \\ &\cong 1 + x + x^2 + x^3 + x + x^2 + x^3 \quad [x^4 - \text{term neglected}] \\ \therefore \frac{1+x}{1-x} &\cong 1 + 2x + 2x^2 + 2x^3\end{aligned}$$

NOTE:

In *all* the above examples on binomial theorem, the power on the brackets is an integer not a rational number. i.e. of the form $\frac{p}{q}$.

EXPANSION OF $(1+x)^n$ if $|x| < 1$ AND **n IS RATIONAL** (i.e. of the form p/q)

The binomial theorem used is;

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

Note:

$|x| < 1$ means $-1 < x < +1$ (i.e. validity range of x for expansion)

Application

- Expansion of expressions
- Finding the n^{th} root of a number
- Finding errors in numbers

Examples

- 1) Expand $(1 - 3x)^{\frac{1}{4}}$ in ascending powers of x as far as the term in x^3 . Hence evaluate $\sqrt[4]{13}$ correct to 3 s.f.

Solution

Using $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$ for $|x| < 1$

$$\begin{aligned}\Rightarrow (1 - 3x)^{\frac{1}{4}} &= 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}(-\frac{3}{4})}{2!}(-3x)^2 + \frac{\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})}{3!}(-3x)^3 + \dots \\ &= 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{189}{128}x^3\end{aligned}$$

$(1 - 3x)^{\frac{1}{4}}$ is valid for expansion if $|-3x| < 1$ or $|3x| < 1$ or $-1 < 3x < 1$
 $-\frac{1}{3} < x < \frac{1}{3}$ is the validity interval for expansion.

Since the value of x to be used to estimate $\sqrt[4]{13}$ in the expansion in the range $-\frac{1}{3} < x < \frac{1}{3}$, then

$$\begin{aligned}\sqrt[4]{13} &= \sqrt[4]{(16 - 3)} \quad [\because 13 \text{ is expressed as a difference of a no. with exact } 4^{\text{th}} \text{ root}] \\ &= \sqrt[4]{(16 - 3)} = \sqrt[4]{16 \left(1 - \frac{3}{16}\right)} = 2 \left(1 - \frac{3}{16}\right)^{\frac{1}{4}}\end{aligned}$$

Now expressing x as a rational for the interval $-\frac{1}{3} < x < \frac{1}{3}$

Comparing $\sqrt[4]{\left(1 - \frac{3}{16}\right)}$ with $\sqrt[4]{(1 - 3x)}$

$$\Rightarrow \frac{3}{16} \equiv 3x \therefore x = \frac{1}{16}, \text{ which is in the range } -\frac{1}{3} < x < \frac{1}{3}$$

From $\sqrt[4]{(1 - 3x)} = 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{189}{128}x^3$, for $x = \frac{1}{16}$, then

$$\begin{aligned}\sqrt[4]{\left(1 - \frac{3}{16}\right)} &\cong 1 - \frac{3}{4}\left(\frac{1}{16}\right) - \frac{27}{32}\left(\frac{1}{16}\right)^2 - \frac{189}{128}\left(\frac{1}{16}\right)^3 \cong 0.94946812 \\ \therefore \sqrt[4]{13} &= 2 \times 0.94946812 \approx 1.90\end{aligned}$$

- 2) Expand $(1 - 16x)^{\frac{1}{4}}$ as far as x^3 . Hence evaluate $\sqrt[4]{39}$ correct to 5 s.f. Take $x = \frac{1}{10000}$

Solution

$$\begin{aligned}(1 - 16x)^{\frac{1}{4}} &= 1 + \frac{1}{4}(-16x) + \frac{\frac{1}{4}(-\frac{3}{4})}{2!}(-16x)^2 + \frac{\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})}{3!}(-16x)^3 + \dots \\ &= 1 - 4x - 24x^2 - 224x^3\end{aligned}$$

$$\Rightarrow \left(1 - \frac{16}{10000}\right)^{\frac{1}{4}} = 1 - \frac{4}{10000} - \frac{24}{10000^2} - \frac{224}{10000^3}$$

$$\left(\frac{9984}{10000}\right)^{\frac{1}{4}} \cong 0.99996$$

$$\frac{\sqrt[4]{256 \times 39}}{10} \cong 0.99996 \therefore \sqrt[4]{39} \cong \frac{0.99996 \times 10}{\sqrt[4]{256}} \cong \frac{0.99996 \times 10}{4}$$

$$\therefore \sqrt[4]{39} \cong 2.4999$$

NOTE:

- 1) In general, if x is small so that successive powers of x quickly become negligible in value, then the sum of the first few terms in the expansion of $(1+x)^n$ gives the approximate value for $(1+x)^n$. The number of terms required to obtain a good approximation depends on;
 - a) The value of x (the smaller x is, the fewer are the terms needed to obtain a good approximation)
 - b) The accuracy required (an answer correct to 3 s.f needs fewer terms than the answer to 6 s.f)
- 2) When finding an approximation, the binomial expansion of $(1+x)^n$ and not $(a+x)^n$ should be used if n is rational. *e.g.* Find an approximate value of $(3.006)^{\frac{1}{5}}$. here use $3^{\frac{1}{5}}(1+0.002)^{\frac{1}{5}}$
- 3) Expand $(1-x)^{\frac{1}{3}}$ in ascending powers of x as far as the fourth term. Taking $x = \frac{1}{1000}$, find $\sqrt[3]{37}$ to 6. sf.

Solution

$$\begin{aligned}(1-x)^{\frac{1}{3}} &= [1+(-x)]^{\frac{1}{3}} = 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}(-x)^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}(-x)^3 \\ &= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3\end{aligned}$$

Putting $x = \frac{1}{1000}$

$$\Rightarrow \left(1 - \frac{1}{1000}\right)^{\frac{1}{3}} = 1 - \frac{1}{9 \times 10^6} - \frac{5}{81 \times 10^9} = \frac{\sqrt[3]{999}}{10} = \frac{2998}{2999}$$

$$\therefore \sqrt[3]{999} = \sqrt[3]{27 \times 37} = \frac{2998}{2999}$$

$$\therefore \sqrt[3]{37} = \frac{29980}{2999 \times \sqrt[3]{27}} \cong 3.33222$$

- 4) Expand $(1+8x)^{\frac{1}{2}}$ as far as x^2 . Hence find $\sqrt{3}$ to 3. p.

Solution

$$\begin{aligned}(1+8x)^{\frac{1}{2}} &\cong 1 + \frac{1}{2}(8x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(8x)^2 \\ &\cong 1 + 4x - 8x^2\end{aligned}$$

Since the expansion is valid for $|8x| < 1$ or $-1 < 8x < +1$ or $-\frac{1}{8} < x < \frac{1}{8}$, x 's suitable substitution should be selected to find $\sqrt{3}$ but within this interval.

$$\text{From } \sqrt{3} = \sqrt{(4-1)} = 2\sqrt{\left(1-\frac{1}{4}\right)}$$

$$\text{Comparing } 1+8x \equiv 1-\frac{1}{4} \Rightarrow x = -\frac{1}{32}$$

$$\text{From } (1+8x)^{\frac{1}{2}} \cong 1+4x-8x^2, \text{ for } x = -\frac{1}{32}$$

$$\sqrt{\left(1-\frac{1}{4}\right)} \cong 1 - \frac{1}{8} - \frac{8}{32^2}$$

$$\therefore \frac{\sqrt{3}}{2} \cong \frac{111}{128} \text{ or } \sqrt{3} \cong \frac{111}{64}$$

Finding suitable value of x for substitution

Let $x = \frac{1}{a}$, where 'a' has an exact square root

$$\Rightarrow 1+8x = 1 + \frac{8}{a} = \frac{a+8}{a} = 3N, N = \text{no. also with exact sq. root}$$

Using try and error method

For $a = 4$, $1 + 8x = \frac{4+8}{4} = \frac{12}{4} = \frac{3 \times 4}{4}$. This is convenient but $-\frac{1}{8} < x < \frac{1}{8}$, hence $x = \frac{1}{4}$ is not in the range hence rejected.

For $a = 16$, $1 + 8x = \frac{16+8}{16} = \frac{24}{16} = \frac{3 \times 8}{16}$, 8 has no exact square root

For $a = 25$, $1 + 8x = \frac{25+8}{25} = \frac{33}{25} = \frac{3 \times 11}{25}$, 25 has no exact square root

For $a = 100$, $1 + 8x = \frac{100+8}{100} = \frac{108}{100} = \frac{3 \times 36}{100}$, 36 has an exact square root

$\therefore x = \frac{1}{100}$ for $-\frac{1}{8} < x < \frac{1}{8}$ can be use.

From $(1 + 8x)^{\frac{1}{2}} \cong 1 + 4x - 8x^2$, $x = \frac{1}{100}$

$$\left(1 + \frac{8}{100}\right)^{\frac{1}{2}} \cong 1 + \frac{4}{100} - \frac{8}{100^2}$$

$$\frac{\sqrt{108}}{10} \cong 1.0392 \therefore \sqrt{3} \cong \frac{10.392}{\sqrt{36}} \cong 1.732$$

- 5) Expand $(1 - 3x)^{\frac{1}{4}}$ in ascending powers of x up to the term in x^3 . Hence evaluate $\sqrt[4]{13}$

Solution

$$\begin{aligned}(1 - 3x)^{\frac{1}{4}} &\cong 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}(-\frac{3}{4})}{2!}(-3x)^2 + \frac{\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})}{3!}(-3x)^3 \\ &\cong 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{819}{128}x^3\end{aligned}$$

Now to find $\sqrt[4]{13}$, we need a suitable value of x -value in the range $|-3x| < 1$ or $-\frac{1}{3} < x < \frac{1}{3}$

Br try and error method

Let $x = \frac{1}{a}$, $a = \text{no. with exact } 4^{\text{th}} \text{ root}$

$$\Rightarrow 1 - 3x = 1 - \frac{3}{a} = \frac{a-3}{a} = \frac{13-N}{a} \quad N = \text{no. with exact } 4^{\text{th}} \text{ root}$$

$$\text{For } a = 16, 1 - 3x = \frac{16-3}{16} = \frac{13}{16} = \frac{13(1)^4}{16}$$

Hence $x = \frac{1}{16}$ can be used

$$\begin{aligned}(1 - 3x)^{\frac{1}{4}} &\cong 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}(-\frac{3}{4})}{2!}(-3x)^2 + \frac{\frac{1}{4}(-\frac{3}{4})(-\frac{7}{4})}{3!}(-3x)^3 \\ \left(1 - \frac{3}{16}\right)^{\frac{1}{4}} &\cong 1 - \frac{3}{4}\left(\frac{3}{16}\right) - \frac{27}{32}\left(\frac{3}{16}\right)^2 - \frac{819}{128}\left(\frac{3}{16}\right)^3 \\ &\cong 0.9495\end{aligned}$$

$$\therefore \sqrt[4]{13} \cong 0.9495 \times \sqrt[4]{16} \cong 1.899$$

- 6) Expand $(1 - x)^{\frac{1}{3}}$ in ascending powers of x as far as 4^{th} term. By taking the first two terms of x find the value of $\sqrt[3]{37}$, correct to 6 s. f.

Solution

$$(1 - x)^{\frac{1}{3}} \cong 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}(-x)^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}(-x)^3$$

$$\cong 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$$

Finding a suitable value of x for approximation of $\sqrt[3]{37}$

Let $x = \frac{1}{a}$, $a = \text{no. with exact cube root}$

$$\Rightarrow 1 - x = 1 - \frac{1}{a} = \frac{a-1}{a} = \frac{37N}{a}, N = \text{no. with exact cube root}$$

For $a = 8$, $1 - x = \frac{8-1}{8} = \frac{7}{8}$, this does not contain 37

For $a = 64$, $1 - x = \frac{63}{64}$, which does not contain 37

⋮

For $a = 1000$, $1 - x = \frac{999}{1000} = \frac{27 \times 37}{1000}$, this holds

Taking $x = \frac{1}{1000}$ in the interval $-1 < x < 1$

$$(1 - x)^{\frac{1}{3}} \cong 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$$

$$\left(1 - \frac{1}{1000}\right)^{\frac{1}{3}} \cong 1 - \frac{1}{3}\left(\frac{1}{1000}\right) \cong \frac{2999}{3000} \cong \frac{\sqrt[3]{999}}{10} \cong \frac{2999}{3000}$$

$$\sqrt[3]{37 \times 27} \cong \frac{2999}{300}$$

$$\therefore \sqrt[3]{37} \cong \frac{2999}{300} \times \frac{1}{3} \cong 3.33222$$

NOTE:

Examples (4), (5) and (6) illustrate how you can use try and error method to obtain a suitable value of x for approximation of n^{th} root using binomial expansion.

See example below.

- 7) Expand $(1 + x)^{\frac{1}{3}}$ up to term in x^2 . Hence evaluate $\sqrt[3]{9}$ correct to 3 d.p.

Solution

$$(1 + x)^{\frac{1}{3}} \cong 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2 \cong 1 + \frac{1}{3}x - \frac{1}{9}x^2$$

To find $\sqrt[3]{9}$, a suitable x value must be got.

Let $x = \frac{1}{a}$, $a = \text{no. with exact cube root}$

$$\Rightarrow 1 + x = 1 + \frac{1}{a} = \frac{a+1}{a} = \frac{9N}{a}, N = \text{no. with exact cube root}$$

For $a = 8$, $1 + x = \frac{8+1}{8} = \frac{9}{8} = \frac{9(1)^3}{8}$, hence $x = \frac{1}{8}$ can be used

$$(1 + x)^{\frac{1}{3}} \cong 1 + \frac{1}{3}x - \frac{1}{9}x^2$$

$$\left(1 + \frac{1}{8}\right)^{\frac{1}{3}} \cong 1 + \frac{1}{3}\left(\frac{1}{8}\right) - \frac{1}{9}\left(\frac{1}{8}\right)^2 \cong \frac{601}{576}$$

$$\therefore \sqrt[3]{9} \cong \frac{601}{576} \times 2 \cong 2.087$$

Task:

Expand $(1 - 3x)^{\frac{1}{3}}$ in ascending powers of x as far as the term in x^3 . Hence evaluate $\sqrt[3]{5}$

Hint: $x = \frac{1}{8}$

More examples on Binomial theorem

1. Use binomial theorem to find

- a) $\sqrt{1.001}$ to 6.p
- b) $\frac{1}{(1.02)^2}$ to 4d.p
- c) $\sqrt{0.998}$ to 6.p
- d) $\frac{1}{\sqrt{0.98}}$ to 4d.p

Solution

- a) $\sqrt{1.001} = (1 + 0.001)^{\frac{1}{2}}$; let $x = 0.001$
 $(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$
 Now $x = 0.001$
 $\Rightarrow (1 + 0.001)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(0.001) - \frac{1}{8}(0.001)^2 + \frac{1}{16}(0.001)^3 \cong 1.000500$
- b) $\frac{1}{(1.02)^2} = (1 + 0.02)^{-2}$; let $x = 0.02$
 $(1+x)^{-2} \cong 1 - 2x + \frac{-2(-3)}{2!}x^2 + \frac{-2(-3)(-4)}{3!}x^3 \cong 1 - 2x + 3x^2 - 4x^3$
 Now $x = 0.02$
 $\Rightarrow (1 + 0.02)^{-2} \cong 1 - 2(0.02) + 3(0.02)^2 - 4(0.02)^3$
 $\therefore \frac{1}{(1.02)^2} \cong 0.9612$
- c) $\sqrt{0.998} = \sqrt{1 - 0.002} = (1 - 0.002)^{\frac{1}{2}}$
 $(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$
 Now $x = -0.002$
 $\Rightarrow (1 - 0.002)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(-0.002) - \frac{1}{8}(-0.002)^2 + \frac{1}{16}(-0.002)^3$
 $\therefore \sqrt{0.998} \cong 0.998999$
- d) $\frac{1}{\sqrt{0.98}} = (0.98)^{-\frac{1}{2}} = (1 - 0.02)^{-\frac{1}{2}}$; let $x = -0.02$
 $(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$
 Now $x = -0.02$
 $\Rightarrow (1 - 0.02)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(-0.02) - \frac{1}{8}(-0.02)^2 + \frac{1}{16}(-0.02)^3$
 $\therefore \frac{1}{\sqrt{0.98}} \cong 1.0102$

Task:

Use binomial theorem to find $\sqrt[3]{1.03}$ correct to 5s.f

2. Use Binomial theorem to evaluate

- a) $\sqrt{23}$ to 3 d.p
- b) $\sqrt{37}$ to 5 s.f
- c) $\sqrt[3]{8.01}$ to 5 s.f
- d) $\sqrt[3]{64.08}$ to 5 s.f

Solution

- a) $\sqrt{23} = \sqrt{25 - 2} = \sqrt{25\left(1 - \frac{2}{25}\right)} = 5\left(1 - \frac{2}{25}\right)^{\frac{1}{2}}$
 From $(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$$\begin{aligned}\therefore \sqrt{23} &\cong 5 \left[1 + \frac{1}{2} \left(-\frac{2}{25} \right) - \frac{1}{8} \left(-\frac{2}{25} \right)^2 \right] \text{ [higher power terms ignored]} \\ &\cong 4.796\end{aligned}$$

NOTE:

$$\left| -\frac{2}{25} \right| < 1 \text{ i.e. } -1 < \frac{2}{25} < 1$$

$$\begin{aligned}\text{b) } \sqrt{37} &= \sqrt{36+1} = \sqrt{36 \left(1 + \frac{1}{36} \right)} = 6 \left(1 + \frac{1}{36} \right)^{\frac{1}{2}} \\ \therefore \sqrt{37} &\cong 6 \left[1 + \frac{1}{2} \left(\frac{1}{36} \right) - \frac{1}{8} \left(\frac{1}{36} \right)^2 \right] \text{ [higher power terms ignored]} \cong 6.0828\end{aligned}$$

NOTE:

$\sqrt{36+1}$, could not be expanded directly by binomial because $|36| > 1$ yet the required interval of $(1+x)^n$ expansion is $|x| < 1$.

$$\begin{aligned}\text{c) } \sqrt[3]{8.01} &= \sqrt[3]{8+0.01} = \sqrt[3]{8 \left(1 + \frac{0.01}{8} \right)} = 2 \left(1 + \frac{0.01}{8} \right)^{\frac{1}{3}} \\ &\cong 2 \left[1 + \frac{1}{3} \left(\frac{0.01}{8} \right) + \frac{\left(\frac{1}{3} \right) \left(-\frac{2}{3} \right)}{2!} \left(\frac{0.01}{8} \right)^2 \right] \\ \therefore \sqrt[3]{8.01} &\cong 2 \left[1 + \frac{0.01}{24} \right] \cong 2.0008 \\ \text{d) } \sqrt[3]{64.08} &= \sqrt[3]{64+0.08} = \sqrt[3]{64 \left(1 + \frac{0.08}{64} \right)} = 4 \left(1 + \frac{0.08}{64} \right)^{\frac{1}{3}} \\ \therefore \sqrt[3]{64.08} &\cong 4 \left[1 + \frac{0.08}{64 \times 3} \right] \text{ [higher power terms ignored]} \cong 4.0017\end{aligned}$$

Proofs in Binomial expansion

1. Show that, if x is small enough for terms in x^3 and higher power terms neglected,

$$\sqrt{\frac{1-x}{1+x}} \cong 1 - x + \frac{x^2}{2}$$

By putting $x = \frac{1}{8}$, show that $\sqrt{7} \cong 2 \frac{83}{128}$

Solution

$$\begin{aligned}\sqrt{\frac{1-x}{1+x}} &= [(1-x)(1+x)^{-1}]^{\frac{1}{2}} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} \\ (1-x)^{\frac{1}{2}} &\cong 1 + \frac{1}{2}(-x) + \frac{\left(\frac{1}{2} \right) \left(-\frac{1}{2} \right)}{2!} (-x)^2 \\ &\cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 \\ (1+x)^{-\frac{1}{2}} &\cong 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right)}{2!} x^2 \\ &\cong 1 - \frac{1}{2}x + \frac{3}{8}x^2 \\ \therefore \sqrt{\frac{1-x}{1+x}} &\cong \left[1 + \frac{1}{2}x - \frac{1}{8}x^2 \right] \left[1 - \frac{1}{2}x + \frac{3}{8}x^2 \right] \\ &\cong 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 \cong 1 - x + \frac{x^2}{2}\end{aligned}$$

[expansion of brackets yield terms in x^3 and
[higher power terms which have been ignored]

Now $x = \frac{1}{8}$

$$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} \cong 1 - \frac{1}{8} + \frac{1}{2}\left(\frac{1}{8}\right)^2 \cong \sqrt{\left(\frac{7}{9}\right)} \cong \frac{113}{128}$$

$$\therefore \sqrt{7} \cong 3 \times \frac{113}{1128} = 2 \frac{83}{128}$$

Task:

If x is so small that its cube and higher powers can be neglected, show that

$$\sqrt{\frac{1+x}{1-x}} \cong 1 + x + \frac{1}{2}x^2. \text{ By putting } x = \frac{1}{17}, \text{ show that } \sqrt{2} \cong 1 \frac{359}{867}$$

2. Show that the first 3 – terms in the expansion in ascending powers of x of $(1+8x)^{\frac{1}{4}}$ are the same as the first 3 – terms in the expansion $\frac{1+5x}{1+3x}$. Use the corresponding approximation $(1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x}$ to obtain approximation to $(1.16)^{\frac{1}{4}}$ as a rational fraction in its lowest form.

Solution

$$(1+8x)^{\frac{1}{4}} \cong 1 + \frac{1}{4}(8x) + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(8x)^2$$

$$\cong 1 + 2x - 6x^2$$

$$\frac{1+5x}{1+3x} \cong (1+5x)(1+3x)^{-1}$$

$$\cong \left[1 + (-1)(3x) + \frac{(-1)(-2)}{2!}(3x)^2\right](1+5x)$$

$$\cong (1-3x+9x^2)(1+5x)$$

$$\cong 1-3x+9x^2+5x-15x^2$$

$$\cong 1+2x-6x^2$$

$$\therefore (1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x} \cong 1+2x-6x^2$$

$$\text{Using } (1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x} \cong 1+2x-6x^2$$

$$(1.16)^{\frac{1}{4}} = (1+0.16)^{\frac{1}{4}}$$

$$\text{By comparison, } 8x = 0.16 \therefore x = 0.02$$

$$\Rightarrow (1+0.16)^{\frac{1}{4}} \cong \frac{1+5(0.02)}{1+3(0.02)}$$

$$\cong \frac{1.1}{1.06} = \frac{59}{53}$$

3. a) It is given that $f(x) = \frac{1}{(1+x)^2} + \sqrt{4+x}$. Show that if x^3 and higher powers are ignored, $f(x) \cong a + bx + cx^2$, and find the values of a , b and c
- b) Show that $3^n = \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^n\binom{n}{n}$. Hence evaluate 3^6
- c) Given that the first three terms in the expansion in ascending powers of x of $(1-8x)^{\frac{1}{4}}$ are the same as the first 3-terms in the expansion of $\frac{1+ax}{1+bx}$. Find the values of a and b . hence find an approximation to $(0.6)^{\frac{1}{4}}$ in the form $\frac{p}{q}$.

Solution

a) $f(x) = \frac{1}{(1+x)^2} + \sqrt{4+x} = (1+x)^{-2} + \sqrt{4\left(1+\frac{x}{4}\right)}$

$$(1+x)^{-2} \cong 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 \cong 1 - 2x + 3x^2$$

$$\begin{aligned}\sqrt{4\left(1+\frac{x}{4}\right)} &= 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} \cong 2\left[1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2\right] \\ &\cong 2\left[1+\frac{x}{8}-\frac{1}{128}x^2\right] \cong 2+\frac{x}{4}-\frac{1}{64}x^2\end{aligned}$$

$$\therefore f(x) \cong 1-2x+3x^2+2+\frac{x}{4}-\frac{1}{64}x^2$$

$$\cong 3-x+\frac{191}{64}x^2$$

$$3-x+\frac{191}{64}x^2 \equiv a+bx+cx^2, \text{ comparing corresponding coefficients;}$$

$$\therefore a=3, b=-1, c=\frac{191}{64}$$

$$\text{b) } 3^n = \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^n \binom{n}{n}$$

$$3^n = (1+2)^n; \text{ By binomial theorem}$$

$$3^n = 1 + {}^nC_1 \cdot (2)^1 + {}^nC_2 \cdot (2)^2 + \dots + {}^nC_n (2)^n$$

$$3^n = \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^n \binom{n}{n}$$

$$\text{For } n=6, 3^6 = \binom{6}{0} + (2)^1 \binom{6}{1} + (2)^2 \binom{6}{2} + \dots + 2^6 \binom{6}{6}$$

$$3^6 = 1 + 12 + 60 + 8 \cdot {}^6C_3 + 16 \cdot {}^6C_4 + 32 \cdot {}^6C_5 + 64 \cdot {}^6C_6$$

$$= 729$$

$$\text{c) } (1-8x)^{\frac{1}{4}} \cong \frac{1+ax}{1+bx}$$

$$(1-8x)^{\frac{1}{4}} \cong 1 + \frac{1}{4}(-8x) + \frac{\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)}{2!}(-8x)^2$$

$$\cong 1-2x-6x^2$$

$$\frac{1+ax}{1+bx} \cong (1+ax)(1+bx)^{-1}$$

$$\cong \left[1 + (-1)(bx) + \frac{(-1)(-2)}{2!}(bx)^2\right](1+ax)$$

$$\cong (1-bx+b^2x^2)(1+ax)$$

$$\cong 1-bx+b^2x^2+ax-abx^2$$

$$[\text{terms in } x^3 \text{ ignored } \therefore \text{ do not make the } 1^{\text{st}} \text{ 3-terms}]$$

$$\cong 1+(a-b)x+(b^2-ab)x^2$$

$$\therefore 1-2x-6x^2 \equiv 1+(a-b)x+(b^2-ab)x^2$$

By comparison of coefficients of corresponding terms,

$$\text{For } x: -2 = a-b \dots\dots\dots (1)$$

$$\text{For } x^2: -6 = b^2-ab = -b(a-b) \dots\dots\dots (2)$$

$$\text{Eqn. (1) } \div \text{ eqn. (2)}$$

$$-\frac{2}{-6} = \frac{(a-b)}{-b(a-b)} = \frac{1}{-b} \therefore b = -3$$

$$\text{From (1) } a = b-2 = -3-2 = -5$$

$$\therefore (1-8x)^{\frac{1}{4}} \cong \frac{1-5x}{1-3x} \cong 1-2x-6x^2$$

$$(0.6)^{\frac{1}{4}} = (1-0.4)^{\frac{1}{4}} = (1-8x)^{\frac{1}{4}}$$

$$\Rightarrow 0.4 = 8x \therefore x = 0.05$$

$$\text{From } (1-8x)^{\frac{1}{4}} \cong \frac{1-5x}{1-3x} \cong 1-2x-6x^2, \text{ For } x = 0.05$$

$$(0.6)^{\frac{1}{4}} \cong \frac{1-5(0.05)}{1-3(0.05)} = \frac{0.75}{0.85} = \frac{15}{17}$$

Task:

1. Expand $\sqrt{\frac{1+2x}{1-2x}}$ as a series of ascending powers of x up to and including terms in x^2 . Using $x = \frac{1}{100}$, find an approximation of $\sqrt{51}$ stating the number of significant figures to which your answer is accurate. **ANSWER: 7.141** (4 s.f)
2. Find the first three terms of the binomial expansion $\frac{2(1+x)}{\sqrt{1-\frac{x}{4}}}$. Hence evaluate $\frac{10}{\sqrt{15}}$

correct to 3d.p.

HINT:

$$\begin{aligned}\frac{2(1+x)}{\sqrt{1-\frac{x}{4}}} &= 2(1+x) \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}} \cong 2(1+x) \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2\right] \\ &\cong 2(1+x) \left[1 + \frac{x}{8} + \frac{3}{128}x^2\right] \\ &\cong 2 \left[1 + \frac{x}{8} + \frac{3}{128}x^2 + x + \frac{x^2}{8}\right] \\ &\cong 2 \left(1 + \frac{9}{8}x + \frac{19}{128}x^2\right)\end{aligned}$$

Now finding suitable value of x for evaluation of $\frac{10}{\sqrt{15}}$

$$\begin{aligned}\Rightarrow \frac{2(1+x)}{\sqrt{1-\frac{x}{4}}} &= \frac{10}{\sqrt{15}} \text{ or } \frac{(1+x)}{\sqrt{1-\frac{x}{4}}} = \frac{5}{\sqrt{15}} \\ \Rightarrow \frac{(1+x)^2}{1-\frac{x}{4}} &= \frac{25}{15} = \frac{5}{3} \text{ or } \frac{4(1+x)^2}{4-x} = \frac{5}{3} \\ \Rightarrow 12 + 24x + 12x^2 &= 20 - 5x \text{ or } 12x^2 + 29x - 8 = 0 \\ \text{Solving } x &= \frac{1}{4}, -\frac{8}{3}\end{aligned}$$

Since the expansion $\sqrt{1-\frac{x}{4}}$ is valid for $\left|\frac{x}{4}\right| < 1$ or $|x| < 4$, then taking $x = \frac{1}{4}$

$$\begin{aligned}\Rightarrow \frac{2\left(1+\frac{1}{4}\right)}{\sqrt{1-\frac{1}{16}}} &\cong 2 \left[1 + \frac{9}{8}\left(\frac{1}{4}\right) + \frac{19}{128}\left(\frac{1}{4}\right)^2\right] \\ \Rightarrow \frac{\frac{10}{4}}{\sqrt{\left(\frac{15}{16}\right)}} &\cong \frac{264 \times 2}{2048} \\ \therefore \frac{10}{\sqrt{15}} &\cong 2.581\end{aligned}$$

3. Determine the binomial expansion of $\left(1 + \frac{x}{2}\right)^4$. Hence evaluate $(2.1)^4$ correct to 2d.p (**Uneb 2005**)

HINT:

$$\begin{aligned}\left(1 + \frac{x}{2}\right)^4 &= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} \\ (2.1)^4 &= (2 + 0.1)^4 = \left[2\left(1 + \frac{0.1}{2}\right)\right]^4 = 16\left(1 + \frac{0.1}{2}\right)^4 \\ \text{But } \frac{x}{2} &= \frac{0.1}{2} \therefore x = 0.1 \\ \Rightarrow 16\left(1 + \frac{0.1}{2}\right)^4 &\cong 16\left(1 + 2(0.1) + \frac{3}{2}(0.1)^2 + \frac{(0.1)^3}{2}\right) \cong 19.448\end{aligned}$$

4. Expand $\sqrt{\frac{1+5x}{1-5x}}$ as far as and including the term in x^3 . Taking the first three terms and $x = \frac{1}{9}$ evaluate $\sqrt{14}$

5. Show that $\sqrt{1+4ax} \cong 1 + 2ax - 2a^2x^2 + 4a^3x^3 + \dots$, and deduce the expansion for $\sqrt{1-4x}$ up to the term in x^4 . Letting $x = \frac{1}{10}$, evaluate $\sqrt{6}$ to 5 d.p.
HINT: comparing $\sqrt{1+4ax}$ with $\sqrt{1-4x}$, $a = -1$
6. Expand $(1-x)^{\frac{1}{3}}$ in ascending powers of x up to term in x^3 . Hence evaluate $\sqrt[3]{998}$.
Hint: $\sqrt[3]{998} = \sqrt[3]{1000-2} = \sqrt[3]{1000(1-0.002)} = 10 \sqrt[3]{1-0.002}$
7. If x is sufficiently small and allow any terms in x^5 or higher power terms be neglected, show that $(1+x)^6(1-2x^3)^{10} \cong 1 + 6x + 15x^2 - 105x^4$
8. Expand using binomial theorem $(1+4x)^{\frac{1}{2}}$ up to the fourth term. Hence evaluate $\sqrt{6}$ to 3 d.p using $x = \frac{1}{100}$
9. a) Expand $\sqrt{\frac{1+x}{1-x}}$ in ascending powers of x to a term in x^2 .
 b) (i) Using the expansion of $\sqrt{1+x}$ up to the term in x^3 , find the value of $\sqrt{1.08}$ to 4 d.p.
 (ii) Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$. Hence evaluate $\sqrt{3}$ to 3 s.f. (**Unib 2010**)
10. show that if x is so small in comparison with unity that x^3 and higher power terms can be neglected, $\frac{(1-4x)^{\frac{1}{2}}(1+3x)^{\frac{1}{3}}}{\sqrt{1+x}} \cong 1 - \frac{3}{2}x - \frac{33}{8}x^2$

PERCENTAGES AND ERRORS IN BINOMIAL APPROXIMATION

The knowledge of measurement is applicable.

Examples

- 1) The radius of a cylinder is reduced by 3% and its height increased by 4%. Determine the appropriate percentage change in;
 a) Its volume
 b) In curved surface area
 (Neglect product of small changes)

Solution

- a) Let r and h be the original values of radius and height respectively
 Volume of cylinder, $V = \pi r^2 h$
 New values of r and h are respectively; $\left(\frac{100-3}{100}\right)r$, $\left(\frac{100+4}{100}\right)h$
 New volume, $V' = \pi \left[\left(\frac{100-3}{100}\right)r\right]^2 \left(\frac{100+4}{100}\right)h$

$$= \pi(1-0.03)^2 r^2 \cdot (1+0.04)h$$

$$= \pi r^2 h \cdot (1-0.03)^2 \cdot (1+0.04)$$
 But $(1-0.03)^2 \cong 1 + 2(-0.03)$

$$\cong 1 - 0.06 \text{ (Neglecting product of small changes)}$$

$$\therefore V' = \pi r^2 h \cdot (1-0.06) \cdot (1+0.04)$$

$$= \pi r^2 h \cdot [1 + 0.04 - 0.06] \text{ (Neglecting product of small changes)}$$

$$= 0.98\pi r^2 h$$

$$\% \text{ge change in } V = \left(\frac{V-V'}{V}\right) \times 100$$

$$= \left(\frac{\pi r^2 h - 0.98\pi r^2 h}{\pi r^2 h}\right) \times 100 = 2\%$$

$\therefore$ Volume reduced by 2%

- b. Curved surface area of a cylinder, $A = 2\pi rh$

$$\text{New curved surface area, } A' = 2\pi \left[\left(\frac{100-3}{100} \right) r \cdot \left(\frac{100+4}{100} \right) h \right]$$

$$\Rightarrow A' = 2\pi rh \cdot (1 - 0.03) \cdot (1 + 0.04)$$

$$= 2\pi rh[1 + 0.04 - 0.03] \text{ [Products of higher smaller terms ignored]}$$

$$\therefore A' = 2\pi rh \cdot (1.01)$$

$$\% \text{ge change in } A = \left(\frac{A-A'}{A} \right) \times 100$$

$$= \left(\frac{2\pi rh \cdot (1.01) - 2\pi rh}{2\pi rh} \right) \times 100 = 1\%$$

$\therefore$ Curved surface area increased by 1%

- 2) Pressure P and volume V are related by the expression $PV^3 = C$ where C is a constant. Find the appropriate percentage change in C when P is increased by 2% and volume decreased by 0.8%.

Solution

Let P and V be original values of pressure and volume respectively.

$$\text{New value of } P = \left(\frac{100+2}{100} \right) P = (1 + 0.02)P$$

$$\text{New value of } V = \left(\frac{100-0.8}{100} \right) V = (1 - 0.008)V$$

$$\text{Now new value of } C = C' = (1 + 0.02)P[(1 - 0.008)V]^3$$

$$\Rightarrow C' = PV^3 \cdot (1 + 0.02) \cdot (1 - 0.008)^3$$

But $(1 - 0.008)^3 \cong 1 - 3(0.008)$, higher power terms ignored

$$\therefore C' = PV^3 \cdot (1 + 0.02) \cdot (1 - 3(0.008))$$

$$= PV^3 \cdot [1 - 3(0.008) + 0.02] \text{ , products of smaller terms ignored}$$

$$= PV^3 \cdot (0.996)$$

$$\therefore \% \text{ge change in } C = \left(\frac{C-C'}{C} \right) \times 100 = \left(\frac{PV^3 - PV^3 \cdot (0.996)}{PV^3} \right) \times 100$$

$$\therefore \% \text{ge change in } C \text{ (Reduction)} = 0.4\%$$

- 3) An error is made in measuring the radius of a sphere. Find the percentage error in surface area.

Solution

Let the error lead to increase in radius.

Area of sphere, $A = 4\pi r^2$ where A and r are original values.

$$\text{New value of } r = \left(1 + \frac{2}{100} \right) r = (1 + 0.02)r$$

$$\text{New value of } A = A' = 4\pi[(1 + 0.02)r]^2$$

$$= 4\pi r^2 (1 + 0.02)^2$$

$$= 4\pi r^2 (1 + 2(0.02)) = 4\pi r^2 (1.04)$$

$$\therefore \% \text{ge change in } A = \left(\frac{A-A'}{A} \right) \times 100$$

$$= \left(\frac{4\pi r^2 (1.04) - 4\pi r^2}{4\pi r^2} \right) \times 100 = 4\%$$

NOTE:

If error was to lead to a decrease in r you get the same percentage change 4%.

- 4) An error of $2\frac{1}{2}\%$ is made in measurement of the area of a circle. What percentage error results in the radius and circumference?

Solution

Area $A = \pi r^2$, where A and r are original values of area and radius of circle respectively.

New value of $A = \left(\frac{100+2.5}{100}\right)A = (1 + 0.025)A$ [let error lead to increase in A]

$$\text{From } A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$$

$$\text{Now new value of } r = \sqrt{\frac{(1+0.025)A}{\pi}} = \sqrt{\frac{A}{\pi}} \cdot (1 + 0.025)^{\frac{1}{2}}$$

$$= \sqrt{\frac{A}{\pi}} \cdot \left[1 + \frac{1}{2}(0.025)\right], \text{ higher power terms neglected} = 1.0125 \sqrt{\frac{A}{\pi}}$$

$$\% \text{ge change in } r = \left(\frac{1.0125 \sqrt{\frac{A}{\pi}} - \sqrt{\frac{A}{\pi}}}{\sqrt{\frac{A}{\pi}}} \right) \times 100 = 1.25\%$$

Circumference $C = 2\pi r$, where C and r are original values of circumference and radius respectively. Since the error of $2\frac{1}{2}\%$ leads to an increase in r by 1.25%, then;

$$\text{New value of } r = \left(\frac{100+1.25}{100}\right)r = (1 + 0.0125)r$$

$$\therefore \text{New value of } C = 2\pi(1 + 0.0125)r = 2\pi r(1 + 0.0125)$$

$$\therefore \% \text{ge change in } C = \left(\frac{2\pi r(1+0.0125) - 2\pi r}{2\pi r} \right) \times 100 = 1.25\%$$

Alternatively

$$A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$$

$$C = 2\pi r \Rightarrow C = 2\pi \sqrt{\frac{A}{\pi}} = 2\sqrt{\pi} \cdot \sqrt{A} = 2\sqrt{\pi A}$$

New value of A due to error $= (1 + 0.025)A$

$$\text{New value of } C = C' = 2\sqrt{\pi} \cdot \sqrt{(1 + 0.025)A}$$

$$= 2\sqrt{\pi A} \cdot (1 + 0.025)^{\frac{1}{2}}$$

$$\cong 2\sqrt{\pi A} \cdot \left(1 + \frac{0.025}{2}\right) = 2\sqrt{\pi A}(1 + 0.0125), \text{ higher power terms neglected}$$

$$\therefore \% \text{ge change in } C = \left(\frac{2\sqrt{\pi A}(1+0.0125) - 2\sqrt{\pi A}}{2\sqrt{\pi A}} \right) \times 100 = 1.25\% \text{ as before}$$

- 5) One side of a rectangle is 3-times the other. If the perimeter increases by 2%, what is the percentage increase in area?

Solution

$$\text{length} = 3w$$

$P = \text{perimeter}$

 $\text{width} = w \quad P = 2(3w + w) = 8w$

$$\text{New value of } P = P' = \left(\frac{100+2}{100}\right)P = (1 + 0.02)P$$

$$\text{Now area, } A = 3w^2$$

$$\text{From } P = 8w \Rightarrow w = \frac{P}{8}$$

$$\therefore A = 3 \left(\frac{P}{8}\right)^2 = \frac{3}{64}P^2$$

$$\text{New value of } A \text{ (due to increase in } P) = A' = \frac{3}{64}[(1 + 0.02)P]^2$$

$$= \frac{3}{64}P^2((1 + 0.02))^2 = \frac{3}{64}P^2[1 + 2(0.02)], \text{ higher power terms neglected}$$

$$= 1.04 \left(\frac{3}{64} P^2 \right)$$

$$\therefore \% \text{ge in area} = \left[\frac{(1.04-1) \frac{3}{64} P^2}{\frac{3}{64} P^2} \right] \times 100 = 4\%$$

Alternatively

$$P = 8w \Rightarrow w = \frac{P}{8}; \text{New } P = (1 + 0.02)P$$

Finding percentage change in w due to 2% increase in P

$$\text{New } w = w' = \frac{(1+0.02)P}{8}$$

$$\% \text{ge change in } w = \left(\frac{\frac{(1+0.02)P}{8} - \frac{P}{8}}{\frac{P}{8}} \right) \times 100 = 2\%$$

$$\text{Now area, } A = 3w^2$$

$$\begin{aligned} \text{New area due to increase in } w &= 3[(1 + 0.02)w]^2 \\ &= 3w^2(1 + 0.02)^2 \cong 3w^2[1 + 2(0.02)] \\ &= 3w^2 \cdot (1.04) \end{aligned}$$

$$\therefore \% \text{ge in area} = \left(\frac{3w^2 \cdot (1.04) - 3w^2}{3w^2} \right) \times 100 = 4\%$$

- 6) The height cylinder 10cm and its radius is 4cm. find an approximate increase in volume when the radius increases to 4.02.

Solution

Volume, $V = \pi r^2 h$; assuming h is constant

$$\Rightarrow r = 4\text{cm}, h = 10\text{cm}$$

$$\text{Finding \%ge change in } r = \left(\frac{4.02-4}{4} \right) \times 100 = 0.5\%$$

$$\begin{aligned} \text{Now new value of } V - V' &= \pi \left[\left(\frac{100+0.5}{100} \right) r \right]^2 \cdot h \\ &= \pi r^2 h (1 + 0.005)^2 \cong \pi r^2 h [1 + 2(0.005)] \end{aligned}$$

$$V' = 1.01\pi r^2 h$$

$$\begin{aligned} \therefore \text{Increase in volume} &= 1.01\pi r^2 h - \pi r^2 h \\ &= 0.01\pi r^2 h = 0.01\pi \times 4^2 \times 10 \\ &= 1.6\pi \text{ cm}^3 \end{aligned}$$

- 7) The base radius of a right circular cone increases and the volume changes by 2%. If the height remains constant, find the percentage increase in the circumference of the base. (Uneb 2011)

Solution

Let radius increase by $x\%$

Let $V = \frac{1}{3}\pi r^2 h$, where r and V are original values of radius and volume respectively

$$\begin{aligned} \Rightarrow \text{New volume, } V' &= \frac{1}{3}\pi \left[\left(\frac{100+x}{100} \right) r \right]^2 h \\ &= \frac{1}{3}\pi r^2 h (1 + 0.01x)^2 \\ &= \frac{\pi r^2 h}{3} (1 + 0.02x) \quad [\text{Higher power terms neglected}] \end{aligned}$$

$$\text{Now \%ge change in volume} = \left(\frac{V' - V}{V} \right) \times 100 = 2$$

$$\therefore \left(\frac{\frac{\pi r^2 h}{3} (1 + 0.02x) - \frac{\pi r^2 h}{3}}{\frac{\pi r^2 h}{3}} \right) \times 100 = 2$$

$$\Rightarrow 0.02x = 0.02 \therefore x = 1$$

$\therefore$ Increase in radius is by 1%

Now circumference, $C = 2\pi r$

New value of C after increase in radius by 1% = $C' = 2\pi \left(\frac{100+1}{100}\right)r = 2\pi r(1.01)$

$$\therefore \% \text{ge increase in } C = \left(\frac{2\pi r(1.01) - 2\pi r}{2\pi r}\right) \times 100 = 1\%$$

- 8) If L is the length of a pendulum and $t(s)$ is the time of one complete swing. It is known that

$L = kt^2$. If the length of a pendulum increases by $x\%$, x being small, find the corresponding increase in the time of swing.

Solution

$L = kt^2$, taking k a constant

$$\text{Increase in length} = \left(\frac{100+x}{100}\right)L = (1 + 0.01x)L$$

$$\text{Now } t = \sqrt{\frac{L}{k}}$$

$$\begin{aligned} \therefore \text{New value of } t = t' &= \sqrt{\frac{(1+0.01x)L}{k}} = \sqrt{\frac{L}{k}} \cdot (1 + 0.01x)^{\frac{1}{2}} \\ &\cong \sqrt{\frac{L}{k}} \cdot \left[1 + \frac{1}{2}(0.01x)\right] \cong \sqrt{\frac{L}{k}} \cdot (1 + 0.005x) \end{aligned}$$

$$\begin{aligned} \therefore \% \text{ge increase in } t &= \left(\frac{t' - t}{t}\right) \times 100 \\ &= \left(\frac{\sqrt{\frac{L}{k}}(1+0.005x) - \sqrt{\frac{L}{k}}}{\sqrt{\frac{L}{k}}}\right) \times 100 = 0.5x\% \end{aligned}$$

- 9) The period T of a pendulum is calculated from the formula $T = 2\pi \sqrt{\left(\frac{l}{g}\right)}$ where l is the length of a pendulum and g is acceleration due to gravity. Find the percentage change in the period caused by lengthening the pendulum by 2%.

Solution

$$T = 2\pi \sqrt{\left(\frac{l}{g}\right)}, \text{ new value of } l = \left(\frac{100+2}{100}\right)l = (1 + 0.02)l$$

$$\begin{aligned} \therefore \text{New } T = T' &= 2\pi \sqrt{\left(\frac{(1+0.02)l}{g}\right)} \\ &= 2\pi \sqrt{\left(\frac{l}{g}\right)} \cdot (1 + 0.02)^{\frac{1}{2}} \\ &= 2\pi \sqrt{\left(\frac{l}{g}\right)} \cdot \left(1 + \frac{0.02}{2}\right) \text{ [higher power terms neglected]} \end{aligned}$$

$$\therefore T' = 2\pi \sqrt{\left(\frac{l}{g}\right)} \cdot (1.01)$$

$$\begin{aligned} \therefore \% \text{ge change in } T &= \left(\frac{T' - T}{T}\right) \times 100 \\ &= \left(\frac{2\pi \sqrt{\left(\frac{l}{g}\right)} \cdot (1.01) - 2\pi \sqrt{\left(\frac{l}{g}\right)}}{2\pi \sqrt{\left(\frac{l}{g}\right)}}\right) \times 100 = 1\% \end{aligned}$$

- 10) Find the approximate percentage change in the square of a quantity when the quantity itself changes by 0.1%. hence find the approximate value for $(10.01)^2$

Solution

Let the quantity be x

$$\Rightarrow y = x^2$$

Let percentage change in square of quantity be an increase

$$\therefore \text{New value of } x = \left(\frac{100+0.1}{100}\right)x = (1 + 0.001)x$$

Now percentage change in square of quantity. i.e. $y = \left(\frac{y' - y}{y}\right) \times 100$

$$\begin{aligned}\text{But } y' &= [(1 + 0.001)x]^2 = (1 + 0.001)^2 x^2 \\ &\cong [1 + 2(0.001)]x^2 \cong 1.002x^2\end{aligned}$$

$\therefore$ Required %age change in square of a quantity

$$= \left(\frac{1.002x^2 - x^2}{x^2}\right) \times 100 = 0.2\%$$

$$\text{Now let } y = (10.01)^2 = (10 + 0.01)^2$$

Hence if $x = 10$, then x increases by $0.01 = 1\%$

$$\text{Since } y' = x^2 \cong 1.002x^2, x = 10$$

$$\Rightarrow y' = 1.002 \times 10^2 = 100.2$$

$$\left[\begin{aligned} \text{Alt: } (10.01)^2 &= (10 + 0.01)^2 = (10.01)^2 = (10 + 0.01)^2 \\ &= \left[10 \left(1 + \frac{0.01}{10}\right)\right]^2 = 100(1 + 0.001)^2 = 10^2 \times (1 + 2(0.001)) = 100.2 \end{aligned} \right]$$

NOTE:

The above examples on percentages and errors in binomial expansion can also be done using the knowledge of *differentiation*. i.e. *Small increments and percentages*

Task:

1. The modulus of rigidity G is given by $G = \frac{R^4 \theta}{L}$ where R is radius, θ is the angle of twist and L is length. Find the approximate percentage in G when R is measured 1.5% too large and θ is measured 5% too small. **Answer: 1%**
2. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. If the volume increases by $\frac{2\pi}{3} \text{ cm}^3 / \text{min}$ and height increase by 0.03 cm/min . Find the rate of change of the radius r when $r = 10 \text{ cm}$ and $h = 5 \text{ cm}$. **Answer: 5**
3. If the radius of a spherical bubble increases from 1 cm to 1.02 cm , find the approximate increase in volume. **Answer: $0.08\pi \text{ cm}^3$**
4. If $R = ar^n$ and an error of $x\%$ is made in measuring r , prove that an error of $nx\%$ will result in R
5. The volume of a sphere is increased by 3% . Find the percentage increase in the radius. **Answer: 1%**

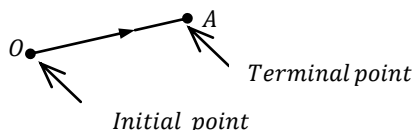
VECTORS

Definition:

A quantity that can be specified with magnitude and direction *e.g.* force, acceleration, displacement, velocity *etc* is a vector.

Representation of a vector

A vector is represented symbolically as $\vec{a}$ or $\mathbf{a}$ or $\underline{a}$
Graphically, it is represented by a line



The length of a line is the magnitude and the arrow on the line is the direction of the vector.

Classification of vectors

Vectors are classified depending on their behavior and properties.

a) **Null vector or Zero vector**

A vector with a null/zero magnitude, and is denoted as $\mathbf{0}$

b) **Unit vector**

A vector with a unit magnitude is a unit vector. It is denoted with a cap *i.e.* for a vector, its unit vector is $\hat{a}$

NOTE

$$\frac{|\text{VECTOR}|}{|\text{VECTOR}|} = \text{UNIT VECTOR}$$

c) **Like/unlike vectors**

Like vectors are vectors with the same direction regardless of their magnitude



Unlike vectors are vectors with different directions



d) **Negative vectors**

Negative vectors are vectors with same magnitude but opposite in direction. Consider the figure below



$$\vec{AB} = -\vec{BA} \quad \text{Application: subtraction of vectors}$$

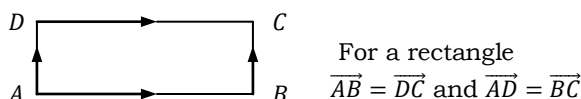
e) **Equal vectors**

Equal vectors are vectors with same magnitude and direction

Application

- Rhombus
- Parallelogram
- Rectangles
- Squares

Consider the figure below



Note

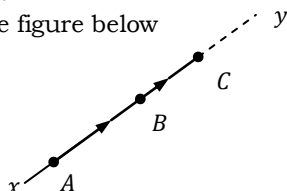
$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$, $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. Thus equal vectors are also parallel but with the same magnitude and direction.

f) Collinear/ parallel vectors

Are vectors with the same line of action or having line of action parallel to one another.

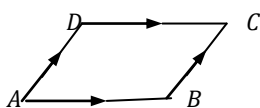
Consider the figure below

a)



$\overrightarrow{AB}$ and $\overrightarrow{BC}$, act on point B, but both have xy line of action, hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$

b)



ABCD is a parallelogram.

$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

$\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. Thus $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are collinear and also $\overrightarrow{AD}$ and

$\overrightarrow{BC}$ are collinear/parallel.

Condition for collinear/parallel vectors

From fig (a)

$$\overrightarrow{AB} = k\overrightarrow{BC} \text{ or } \overrightarrow{AB} = \lambda\overrightarrow{AC}$$

From fig (b)

$$\overrightarrow{AB} = k\overrightarrow{DC} \text{ or } \overrightarrow{AD} = \lambda\overrightarrow{BC}$$

Where k and λ are scalars or constants

g) Position vectors

Are vectors that specify position of a point with respect to a fixed point O (Origin)

Application

Used mainly in coordinate system, for locating a point in x – y or x – y – z planes

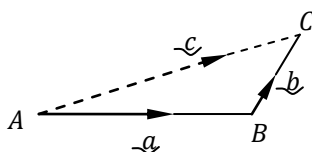
h) Coplanar vectors

Are vectors which lie in the same plane or vectors paralleled to the same plane

ADDITION OF VECTORS

Vectors are added by finding their *resultant*, a single or equivalent vector. Consider the following cases of vector addition.

1) Case I: Addition of TWO vectors.



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

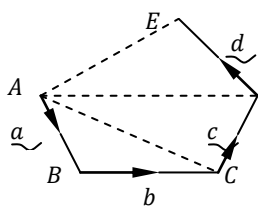
$$\text{or } \underline{a} + \underline{b} = \underline{c}$$

This is called *triangle law* of addition of vectors.

In general, to find the sum of two vectors $\mathbf{a}$ and $\mathbf{b}$, draw them as a chain, starting the second where the first ends; the sum $\mathbf{c}$ is given by the single vector joining the start of the first to the end of the second.

2) **Case II:** Addition of a number of vectors: $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \dots$

A)



a) Draw the vectors as a chain

b) Then;

$$\mathbf{a} + \mathbf{b} = \overrightarrow{AC}, \overrightarrow{AC} + \mathbf{c} = \overrightarrow{AD}$$

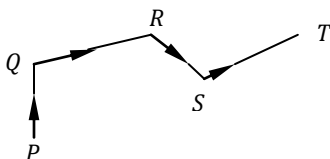
$$\therefore \mathbf{a} + \mathbf{b} + \mathbf{c} = \overrightarrow{AD}$$

$$\overrightarrow{AD} + \mathbf{d} = \overrightarrow{AE}$$

$$\therefore \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \overrightarrow{AE}$$

i.e. The sum of all the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ is given by the single vector joining the start of the first to the end of the last i.e. $\overrightarrow{AE}$

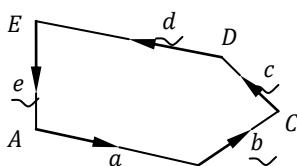
B)



$$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} = \overrightarrow{PT}$$

3) **Case III:** Sum of vectors = 0 for a closed figure

Consider the figure below



$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = 0$$

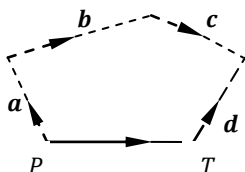
$$\text{i.e. } \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} + \mathbf{e} = \mathbf{0}$$

NOTE

The end of the last vector $\mathbf{B}$ coincides with the beginning of the first, so the resultant vector is zero with no magnitude.

COMPONENTS OF A GIVEN VECTOR

A single vector can be replaced by any number of component vectors so long as they form a chain in the vector diagram, beginning at initial point and ending at the terminal point.



$$\overrightarrow{PT} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$$

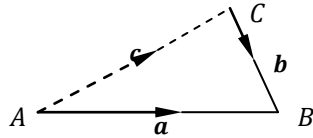
See examples immediately after subtraction of vectors.

SUBTRACTION OF VECTORS

During the addition of vectors in a specified direction it may be encountered that the direction of a vector needs to be changed in order for it to be added. Once a direction of vector is changed, a *negative vector* is formed, hence addition, subtraction of a vector occurs.

NOTE

Triangle law is still used for subtraction of *two* vectors.



$$\vec{AC} = \vec{AB} + \vec{BC}$$

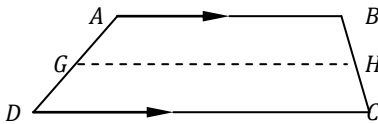
$$\text{But } \vec{BC} = -\vec{CB}$$

$$\therefore \vec{AC} = \vec{AB} - \vec{CB} \text{ or } \mathbf{c} = \mathbf{a} - \mathbf{b}$$

Examples on components of vectors

1. ABCD is a quadrilateral, with G and H the midpoints of DA and BC respectively. Show that $\vec{AB} + \vec{DC} = 2\vec{GH}$

Solution



$$\text{By chain vectors, } \vec{AB} = \vec{AG} + \vec{GH} + \vec{HB}$$

$$\text{Also } \vec{DC} = \vec{DH} + \vec{GH} + \vec{HC}$$

$$\begin{aligned} \text{Now } \vec{AB} + \vec{DC} &= \vec{AG} + \vec{GH} + \vec{HB} + \vec{DH} + \vec{GH} + \vec{HC} \\ &= 2\vec{GH} + (\vec{AG} + \vec{DH}) + (\vec{HB} + \vec{HC}) \end{aligned}$$

$$\text{But G is the midpoint of AD, } \Rightarrow \vec{AG} = \vec{GD} \text{ or } \vec{AG} = -\vec{DG} \text{ (negative vector)}$$

$$\text{Also H is the midpoint of BC, } \Rightarrow \vec{BH} = \vec{HC} \text{ or } -\vec{HB} = \vec{BH} = \vec{HC}$$

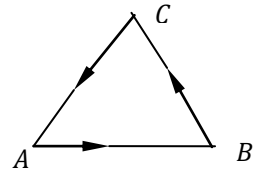
$$\therefore \vec{AB} + \vec{DC} = 2\vec{GH} + (-\vec{DG} + \vec{DG}) + (-\vec{HB} + \vec{HB}) = 2\vec{GH}$$

2. Points L, M, N are midpoints of the sides AB, BC, CA of the triangle ABC. Show that;

a. $\vec{AB} + \vec{BC} + \vec{CA} = 0$

b. $2\vec{AB} + 3\vec{BC} + \vec{CA} = \vec{LC}$

c. $\vec{AM} + \vec{BN} + \vec{CL} = 0$



Solution

a. $\vec{AB} + \vec{BC} + \vec{CA} = 0$ (For a closed figure)

b. To show that $2\vec{AB} + 3\vec{BC} + \vec{CA} = \vec{LC}$

$$\vec{CA} = \vec{CL} + \vec{LA}$$

$$\therefore 2\vec{AB} + 3\vec{BC} + \vec{CA} = 4\vec{AL} + 3\vec{BL} + 3\vec{LC} + \vec{CL} + \vec{LA}$$

$$= 4\vec{AL} + 3(-\vec{LB}) + 3\vec{LC} + (-\vec{LC}) + (-\vec{AL})$$

$$= 3\vec{AL} - 3\vec{LB} + 2\vec{LC}$$

$$\text{Since } AL = LB, 3\vec{AL} - 3\vec{LB} = 0$$

$$\therefore 2\vec{AB} + 3\vec{BC} + \vec{CA} = \vec{LC}$$

c. To show that $\vec{AM} + \vec{BN} + \vec{CL} = 0$

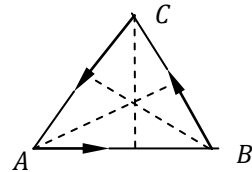
From the figure (b) above

$$\vec{AM} = \vec{AB} + \vec{BM}, \vec{BN} = \vec{BC} + \vec{CN}, \vec{CL} = \vec{CA} + \vec{AL}$$

$$\vec{AM} + \vec{BN} + \vec{CL} = \vec{AB} + \vec{BM} + \vec{BC} + \vec{CA} + \vec{AL} + \vec{CN}$$

$$= (\vec{AB} + \vec{BC} + \vec{CA}) + (\vec{BM} + \vec{AL} + \vec{CN})$$

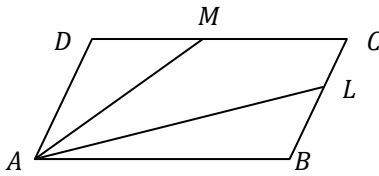
$$= 0 + \frac{1}{2}(\vec{BC} + \vec{AB} + \vec{CA}) = \frac{1}{2}(\vec{AB} + \vec{BC} + \vec{CA}) = 0$$



Task

Prove that the line joining the midpoint of sides of a triangle is parallel to the third side and half its length.

3. ABCD is a parallelogram. If L, M are the midpoints of BC and CD, express $\vec{AL}$ and $\vec{AM}$ in terms of $\vec{AB}$ and $\vec{AD}$. Show that $\vec{AL} + \vec{AM} = \frac{3}{2}\vec{AC}$

Solution


$$\vec{AL} = \vec{AB} + \vec{BL}$$

$$\text{But } \vec{BL} = \frac{1}{2}\vec{BC} = \frac{1}{2}\vec{AD} \quad [\because \vec{AD} = \vec{BC}]$$

$$\therefore \vec{AL} = \vec{AB} + \frac{1}{2}\vec{BC}$$

$$\vec{AM} = \vec{AD} + \vec{DM}$$

$$\text{But } \vec{DM} = \frac{1}{2}\vec{DC}, \vec{DC} = \vec{AB}, \therefore \vec{DM} = \frac{1}{2}\vec{AB}$$

$$\Rightarrow \vec{AM} = \vec{AD} + \frac{1}{2}\vec{AB}$$

$$\text{Now to show: } \vec{AL} + \vec{AM} = \frac{3}{2}\vec{AC}$$

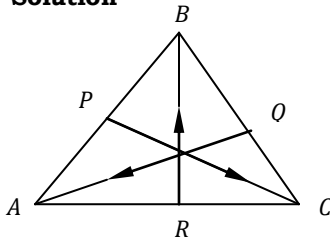
$$\Rightarrow \vec{AB} + \frac{1}{2}\vec{AD} + \vec{AD} + \frac{1}{2}\vec{AB} = \vec{AL} + \vec{AM}$$

$$\Rightarrow \vec{AL} + \vec{AM} = \frac{3}{2}\vec{AB} + \frac{3}{2}\vec{AD} = \frac{3}{2}(\vec{AB} + \vec{AD})$$

$$\text{But } \vec{AB} + \vec{AD} = \vec{AB} + \vec{BC} = \vec{AC}$$

$$\therefore \vec{AL} + \vec{AM} = \frac{3}{2}\vec{AC}$$

4. If P, Q, R are midpoints of the sides AB, BC and CA of the triangle ABC and O be a point within it, prove that $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OP} + \vec{OQ} + \vec{OR}$

Solution


$$\vec{OA} = \vec{OP} + \vec{PA}$$

$$\vec{OB} = \vec{OQ} + \vec{QB}$$

$$\vec{OC} = \vec{OR} + \vec{RC}$$

$$\therefore \vec{OA} + \vec{OB} + \vec{OC} = \vec{OP} + \vec{PA} + \vec{OQ} + \vec{QB} + \vec{OR} + \vec{RC}$$

$$= (\vec{OP} + \vec{OQ} + \vec{OR}) + (\vec{PA} + \vec{QB} + \vec{RC})$$

$$\text{But } \vec{PA} = \frac{1}{2}\vec{BA}, \vec{QB} = \frac{1}{2}\vec{CB}, \vec{RC} = \frac{1}{2}\vec{AC}$$

$$\Rightarrow \vec{PA} + \vec{QB} + \vec{RC} = \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{CB} + \frac{1}{2}\vec{AC} = \frac{1}{2}(\vec{BA} + \vec{CB} + \vec{AC}) = \frac{1}{2} \times 0 = 0$$

$$\therefore \vec{OA} + \vec{OB} + \vec{OC} = \vec{OP} + \vec{OQ} + \vec{OR}$$

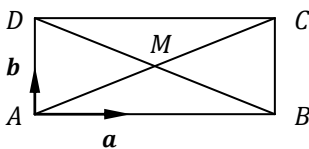
NOTE

Parameters can also be applied in solving the above examples through forming parametric equations.

5. ABCD is a rectangle. By vectors show that the diagonals bisect each other.

Solution

Points A, M and C are collinear, and B, M and D are also collinear.



$$\text{Let } \vec{AB} = \mathbf{a} \text{ and } \vec{AD} = \mathbf{b}$$

Now $\vec{AM} = \lambda\vec{AC}$ and $\vec{BM} = \mu\vec{BD}$ where λ, μ are constants.

$$\vec{AM} = \vec{AB} + \vec{BM} = \vec{AB} + \mu\vec{BD}$$

$$\text{But } \vec{BD} = \vec{BA} + \vec{AD} = -\mathbf{a} + \mathbf{b}$$

$$\therefore \vec{AM} = \mathbf{a} + \mu(-\mathbf{a} + \mathbf{b})$$

$$\text{Also } \vec{AM} = \lambda\vec{AC}; \vec{AC} = \vec{AB} + \vec{BC} = \mathbf{a} + \mathbf{b} \quad [\because \vec{AD} = \vec{BC} = \mathbf{b}]$$

$$\therefore \vec{AM} = (\mathbf{a} + \mathbf{b})\lambda \dots\dots\dots (1)$$

$$\text{Also } \vec{AM} = \mathbf{a} + \mu(-\mathbf{a} + \mathbf{b}) = (1 - \mu)\mathbf{a} + \mu\mathbf{b} \dots\dots\dots (2)$$

Since (1) and (2) are the same

$$\Rightarrow (\mathbf{a} + \mathbf{b})\lambda = \mathbf{a} + \mu(-\mathbf{a} + \mathbf{b}) = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$$

Equating coefficients of corresponding vectors

$$\text{For } \mathbf{a}: \lambda = 1 - \mu \dots\dots\dots (3)$$

$$\text{For } \mathbf{b}: \lambda = \mu \dots\dots\dots (4)$$

Solving (3) and (4) simultaneously

$$\mu = 1 - \mu \text{ or } \mu = \frac{1}{2} = \lambda$$

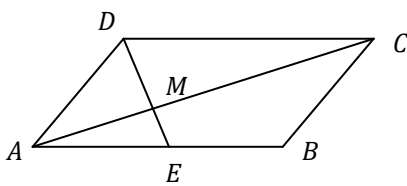
$$\text{Since } \overrightarrow{AM} = \lambda \overrightarrow{AC} \text{ and } \overrightarrow{BM} = \mu \overrightarrow{BD}, \text{ then } \overrightarrow{AM} = \frac{1}{2} \overrightarrow{AC} \text{ and } \overrightarrow{BM} = \frac{1}{2} \overrightarrow{BD}$$

$\therefore M$ is the midpoint of the diagonals, and hence diagonals bisect.

6. If $ABCD$ is a parallelogram and E is the midpoint of AB , show by vector method that DE trisects and is trisected by AC .

Solution

Points A , M and C are collinear and D , M and E are also collinear



$$\text{Let } \overrightarrow{AB} = \mathbf{a} \text{ and } \overrightarrow{AD} = \mathbf{b}$$

$$\overrightarrow{AM} = \lambda \overrightarrow{AC}, \overrightarrow{DM} = k \overrightarrow{DE}, \text{ where } k, \lambda \text{ are constants}$$

$$\overrightarrow{AM} = \lambda \overrightarrow{AC}; \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$= \overrightarrow{AB} + \overrightarrow{AD} = \mathbf{a} + \mathbf{b} [\because \overrightarrow{AD} = \overrightarrow{BC} = \mathbf{b}]$$

$$\therefore \overrightarrow{AM} = (\mathbf{a} + \mathbf{b})\lambda \dots\dots\dots (1)$$

$$\overrightarrow{DM} = k \overrightarrow{DE}; \overrightarrow{DM} = \overrightarrow{DA} + \overrightarrow{AM} = -\mathbf{b} + \overrightarrow{AM}$$

$$\Rightarrow \overrightarrow{DM} = -\mathbf{b} + (\mathbf{a} + \mathbf{b})\lambda \dots\dots\dots (2)$$

$$\text{Now } \overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = -\mathbf{b} + \frac{1}{2} \overrightarrow{AB} = -\mathbf{b} + \frac{1}{2} \mathbf{a}$$

$$\therefore \overrightarrow{DM} = k \overrightarrow{DE} \text{ gives;}$$

$$-\mathbf{b} + (\mathbf{a} + \mathbf{b})\lambda = k \left(-\mathbf{b} + \frac{1}{2} \mathbf{a} \right) \text{ or } (\lambda - 1)\mathbf{b} + \lambda \mathbf{a} = -k\mathbf{b} + \frac{k}{2} \mathbf{a}$$

Equating coefficients of corresponding vectors

$$\text{For } \mathbf{a}: \lambda = \frac{k}{2} \dots\dots\dots (3)$$

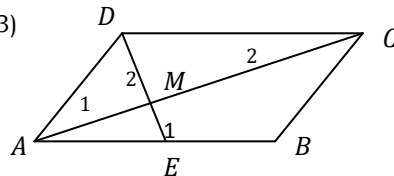
$$\text{For } \mathbf{b}: \lambda - 1 = -k \dots\dots\dots (4)$$

Eqn. (3) in to eqn. (4) gives

$$\frac{k}{2} - 1 = -k \Rightarrow \frac{3}{2}k = 1 \text{ or } k = \frac{2}{3},$$

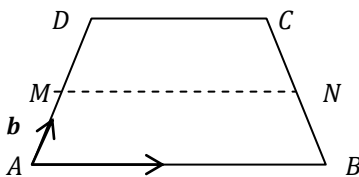
$$\text{and from (1) } \lambda = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$$\therefore \overrightarrow{AM} = \frac{1}{3} \overrightarrow{AC} \text{ and } \overrightarrow{DM} = \frac{2}{3} \overrightarrow{DE}, \text{ hence } DE \text{ trisects and is trisected by } AC.$$



7. Show that the line joining the midpoints of two non parallel sides of a trapezium is parallel to the other sides and half their sum.

Solution



$$\text{Let } \overrightarrow{AB} = \mathbf{a} \text{ and } \overrightarrow{AD} = \mathbf{b}$$

$$\text{Since } \overrightarrow{AB} \parallel \overrightarrow{DC}, \text{ then } \overrightarrow{DC} = \lambda \overrightarrow{AB} = \lambda \mathbf{a}$$

Now expressing $\overrightarrow{MN}$ in terms of $\mathbf{a}$ and $\mathbf{b}$

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{BN}$$

$$\overrightarrow{MA} = \frac{1}{2} \overrightarrow{DA}, \overrightarrow{BN} = \frac{1}{2} \overrightarrow{BC}$$

$$\Rightarrow \overrightarrow{MN} = \frac{1}{2} \mathbf{b} + \mathbf{a} + \frac{1}{2} \overrightarrow{BC}$$

$$\text{But } \overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC} = -\mathbf{a} + \mathbf{b} + \lambda \mathbf{a}$$

$$\Rightarrow \overrightarrow{MN} = -\frac{1}{2} \mathbf{b} + \mathbf{a} + \frac{1}{2} (-\mathbf{a} + \mathbf{b} + \lambda \mathbf{a})$$

$$= -\frac{1}{2}\mathbf{b} + \mathbf{a} - \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} + \frac{1}{2}\lambda\mathbf{a} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\lambda\mathbf{a} = \frac{1}{2}(1 + \lambda)\mathbf{a}$$

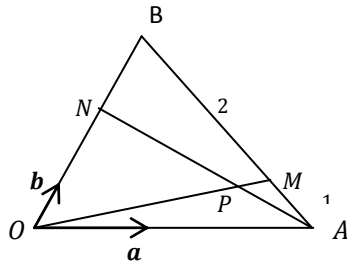
Since $\frac{1}{2}(1 + \lambda)$ is a scalar, let $\frac{1}{2}(1 + \lambda) = \mu$

$\therefore \overrightarrow{MN} = \mu\mathbf{a}$, which shows that $\overrightarrow{MN} \parallel \overrightarrow{AB}$ and $\overrightarrow{DC}$

From $\overrightarrow{MN} = \frac{1}{2}(1 + \lambda)\mathbf{a} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\lambda\mathbf{a} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{DC})$, as required.

8. OAB is a triangle M and N are points on AB and OB respectively, such that $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$. If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, find the position vector of P , the intersection of AN and OM , in terms of $\mathbf{a}$ and $\mathbf{b}$

Solution:



$$\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB} \Rightarrow \frac{AM}{AB} = \frac{1}{3} \text{ or } AM:AB = 1:3$$

$$\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB} \Rightarrow \frac{ON}{OB} = \frac{3}{4}$$

Points O, P, M and A, P, N are collinear.

Let $\overrightarrow{OP} = \lambda\overrightarrow{OM}$; $\overrightarrow{AP} = \mu\overrightarrow{AN}$

Now expressing $\overrightarrow{OP}$ in terms of $\lambda, \mu, \mathbf{a}$ and $\mathbf{b}$

$$\overrightarrow{OP} = \lambda\overrightarrow{OM}; \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$$

$$\Rightarrow \overrightarrow{OP} = \lambda\left(\overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}\right) \text{ But } \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \mathbf{b} - \mathbf{a}$$

$$\therefore \overrightarrow{OP} = \lambda\left(\mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})\right) = \frac{2}{3}\lambda\mathbf{a} + \frac{1}{3}\lambda\mathbf{b} \dots\dots\dots (1)$$

$$\text{Also } \overrightarrow{AP} = \mu\overrightarrow{AN}; \overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = \overrightarrow{OP} - \mathbf{a}$$

$$\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON}; \overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$$

$$\therefore \overrightarrow{AN} = \overrightarrow{AO} + \frac{3}{4}\overrightarrow{OB} = -\mathbf{a} + \frac{3}{4}\mathbf{b}$$

$$\Rightarrow \overrightarrow{OP} - \mathbf{a} = \mu\left(-\mathbf{a} + \frac{3}{4}\mathbf{b}\right) = (1 - \mu)\mathbf{a} + \frac{3}{4}\mu\mathbf{b}$$

$$\therefore \overrightarrow{OP} = (1 - \mu)\mathbf{a} + \frac{3}{4}\mu\mathbf{b} \dots\dots\dots (2)$$

Eqn. 1 = eqn. 2

$$\frac{2}{3}\lambda\mathbf{a} + \frac{1}{3}\lambda\mathbf{b} = (1 - \mu)\mathbf{a} + \frac{3}{4}\mu\mathbf{b}$$

$$\frac{2}{3}\lambda = 1 - \mu \dots\dots\dots (3)$$

[Equating coefficients of $\mathbf{a}$]

$$\frac{1}{3}\lambda = \frac{3}{4}\mu \dots\dots\dots (4)$$

[Equating coefficients of $\mathbf{b}$]

$$\text{From eqn. (3)} \mu = 1 - \frac{2}{3}\lambda \dots\dots\dots (4)$$

Eqn. (5) in to eqn. (4) gives

$$\frac{1}{3}\lambda = \frac{3}{4}\left(1 - \frac{2}{3}\lambda\right) = \frac{3}{4} - \frac{1}{2}\lambda \Rightarrow \frac{1}{3}\lambda + \frac{1}{2}\lambda = \frac{3}{4}$$

$$\therefore \frac{5}{6}\lambda = \frac{3}{4} \Rightarrow \lambda = \frac{9}{10}$$

$$\text{From eqn. (4), } \mu = 1 - \frac{2}{3} \times \frac{9}{10} = \frac{2}{5}$$

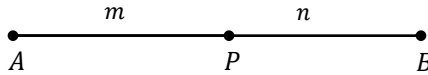
$$\overrightarrow{OP} = \frac{2}{3}\lambda\mathbf{a} + \frac{1}{3}\lambda\mathbf{b} = \frac{2}{3}\left(\frac{9}{10}\right)\mathbf{a} + \frac{1}{3}\left(\frac{9}{10}\right)\mathbf{b} = \frac{3}{5}\mathbf{a} + \frac{3}{10}\mathbf{b}$$

Task

Show that the diagonals of a square, rhombus and parallelogram bisect each other.

POSITION OF A POINT DIVIDING A LINE AB IN THE RATIO $m:n$ (Ratio Theorem)

Let the line AB be divided by a point P in the ratio $m:n$



$$\text{Ratio } \overrightarrow{AP}:\overrightarrow{PB} = m:n \text{ or } \frac{m}{n} = \frac{\overrightarrow{AP}}{\overrightarrow{PB}} \Rightarrow n\overrightarrow{AP} = m\overrightarrow{PB}$$

By position vectors,

$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\overrightarrow{OA} + \overrightarrow{OP}, \overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB} = \overrightarrow{OP} - \overrightarrow{OB}$$

$$\Rightarrow n(-\overrightarrow{OA} + \overrightarrow{OP}) = m(\overrightarrow{OP} - \overrightarrow{OB})$$

$$\Rightarrow n\overrightarrow{OP} + m\overrightarrow{OP} = m\overrightarrow{OB} + n\overrightarrow{OA} \therefore \overrightarrow{OP}(m+n) = n\overrightarrow{OA} + m\overrightarrow{OB}$$

$$\Rightarrow \overrightarrow{OP} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$$

If the position vectors of A and B are respectively $\mathbf{a}$ and $\mathbf{b}$, then $OA = \mathbf{a}, OB = \mathbf{b}$. The ratio theorem becomes,

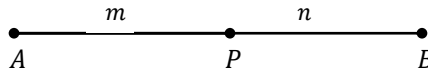
$$\overrightarrow{OP} = \frac{n}{m+n}\mathbf{a} + \frac{m}{m+n}\mathbf{b}$$

NOTE

- P is a point that divides AB either *internally* or *externally* in the ratio the ratio $m:n$

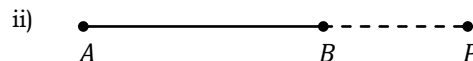
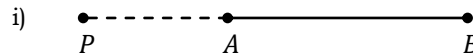
- For *internal division*, the ratio is represented as

$$\overrightarrow{AP}:\overrightarrow{PB} = m:n [\text{Both ratio values} + \text{ve}]$$



- For *external division*, the ratio is represented as

$$\overrightarrow{AP}:\overrightarrow{PB} = m:-n \text{ or } -m:n [\text{Both ratio values of opposite sign}]$$



$$\overrightarrow{AP}:\overrightarrow{PB} = m:n$$

- The knowledge of position vectors is applicable in ratio theorem.

Examples

- Find the position vector (p.v) of a point which divides the line segment AB in the ratio

- 8:3

- 5:-4

Solution

- Ratio 8:3 = $m:n$ [Internal division]

$$\overrightarrow{AP}:\overrightarrow{PB} = 8:3 \text{ or } \frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{8}{3} \Rightarrow 3\overrightarrow{AP} = 8\overrightarrow{PB}$$

$$\text{By position vectors, } 3(\overrightarrow{OP} - \overrightarrow{OA}) = 8(\overrightarrow{OB} - \overrightarrow{OP}) \Rightarrow 3\overrightarrow{OP} - 3\overrightarrow{OA} = 8\overrightarrow{OB} - 8\overrightarrow{OP}$$

$$11\overrightarrow{OP} = 3\overrightarrow{OA} + 8\overrightarrow{OB} \Rightarrow \overrightarrow{OP} = \frac{3}{11}\overrightarrow{OA} + \frac{8}{11}\overrightarrow{OB}$$

$$\text{Let } \overrightarrow{OA} = \mathbf{a} \text{ and } \overrightarrow{OB} = \mathbf{b} \text{ then } \overrightarrow{OP} = \frac{3}{11}\mathbf{a} + \frac{8}{11}\mathbf{b}$$

b) Ratio 5: -4 = m: n [External division]

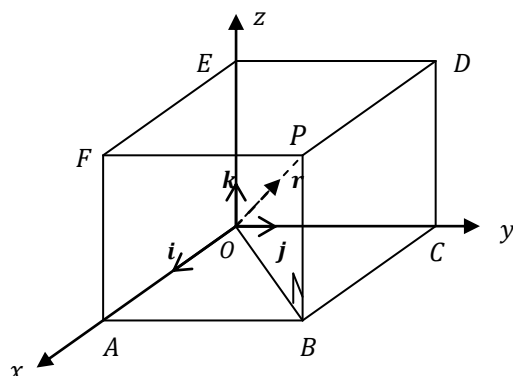
$$\overrightarrow{AP} : \overrightarrow{PB} = 5: -4 \text{ or } \frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{5}{-4} \Rightarrow -4\overrightarrow{AP} = 5\overrightarrow{PB}$$

$$\text{By position vectors, } -4(\overrightarrow{OP} - \overrightarrow{OA}) = 5(\overrightarrow{OB} - \overrightarrow{OP}) \Rightarrow -4\overrightarrow{OP} + 4\overrightarrow{OA} = 5\overrightarrow{OB} - 5\overrightarrow{OP}$$

$$\overrightarrow{OP} = 5\overrightarrow{OB} - 4\overrightarrow{OA} \therefore \overrightarrow{OP} = -4\mathbf{a} + 5\mathbf{b}$$

VECTOR “r” IN TERMS OF UNIT VECTORS $\mathbf{i}, \mathbf{j}$ AND $\mathbf{k}$

Consider the figure below



$$\overrightarrow{OP} = \mathbf{r} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$$

$$\text{Let } |\overrightarrow{OA}| = x \text{ units,}$$

$$|\overrightarrow{AB}| = y \text{ units, } |\overrightarrow{BP}| = z \text{ units}$$

$$\text{From vector} = |\text{vector}| \times \text{unit vector,}$$

$$\overrightarrow{OA} = x.\mathbf{i}, \overrightarrow{AB} = y.\mathbf{j} \text{ and } \overrightarrow{BP} = z.\mathbf{k}$$

$$\therefore \overrightarrow{OP} = x.\mathbf{i} + y.\mathbf{j} + z.\mathbf{k}$$

A point $P(x, y, z)$ has a p.v

$$\overrightarrow{OP} = \mathbf{r} = x.\mathbf{i} + y.\mathbf{j} + z.\mathbf{k}$$

$$\text{or } \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

MODULUS/MAGNITUDE/LENGTH OF A VECTOR $\mathbf{r}$

From $\triangle OPB$, by Pythagoras theorem,

$$|\overrightarrow{OP}|^2 = |\mathbf{r}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{AP}|^2; \text{ But } \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = x.\mathbf{i} + y.\mathbf{j}$$

$$|\overrightarrow{OB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 [\because \triangle OAB \text{ is right angled}]$$

$$= x^2 + y^2$$

$$\therefore |\overrightarrow{OP}|^2 = x^2 + y^2 + z^2 \Rightarrow |\overrightarrow{OP}| = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

DISTANCE BETWEEN TWO VECTORS

If $\mathbf{r}_1 = x_1.\mathbf{i} + y_1.\mathbf{j} + z_1.\mathbf{k}$ and $\mathbf{r}_2 = x_2.\mathbf{i} + y_2.\mathbf{j} + z_2.\mathbf{k}$, distance between $\mathbf{r}_1$ and $\mathbf{r}_2$ is

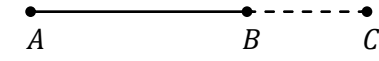
$$\text{Distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

More examples on Ratio Theorem

1. Given the position vector $\vec{OA} = (3, -2, 5)$ and $\vec{OB} = (9, 1, -1)$. Find the position vector of point C such that C divides $\vec{AB}$ internally in the ratio 5: -3 (Uneb 2003)

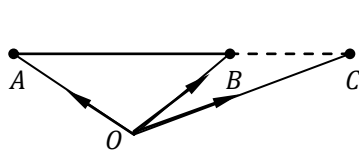
Solution

Method I



$$\begin{aligned} \vec{AC} : \vec{CB} &= 5 : -3 \Rightarrow \frac{\vec{AC}}{\vec{CB}} = \frac{5}{-3} \text{ or } -3\vec{AC} = 5\vec{CB} \\ \Rightarrow -3(\vec{OC} - \vec{OA}) &= 5(\vec{OB} - \vec{OC}) \\ \Rightarrow -3\vec{OC} + 3\vec{OA} &= 5\vec{OB} - 5\vec{OC} \therefore 2\vec{OC} = 5\vec{OB} - 3\vec{OA} \\ \Rightarrow \vec{OC} &= \frac{5}{2}\vec{OB} - \frac{3}{2}\vec{OA} = \frac{5}{2}\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{45}{2} - \frac{9}{2} \\ \frac{5}{2} + \frac{6}{2} \\ -\frac{5}{2} - \frac{15}{2} \end{pmatrix} = \begin{pmatrix} 18 \\ \frac{11}{2} \\ -10 \end{pmatrix} \therefore \vec{OC} = 18\mathbf{i} + \frac{11}{2}\mathbf{j} - 10\mathbf{k} \end{aligned}$$

Method II



By Ratio Theorem $\vec{AC} : \vec{CB} = 5 : -3 = m : n$

$$\begin{aligned} \vec{OC} &= \frac{n}{m+n}\vec{OA} + \frac{m}{m+n}\vec{OB} \\ \therefore \vec{OC} &= -\frac{3}{5+(-3)}\vec{OA} + \frac{5}{5+(-3)}\vec{OB} = \frac{5}{2}\vec{OB} - \frac{3}{2}\vec{OA} \\ &= \frac{5}{2}\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \\ \therefore \vec{OC} &= \begin{pmatrix} \frac{45}{2} - \frac{9}{2} \\ \frac{5}{2} + \frac{6}{2} \\ -\frac{5}{2} - \frac{15}{2} \end{pmatrix} = \begin{pmatrix} 18 \\ \frac{11}{2} \\ -10 \end{pmatrix} \Rightarrow \vec{OC} = 18\mathbf{i} + \frac{11}{2}\mathbf{j} - 10\mathbf{k} \end{aligned}$$

2. A and B are points whose position vectors are $\mathbf{a} = 2\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively. Determine the p.v of the point that divides AB internally in the ratio 4: -1 (uneb 1993)

Solution

Method I

$$\begin{aligned} \text{Ratio of division is } \vec{AP} : \vec{PB} &= 4 : -1 \text{ or } \frac{\vec{AP}}{\vec{PB}} = \frac{4}{-1} \Rightarrow -\vec{AP} = 4\vec{PB} \\ \Rightarrow -\vec{OP} + \vec{OA} &= 4\vec{OB} - 4\vec{OP} \\ \Rightarrow 3\vec{OP} &= 4\vec{OB} - \vec{OA} \therefore \vec{OP} = \frac{4}{3}\vec{OB} - \frac{1}{3}\vec{OA} = \frac{4}{3}(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - \frac{1}{3}(2\mathbf{i} + \mathbf{k}) \\ \Rightarrow \vec{OP} &= \left(\frac{4}{3} - \frac{2}{3}\right)\mathbf{i} - \frac{4}{3}\mathbf{j} + \left(\frac{12}{3} - \frac{1}{3}\right)\mathbf{k} = \frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{11}{3}\mathbf{k} \end{aligned}$$

Method II

Using Ratio Theorem

$$\begin{aligned} \vec{OP} &= \frac{n}{m+n}\vec{OA} + \frac{m}{m+n}\vec{OB}, \text{ Ratio: } \vec{AP} : \vec{PB} = 4 : -1 = m : n \\ \therefore \vec{OP} &= \frac{-1}{4+(-1)}\vec{OA} + \frac{4}{4+(-1)}\vec{OB} = -\frac{1}{3}\vec{OA} + \frac{4}{3}\vec{OB} \\ &= -\frac{1}{3}(2\mathbf{i} + \mathbf{k}) + \frac{4}{3}(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{11}{3}\mathbf{k} \end{aligned}$$

NOTE

You can work the vectors in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ or write the given vectors in terms of column vectors. *i.e.* For $\mathbf{i} - \mathbf{j} + 3\mathbf{k} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

3. Given that the position vectors of A, B and C are

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{OC} = \begin{pmatrix} 7 \\ 10 \\ -7 \end{pmatrix}$$

i) Prove that A, B and C are collinear

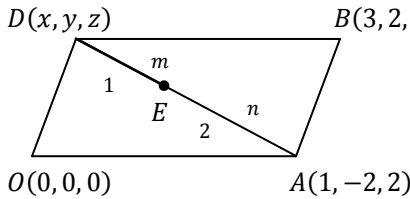
ii) If $OABD$ is a parallelogram, find the *p.v* of E and F such that E divides DA in the ratio $1:2$ and F divides it externally in the ratio $1:2$ (**Unneb 2007**)

Solution

i) Shall be tackled later after looking at collinear vectors

ii) For a parallelogram, $\overrightarrow{OD} = \overrightarrow{AB}$ and $\overrightarrow{OA} = \overrightarrow{DB}$

$$\begin{aligned} D(x, y, z) \quad B(3, 2, -1) &\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \\ &\therefore D(x, y, z) = D(2, 4, -3) \end{aligned}$$



Now finding coordinates of E that divides DA internally in the ratio $1:2$

Using Ratio Theorem

$$\overrightarrow{OE} = \frac{n}{m+n} \overrightarrow{OD} + \frac{m}{m+n} \overrightarrow{OA} \quad [\text{state from } D \text{ to } A]$$

$$= \frac{2}{2+1} \overrightarrow{OD} + \frac{1}{2+1} \overrightarrow{OA}$$

$$= \frac{2}{3} \overrightarrow{OD} + \frac{1}{3} \overrightarrow{OA} = \frac{2}{3} \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\text{or } \overrightarrow{OE} = \frac{2}{3}(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) + \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

$$\begin{pmatrix} \frac{4}{3} + \frac{1}{3} \\ \frac{8}{3} - \frac{2}{3} \\ -\frac{6}{3} + \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{6}{3} \\ -\frac{4}{3} \end{pmatrix} \therefore \overrightarrow{OE} = \frac{5}{3}\mathbf{i} + 2\mathbf{j} + \frac{4}{3}\mathbf{k}$$

Now finding E , for external division



$$\text{Ratio } \overrightarrow{DE} : \overrightarrow{EA} = 1 : -2 = m : n$$

Using Ratio Theorem

$$\overrightarrow{OE} = \frac{n}{m+n} \overrightarrow{OD} + \frac{m}{m+n} \overrightarrow{OA}$$

$$\Rightarrow \overrightarrow{OE} = \frac{-2}{1+(-2)} \overrightarrow{OD} + \frac{1}{1+(-2)} \overrightarrow{OA} = 2\overrightarrow{OD} - \overrightarrow{OA}$$

$$\overrightarrow{OE} = 2 \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4-1 \\ 8+2 \\ -6-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -8 \end{pmatrix} \therefore \overrightarrow{OE} = 3\mathbf{i} + 10\mathbf{j} - 8\mathbf{k}$$

Examples on Distance between two points

1. Prove that the points $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ are vertices of a right angled Δ .

Solution

Let $P(2, -1, 1)$, $Q(1, -3, -5)$, $R(3, -4, -4)$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix} \Rightarrow |\overrightarrow{PQ}| = \sqrt{(-1)^2 + 2^2 + (-6)^2} = \sqrt{41}$$

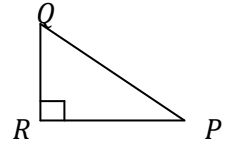
$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow |\overrightarrow{QR}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} \Rightarrow |\overrightarrow{PR}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{35}$$

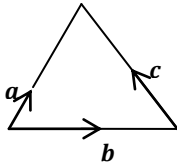
For a right angled triangle,

sum of squares of two sides = square of the longer side

$$\Rightarrow |\overrightarrow{QR}|^2 + |\overrightarrow{PR}|^2 = |\overrightarrow{PQ}|^2$$



2. Prove that the vectors $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$, $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ form the sides of an equilateral triangle.

Solution


Let $\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

$$|\mathbf{a}| = \sqrt{6^2 + (-2)^2 + 3^2} = 7 \text{ units}$$

$$|\mathbf{b}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7 \text{ units}$$

$$|\mathbf{c}| = \sqrt{3^2 + 6^2 + (-2)^2} = 7 \text{ units}$$

$\therefore$ Vectors form sides of an equilateral triangle.

General examples

3. Show that the points A, B and C with position vectors

a) $4\mathbf{i} - 8\mathbf{j} - 13\mathbf{k}$, $3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

b) $5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $7\mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$

c) $7\mathbf{i} + 10\mathbf{k}$, $-\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}$

, are vertices of a triangle.

Solution

- a) $A(4, -8, -13)$, $B(3, -2, -3)$ $C(3, -1, -2)$

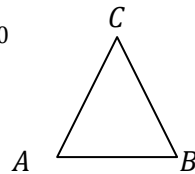
For a closed figure of three sides, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -8 \\ -13 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 10 \end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\text{and } \overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \begin{pmatrix} 4 \\ -8 \\ -13 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -9 \\ -11 \end{pmatrix}$$

$$\text{Now } \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \begin{pmatrix} -1 \\ 6 \\ 10 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -9 \\ -11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

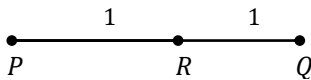


- b) Left as an exercise

- c) Left as an exercise

4. Given that the p.v.s of P and Q are $4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{i} + 2\mathbf{k}$ respectively, find the coordinates of the point R such that $PQ:PR = 2:1$

Solution



Then R is the midpoint of PQ or R divides PQ internally in the ratio $1:1$

Method I

$$\begin{aligned} \overrightarrow{PQ}:\overrightarrow{PR} &= 2:1 \Rightarrow \frac{\overrightarrow{PQ}}{\overrightarrow{PR}} = \frac{2}{1} \text{ or } \overrightarrow{PQ} = 2\overrightarrow{PR} \\ \Rightarrow \overrightarrow{OQ} - \overrightarrow{OP} &= 2(\overrightarrow{OR} - \overrightarrow{OP}) = 2\overrightarrow{OR} - 2\overrightarrow{OP} \\ \overrightarrow{OP} &= 2\overrightarrow{OR} - \overrightarrow{OQ} \Rightarrow \overrightarrow{OR} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}) \\ \therefore \overrightarrow{OR} &= \frac{1}{2} \left[\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ \frac{7}{2} \end{pmatrix} \therefore R \left(\frac{5}{2}, -\frac{3}{2}, \frac{7}{2} \right) \end{aligned}$$

Method II

By ratio theorem, $\overrightarrow{PQ}:\overrightarrow{RQ} = 1:1 \equiv m:n$

$$\begin{aligned} \overrightarrow{OR} &= \frac{n}{m+n} \overrightarrow{OP} + \frac{m}{m+n} \overrightarrow{OQ} \\ &= \frac{1}{2} \overrightarrow{OP} + \frac{1}{2} \overrightarrow{OQ} = \frac{1}{2} (\overrightarrow{OP} + \overrightarrow{OQ}) = \frac{1}{2} \left[\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ \frac{7}{2} \end{pmatrix} \\ \therefore R &\left(\frac{5}{2}, -\frac{3}{2}, \frac{7}{2} \right) \end{aligned}$$

5. The points A and B have p.v.s $\mathbf{a}$ and $\mathbf{b}$ respectively relative to origin, where $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + s\mathbf{j} + t\mathbf{k}$. Find the possible values of s and t if $|\overrightarrow{AB}| = 7$ and $s = 2t$

Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ s \\ t \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ s-1 \\ t+3 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{(-6)^2 + (s-1)^2 + (t+3)^2} = 7$$

Squaring both sides

$$36 + (s-1)^2 + (t+3)^2 = 49$$

$$(s-1)^2 + (t+3)^2 = 13 \dots\dots\dots (1)$$

Now

$$s = 2t \dots\dots\dots (2)$$

eqn, (2) in to (1)

$$(2t-1)^2 + (t+3)^2 = 13$$

$$4t^2 - 4t + 1 + t^2 + 6t + 9 = 13 \Rightarrow 5t^2 + 2t - 3 = 0$$

$$5t^2 + 5t - 3t - 3 = 0$$

$$5t(t+1) - 3(t+1) = 0 \Rightarrow (5t-3)(t+1) = 0$$

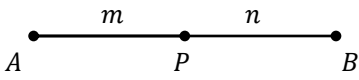
$$\therefore t = -1, \frac{3}{5}$$

$$\text{For } t = -1, s = 2t = -2 \text{ and for } t = \frac{3}{5}, s = 2t = \frac{6}{5}$$

$$\therefore s = \frac{6}{5}, t = \frac{3}{5}; s = -2, t = -1$$

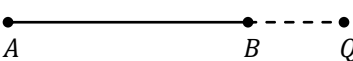
6. A and B are points with p.v.s $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $7\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ respectively. Find the coordinates of P and Q which divide AB
- Internally in the ratio 1:2
 - Externally in the ratio 3:1

Solution

a)  Ratio: $\overrightarrow{AP} : \overrightarrow{PB} = 1:2 \Rightarrow \frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{1}{2}$ or $2\overrightarrow{AP} = \overrightarrow{PB}$

$$\Rightarrow 2(\overrightarrow{OP} - \overrightarrow{OA}) = \overrightarrow{OB} - \overrightarrow{OP} \therefore 3\overrightarrow{OP} = \overrightarrow{OB} + 2\overrightarrow{OA}$$

$$\Rightarrow OP = \frac{1}{3}(7\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + \frac{2}{3}(\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k} \therefore P(3, -1, 3)$$

b) 

Ratio: $\overrightarrow{AQ} : \overrightarrow{QB} = 3:-1$ (check ratio values)

$$\frac{\overrightarrow{AQ}}{\overrightarrow{QB}} = \frac{3}{-1} \Rightarrow -\overrightarrow{AQ} = 3\overrightarrow{QB}$$

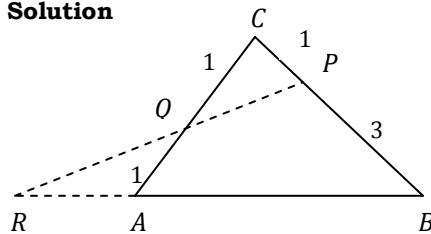
$$\Rightarrow 2(\overrightarrow{OQ} - \overrightarrow{OA}) = 3(\overrightarrow{OB} - \overrightarrow{OQ})$$

$$2\overrightarrow{OQ} = 3\overrightarrow{OB} - \overrightarrow{OA} \Rightarrow \overrightarrow{OQ} = \frac{3}{2}(7\mathbf{i} - \mathbf{j} - 2\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

$$\overrightarrow{OQ} = \frac{19}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - 7\mathbf{k} \therefore Q\left(\frac{19}{2}, -\frac{1}{2}, -7\right)$$

7. The vertices A, B and C of a triangle have p.v.s $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ respectively. The point P divides BC internally in the ratio 3:1, the point Q is the midpoint of CA and the point R divides AB externally in the ratio 1:3. If the p.v.s of P, Q and R are $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ respectively, express $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. find also the ratio $PR:QR$.

Solution



Given: $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}, \overrightarrow{OC} = \mathbf{c}$

$\overrightarrow{BP} : \overrightarrow{PC} = 3:1, \overrightarrow{AQ} : \overrightarrow{QC} = 1:1$

$\overrightarrow{AR} : \overrightarrow{RB} = -1:3$

$\overrightarrow{OP} = \mathbf{p}, \overrightarrow{OQ} = \mathbf{q}, \overrightarrow{OR} = \mathbf{r}$

From $\overrightarrow{AR} : \overrightarrow{RB} = -1:3$

$$\Rightarrow \frac{\overrightarrow{AR}}{\overrightarrow{RB}} = -\frac{1}{3} \text{ or } 3\overrightarrow{AR} = -\overrightarrow{RB}$$

$$\Rightarrow 3(\overrightarrow{OR} - \overrightarrow{OA}) = -(\overrightarrow{OB} - \overrightarrow{OR}) = -\overrightarrow{OB} + \overrightarrow{OR}$$

$$\Rightarrow 2\overrightarrow{OR} = 3\overrightarrow{OA} - \overrightarrow{OB} \therefore \overrightarrow{OR} = \frac{3}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} \dots\dots\dots (1)$$

From $\overrightarrow{BP} : \overrightarrow{PC} = 3:1 \Rightarrow \frac{\overrightarrow{BP}}{\overrightarrow{PC}} = \frac{3}{1} \text{ or } \overrightarrow{BP} = 3\overrightarrow{PC}$

$$\Rightarrow \overrightarrow{OP} - \overrightarrow{OB} = 3(\overrightarrow{OC} - \overrightarrow{OP}) = 3\overrightarrow{OC} - 3\overrightarrow{OP}$$

$$\Rightarrow 4\overrightarrow{OP} = \overrightarrow{OB} + 3\overrightarrow{OC} = \mathbf{b} + 3\mathbf{c}$$

$$\therefore \overrightarrow{OP} = \frac{1}{4}\mathbf{b} + \frac{3}{4}\mathbf{c} \dots\dots\dots (2)$$

From $\overrightarrow{AQ} : \overrightarrow{QC} = 1:1 \Rightarrow \overrightarrow{AQ} = \overrightarrow{QC}$

$$\Rightarrow \overrightarrow{OQ} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OQ}$$

$$\Rightarrow 2\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{OC} \therefore \overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}$$

$$\therefore q = \frac{1}{2}a + \frac{1}{2}c \dots\dots\dots (3)$$

Solving (1) , (2) and (3) simultaneously by expressing a, b and c in terms of p and q

$$p = \frac{1}{4}b + \frac{3}{4}c \text{ or } 4p = b + 3c \dots\dots\dots (4)$$

$$q = \frac{1}{2}a + \frac{1}{2}c \text{ or } 2q = a + c \dots\dots\dots (5)$$

Eliminating c from (4) and (5)

$$\begin{array}{l} 4p = b + 3c \\ 2q = a + c \end{array} \quad \times 3 \quad \Rightarrow \quad \begin{array}{l} 4p = b + 3c \\ 6q = 3a + 3c \\ \hline 4p - 6q = -3a + b \end{array} \dots\dots\dots (6)$$

$$\text{From (1) } \overrightarrow{OR} = r = \frac{3}{2}a - \frac{1}{3}b \text{ or } 2r = 3a - b \dots\dots\dots (7)$$

Eliminating a and b from (6) and (7), adding the equations

$$\Rightarrow 4p - 6q + 2r = 0 \text{ or } 2p - 3q + r = 0$$

$$\therefore r = 3q - 2p$$

$$\text{Ratio } \overrightarrow{PR} : \overrightarrow{QR} = \frac{\overrightarrow{PR}}{\overrightarrow{QR}}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (3q - 2p) - p = 3(q - p)$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = (3q - 2p) - q = 2(q - p)$$

$$\therefore \frac{\overrightarrow{PR}}{\overrightarrow{QR}} = \frac{3(q-p)}{2(q-p)} = \frac{3}{2}$$

$$\therefore \overrightarrow{PR} : \overrightarrow{QR} = 3:2 \text{ (Internal division)}$$

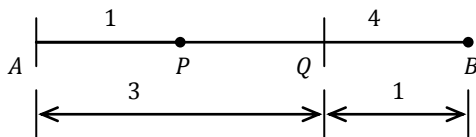
8. A, P, Q, B are points on a straight line such that $AP:PB = 1:4, AQ:QB = 3:1$.

Find;

a) $AP:PQ:QB$

b) $AQ:QP$

Solution



Let p.v of A and B be a and b respectively.

$$\text{For } AP:PQ = 1:4 \Rightarrow \overrightarrow{OP} = \frac{4}{5}a + \frac{1}{5}b$$

$$\text{For } AQ:QB = 3:1 \Rightarrow \overrightarrow{OQ} = \frac{1}{4}a + \frac{3}{4}b$$

$$\text{a) } \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \frac{4}{5}a + \frac{1}{5}b - a = \frac{1}{5}(b - a)$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \left(\frac{1}{4}a + \frac{3}{4}b\right) - \left(\frac{4}{5}a + \frac{1}{5}b\right) = -\frac{11}{20}a + \frac{11}{20}b = \frac{11}{20}(b - a)$$

$$\overrightarrow{QB} = \overrightarrow{OB} - \overrightarrow{OQ} = b - \left(\frac{1}{4}a + \frac{3}{4}b\right) = -\frac{1}{4}a + \frac{1}{4}b = \frac{1}{4}(b - a)$$

$$\text{Now } \overrightarrow{AP} : \overrightarrow{PQ} = \frac{\overrightarrow{AP}}{\overrightarrow{PQ}} = \frac{\frac{1}{5}(b-a)}{\frac{11}{20}(b-a)} = \frac{4}{11} \therefore \overrightarrow{AP} : \overrightarrow{PQ} = 4:11$$

$$\text{Also } \overrightarrow{PQ} : \overrightarrow{QB} = \frac{\frac{11}{20}(b-a)}{\frac{1}{4}(b-a)} = \frac{11}{5} \therefore \overrightarrow{PQ} : \overrightarrow{QB} = 11:5$$

Combining the ratios using the ratio algebra,

$$AP:PQ = 4:11 ; PQ:QB = 11:5 \therefore AP:PQ:QB = 4:11:5$$

$$\text{b) } \overrightarrow{AQ} : \overrightarrow{QP} = \frac{\overrightarrow{AQ}}{\overrightarrow{QP}}$$

$$\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = \left(\frac{1}{4}a + \frac{3}{4}b\right) - a = \frac{3}{4}(b - a)$$

$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \left(\frac{4}{5}a + \frac{1}{5}b\right) - \left(\frac{1}{4}a + \frac{3}{4}b\right) = -\frac{11}{20}(b - a)$$

$$\therefore \overrightarrow{AQ} : \overrightarrow{QP} = \frac{\overrightarrow{AQ}}{\overrightarrow{QP}} = \frac{\frac{3}{4}(b-a)}{-\frac{11}{20}(b-a)} = \frac{3}{4} \times \frac{20}{-11} = \frac{15}{-11} \therefore AQ : QP = 15 : -11$$

NOTE:

Vectors $\overrightarrow{OP} = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$ and $\overrightarrow{OQ} = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$ have been deduced using *Ratio Theorem*

Task

1. If the points A and B have p.vs $\mathbf{a}$ and $\mathbf{b}$ respectively, find in terms of $\mathbf{a}$ and $\mathbf{b}$ the p.vs of

- The point P which divides AB internally in the ratio 5:3
- The point Q which divides AB externally in the ratio 3:1
- The point R such that A is the midpoint of BR

ANSWER: a) $\frac{1}{8}(3\mathbf{a} + 5\mathbf{b})$, b) $\frac{1}{2}(3\mathbf{b} - \mathbf{a})$ c) $2\mathbf{a} - \mathbf{b}$

2. Write down, in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ the position vectors of the points $A(2, -5, 3)$ and $B(7, 0, -2)$. Hence find the coordinates of C which divides AB internally in the ratio 2:3 and point D which divides it externally in the ratio 3:8.

ANSWER: $2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$; $7\mathbf{i} - 2\mathbf{k}$; $(4, -3, 1)$, $(-1, -8, 6)$

3. The points A, B, C and D have coordinates $A(-7, 9)$, $B(3, 4)$, $C(1, 2)$ and $D(-2, -9)$. Find the position vector of a point P that divides AB in the ratio 2:3 and Q that divides it in the ratio 1:-4

4. **ANSWER:** Show that the points A, B and C with p.vs $3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $8\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ and $11\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ respectively are vertices of a triangle. (*Uneb 2011*)

5. Given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{d} = -4\mathbf{j} + 8\mathbf{k}$. Express $\mathbf{d}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Answer: Left as an exercise

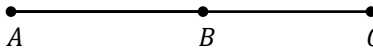
[*HINT: Let $\mathbf{d} =$*

$\lambda\mathbf{a} + \mu\mathbf{b} + \gamma\mathbf{c}$, Form and solve 3 simultaneous equations for the scalars]

COLLINEARITY OF VECTORS

Vectors are either collinear or parallel.

Consider three collinear points A, B and C



vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are collinear if
where λ , μ and γ are
Scalars/constants and

$\lambda\overrightarrow{AB} + \mu\overrightarrow{BC} + \gamma\overrightarrow{AC} = 0$

$\lambda + \mu + \gamma = 0$

When vectors are given in the form $i - j - k$, 3-equations are formulated containing only scalars. The 3 equations are solved for the scalars using cross multiplication, and when put in ,

$\lambda + \mu + \gamma = 0$

and satisfies then the vectors are collinear.

Alternatively, (commonly used)

For $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ to be collinear/parallel, then
, where λ , μ are scalars.

$\overrightarrow{AB} = \lambda\overrightarrow{BC} = \mu\overrightarrow{AC}$

Examples

1. Show that the points with the position vectors

- a) $i + 2j + 5k, 3i + 2j + k$ and $2i + 2j + 3k$
 b) $i + j + k, 4i + 3j$ and $10i + 7j - 2k$
 , are collinear.

Solution

- a) Let $A(1, 2, 5), B(3, 2, 1), C(2, 2, 3)$
 For A, B and C to be collinear, $\overrightarrow{AB} = \lambda \overrightarrow{AC}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Now $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ gives

$$\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \Rightarrow 2 = \lambda \dots\dots\dots (1)$$

$$\Rightarrow -4 = -2\lambda \text{ or } \lambda = 2 \dots\dots\dots (2)$$

Since the $\lambda = 2$ is consistent for the two equations formulated, the points are collinear.

Alternatively (Not commonly used)

Let $A(1, 2, 5), B(3, 2, 1), C(2, 2, 3)$ have position vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.

For the vectors to be collinear, then $\lambda \overrightarrow{OA} + \mu \overrightarrow{OB} + \gamma \overrightarrow{OC} = 0$

$$\Rightarrow \lambda \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 0$$

$$\Rightarrow \lambda + 3\mu + 2\gamma = 0 \dots\dots\dots (1)$$

$$\Rightarrow 2\lambda + 2\mu + 2\gamma = 0 \dots\dots\dots (2)$$

$$\Rightarrow 5\lambda + \mu + \gamma = 0 \dots\dots\dots (3)$$

$$\text{Also } \lambda + \mu + \gamma = 0 \dots\dots\dots (4)$$

Expressing λ, μ in terms of γ only

Considering eqn. (1) and eqn. (2), eliminating λ

$$\lambda + 3\mu + 2\gamma = 0 \quad \times 2$$

$$2\lambda + 2\mu + 2\gamma = 0 \quad \times 1$$

$$4\mu + 2\gamma = 0 \Rightarrow \mu = -\frac{1}{2}\gamma \dots\dots\dots (5)$$

Eqn. (5) in to eqn. (1) gives

$$\lambda + 3\left(-\frac{1}{2}\gamma\right) + 2\gamma = 0 \Rightarrow \lambda = -\frac{1}{2}\gamma \dots\dots\dots (6)$$

Eqn. (5), (6) in to (3) and (4) for confirmation

From eqn. (3),

$$5\lambda + \mu + \gamma = 0 \Rightarrow -\frac{5}{2}\gamma - \frac{1}{2}\gamma + 3\gamma = 0 \text{ correct}$$

From eqn. (4) $\lambda + \mu + \gamma = 0$

$$\Rightarrow -\frac{1}{2}\gamma - \frac{1}{2}\gamma + \gamma = 0 \text{ correct . Hence points are collinear.}$$

- b) Left as an exercise

2. Show that the points R, P and T with position vectors are

a) $\frac{1}{2}(3\mathbf{a} - \mathbf{b}), \frac{1}{70}(53\mathbf{a} + 17\mathbf{b})$ and $\frac{1}{5}(\mathbf{a} + 4\mathbf{b})$

b) $7\mathbf{a} + \mathbf{b} + \mathbf{c}, 10\mathbf{a} + 2\mathbf{b} + 9\mathbf{c}$ and $\mathbf{a} - \mathbf{b} + 3\mathbf{c}$

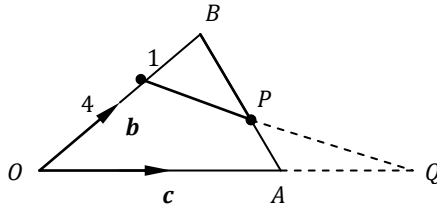
c) $\frac{1}{2}(17\mathbf{a} + 20\mathbf{b} + 3\mathbf{c}), 10\mathbf{a} + 11\mathbf{b} + 2\mathbf{c}$ and $8\mathbf{a} + \frac{29}{3}\mathbf{b} + \frac{4}{3}\mathbf{c}$, are collinear.

Solution

- a) For R, P and T to be collinear, $\overrightarrow{RP} = \lambda \overrightarrow{RT}$
- $$\begin{aligned}\overrightarrow{RP} &= \overrightarrow{OP} - \overrightarrow{OR} = \frac{1}{70}(53\mathbf{a} + 17\mathbf{b}) - \frac{1}{2}(3\mathbf{a} - \mathbf{b}) \\ &= -\frac{26}{35}\mathbf{a} + \frac{26}{35}\mathbf{b} = \frac{26}{35}(\mathbf{b} - \mathbf{a}) \\ \overrightarrow{RT} &= \overrightarrow{OT} - \overrightarrow{OR} = \frac{1}{5}(\mathbf{a} + 4\mathbf{b}) - \frac{1}{2}(3\mathbf{a} - \mathbf{b}) \\ &= -\frac{13}{10}\mathbf{a} + \frac{13}{10}\mathbf{b} = \frac{13}{10}(\mathbf{b} - \mathbf{a}) \\ \text{Now } \overrightarrow{RP} &= \lambda \overrightarrow{RT} \text{ yields} \\ \frac{26}{35}(\mathbf{b} - \mathbf{a}) &= \lambda \left[\frac{13}{10}(\mathbf{b} - \mathbf{a}) \right] \Rightarrow \frac{26}{35} = \frac{13}{10}\lambda \therefore \lambda = \frac{2}{3} (\text{scalar}) \\ \therefore \overrightarrow{RP} &= \frac{2}{3}\overrightarrow{RT}, \text{ if R, P and T are collinear}\end{aligned}$$
- b) $\overrightarrow{RP} = \lambda \overrightarrow{RT}$
- $$\begin{aligned}\overrightarrow{RP} &= \overrightarrow{OP} - \overrightarrow{OR} = (10\mathbf{a} + 11\mathbf{b} + 2\mathbf{c}) - (7\mathbf{a} + \mathbf{b} + \mathbf{c}) = 3\mathbf{a} + \mathbf{b} + 2\mathbf{c} \\ \overrightarrow{RT} &= \overrightarrow{OT} - \overrightarrow{OR} = (\mathbf{a} - \mathbf{b} + 3\mathbf{c}) - (7\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ &= -6\mathbf{a} - 2\mathbf{b} - 4\mathbf{c} = -2(3\mathbf{a} + \mathbf{b} + 2\mathbf{c}) \\ \Rightarrow (3\mathbf{a} + \mathbf{b} + 2\mathbf{c}) &= -2(3\mathbf{a} + \mathbf{b} + 2\mathbf{c})\lambda \\ \Rightarrow 1 &= -2\lambda \therefore \lambda = -\frac{1}{2} (\text{scalar}) \\ \therefore \overrightarrow{RP} &= -\frac{1}{2}\overrightarrow{RT}, \text{ if R, P and T are collinear}\end{aligned}$$
3. Given that $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + t\mathbf{j}$ and $\mathbf{c} = 2\lambda\mathbf{i} - \mathbf{j}$. Find the value of λ and t if
- $\mathbf{a}$ and $\mathbf{b}$ are parallel
 - $\mathbf{a}$ and $\mathbf{c}$ are collinear

Solution

- a) For $\mathbf{a}$ and $\mathbf{b}$ to be parallel, $\mathbf{a} = \mu\mathbf{b}$
- $$\Rightarrow (2\mathbf{i} + 6\mathbf{j}) = \mu(3\mathbf{i} + t\mathbf{j}) = 3\mu\mathbf{i} + t\mu\mathbf{j}$$
- Equating coefficients of unit vectors
- For $\mathbf{i}$: $2 = 3\mu \Rightarrow \mu = \frac{2}{3}$
- For $\mathbf{j}$: $6 = t\mu \Rightarrow 6 = \frac{2}{3}t \Rightarrow t = 9$
- b) $\mathbf{a}$ and $\mathbf{c}$ are collinear/parallel if $\mathbf{a} = k\mathbf{c}$
- $$\Rightarrow 2\mathbf{i} + 6\mathbf{j} = k(2\lambda\mathbf{i} - \mathbf{j}) = 2\lambda k\mathbf{i} - k\mathbf{j}$$
- Equating coefficients of unit vectors
- For $\mathbf{i}$: $2 = 2\lambda k$ (1)
- For $\mathbf{j}$: $6 = -k \Rightarrow k = -6$ (2)
- Eqn. (2) in to (1)
- $$2 = -12\lambda \Rightarrow \lambda = -\frac{1}{6} \therefore k = -6, \lambda = -\frac{1}{6}$$
4. Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, point R is on $\overrightarrow{OB}$ such that $\overrightarrow{OR}:\overrightarrow{RB} = 4:1$. Point P is on BA such that $\overrightarrow{BP}:\overrightarrow{PA} = 2:3$, and when RP and OA are both produced they meet at point Q. Find
- $\overrightarrow{OR}$ and $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$
 - $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ (Uneb 2006)

Solution


$$\begin{aligned} \text{a) From } \overrightarrow{OR} : \overrightarrow{RB} &= 4:1 \Rightarrow \frac{\overrightarrow{OR}}{\overrightarrow{RB}} = \frac{4}{1} \\ \overrightarrow{OR} &= 4\overrightarrow{RB} \\ \Rightarrow \overrightarrow{OR} &= 4(\overrightarrow{OB} - \overrightarrow{OR}) \Rightarrow \overrightarrow{OR} = \\ &\frac{4}{5}\overrightarrow{OB} \\ \therefore \overrightarrow{OR} &= \frac{4}{5}\mathbf{b} \end{aligned}$$

$$\begin{aligned} \text{From } \overrightarrow{BP} : \overrightarrow{PA} &= 2:3 \Rightarrow \frac{\overrightarrow{BP}}{\overrightarrow{PA}} = \frac{2}{3} \Rightarrow 3\overrightarrow{BP} = 2\overrightarrow{PA} \\ \Rightarrow 3(\overrightarrow{OP} - \overrightarrow{OB}) &= 2(\overrightarrow{OA} - \overrightarrow{OP}) \Rightarrow 5\overrightarrow{OP} = 2\overrightarrow{OA} + 3\overrightarrow{OB} \\ \Rightarrow \overrightarrow{OP} &= \frac{2}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OB} \therefore \overrightarrow{OP} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b} \end{aligned}$$

b) Since O, A and Q are collinear, then

$$\overrightarrow{OQ} = \lambda \overrightarrow{OA} = \lambda \mathbf{a} \dots\dots\dots (1)$$

Since R, P and Q are also collinear, then

$$\overrightarrow{RQ} = \mu \overrightarrow{RP} \Rightarrow (\overrightarrow{OQ} - \overrightarrow{OR}) = \mu(\overrightarrow{OP} - \overrightarrow{OR})$$

$$\therefore \overrightarrow{OQ} = \overrightarrow{OR} + \mu(\overrightarrow{OP} - \overrightarrow{OR}) = (1 - \mu)\overrightarrow{OR} + \mu\overrightarrow{OP}$$

$$= (1 - \mu) \cdot \frac{4}{5}\mathbf{b} + \mu\left(\frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}\right) = \left(\frac{4-4\mu}{5} + \frac{3}{5}\mu\right)\mathbf{b} + \frac{2}{5}\mu\mathbf{a}$$

$$\therefore \overrightarrow{OQ} = \left(\frac{4-\mu}{5}\right)\mathbf{b} + \frac{2}{5}\mu\mathbf{a} \dots\dots\dots (2)$$

eqn. (2) = eqn. (1)

$$\Rightarrow \lambda \mathbf{a} = \frac{2}{5}\mu\mathbf{a} + \left(\frac{4-\mu}{5}\right)\mathbf{b}$$

Equating corresponding vectors

$$\lambda = \frac{2}{5}\mu \dots\dots\dots (3)$$

$$0 = \frac{4-\mu}{5} \Rightarrow \mu = 4 \dots\dots\dots (4)$$

$$\therefore \lambda = \frac{8}{5}$$

$$\text{From eqn. (1), } \overrightarrow{OQ} = \lambda \overrightarrow{OA} = \lambda \mathbf{a} = \frac{8}{5}\mathbf{a}$$

5. The vertices of a triangle are $P(2, -1, 5)$, $Q(7, 1, -3)$ and $R(2, -2, 0)$. Find the coordinates of S if PQRS is a rectangle.

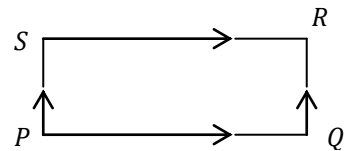
Solution

using $\overrightarrow{PQ} = \overrightarrow{SR}$ or $\overrightarrow{PS} = \overrightarrow{QR}$

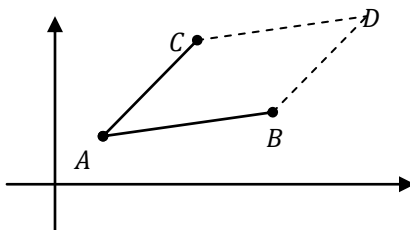
$\overrightarrow{PQ} = \overrightarrow{SR}$, gives $\overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{OR} - \overrightarrow{OS}$

$$\therefore \overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{OP} - \overrightarrow{OQ}$$

$$= \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 8 \end{pmatrix} \therefore S(-3, -4, 8)$$



6. The vertices of a parallelogram are $A(1, 1)$, $B(5, 3)$, $C(3, 7)$ and D. find the coordinates of D.

Solution


For a parallelogram, $\overrightarrow{AB} = \overrightarrow{CD}$ or $\overrightarrow{AC} = \overrightarrow{BD}$

Using $\overrightarrow{AB} = \overrightarrow{CD}$

$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OC} \Rightarrow \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{OD} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix} \therefore D(7, 9)$$

PRODUCT OF TWO VECTORS

There are two types of products

- Scalar or Dot product
- Vector or Cross product

SCALAR/DOT PRODUCT

Consider two position vectors $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$ making an angle of θ with each other. The scalar product is defined as

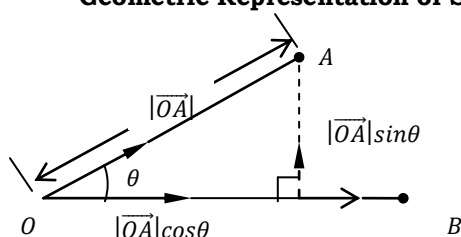
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta \text{ where } 0 \leq \theta \leq \pi$$

The product will either be +ve or -ve or zero depending on the value of θ .

NOTE

Two vectors can be dot multiplied if they are in *same direction*.

Geometric Representation of Scalar product



$|\vec{OA}| \cos \theta$ is the projection of $\vec{OA}$ and $\vec{OB}$
 $\vec{OA} \cos \theta$ and $\mathbf{b}$ are in the same direction.
 Multiplying their scalars

$$\Rightarrow |\vec{OA}| \cos \theta \times |\mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \cos \theta = \mathbf{a} \cdot \mathbf{b}$$

$$\therefore \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

PROPERTIES OF SCALAR PRODUCT

The properties below are very useful in the computations involving product of vectors.

- 1) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ commutative property
- 2) $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda (\mathbf{a} \cdot \mathbf{b})$ where $\lambda = \text{scalar}$
- 3) $(\lambda \mathbf{a}) \cdot (\mu \mathbf{b}) = \mu \lambda (\mathbf{a} \cdot \mathbf{b}) = (\mu \lambda) \cdot \mathbf{a} \cdot \mathbf{b}$
- 4) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ Distributive property
- 5) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} + \mathbf{b})^2 = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$
 $= a^2 + 2\mathbf{a} \cdot \mathbf{b} + b^2$ (a and b are now scalars)
- 6) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$
 $= a^2 - b^2$ [$\mathbf{a} \neq \mathbf{a}$]. Here a and b are scalars
- 7) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = (\mathbf{a} - \mathbf{b})^2 = \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$
 $= a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2$ (a and b are now scalars)
- 8) For $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$
 - a) If $\theta = 90^\circ$ i.e. $\mathbf{a}$ is perpendicular to $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = 0$
 $[\because \cos 90^\circ = 0]$. Hence $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if

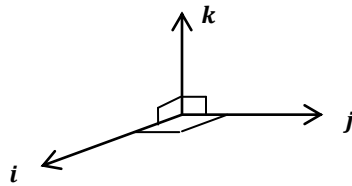
$$\mathbf{a} \cdot \mathbf{b} = 0$$

- b) If $\theta = 180^\circ$ i.e. $\mathbf{a}$ and $\mathbf{b}$ are in opposite direction, then
 $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}| |\mathbf{b}|$ [$\because \cos 180^\circ = -1$]
- c) Angle between $\mathbf{a}$ and $\mathbf{b}$ is given from

$$\cos \theta = \frac{|\mathbf{a}| |\mathbf{b}|}{\mathbf{a} \cdot \mathbf{b}}$$

MULTIPLICATION OF UNIT VECTORS i, j, k

Unit vectors are mutually perpendicular



$$i \cdot i = i^2 = |i||i|\cos 0 = 1$$

$$j \cdot j = j^2 = |j||j|\cos 0 = 1$$

$$k \cdot k = k^2 = |k||k|\cos 0 = 1$$

$$k \cdot i = |k||i|\cos 90 = 0$$

$$i \cdot j = |i||j|\cos 90 = 0$$

$$k \cdot j = |k||j|\cos 90 = 0$$

The above results in conjunction

with the above properties of dot product are useful in vector product computations.

Example

1. Given that $a = 2i + j + k$, $b = i + j - 3k$ and $c = i + 2j - 3k$. Find

- $a \cdot b$
- $a \cdot (b + c)$
- $(a \cdot b) \cdot (a - b)$

Solution

a) $a \cdot b = (2i + j + k) \cdot (i + j - 3k) = 2i \cdot i + 2i \cdot j - 6i \cdot k$

$$+ j \cdot i + j \cdot j - 3j \cdot k + k \cdot i + k \cdot j - 3k \cdot k$$

$$= 2 + 0 - 0 + 0 + 1 + 0 + 0 + 0 - 3 = 2 + 1 - 3 = 0$$

b) $a \cdot (b + c)$; $b + c = (i + j - 3k) + (i + 2j - 3k) = 2i + 2j - 6k$

$$\Rightarrow a \cdot (b + c) = (2i + j + k) \cdot (2i + 2j - 6k)$$

$$= i \cdot i + 2i \cdot j - 3i \cdot k + j \cdot i + j \cdot j - 3j \cdot k - 3k \cdot i - 6k \cdot j + 9k \cdot k$$

$$= 2 + 0 + 0 + 0 + 2 - 0 + 0 + 0 - 6 = 2 + 2 - 6 = -2$$

c) $(a \cdot b) \cdot (a - b)$; $a - b = (2i + j + k) - (i + j - 3k) = i + 0j + 4k$

But $a \cdot b = 0 \Rightarrow (a \cdot b) \cdot (a - b) = 0 \cdot (i + 0j + 4k) = 0$

NOTE

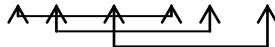
 When dotting two vectors in multiplication always dot the corresponding vectors only. *i.e.* i^2, j^2 and k^2 , since other products of not similar unit vectors yield zero results

2. Given that $a = 4i + 5j - 7k$, $b = 7i + 2j + k$ and $c = 3i - 4j + 9k$. Find

- $a \cdot b$
- $b \cdot c$
- $a \cdot (a + b)$
- $(a \cdot b) \cdot (b + c)$

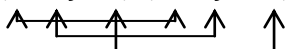
Solution

a) $a \cdot b = (4i + 5j - 7k) \cdot (7i + 2j + k)$



$$= 28 + 10 - 7 = 31$$

b) $b \cdot c = (7i + 2j + k) \cdot (3i - 4j + 9k)$



$$= 21 - 8 + 9 = 22$$

c) $a \cdot (a + b)$; $a + b = (4i + 5j - 7k) + (7i + 2j + k) = 11i + 7j - 6k$

$$\Rightarrow \mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = (4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}) \cdot (11\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}) = 44 + 35 + 42 = 121$$

Alternatively

$$\text{Using } \mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{a} \cdot \mathbf{a} = (4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}) \cdot (4\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}) = 16 + 25 + 49 = 90$$

$$\therefore \mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = 90 + 31 = 121$$

$$\begin{aligned} \text{d) } (\mathbf{a} \cdot \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) &= 31 \times [(7\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + (3\mathbf{i} - 4\mathbf{j} + 9\mathbf{k})] \\ &= 31(10\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}) = 310\mathbf{i} - 62\mathbf{j} + 310\mathbf{k} \end{aligned}$$

NOTE

When two vectors are dot multiplied the answer is a constant value **not** a vector.

3. Given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 8$, calculate:

a) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

b) Hence find $|\mathbf{a} + \mathbf{b}|$ and $|\mathbf{a} - \mathbf{b}|$

Solution

a) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2(8)$

$$\text{But if } \mathbf{a} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, |\mathbf{a}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\text{, and } \mathbf{a} \cdot \mathbf{a} = (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) \cdot (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) = x_1^2 + x_2^2 + x_3^2$$

$$\therefore |\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = 49, |\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b} = 16$$

$$\Rightarrow (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 49 + 16 + 16 = 81$$

b) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$
 $= \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = 49 - 2 \times 8 + 16 = 49$

$$\text{From } |\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} \Rightarrow |\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$$

$$\therefore |\mathbf{a} + \mathbf{b}| = \sqrt{(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})} = \sqrt{81} = 9$$

$$\text{Also } |\mathbf{a} - \mathbf{b}| = \sqrt{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})} = \sqrt{49} = 7$$

Task

Given that $|\mathbf{x}| = 5$, $|\mathbf{y}| = 10$ and $\mathbf{x} \cdot \mathbf{y} = 22$, calculate $|\mathbf{x} + \mathbf{y}|$ and $|\mathbf{x} - \mathbf{y}|$

Answer: 13, 9

4. Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \cdot \mathbf{c} = 3$, $\mathbf{b} \cdot \mathbf{c} = 4$. Given that the vector $\mathbf{d} = \mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{c}$, find the value of λ .

Solution

$\mathbf{d}$ is perpendicular to $\mathbf{c}$ if $\mathbf{d} \cdot \mathbf{c} = 0$ [see properties]

$$\mathbf{d} = \mathbf{a} + \lambda\mathbf{b} \Rightarrow (\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \lambda\mathbf{b} \cdot \mathbf{c} = 0 \quad [\text{see properties}]$$

$$\Rightarrow 3 + 4\lambda = 0 \therefore \lambda = -\frac{3}{4}$$

5. Given that $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + \lambda\mathbf{j}$. Find the value of λ such that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Solution

For $\mathbf{a}$ and $\mathbf{b}$ to be perpendicular $\mathbf{a} \cdot \mathbf{b} = 0$

$$\Rightarrow (2\mathbf{i} + 6\mathbf{j}) \cdot (3\mathbf{i} + \lambda\mathbf{j}) = 0$$

$$\Rightarrow 6 + 6\lambda = 0 \therefore \lambda = -1$$

6. Find the angle between vectors

a) $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

b) $\mathbf{j} + 7\mathbf{k}$ and $-5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

c) $2\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$ and $6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$

Solution

a) Let $\mathbf{a} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

From dot product, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$\mathbf{a} \cdot \mathbf{b} = (8\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 16 - 2 - 4 = 10$$

$$|a| = \sqrt{8^2 + (-1)^2 + 4^2} = 9, \quad |b| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\therefore 10 = 9 \times 3 \cos \theta \text{ or } \cos \theta = \frac{10}{27}$$

$$\Rightarrow \theta = \cos^{-1} \left(\frac{10}{27} \right) = 68.3^\circ$$

b) Left as an exercise **Answer:** 60°

c) Let $a = 2i - 3j + 8k, b = 6i - 4j - 3k$

From dot product, $a \cdot b = |a||b|\cos\theta$

$$a \cdot b = (2i - 3j + 8k) \cdot (6i - 4j - 3k) = 12 + 12 - 24 = 0$$

$$\therefore 0 = |a||b|\cos\theta \text{ or } \cos\theta = 0 \Rightarrow \theta = \cos^{-1}(0) = 90^\circ$$

Proofs involving dot product

1. Given that $|a| = 3, |b| = 8$ and $|a - b| = 7$, show that the angle between a and b is 60° .

Solution

By dot product, $a \cdot b = |a||b|\cos\theta$

But $|a|^2 = a \cdot a, |b|^2 = b \cdot b$ and $|a - b|^2 = (a - b) \cdot (a - b)$

$$\text{From } |a - b|^2 = 49 = (a - b) \cdot (a - b) = a \cdot a - 2a \cdot b + b \cdot b$$

$$\Rightarrow 49 = 3^2 - 2a \cdot b + 8^2 = 73 - 2a \cdot b$$

$$\Rightarrow -24 = -2a \cdot b \therefore a \cdot b = 12$$

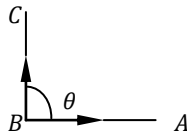
$$a \cdot b = |a||b|\cos\theta = 12$$

$$\Rightarrow 12 = 3 \times 8 \cos\theta \text{ or } \cos\theta = \frac{1}{2}$$

$$\therefore \cos^{-1} \left(\frac{1}{2} \right) = 60^\circ$$

2. The points A, B and C have p. vs $a = 5i + 3j + 2k, b = 2i - j + 3k$ and $c = 7i - 3j + 10k$ respectively. Show that $\angle ABC$ is a right angle.

Solution



NOTE: indicated vectors originate from B

Given: $\overrightarrow{OA} = a, \overrightarrow{OB} = b, \overrightarrow{OC} = c$

$$\overrightarrow{BA} = d = \overrightarrow{OA} - \overrightarrow{OB}$$

$$= (5i + 3j + 2k) - (2i - j + 3k)$$

$$= 3i + 4j - k$$

$$\overrightarrow{BC} = e = \overrightarrow{OC} - \overrightarrow{OB} = (7i - 3j + 10k) - (2i - j + 3k)$$

$$= 5i - 2j + 7k$$

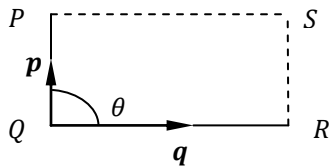
For d and e to be perpendicular, $d \cdot e = 0$, by dot product $d \cdot e = |d||e|\cos\theta$

$$\Rightarrow d \cdot e = (3i + 4j - k) \cdot (5i - 2j + 7k) = 15 - 8 - 7 = 0$$

$$\therefore d \cdot e = 0 \Rightarrow \cos\theta = 0 \therefore \theta = 90^\circ$$

3. The vertices of a Δ are $P(2, -1, 5), Q(7, 1, -3)$ and $R(13, -2, 0)$. Show that $\angle PQR = 90^\circ$. Find the coordinates of S if PQRS is a rectangle. (Uneb 2002)

Solution



$$\text{Let } \overrightarrow{PQ} = p = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}$$

$$\overrightarrow{QR} = q = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix} 13 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix}$$

For $\mathbf{p}$ and $\mathbf{q}$ to be perpendicular, $\mathbf{p} \cdot \mathbf{q} = 0$

$$\Rightarrow \mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = 30 - 6 - 24 = 0$$

$$\text{or } \mathbf{p} \cdot \mathbf{q} = (5\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}) \cdot (6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = 30 - 6 - 24 = 0$$

$$\therefore \cos \theta = 0 \text{ or } \theta = 90^\circ$$

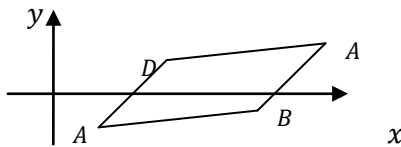
For a rectangle, $\overrightarrow{QR} = \overrightarrow{PS}$

$$\Rightarrow \mathbf{q} = \overrightarrow{OS} - \overrightarrow{OP} \therefore \overrightarrow{OS} = \mathbf{q} + \overrightarrow{OP}$$

$$\Rightarrow \overrightarrow{OS} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 8 \end{pmatrix} \therefore S(8, -4, 8)$$

4. ABCD is a quadrilateral with $A(2, -2)$, $B(5, -1)$, $C(6, 2)$ and $D(3, 1)$. Show that the quadrilateral is a rhombus.

Solution



For a rhombus

- All sides are equal
- Diagonals meet at 90°
- All opposite angles are equal but not right angles

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$|\overrightarrow{AB}| = |\overrightarrow{DC}| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$|\overrightarrow{AD}| = |\overrightarrow{BC}| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

Hence all sides are equal = $\sqrt{10}$ units

Now diagonals are $\overrightarrow{BD}$ and $\overrightarrow{AC}$

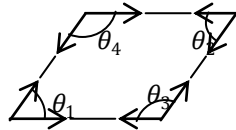
$$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

For $\overrightarrow{BD}$ and $\overrightarrow{AC}$ to be perpendicular, $\overrightarrow{BD} \cdot \overrightarrow{AC} = 0$

$$\overrightarrow{BD} \cdot \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 8 - 8 = 0. \text{ Hence diagonals meet at } 90^\circ$$

Now angles are as in the figure.



$\angle \theta_1 = \text{angle between } \overrightarrow{AD} \text{ and } \overrightarrow{AB}$

$$\Rightarrow \overrightarrow{AD} \cdot \overrightarrow{AB} = |\overrightarrow{AD}| |\overrightarrow{AB}| \cos \theta_1$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \sqrt{10} \times \sqrt{10} \cos \theta$$

$$\cos \theta_1 = \frac{3+3}{10} \Rightarrow \theta_1 = 53.1^\circ$$

$\angle \theta_2 = \text{angle between } \overrightarrow{CB} \text{ and } \overrightarrow{CD}$

$$\Rightarrow \overrightarrow{CB} \cdot \overrightarrow{CD} = |\overrightarrow{CB}| |\overrightarrow{CD}| \cos \theta_2$$

$$\text{But } \overrightarrow{CB} = -\overrightarrow{BC}, \overrightarrow{CD} = -\overrightarrow{DC}$$

$$\therefore |\overrightarrow{CB}| = |\overrightarrow{BC}| = |\overrightarrow{CD}| = |\overrightarrow{DC}| = \sqrt{10}$$

$$\begin{aligned}
 \therefore \vec{CB} \cdot \vec{CD} &= \vec{BC} \cdot \vec{DC} = |\vec{BC}| |\vec{DC}| \cos \theta_2 \\
 \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= \sqrt{10} \times \sqrt{10} \cos \theta_2 \\
 \therefore \theta_1 = \theta_2 &= 53.1^\circ \\
 < \theta_3 &= \text{angle between } \vec{AD} \text{ and } \vec{AB} \\
 \Rightarrow \vec{BC} \cdot \vec{BA} &= |\vec{BC}| |\vec{BA}| \cos \theta_3 \\
 \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \end{pmatrix} &= \sqrt{10} \times \sqrt{10} \cos \theta_3 \quad [\because |\vec{BA}| = |\vec{AB}|] \\
 -3 - 3 &= 10 \cos \theta_3 \text{ or } 10 \cos \theta_3 = -0.6 \therefore \theta_3 = 126.9^\circ \\
 < \theta_3 &= \text{angle between } \vec{DA} \text{ and } \vec{DC} = 126.9^\circ \\
 \text{Hence this is a rhombus.}
 \end{aligned}$$

5. The points $P(4, -6, 1)$, $Q(2, 8, 4)$ and $R(3, 7, 14)$ lie in the same plane. Show that the angle formed between $\vec{PQ}$ and $\vec{QR}$ is $\cos^{-1}\left(\frac{14}{\sqrt{209 \times 102}}\right)$.

Solution

$$\begin{aligned}
 \vec{PQ} &= \vec{OQ} - \vec{OP} = \begin{pmatrix} 2 \\ 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 14 \\ 3 \end{pmatrix} \\
 \vec{QR} &= \vec{OR} - \vec{OQ} = \begin{pmatrix} 3 \\ 7 \\ 14 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 10 \end{pmatrix}
 \end{aligned}$$

Using dot product $\vec{PQ} \cdot \vec{QR} = |\vec{PQ}| |\vec{QR}| \cos \theta$

$$\vec{PQ} \cdot \vec{QR} = \begin{pmatrix} -2 \\ 14 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 10 \end{pmatrix} = -2 - 14 + 30 = 14$$

$$|\vec{PQ}| = \sqrt{(-2)^2 + 14^2 + 3^2} = \sqrt{209}$$

$$|\vec{QR}| = \sqrt{1^2 + (-1)^2 + 10^2} = \sqrt{102}$$

$$\therefore \cos \theta = \left(\frac{14}{\sqrt{209 \times 102}} \right) = \left(\frac{14}{\sqrt{209 \times 102}} \right) \Rightarrow \theta = \cos^{-1} \left(\frac{14}{\sqrt{209 \times 102}} \right)$$

6. Show that A , B and C with p .vs respectively are vertices of a triangle of a ΔABC

- $2i - j + k$, $i - 3j - 5k$ and $3i - 4j - 4k$
- $4i - 5j - 2k$, $8i - 5j + 6k$ and origin
- $7i + 3j + k$, $i + 5j + 2k$ and $3i - j - k$

Solution

- a) Let $A(2, -1, 1)$, $B(1, -3, -5)$, $C(3, -4, -4)$

For A , B and C to be vertices of a ΔABC , then

$$\vec{AB} + \vec{BC} + \vec{CA} = 0 \text{ (for a closed figure)}$$

or

Sum of angles at A , B and C is 180°

Considering the **first case**

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 3 \\ -4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$$

$$\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Now from $\vec{AB} + \vec{BC} + \vec{CA} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$. Hence the vectors are

vertices of a triangle.

Now considering **second case**

$\angle \theta_1 = \text{angle between } \vec{AC} \text{ and } \vec{AB}$

By dot product $\vec{AC} \cdot \vec{AB} = |\vec{AC}| \cdot |\vec{AB}| \cos \theta_1$

$$\Rightarrow \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix} = \sqrt{1^2 + (-3)^2 + (-5)^2} \cdot \sqrt{(-1)^2 + (-2)^2 + (-6)^2} \cos \theta_1$$

$$-1 + 6 + 30 = \sqrt{35} \times \sqrt{41} \cos \theta_1 \Rightarrow \cos \theta_1 = \frac{35}{\sqrt{35 \times 41}} \therefore \theta_1 = 22.49^\circ$$

$\angle \theta_2 = \text{angle between } \vec{BA} \text{ and } \vec{BC}$

$\Rightarrow \vec{BA} \cdot \vec{BC} = |\vec{BA}| |\vec{BC}| \cos \theta_2$, but $\vec{BA} = -\vec{AB}$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \sqrt{1^2 + 2^2 + 6^2} \times \sqrt{2^2 + (-1)^2 + 1^2} \cos \theta_2$$

$$\Rightarrow 2 - 2 + 6 = \sqrt{41} \times \sqrt{6} \cos \theta_2 \Rightarrow \cos \theta_2 = \frac{6}{\sqrt{41 \times 6}} \therefore \theta_2 = 67.51^\circ$$

$\angle \theta_3 = \text{angle between } \vec{CA} \text{ and } \vec{CB}$

But $\vec{CA} = -\vec{AC}$ and $\vec{CB} = -\vec{BC}$

$$\Rightarrow (-\vec{AC}) \cdot (-\vec{BC}) = |\vec{AC}| \cdot |\vec{BC}| \cos \theta_3 \quad [\because |\vec{AC}| = |-\vec{AC}|, |\vec{BC}| = |-\vec{BC}|]$$

$$\Rightarrow \vec{AC} \cdot \vec{BC} = |\vec{AC}| \cdot |\vec{BC}| \cos \theta_3$$

$$\begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \sqrt{1^2 + (-3)^2 + (-5)^2} \times \sqrt{2^2 + (-1)^2 + 1^2} \cos \theta_3$$

$$2 + 3 - 5 = \sqrt{35} \times \sqrt{6} \cos \theta_3 \text{ or } \cos \theta_3 = 0 \therefore \theta_3 = 90^\circ$$

Now $\theta_1 + \theta_2 + \theta_3 = 22.49^\circ + 67.51^\circ + 90^\circ = 180^\circ$, which satisfy the condition.

Hence points are vertices of a triangle.

b) Left as an exercise

c) Left as an exercise

NOTE

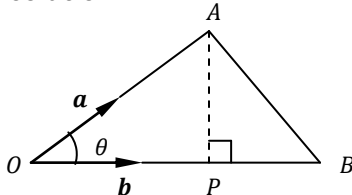
The question can also be done using $\vec{AB} + \vec{BC} + \vec{CA} = 0$

Other Scalar Product Proofs

- The points A and B have p.v.s **a** and **b** with respect to the origin O. show that the area of a triangle OAB is given by

$$\Delta = \frac{1}{2} \sqrt{a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2}, \text{ where } a = |\mathbf{a}|, b = |\mathbf{b}|$$

Solution



$$\text{Area, } \Delta = \frac{1}{2} \times |\vec{OB}| \times \vec{AP}$$

$$\text{From dot product, } \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\text{From } \Delta OAP, \vec{AP} = |\vec{OA}| \sin \theta = a \sin \theta$$

$$\therefore \Delta = \frac{1}{2} \times b \times a \sin \theta = \frac{1}{2} ab \sin \theta$$

$$\text{But } \sin \theta = \sqrt{1 - \cos^2 \theta} \quad [\because \cos^2 \theta + \sin^2 \theta = 1]$$

$$\text{From } \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta = ab \cos \theta$$

$$\Rightarrow \frac{\mathbf{a} \cdot \mathbf{b}}{ab} = \cos \theta$$

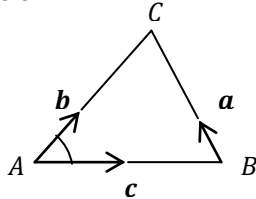
$$\Rightarrow \sin\theta = \sqrt{1 - \left(\frac{a \cdot b}{ab}\right)^2} = \frac{1}{ab} \sqrt{a^2 b^2 - (a \cdot b)^2}$$

$$\therefore \text{Area, } \Delta = \frac{1}{2} ab \sin\theta = \frac{1}{2} ab \times \frac{1}{ab} \sqrt{a^2 b^2 - (a \cdot b)^2}$$

$$\therefore \Delta = \frac{1}{2} \sqrt{a^2 b^2 - (a \cdot b)^2} \text{ As required}$$

2. In a ΔABC with vectors $AC = \mathbf{b}$, $AB = \mathbf{c}$ and $BC = \mathbf{a}$, then $a^2 = b^2 + c^2 - 2bc \cdot \cos A$, the cosine rule where $a = |\mathbf{a}|$, $b = |\mathbf{b}|$

Solution



$$\begin{aligned} a^2 &= \mathbf{a} \cdot \mathbf{a} = (\mathbf{b} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) \\ &= \mathbf{b} \cdot \mathbf{b} - 2\mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c} \\ &= b^2 - 2\mathbf{b} \cdot \mathbf{c} + c^2 \end{aligned}$$

$$\text{But } \mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| |\mathbf{c}| \cos A = bc \cos A$$

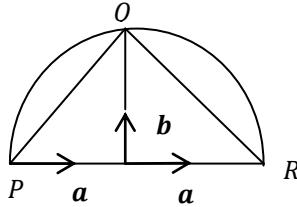
$$\therefore a^2 = b^2 - 2bc \cdot \cos A + c^2$$

or $a^2 = b^2 + c^2 - 2bc \cdot \cos A$ As required

Note: the figure is very important in the derivation.

3. Show that any angle inscribed in a semi circle is a right angle.

Solution



Let O be the centre of the semi circle,

$$\overrightarrow{PO} = \overrightarrow{OR} = \mathbf{a} \text{ and}$$

$$OQ = \mathbf{b}$$

$$\text{Required: } \overrightarrow{QP} \cdot \overrightarrow{QR} = 0 \text{ i.e. } \angle PQR = 90^\circ$$

$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = -\mathbf{a} - \mathbf{b} = -(\mathbf{a} + \mathbf{b})$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \mathbf{a} - \mathbf{b}$$

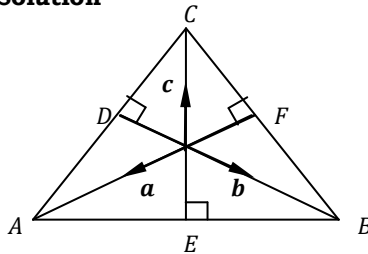
$$\text{Now } \overrightarrow{QP} \cdot \overrightarrow{QR} = -(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = -(\mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b})$$

$$\text{But } \mathbf{a} \cdot \mathbf{a} = a^2, \mathbf{b} \cdot \mathbf{b} = b^2 \text{ and } a^2 = b^2 = \text{radius}$$

$$\Rightarrow \overrightarrow{QP} \cdot \overrightarrow{QR} = -(a^2 - b^2) = (a^2 - a^2) = 0 \therefore \angle PQR = 90^\circ$$

4. In ΔABC , the altitudes through A, B and C intersect at point O. show that if $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$, then $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{c}$. show also further that $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$

Solution



From the figure, points A, O, F; B, O, D and C, O, E are collinear.

$$\Rightarrow \overrightarrow{AF} = \lambda \overrightarrow{OA}, \overrightarrow{BD} = \mu \overrightarrow{OB} \text{ and } \overrightarrow{OE} = k \overrightarrow{OC},$$

where λ, μ and k are either positive/negative constants. Since

$$\overrightarrow{AF} \perp \overrightarrow{BC}, \text{ then } \overrightarrow{AF} \cdot \overrightarrow{BC} = 0 \dots\dots\dots (1)$$

$$\overrightarrow{AD} \perp \overrightarrow{AC}, \text{ then } \overrightarrow{AD} \cdot \overrightarrow{AC} = 0 \dots\dots\dots (2)$$

$$\overrightarrow{CE} \perp \overrightarrow{AB}, \text{ then } \overrightarrow{CE} \cdot \overrightarrow{AB} = 0 \dots\dots\dots (3)$$

From eqn. (1)

$$\overrightarrow{AF} \cdot \overrightarrow{BC} = 0, \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{c} - \mathbf{b}$$

$$\Rightarrow \lambda \overrightarrow{OA} \cdot (\mathbf{c} - \mathbf{b}) = 0 \text{ or } \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$$

$$\therefore \mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$$

From eqn. (2)

$$\overrightarrow{AD} \cdot \overrightarrow{AC} = 0, \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \mathbf{c} - \mathbf{a}$$

$$\Rightarrow \mu \overrightarrow{OB} \cdot (\mathbf{c} - \mathbf{a}) = 0 \text{ or } \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0$$

$$\therefore \mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{a} = 0 \Rightarrow \mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{c}$$

$$\text{From } \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \Rightarrow \mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} = 0 \dots\dots\dots (4)$$

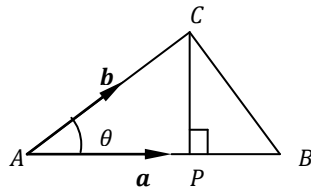
$$\text{From } \mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{c} \Rightarrow \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} = 0 \dots\dots\dots (5)$$

$$\text{Eqn. (4) + eqn. (5)}$$

$$\mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} = 0 \therefore \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c}$$

AREA OF A TRIANGLE, BY DOT PRODUCT

Consider A, B and C be vertices of a triangle ABC



$$\text{Let } \overrightarrow{AB} = \mathbf{a}, \overrightarrow{AC} = \mathbf{b}$$

$$|\overrightarrow{AC}| = |\mathbf{b}|, |\overrightarrow{AB}| = |\mathbf{a}|$$

$$\text{Area, } \Delta = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{AC}|$$

$$\text{But } \overline{CP} = |\overrightarrow{AC}| \sin \theta = |\mathbf{b}| \sin \theta$$

Now finding $\sin \theta$

$$\text{By dot product, } \overrightarrow{AC} \cdot \overrightarrow{AB} = |\overrightarrow{AC}| |\overrightarrow{AB}| \cos \theta$$

$$\Rightarrow \cos \theta = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}| |\mathbf{a}|}$$

$$\Rightarrow \text{Area, } \Delta = \frac{1}{2} \times |\mathbf{a}| \times |\mathbf{b}| \times \sin \theta = \frac{1}{2} \times |\mathbf{a}| \times \sqrt{(1 - \cos^2 \theta)} [\because \cos^2 \theta + \sin^2 \theta = 1]$$

$$= \frac{1}{2} \times |\mathbf{a}| \times |\mathbf{b}| \times \sqrt{\left(1 - \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}| |\mathbf{a}|}\right)^2\right)} = \frac{1}{2} \sqrt{\{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2\}}$$

$$\therefore \text{Area, } \Delta = \frac{1}{2} \sqrt{\{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2\}}$$

Examples

Find the area of the triangle with vertices

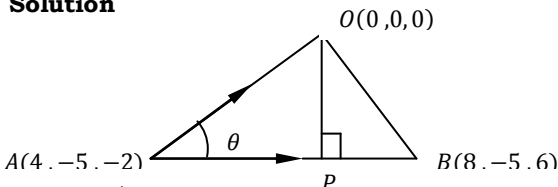
a) $A(4, -5, -2), B(8, -5, 6)$ and origin

b) $OA = \mathbf{i} - \mathbf{j}, OB = 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}, OC = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

c) $A(0, -2, 1), B(1, -1, -2)$ and $C(-1, 1, 0)$

Solution

a)



$$\text{Area, } \Delta = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OP}|$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{4^2 + 0^2 + 8^2} = \sqrt{80}$$

$$|\overrightarrow{OP}| = |\overrightarrow{AO}| \sin \theta; |\overrightarrow{AO}| = \sqrt{4^2 + (-5)^2 + (-2)^2} = \sqrt{45}$$

$$\text{From dot product, } \overrightarrow{AO} \cdot \overrightarrow{AB} = |\overrightarrow{AO}| |\overrightarrow{AB}| \cos \theta$$

$$\overrightarrow{AO} \cdot \overrightarrow{AB} = -\begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} = -(16 + 0 - 16) = 0$$

$$\therefore \overrightarrow{AO} \cdot \overrightarrow{AB} = 0 \Rightarrow \cos \theta = 0 \therefore \theta = 90^\circ$$

$$\therefore \text{Area, } \Delta = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OP}| = \frac{1}{2} \times \sqrt{80} \times \sqrt{45} \times \sin 90^\circ$$

$$= \frac{1}{2} \times \sqrt{16 \times 5} \times \sqrt{9 \times 5} = \frac{1}{2} \times 60 = 30 \text{ sq. units}$$

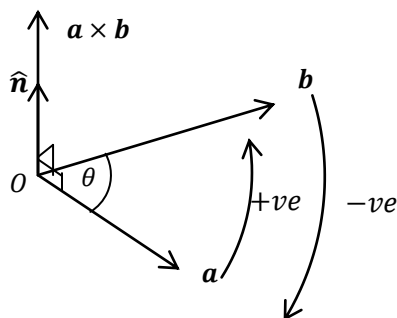
- b) $2\sqrt{2}$ sq. units
 c) **Answer:** $\sqrt{30}$ sq. units

VECTOR/CROSS PRODUCT

For two vectors **a** and **b**, the cross product of **a** and **b** is such that

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta.\hat{\mathbf{n}}, \text{ where } 0 \leq \theta \leq 360^\circ$$

Consider the figure below



$\hat{\mathbf{n}}$ = Normal unit vector $\perp$ **a**, **b** and is resultant of $\mathbf{a} \times \mathbf{b}$

Properties of cross product

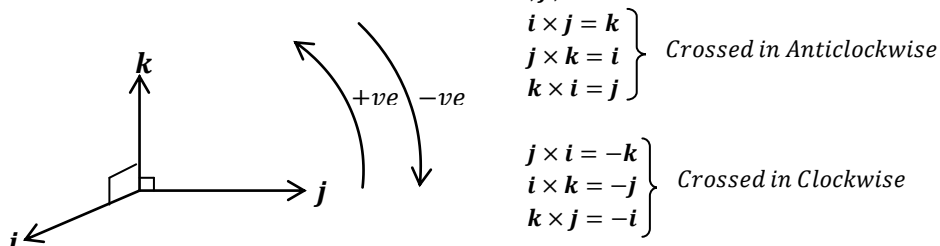
- $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$ [$\because \mathbf{a} \times \mathbf{b}$ is in anticlockwise while $\mathbf{a} \times \mathbf{b}$ is in clockwise direction]
- $m(\mathbf{a} \times \mathbf{b}) = m\mathbf{a} \times \mathbf{b} = \mathbf{a} \times m\mathbf{b}$
- From the definition $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta.\hat{\mathbf{n}}$
 - For $\theta = 0^\circ$, $\mathbf{a} \times \mathbf{b} = 0$. i. e. **a** and **b** are parallel. Thus for **a** and **b** to be parallel or collinear, $\mathbf{a} \times \mathbf{b} = 0$

- For $\theta = 90^\circ$, $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\hat{\mathbf{n}}$, $\mathbf{a} \perp \mathbf{b}$. Thus for perpendicular vectors

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\hat{\mathbf{n}}$$

- $\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = \mathbf{c} \times \mathbf{c} = 0$, since $\theta = 0$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

MULTIPLICATION OF UNIT VECTORS $\mathbf{i}, \mathbf{j}, \mathbf{k}$



Examples

Given that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$ and $\mathbf{c} = 7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$, find;

- $\mathbf{a} \times \mathbf{b}$
- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

Solution

- $\mathbf{a} \times \mathbf{b} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k})$

$$\begin{aligned}
 &= 4\mathbf{i} \times \mathbf{i} + 5\mathbf{i} \times \mathbf{j} - 6\mathbf{i} \times \mathbf{k} + \\
 &\quad 8\mathbf{j} \times \mathbf{i} + 10\mathbf{j} \times \mathbf{j} - 12\mathbf{j} \times \mathbf{k} + \\
 &\quad 12\mathbf{k} \times \mathbf{i} + 15\mathbf{k} \times \mathbf{j} - 18\mathbf{k} \times \mathbf{k} \\
 &= 0 + 5\mathbf{k} + 6\mathbf{j} + \\
 &\quad -8\mathbf{k} + 0 - 12\mathbf{i} + \\
 &\quad 12\mathbf{j} - 15\mathbf{i} - 0 \\
 &= -27\mathbf{i} + 18\mathbf{j} - 3\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= (-27\mathbf{i} + 18\mathbf{j} - 3\mathbf{k}) \times (7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}) \\
 &= 186\mathbf{i} + 222\mathbf{j} - 342\mathbf{k} \quad \text{Left as an exercise}
 \end{aligned}$$

NOTE:

The same results can still be obtained using multiplication of vectors using *Rectangular Components*.

MULTIPLICATION OF VECTORS BY RECTANGULAR COMPONENTS

Consider $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\Rightarrow \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \text{ where the}$$

elements in $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$ give a matrix whose determinant is of a 2×2 square matrix

Example

Referring to the above example

$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$ and $\mathbf{c} = 7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$

$$\begin{aligned}
 \text{a) } \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & -6 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 3 \\ 5 & -6 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 4 & -6 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\
 &= (-12 - 15)\mathbf{i} + (-6 - 12)\mathbf{j} + (5 - 8)\mathbf{k} \\
 &= -27\mathbf{i} + 18\mathbf{j} - 3\mathbf{k} \text{ As before}
 \end{aligned}$$

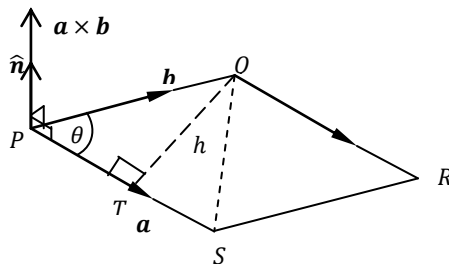
$$\begin{aligned}
 \text{b) } (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -27 & 18 & -3 \\ 7 & 8 & 9 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 18 & -3 \\ 8 & 9 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -27 & -3 \\ 7 & 9 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -27 & 18 \\ 7 & 8 \end{vmatrix} \\
 &= 186\mathbf{i} + 222\mathbf{j} - 342\mathbf{k} \text{ As before}
 \end{aligned}$$

NOTE:

Using Rectangular Components in multiplication of two vectors is easier to handle than the first method.

GEOMETRIC INTERPRETATION OF VECTOR PRODUCT

The magnitude of a vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the *area* of a parallelogram whose adjacent sides are represented by $\mathbf{a}$ and $\mathbf{b}$.



From the definition;
 $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta \cdot \hat{\mathbf{n}}$,
 where $\hat{\mathbf{n}}$ is a unit vector $\perp$
 to $\mathbf{a}$ and $\mathbf{b}$

$$\therefore |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta \cdot |\hat{\mathbf{n}}|; \text{ since } |\hat{\mathbf{n}}| = 1$$

$$\Rightarrow |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

From $\triangle PQT$, $h = |\mathbf{b}|\sin\theta = \text{height of parallelogram}$

$$\Rightarrow |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \times h, \text{ where } |\mathbf{a}| = \text{base of parallelogram}$$

$\therefore |\mathbf{a} \times \mathbf{b}| = \text{base} \times \text{height} = \text{Area of parallelogram}$

But $\Delta s PQS$ and QRS are equal

$$\therefore |\mathbf{a} \times \mathbf{b}| = 2 \times \text{Area of } \Delta PQS \text{ or Area of } \Delta = \frac{1}{2} \times |\mathbf{a} \times \mathbf{b}|$$

Examples

1. Find the unit vector perpendicular to each vector $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. find also the angle between $\mathbf{a}$ and $\mathbf{b}$

Solution

Using $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta$. $\hat{\mathbf{n}} \Rightarrow \hat{\mathbf{n}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}||\mathbf{b}|\sin\theta}$

But $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ [$\therefore |\hat{\mathbf{n}}| = 1$]

$$\Rightarrow \hat{\mathbf{n}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \text{unit vector perpendicular to } \mathbf{a} \text{ and } \mathbf{b}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ 3 & 1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} = -8\mathbf{i} + 8\mathbf{j} + 8\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-8)^2 + 8^2 + 8^2} = 8\sqrt{3}$$

$$\therefore \hat{\mathbf{n}} = \frac{-8\mathbf{i} + 8\mathbf{j} + 8\mathbf{k}}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(-\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Now finding θ

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta, |\mathbf{a}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24}, |\mathbf{b}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$$

$$\Rightarrow \sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{8\sqrt{3}}{\sqrt{24} \times \sqrt{14}} = 8\sqrt{\left(\frac{3}{24 \times 14}\right)} = \frac{8}{\sqrt{112}}$$

$$\therefore \theta = \sin^{-1}\left(\frac{8}{\sqrt{112}}\right) = 4.1^\circ$$

2. Find the unit vector perpendicular to vectors $\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} - 14\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Hence also find the angle between them.

Solution

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 6 & -3 & -14 \\ 3 & 2 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -3 & -14 \\ 2 & -3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 6 & -14 \\ 3 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 6 & -3 \\ 3 & 2 \end{vmatrix} = 37\mathbf{i} - 24\mathbf{j} + 21\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{37^2 + 24^2 + 21^2} = \sqrt{2386}$$

$$\therefore \hat{\mathbf{n}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{1}{\sqrt{2386}}(37\mathbf{i} - 24\mathbf{j} + 21\mathbf{k})$$

$$|\mathbf{a}| = \sqrt{6^2 + (-3)^2 + (-14)^2} = \sqrt{241}, |\mathbf{b}| = \sqrt{3^2 + 2^2 + (-3)^2} = \sqrt{22}$$

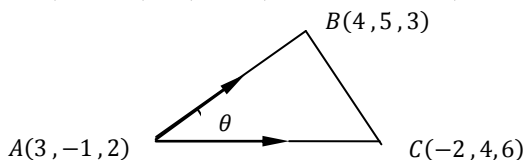
$$\text{From } \sin\theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{\sqrt{2386}}{\sqrt{241} \times \sqrt{22}}$$

$$\therefore \theta = 26.7^\circ$$

3. Find the area of the triangle with vertices
- $(3, -1, 2), (4, 5, 3)$ and $(-2, 4, 6)$
 - $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and origin
 - $(1, -1, 2), (2, 1, -1)$ and $(3, -1, 2)$

Solution

- a) Let $A(3, -1, 2), B(4, 5, 3)$ and $C(-2, 4, 6)$



$$\text{Let } \mathbf{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$$

$$\mathbf{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix}$$

$$\text{Area of } \Delta = \frac{1}{2} |\mathbf{a} \times \mathbf{b}|$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 1 \\ -6 & -1 & 3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 6 & 1 \\ -1 & 3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ -6 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 6 \\ -6 & -1 \end{vmatrix} = 19\mathbf{i} - 9\mathbf{j} + 35\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{19^2 + (-9)^2 + 35^2} = \sqrt{1667}$$

$$\therefore \text{Area} = \frac{1}{2} \times \sqrt{1667} = 20.41 \text{ sq. units}$$

b) $3\sqrt{5}$ sq. units Let

c) Left as an exercise **Answer:** $\sqrt{13}$ sq. units

4. Calculate the area of a parallelogram having diagonals;

a) $\overrightarrow{AC} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{BD} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

b) $\overrightarrow{AB} = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$ and $\overrightarrow{DC} = 4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$

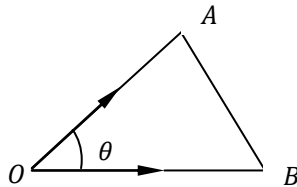
Solution

a) Let O = midpoint of $\overline{AC}$ and $\overline{BD}$

Now there are 4 - triangles of the same area

$\therefore$ Area of $ABCD = 4 \times$ area of one triangle

Consider ΔOCD



$$\overrightarrow{OB} = -\overrightarrow{BO}$$

$$\text{Let } \overrightarrow{OC} = \mathbf{c} \text{ and } \overrightarrow{OB} = \mathbf{b}$$

$$\text{But } \overrightarrow{BO} = \frac{1}{2} \overrightarrow{BD} = \frac{1}{2} (\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$$

$$\Rightarrow \overrightarrow{OB} = \mathbf{b} = \frac{1}{2} (-\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

$$\text{Also } \overrightarrow{OC} = \mathbf{c} = \frac{1}{2} \overrightarrow{AC} = \frac{1}{2} (3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

$$\mathbf{b} \times \mathbf{c} = \frac{1}{2} (-\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \times \frac{1}{2} (3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

$$= \frac{1}{4} [(-\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \times (3\mathbf{i} + \mathbf{j} - 2\mathbf{k})] [\because (am) \times (bn) = mn(a \times b)]$$

$$\mathbf{b} \times \mathbf{c} = \frac{1}{4} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -4 \\ 3 & 1 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & -4 \\ 3 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= \frac{1}{4} (-2\mathbf{i} + 14\mathbf{j} - 10\mathbf{k}) = \frac{1}{2} (-\mathbf{i} + 7\mathbf{j} - 5\mathbf{k})$$

$$\text{Area of } \Delta = \frac{1}{2} |\mathbf{b} \times \mathbf{c}|$$

$$|\mathbf{b} \times \mathbf{c}| = \frac{1}{2} \sqrt{(-1)^2 + 7^2 + (-5)^2} = \frac{1}{2} \sqrt{75} = \frac{1}{2} \times 5\sqrt{3}$$

$$\therefore \text{Area of } \Delta = \frac{1}{2} |\mathbf{b} \times \mathbf{c}| = \frac{1}{2} \times \frac{1}{2} \times 5\sqrt{3} = \frac{5}{4} \sqrt{3} \text{ sq. units}$$

$$\therefore \text{Area of parallelogram} = 4 \times \frac{5}{4} \sqrt{3} = 5\sqrt{3} \text{ sq. units}$$

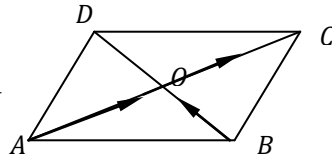
b) Left as an exercise. **Answer:** 30 sq. units

Task

Find the unit vector perpendicular to vectors;

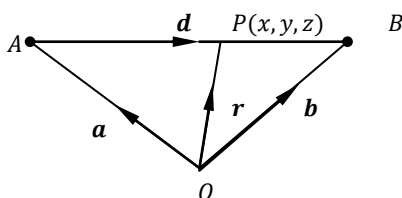
a) $\mathbf{a} = \mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{i}$; **Answer:** $\frac{\sqrt{2}}{2}$

b) $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$; **Answer:** $\frac{\sqrt{2}}{3}$



VECTOR EQUATION OF A STRAIGHT LINE

- a) **To find the vector equation of a line passing through a given point A and parallel to a given vector.**



$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}; \text{ but } \overrightarrow{AP} \text{ is } // \text{ to } \mathbf{d}$$

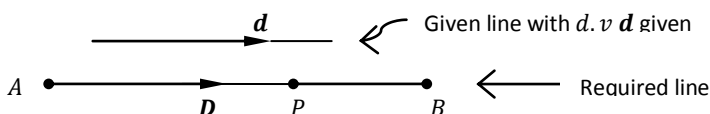
$$\therefore \overrightarrow{AP} = \lambda \mathbf{d}$$

$$\Rightarrow \vec{P} = \mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$$

$$\therefore \mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$$

, which is in *parametric* form of equation of a line

- b) **To find the Cartesian equation of a line through a given point $A(a_1, a_2, a_3)$ and parallel to the given line whose direction cosines are directly proportional to $B(b_1, b_2, b_3)$**



From (a) above, the required equation is

$$\therefore r = a + k d$$

But $\mathbf{d}$ is $//$ $\mathbf{D} \Rightarrow \mathbf{D} = \mu \mathbf{d}$

$$\therefore \mathbf{r} = \mathbf{a} + k \mu \mathbf{d}, \text{ replacing } k\mu \text{ with } \lambda$$

$$\Rightarrow \mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$$

Where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Rightarrow x = a_1 + \lambda b_1 \quad \therefore \lambda = \frac{x-a_1}{b_1}$$

$$\Rightarrow y = a_2 + \lambda b_2 \quad \therefore \lambda = \frac{y - a_2}{b_2}$$

$$\Rightarrow z = a_3 + \lambda b_3 \quad \therefore \lambda = \frac{z - a_3}{b_3}$$

$\therefore$ The Cartesian equation of a line is

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$$

- c) **To find the equation of a line passing through two given points A and B**

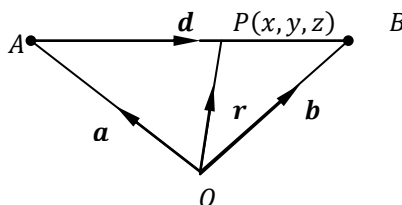
Let $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}; \text{ but } \overrightarrow{AP} \text{ is } // \overrightarrow{AB}$$

$$\therefore \overrightarrow{AP} = \lambda \overrightarrow{AB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

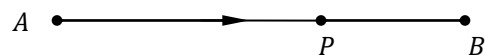
$$\therefore \overrightarrow{OP} = \mathbf{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$



$$\Rightarrow \mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right]$$

VECTOR EQUATION OF A LINE IN NON-PARAMETRIC FORM

Let $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$



$$\overrightarrow{AP} \text{ is } // \overrightarrow{AB} \Rightarrow \overrightarrow{AP} \times \overrightarrow{AB} = 0$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \mathbf{a}, \mathbf{a} = \overrightarrow{OA}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a} = \mathbf{d}$$

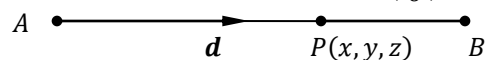
$$(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = 0$$

General examples

1. Find the vector equation of a line through a point $4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and parallel to the vector $-2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$. Express the equation in;
 - a) Cartesian form
 - b) Non-parametric form

Solution

Let the point be $A(4, 2, -3)$, $\mathbf{d} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$



$$\overrightarrow{AP} \text{ is } // \mathbf{d}, \overrightarrow{AP} = \lambda \mathbf{d}$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

$$\Rightarrow \mathbf{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$$

$$\therefore \mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$$

$$\text{Let } \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$$

$$x = 4 - 2\lambda \Rightarrow \lambda = \frac{x-4}{-2}$$

$$y = 2 + \lambda \Rightarrow \lambda = \frac{y-2}{1}$$

$$z = -3 + 6\lambda \Rightarrow \lambda = \frac{z+3}{6}$$

$$\therefore \frac{x-4}{-2} = \frac{y-2}{1} = \frac{z+3}{6}, \text{ cartesian form}$$

Now expressing in Non-parametric form

$$\overrightarrow{AP} \text{ is } // \mathbf{d} \Rightarrow \overrightarrow{AP} \times \mathbf{d} = 0 \text{ [For parallel vectors]}$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$$

$$\therefore \left[\mathbf{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \right] \times \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} = 0 \text{ or } [\mathbf{r} - (4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})] \times (-2\mathbf{i} + \mathbf{j} + 6\mathbf{k}) = 0$$

2. Find the equation of a straight line through points;

a) $A(1, -1, -1)$ and $B(3, 4, 2)$

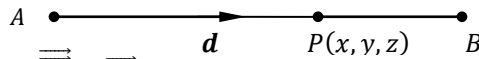
b) $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$

c) $A(1, 2, 5)$ and origin

Express the equation in Cartesian form

Solution

a) $A(1, -1, -1)$ and $B(3, 4, 2)$



$$\overrightarrow{AP} \text{ is } // \overrightarrow{AB} \Rightarrow \overrightarrow{AP} = \lambda \overrightarrow{AB}$$

$$\overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$$

$$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \lambda(\overrightarrow{OB} - \overrightarrow{OA})$$

$$\therefore \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \left[\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right]$$

$$\therefore \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

Now expressing in Cartesian form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

$$\Rightarrow x = 1 + 2\lambda \Rightarrow \lambda = \frac{x-1}{2}$$

$$\Rightarrow y = -1 + 5\lambda \Rightarrow \lambda = \frac{y+1}{5}$$

$$\Rightarrow z = -1 + 3\lambda \Rightarrow \lambda = \frac{z+1}{3}$$

$$\therefore \frac{x-1}{2} = \frac{y+1}{5} = \frac{z+1}{3}$$

b) Left as an exercise.

$$\text{ANSWER: } \frac{x-1}{3} = \frac{y-2}{-5} = \frac{z-3}{-8}$$

c) Left as an exercise.

$$\text{ANSWER: } \frac{x-1}{-1} = \frac{y-2}{-2} = \frac{z-5}{-5}$$

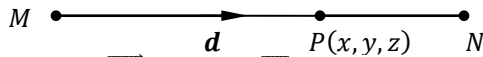
3. Given the points $A(-3, 3, 4)$, $B(5, 7, 2)$ and $C(1, 1, 4)$. Find the vector equation of a line which joins the midpoints of $\overline{AB}$ and $\overline{BC}$.

Solution

Let M and N be the points of $\overline{AB}$ and $\overline{BC}$ respectively.

$$M\left(\frac{-3+5}{2}, \frac{3+7}{2}, \frac{4+2}{2}\right) = M(1, 5, 3)$$

$$N\left(\frac{5+1}{2}, \frac{1+7}{2}, \frac{4+2}{2}\right) = N(3, 4, 3)$$



$$\overrightarrow{MP} // \overrightarrow{MN} \Rightarrow \overrightarrow{MP} = \lambda \overrightarrow{MN}$$

$$\overrightarrow{MP} = \overrightarrow{OP} - \overrightarrow{OM} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-5 \\ z-3 \end{pmatrix}$$

$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{ON} - \overrightarrow{OM} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \\ \therefore \begin{pmatrix} x-1 \\ y-5 \\ z-3 \end{pmatrix} &= \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\end{aligned}$$

4. Given that $A(1, 2, 3)$, $B(4, 3, 2)$, $C(-1, -4, -5)$ and $D(-5, -2, -1)$. Find the vector equation of a line joining points M and N such that M is the midpoint of $\overline{AB}$ and N divides $\overline{CD}$ in the ratio $-4:1$.

Solution



$$M\left(\frac{1+4}{2}, \frac{2+3}{2}, \frac{3+2}{2}\right) = M\left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}\right)$$

Finding the coordinates of N

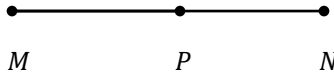
Ratio of division: $CN:ND = -4:1$

$$\frac{\overrightarrow{CN}}{\overrightarrow{ND}} = -\frac{4}{1} \Rightarrow \overrightarrow{CN} = -4\overrightarrow{ND}$$

$$\overrightarrow{ON} - \overrightarrow{OC} = -4(\overrightarrow{OD} - \overrightarrow{ON}) = -4\overrightarrow{OD} + 4\overrightarrow{ON}$$

$$-3\overrightarrow{ON} = \overrightarrow{OC} - 4\overrightarrow{OD} \Rightarrow \overrightarrow{ON} = -\frac{1}{3}\left[\begin{pmatrix} -1 \\ -4 \\ -5 \end{pmatrix} - 4\begin{pmatrix} -5 \\ -2 \\ -1 \end{pmatrix}\right] = -\frac{1}{3}\begin{pmatrix} 19 \\ 4 \\ -1 \end{pmatrix}$$

$$\therefore N\left(-\frac{19}{3}, -\frac{4}{3}, \frac{1}{3}\right)$$



$$\overrightarrow{MP} = \lambda \overrightarrow{MN} \Rightarrow \overrightarrow{OP} - \overrightarrow{OM} = \lambda(\overrightarrow{ON} - \overrightarrow{OM})$$

$$\Rightarrow \mathbf{r} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \end{pmatrix} = \lambda \begin{pmatrix} -\frac{19}{3} \\ -\frac{4}{3} \\ \frac{1}{3} \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} -\frac{53}{6} \\ -\frac{23}{6} \\ -\frac{13}{6} \end{pmatrix} = -\frac{\lambda}{6} \begin{pmatrix} 53 \\ 23 \\ 13 \end{pmatrix}$$

$$\text{Let } -\frac{\lambda}{6} = k, \Rightarrow \mathbf{r} - \begin{pmatrix} 2.5 \\ 2.5 \\ 2.5 \end{pmatrix} = k \begin{pmatrix} 53 \\ 23 \\ 13 \end{pmatrix} \text{ or } \mathbf{r} = \begin{pmatrix} 2.5 \\ 2.5 \\ 2.5 \end{pmatrix} + k \begin{pmatrix} 53 \\ 23 \\ 13 \end{pmatrix}$$

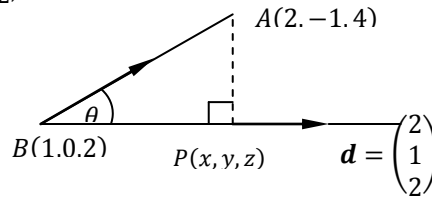
EQUATION OF A PERPENDICULAR AND PERPENDICULAR DISTANCE OF A POINT FROM A GIVEN LINE:

Dot or vector product may be used.

1. Find the equation of perpendicular line from point $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ on to the line $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. Find also perpendicular distance of A from r

Solution

The line $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ passes through the point $B(1,0,2)$ with direction vector $d = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$



Finding the equation of perpendicular AP:

Vectors $\overrightarrow{AP} \perp d$

$$\Rightarrow \overrightarrow{AP} \cdot d = 0$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} x-2 \\ y+1 \\ z-4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x-2 \\ y+1 \\ z-4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0 \text{ or } 2x - 4 + y + 1 + 2z - 8 = 0$$

$$2x + y + 2z = 11 \dots\dots\dots (1)$$

But $P(x, y, z)$ lies on the line

$$\Rightarrow r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow x = 1 + 2\lambda, y = \lambda, z = 2 + 2\lambda \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow 2(1 + 2\lambda) + \lambda + 2(2 + 2\lambda) = 11$$

$$\Rightarrow 2 + 4\lambda + \lambda + 4 + 4\lambda = 11 \therefore \lambda = \frac{5}{9}$$

$$P(x, y, z) = P(1 + 2\lambda, \lambda, 2 + 2\lambda) \Rightarrow P\left(1 + \frac{10}{9}, \frac{5}{9}, 2 + \frac{10}{9}\right) = P\left(\frac{19}{9}, \frac{5}{9}, \frac{28}{9}\right)$$

Now finding equation of $\overrightarrow{AP}$

$$\begin{array}{c} \bullet \text{-----} \bullet \text{-----} \bullet \\ A(2, -1, 4) \quad Q \quad P\left(\frac{19}{9}, \frac{5}{9}, \frac{28}{9}\right) \end{array}$$

$$AQ = \lambda AP \Rightarrow \overrightarrow{OQ} - \overrightarrow{OA} = \lambda(\overrightarrow{OP} - \overrightarrow{OA})$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \lambda \left[\begin{pmatrix} \frac{19}{9}i + \frac{5}{9}j + \frac{28}{9}k \end{pmatrix} - (2i - j + 4k) \right]$$

$$= \lambda \left(\frac{1}{9}i + \frac{14}{9}j - \frac{8}{9}k \right) = \frac{\lambda}{9} (i + 14j - 8k); \text{ let } \frac{\lambda}{9} = \mu$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$$

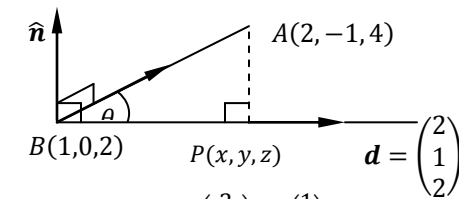
$$\therefore \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$$

Now finding $|\overrightarrow{AP}| = \text{perpendicular distance}$

$$\overrightarrow{AP} = \frac{1}{9}\mathbf{i} + \frac{14}{9}\mathbf{j} - \frac{8}{9}\mathbf{k}$$

$$\therefore |\overrightarrow{AP}| = \sqrt{\left(\frac{1}{9}\right)^2 + \left(\frac{14}{9}\right)^2 + \left(-\frac{8}{9}\right)^2} = \frac{\sqrt{29}}{3} \text{ units}$$

Alternative method of finding perpendicular distance of a point from a line:



$$|\overrightarrow{AP}| = |\overrightarrow{BA}| \sin \theta$$

By cross product,

$$\mathbf{d} \times \overrightarrow{BA} = |\mathbf{d}| |\overrightarrow{BA}| \sin \theta \cdot \hat{\mathbf{n}}$$

$$|\mathbf{d} \times \overrightarrow{BA}| = |\mathbf{d}| |\overrightarrow{BA}| \sin \theta \quad [\because |\hat{\mathbf{n}}| = 1]$$

$$\therefore |\overrightarrow{AP}| = |\overrightarrow{BA}| \sin \theta = \frac{|\mathbf{d} \times \overrightarrow{BA}|}{|\mathbf{d}|}$$

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\mathbf{d} \times \overrightarrow{BA} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \quad [\text{See cross product above how to get this answer}]$$

$$|\mathbf{d} \times \overrightarrow{BA}| = \sqrt{4^2 + (-2)^2 + (-3)^2} = \sqrt{29}; |\mathbf{d}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$\therefore |\overrightarrow{AP}| = \frac{\sqrt{29}}{3} \text{ units}$$

2. Find the equation and length of perpendicular drawn from a point

$$R(-2, 2, -3) \text{ to the line } x - 3 = \frac{y+1}{2} = \frac{z-2}{-4}$$

Solution

Expressing the equation in parametric form

$$\text{Let } x - 3 = \frac{y+1}{2} = \frac{z-2}{-4} = \lambda$$

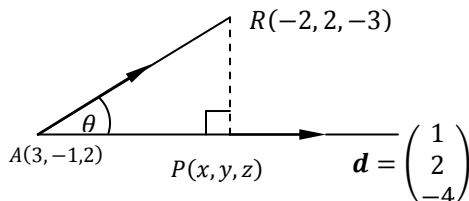
$$\Rightarrow x - 3 = \lambda, x = 3 + \lambda$$

$$\Rightarrow \frac{y+1}{2} = \lambda, y = -1 + 2\lambda$$

$$\Rightarrow \frac{z-2}{-4} = \lambda, z = 2 - 4\lambda$$

$$\therefore \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

This line passes through point A(3, -1, 2) and has a direction vector $\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$



Finding the equation of perpendicular PR:

 Vectors $\vec{PR} \perp \mathbf{d}$

$$\Rightarrow \vec{PR} \cdot \mathbf{d} = 0$$

$$\vec{PR} = \vec{OR} - \vec{OP} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-x \\ 2-y \\ -3-z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2-x \\ 2-y \\ -3-z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 0 \text{ or } -2-x+4-2y+12+4z=0$$

$$-x-2y+4z=-14 \dots\dots\dots (1)$$

 But $P(x, y, z)$ lies on the line

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow x = 3 + \lambda, y = -1 + 2\lambda, z = 2 - 4\lambda \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow -(3 + \lambda) - 2(-1 + 2\lambda) + 4(2 - 4\lambda) = -14$$

$$\Rightarrow -3 - \lambda + 2 - 4\lambda + 8 - 16\lambda = -14 \text{ or } -21\lambda = -21 \therefore \lambda = 1$$

$$P(x, y, z) = P(3 + \lambda, -1 + 2\lambda, 2 - 4\lambda) \Rightarrow P(3 + 1, -1 + 2, 2 - 4) = P(4, 1, -2)$$

 Now finding equation of $\vec{RP}$

$$\begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ R(-2, 2, -3) \quad O \quad P(4, 1, -2) \end{array}$$

$$\vec{RQ} = \lambda \vec{RP} \Rightarrow \vec{OQ} - \vec{OR} = \lambda(\vec{OP} - \vec{OR})$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} = \lambda \left[\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} \right] = \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

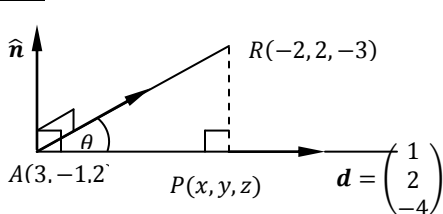
$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore \mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

 Now finding $|\vec{PR}|$ = perpendicular distance

$$\vec{PR} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore |\vec{PR}| = \sqrt{(6)^2 + (-1)^2 + (1)^2} = \sqrt{38} \text{ units}$$

Alternative method of finding perpendicular distance of a point from a line:


$$|\vec{PR}| = |\vec{AR}| \sin \theta$$

By cross product,

$$\mathbf{d} \times \vec{AR} = |\mathbf{d}| |\vec{AR}| \sin \theta \cdot \hat{\mathbf{n}}$$

$$|\mathbf{d} \times \vec{AR}| = |\mathbf{d}| |\vec{AR}| \sin \theta [\because |\hat{\mathbf{n}}| = 1]$$

$$\therefore |\vec{PR}| = |\vec{AR}| \sin \theta = \frac{|\mathbf{d} \times \vec{AR}|}{|\mathbf{d}|}$$

$$\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -5 \end{pmatrix}; \mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

$$\mathbf{d} \times \overrightarrow{AR} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 1 & 2 & -4 \\ -5 & 3 & -5 \end{vmatrix} = 2\mathbf{i} + 25\mathbf{j} + 13\mathbf{k}$$

$$|\mathbf{d} \times \overrightarrow{AR}| = \sqrt{2^2 + (25)^2 + (13)^2} = \sqrt{798}; |\mathbf{d}| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}$$

$$\therefore |\overrightarrow{AP}| = \frac{\sqrt{798}}{\sqrt{21}} = \sqrt{38} \text{ units}$$

NOTE:

A student must know how to identify the point through which the line is passing. If the equation is in Cartesian form. This can easily be deduced from the parametric form.

Examples

Express the following equations in parametric form.

a) $x - 1 = \frac{y-2}{0} = \frac{z-1}{3}$

b) $\frac{x-1}{2} = \frac{4-y}{3} = \frac{3-z}{0}$

c) $2x + 1 = \frac{y-2}{4} = \frac{3-z}{2}$

Solution

a) $x - 1 = \frac{y-2}{0} = \frac{z-1}{3}$

Let $x - 1 = \lambda \Rightarrow x = 1 + \lambda$

For $\frac{y-2}{0}$, $y - 2 = 0 \Rightarrow y = 2 = 2 + 0\lambda$

$\frac{z-1}{3} = \lambda \Rightarrow z = 1 + 3\lambda$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \Rightarrow \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

b) $\frac{x-1}{2} = \frac{4-y}{3} = \frac{3-z}{0}$

Let $\frac{x-1}{2} = \lambda \Rightarrow x = 1 + 2\lambda$

$\frac{4-y}{3} = \lambda \Rightarrow y = 4 - 3\lambda$

For $\frac{3-z}{0}$, $3 - z = 0 \Rightarrow z = 3 = 3 + 0\lambda$

$\therefore \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

c) $2x + 1 = \frac{y-2}{4} = \frac{3-z}{2}$

Let $2x + 1 = \lambda \Rightarrow x = -\frac{1}{2} + \frac{1}{2}\lambda$

$\frac{y-2}{4} = \lambda \Rightarrow y = 2 + 4\lambda$

$\frac{3-z}{2} = \lambda \Rightarrow z = 3 - 2\lambda$

$\therefore \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 4 \\ -2 \end{pmatrix}$

or $\mathbf{r} = \begin{pmatrix} -0.5 \\ 2 \\ 3 \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix} -0.5 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix}$

INTERSECTION OF TWO LINES

Let the lines be;

$$\mathbf{r}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

For two lines to intersect,

$$\boxed{\mathbf{r}_1 = \mathbf{r}_2}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\Rightarrow a_1 + \lambda b_1 = c_1 + \mu d_1 \dots\dots\dots (1)$$

$$\Rightarrow a_2 + \lambda b_2 = c_2 + \mu d_2 \dots\dots\dots (2)$$

$$\Rightarrow a_3 + \lambda b_3 = c_3 + \mu d_3 \dots\dots\dots (3)$$

Solving for the constants λ and μ in any two equations, the values should satisfy the third equation .i.e. the three equations must be consistent for λ and μ .

Examples

Show that the pair of lines intersect, and find the coordinates of point of intersection.

a) $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ and $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

c) $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ and the line passing through $A(2, 2, 5)$ and $B(1, 2, 3)$.

Solution

a) $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$; $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

$$\mathbf{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} ; \mathbf{r}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

For intersection $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

$$\Rightarrow 4 + \lambda = 1 + 2\mu \text{ or } \lambda - 2\mu = -3 \dots\dots\dots (1)$$

$$\Rightarrow -3 - 4\lambda = -1 - 3\mu \text{ or } -4\lambda + 3\mu = 2 \dots\dots\dots (2)$$

$$\Rightarrow -1 + 7\lambda = -10 + 8\mu \text{ or } 7\lambda - 8\mu = -9 \dots\dots\dots (3)$$

Solving eqn. (1) and eqn. (2) simultaneously

$$4 \times \text{eqn. (1)} + \text{eqn. (2)}$$

$$\Rightarrow 4\lambda - 8\mu = -12$$

$$\Rightarrow -4\lambda + 3\mu = 2$$

$$\text{From eq } -5\mu = -10, \mu = 2 \quad 4 = 1$$

$$\therefore \lambda = 1, \mu = 2 \dots\dots\dots (4)$$

subst. eqn. (4) in to eqn. (3)

$$7 - 16 = -9 = \text{RHS. Hence lines intersect.}$$

$$\text{From } \mathbf{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}, \text{ for } \lambda = 1 \text{ then } x = 5, y = -7, z = 6$$

$\therefore (5, -7, 6)$ is the point of intersection.

b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow 1 + 2\lambda = 3 + 2\mu \text{ or } 2\lambda - 2\mu = 2 \dots\dots\dots (1)$$

$$\Rightarrow 2 + 5\lambda = 7 + \mu \text{ or } 5\lambda - \mu = 5 \dots\dots\dots (2)$$

$$\Rightarrow 3 + 2\lambda = 5 + 2\mu \text{ or } 2\lambda - 2\mu = 2 \dots\dots\dots (3)$$

Solving eqn. (1) and eqn. (2) simultaneously

eqn. (1) - eqn. (2)

$$\Rightarrow \lambda - \mu = 1$$

$$\Rightarrow 5\lambda - \mu = 5$$

$$\hline -4\lambda = -4, \lambda = 1$$

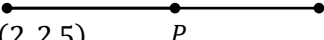
$$\text{From eqn. (1) } \lambda - \mu = 1, \lambda = 1 \Rightarrow \mu = 0 \dots\dots\dots (4)$$

Eqn. (4) in to eqn. (3) $\Rightarrow 2 - 0 = 2$. Hence lines intersect.

$$\text{From } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}, \lambda = 1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$$

$\therefore (3, 7, 5)$ is the point of intersection of the lines.

c) $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3} \Rightarrow \mathbf{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

 $A(2, 2, 5)$ P $B(1, 2, 3)$
 $\vec{Ar} = \lambda \vec{AB}$

$$\Rightarrow \vec{OP} - \vec{OA} = \lambda(\vec{OB} - \vec{OA})$$

$$\Rightarrow \mathbf{r} - \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} = \lambda \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} \right] = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

$$\therefore \mathbf{r}_2 = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \text{ Is the equation through the two given points.}$$

For intersection of lines, $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

$$\Rightarrow 1 + \mu = 2 - \lambda \text{ or } \mu + \lambda = 1 \dots\dots\dots (1)$$

$$\Rightarrow 1 + 3\mu = 5 - 2\lambda \text{ or } 3\mu + 2\lambda = 4 \dots\dots\dots (2)$$

Solving equations simultaneously

$$2 \times \text{eqn. (1)} - \text{eqn. (2)}$$

$$2\lambda + 2\mu = 2$$

$$\hline 2\lambda + 3\mu = 4$$

$$-\mu = -2 \Rightarrow \mu = 2$$

$$\text{From eqn. (1), } \lambda = 1 - 2 = -1$$

$$\text{From } \mathbf{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \mu = 2 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$$

$\therefore (3, 2, 7)$ is the point of intersection.

Alternative approach to these questions:

Expressing the equations in Cartesian form;

a) $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ (1)

$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ (2)

Splitting the equations in to two other equations;

For eqn. (1) $\frac{x-4}{1} = \frac{y+3}{-4}, \frac{y+3}{-4} = \frac{z+1}{7}$

For eqn. (2) $\frac{x-1}{2} = \frac{y+1}{-3}, \frac{y+1}{-3} = \frac{z+10}{8}$

$\Rightarrow \frac{x-4}{1} = \frac{y+3}{-4}$ or $-4x + 16 = y + 3$

$\therefore 4x + y = 13$ (3)

$\Rightarrow \frac{y+3}{-4} = \frac{z+1}{7}$ or $7y + 21 = -4z - 4$

$\therefore 7y + 4z = -25$ (4)

$\Rightarrow \frac{x-1}{2} = \frac{y+1}{-3}$ or $-3x + 3 = 2y + 2$

$\therefore 3x + 2y = 1$ (5)

$\Rightarrow \frac{y+1}{-3} = \frac{z+10}{8}$ or $8y + 8 = -3z - 30$

$\therefore 8y + 3z = -38$ (6)

Solving any *three* equations simultaneously for point of intersection;

Solving eqn. (3) and eqn. (5)

$$2 \times \begin{cases} 4x + y = 13 \\ 3x + 2y = 1 \end{cases} \Rightarrow \begin{cases} 8x + 2y = 26 \\ 3x + 2y = 1 \end{cases}$$

Subtracting the equations; $5x = 25 \Rightarrow x = 5$

From $4x + y = 13$; $x = 5 \Rightarrow 20 + y = 13 \therefore y = -7$

From eqn. (4); $7y + 4z = -25, y = -7 \Rightarrow -49 + 4z = -25 \therefore z = 6$

$\therefore (5, -7, 6)$ is the point of intersection.

This must satisfy the fourth equation if the lines intersect.

$8y + 3z = -38; y = -7, z = 6$

$\Rightarrow 8(-7) + 3(6) = -38 = \text{RHS}$. Hence lines intersect at $(5, -7, 6)$

b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \Rightarrow \frac{x-1}{2} = \frac{y-2}{5} = \frac{z-3}{2}$ (1)

$\mathbf{r} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \frac{x-3}{2} = \frac{y-7}{1} = \frac{z-5}{2}$ (2)

From eqn. (1): $\frac{x-1}{2} = \frac{y-2}{5} \Rightarrow 5x - 2y = 1$ (3)

$\frac{y-7}{1} = \frac{z-5}{2} \Rightarrow 2y - 5z = -11$ (4)

From eqn. (2): $\frac{x-3}{2} = \frac{y-7}{1} \Rightarrow x - 2y = -11$ (5)

$\frac{y-7}{1} = \frac{z-5}{2} \Rightarrow 2y - z = 9$ (6)

Solving any *three* equations simultaneously for point of intersection;

Solving eqn. (3) and eqn. (5)

$$\begin{array}{r} 5x - 2y = 1 \\ - \quad x - 2y = -11 \\ \hline 4x = 12 \Rightarrow x = 3 \end{array}$$

From $x - 2y = -11, x = 3 \Rightarrow 3 - 2y = -11 \therefore y = 7$

From eqn. (4): $2y - 5z = -11, y = 7 \Rightarrow 14 - 5z = -11 \therefore z = 5$

$\therefore (3, 7, 5)$ is the point of intersection.

This must satisfy eqn. (6)

$2y - z = 9 \Rightarrow 14 - 5 = 9 = \text{RHS}$. Hence lines intersect at $(3, 7, 5)$

2. Prove that the lines $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ and $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ intersect. Find the point of intersection of the lines. **Answer:** $(5, -7, 6)$
3. Find the equation of a straight line perpendicular to both lines $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}$ and $\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$ and passing through their point of intersection.

Solution

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}, \quad \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}, \quad \mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

For intersection to occur, $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow 1 + \lambda = -2 + 2\mu \text{ or } \lambda - 2\mu = -3 \dots\dots\dots (1)$$

$$\Rightarrow 1 + 2\lambda = 5 - \mu \text{ or } 2\lambda + \mu = 4 \dots\dots\dots (2)$$

$$\Rightarrow -2 + 3\lambda = -3 + 2\mu \text{ or } 3\lambda - 2\mu = -1 \dots\dots\dots (3)$$

Solving (1) and (2) simultaneously

$$2 \times \text{eqn. (1)} - \text{eqn. (2)}$$

$$2\lambda - 4\mu = -6$$

$$-2\lambda + \mu = 4$$

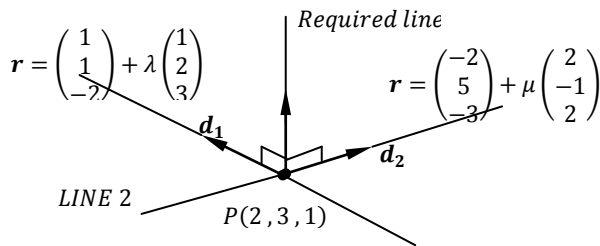
$$-5\mu = -10 \Rightarrow \mu = 2$$

From eqn. (1) $\lambda - 2\mu = -3, \mu = 2$ solving $\lambda = 1$

[Confirming, from (3), $3\lambda - 2\mu = -1 \Rightarrow 3 - 4 = -1$, hence lines intersect]

$$\text{From } \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ for } \lambda = 1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\therefore (2, 3, 1)$ is the point of intersection.



Required line is $\perp$ to $\hat{\mathbf{n}}$ and perpendicular to direction vectors of the given lines.

$$\hat{\mathbf{n}} = \frac{\mathbf{d}_1 \times \mathbf{d}_2}{|\mathbf{d}_1 \times \mathbf{d}_2|}$$

$$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = 7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$$

$$\therefore \hat{\mathbf{n}} = \frac{\mathbf{d}_1 \times \mathbf{d}_2}{|\mathbf{d}_1 \times \mathbf{d}_2|} = \frac{7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}}{\sqrt{90}}$$

$$\therefore \mathbf{d} = d.v \text{ of required line} = \frac{7\mathbf{i}+4\mathbf{j}-5\mathbf{k}}{\sqrt{90}} \cdot \gamma \quad [\because \text{for } \mathbf{d} \text{ and } \hat{\mathbf{n}} \text{ to be } \mathbf{d} = \gamma \hat{\mathbf{n}}] \quad //$$

Required line: $\mathbf{r} = \mathbf{a} + \alpha \mathbf{d}$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \frac{\alpha\gamma}{\sqrt{90}} \cdot \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix} \text{ or } \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}, \text{ where } \frac{\alpha\gamma}{\sqrt{90}} = \beta$$

NOTE:

To avoid over manipulation of constants when finding the direction vector of the required line after crossing $\mathbf{d}_1$ and $\mathbf{d}_2$, then $\mathbf{d}_1 = \hat{\mathbf{n}}$ since for parallel lines the ratio of direction vectors for both lines must be the same.

(see notes on parallel lines)

3. Find the equation of the line through the point $(1, 3, -2)$ and perpendicular to the line passing through points $A(0, 0, -8)$, $B(1, 2, -3)$ and $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

Solution

Equation of line through A and B;

$$\begin{array}{c} \bullet \text{-----} \bullet \text{-----} \bullet \\ A(0, 0, -8) \quad P \quad B(1, 2, -3) \end{array}$$

$$\overrightarrow{AP} = \mu \overrightarrow{AB}, \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$$

$$\therefore \mathbf{r}_1 = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

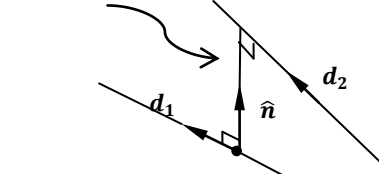
$$\text{For } \mathbf{r}_1, \mathbf{d}_1 = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}, \text{ for } \mathbf{r}_2, \mathbf{d}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

Direction vector or perpendicular is

$$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 1 & 2 & 5 \\ -2 & 3 & -1 \end{vmatrix} = -17\mathbf{i} - 9\mathbf{j} + 4\mathbf{k}$$

$$\text{Required line: } \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -17 \\ -9 \\ 4 \end{pmatrix}$$

Required line



$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

4. Given the two lines are parametrically given by:

$$\mathbf{r}_1 = -\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \text{ and}$$

$$\mathbf{r}_2 = -3\mathbf{i} + p\mathbf{j} + 7\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

If the lines intersect, find the

- the value of t, s and p .
- Coordinates of the points of intersection.

Solution

For lines to intersect, $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow -\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = -3\mathbf{i} + p\mathbf{j} + 7\mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

$$\Rightarrow \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ p \\ 7 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow -1 + t = -3 + s \text{ or } t - s = -2 \dots\dots\dots (1)$$

$$\Rightarrow 2 - 2t = p - s \text{ or } 2t - s = 2 - p \dots\dots\dots (2)$$

$$\Rightarrow 1 + 3t = 7 + 2s \text{ or } 3t - 2s = 6 \dots\dots\dots (3)$$

Solving eqn. (1) and eqn. (3) simultaneously

$$2 \times \begin{array}{l} t - s = -2 \Rightarrow 2t - 2s = -4 \\ 3t - 2s = 6 \Rightarrow 3t - 2s = 6 \end{array}$$

Subtracting; $-t = -10 \therefore t = 10$

From $t - s = -2, t = 10 \Rightarrow s + 10 + 2 = 12$

$\therefore s = 12, t = 10$

From $2t - s = 2 - p \Rightarrow 21 - 12 = 2 - p \therefore p = -6$

$$\text{From } \mathbf{r}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, t = 10 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + 10 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -18 \\ 31 \end{pmatrix}$$

$\therefore (9, -18, 31)$ is the point of intersection.

5. Given that the lines $\mathbf{r}_1 = \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 2 \\ 12 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix}$ intersect, find the value of p . hence find the point of intersection.

Solution

For point of intersection, $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow \begin{pmatrix} p \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix}$$

$$\Rightarrow p + \mu = 2 + \lambda \dots\dots\dots (1)$$

$$\Rightarrow 2 + 5\mu = 12 \Rightarrow \mu = 2 \dots\dots\dots (2)$$

$$\Rightarrow 1 + 3\mu = 5 + \lambda p \dots\dots\dots (3)$$

eqn. (2) in to eqn. (1)

$$\Rightarrow 1 + 6 = 5 + \lambda p, \lambda p = 1 \text{ or } \lambda = \frac{1}{p} \dots\dots\dots (4)$$

eqn. (2), (4) in to eqn. (1)

$$\Rightarrow p + 2 = 2 + \lambda, p = \lambda \dots\dots\dots (5)$$

eqn. (5) in to eqn. (4)

$$\Rightarrow \lambda = \frac{1}{\lambda}, \lambda^2 = 1 \therefore \lambda = \pm 1$$

$$\therefore p = \pm 1$$

6. Show that the lines $\mathbf{r}_1 = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + m \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + n \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ do not intersect and not parallel. [i.e. are skew]

Solution

For lines to intersect, $\mathbf{r}_1 = \mathbf{r}_2$

$$\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + m \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + n \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$\Rightarrow 3 + m = 1 + 2n \text{ or } m - 2n = -2 \dots\dots\dots (1)$$

$$\Rightarrow 5 + 2m = 2 + 3n \text{ or } 2m - 3n = -3 \dots\dots\dots (2)$$

$$\Rightarrow 7 + m = 3 + 5n \text{ or } m - 5n = -4 \dots\dots\dots (3)$$

Solving eqn. (1) and eqn. (2) simultaneously

$$m = 0, n = 1 \dots\dots\dots (4)$$

eqn. (4) in to eqn. (5)

$0 - 5 \neq -4$. Hence lines do not intersect.

Considering a case of parallel lines

For parallel lines, the direction ratio values are similar.

For $\mathbf{r}_1 = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + m \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, direction ratio: 1:2:1

For $\mathbf{r}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + n \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, direction ratio: 2:3:5

Since the direction ratios are not the same, lines are not parallel. Thus are skew.

7. Show that the lines are parallel

a) $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}$

b) $\mathbf{r}_1 = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$

Solution

a) For $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$, direction ratio: -2:1:5

For $\mathbf{r}_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix}$, direction ratio: 2:-1:-5, dividing through by -1

$\Rightarrow$ direction ratio is -2:1:5

Since the direction ratios are the same the lines are parallel.

b) For $\mathbf{r}_1 = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$, direction ratio: 2:-3:-5

For $\mathbf{r}_2 = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$, direction ratio: -2:3:5, dividing through by -1

$\Rightarrow$ direction ratio is 2:-3:-5

The ratios are the same, hence lines are parallel.

NOTE:

Also for parallel lines $\mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{0}$ can be applied.

5. Given the equation of the lines in the form

$\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2}$, show that (8, 14, 11) lie on this line.

Solution

Let $\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2} = \lambda$; $\Rightarrow \frac{x-2}{3} = \lambda, \frac{y-4}{5} = \lambda, \frac{z-7}{2} = \lambda$

For (8, 14, 11), λ must be consistent

$\frac{x-2}{3} = \frac{8-2}{3} = 2 = \lambda$

$\frac{y-4}{5} = \frac{14-4}{5} = 2 = \lambda$

$\frac{z-7}{2} = \frac{11-7}{2} = 2 = \lambda$

Since $\lambda = 2$, consistent for (8, 14, 11), hence point lies on line

6. Given the equation of line: $\mathbf{r} = (2 + 3\lambda)\mathbf{i} + (4 + 5\lambda)\mathbf{j} + (7 + 2\lambda)\mathbf{k}$. Show that (-4, -6, 3) lie on the line.

Solution

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\Rightarrow x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (2 + 3\lambda)\mathbf{i} + (4 + 5\lambda)\mathbf{j} + (7 + 2\lambda)\mathbf{k}$$

Equating unit vectors,

$$x = 2 + 3\lambda, \quad y = 4 + 5\lambda, \quad z = 7 + 2\lambda$$

Point $(-4, -6, 3)$ must give the same value of λ in the equations formulated.

$$\Rightarrow -4 = 2 + 3\lambda \quad \therefore \lambda = -2$$

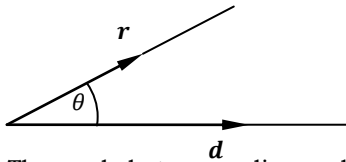
$$\Rightarrow -6 = 4 + 5\lambda \quad \therefore \lambda = -2$$

$$\Rightarrow 3 = 7 + 2\lambda \quad \therefore \lambda = -2$$

Since $\lambda = -2$, consistent for $(-4, -6, 3)$, hence point lies on line

ANGLE BETWEEN A VECTOR AND A LINE

Consider the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ and the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



The angle between a line and a vector is the angle between *vector* and *direction vector* of the line. *i.e.*

By dot product, $\mathbf{d} \cdot \mathbf{r} = |\mathbf{d}||\mathbf{r}|\cos\theta$

$$\text{Let } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\mathbf{d} \cdot \mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = a_1d_1 + a_2d_2 + a_3d_3$$

$$|\mathbf{d}| = \sqrt{d_1^2 + d_2^2 + d_3^2}, \quad |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \cos\theta = \frac{\mathbf{d} \cdot \mathbf{r}}{|\mathbf{d}||\mathbf{r}|}$$

Example

1. Find the angle between

a. Vector $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

b. Vector $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and line $\frac{x-1}{2} = \frac{4-y}{1} = \frac{3-z}{2}$

c. Vector $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ and line $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

Solution

- a. By dot product, $\mathbf{d} \cdot \mathbf{r} = |\mathbf{d}||\mathbf{r}|\cos\theta$

$$\text{Let } \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{r} \cdot \mathbf{d} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2 + 6 + 3 = 11$$

$$|d| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, |r| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

$$\therefore \cos \theta = \frac{dr}{|d||r|} = \frac{11}{\sqrt{14} \times \sqrt{14}} = \frac{11}{14} \Rightarrow \theta = 38.2^\circ$$

b. $r = 3i - 2j + k, d = ?$

$$\text{Let } \frac{x-1}{2} = \frac{4-y}{1} = \frac{3-z}{2} = \lambda$$

$$\frac{x-1}{2} = \lambda \Rightarrow x = 1 + 2\lambda$$

$$\frac{4-y}{1} = \lambda \Rightarrow y = 4 - \lambda$$

$$\frac{3-z}{2} = \lambda \Rightarrow z = 3 - 2\lambda$$

$$\therefore r = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \Rightarrow d = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

$$\Rightarrow r \cdot d = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 6 + 2 - 2 = 6$$

$$\Rightarrow |d| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3, |r| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

$$\therefore \cos \theta = \frac{dr}{|d||r|} = \frac{6}{3 \times \sqrt{14}} = \frac{11}{14} \Rightarrow \theta = 57.7^\circ$$

c. Left as an exercise

2. Find the value of λ if

a. Vector $r = 2i + \lambda j + 3k$ and line $r = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ are perpendicular

b. Vector $r = i + \lambda j + k$ and line $r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ make an angle of 60°

Solution

a. $r = 2i + \lambda j + 3k = \begin{pmatrix} 2 \\ \lambda \\ 3 \end{pmatrix}, r = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

By dot product,

$$d \cdot r = |d||r|\cos \theta, \text{ but } \theta = 90^\circ [\because \text{vector is perpendicular to line}]$$

$$\therefore r \cdot d = 0 \Rightarrow \begin{pmatrix} 2 \\ \lambda \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0 \text{ or } 2 + 2\lambda + 9 = 0 \therefore \lambda = -\frac{11}{2}$$

b. $r = i + \lambda j + k = \begin{pmatrix} 1 \\ \lambda \\ 1 \end{pmatrix}, r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow d = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

By dot product, $d \cdot r = |d||r|\cos \theta$, but $\theta = 60^\circ$

$$\Rightarrow d \cdot r = |d||r|\cos 60^\circ = |d||r| \times \frac{1}{2} \quad [\because \cos 60^\circ = \frac{1}{2}]$$

$$\therefore \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \lambda \\ 1 \end{pmatrix} = \sqrt{1^2 + (-2)^2 + 1^2} \times \sqrt{1^2 + \lambda^2 + 1^2} \times \frac{1}{2}$$

$$\Rightarrow 1 - 2\lambda + 1 = \frac{\sqrt{6}}{2} \times \sqrt{2 + \lambda^2} \text{ or } 4 - 4\lambda = \sqrt{12 + 6\lambda^2}$$

squaring both sides

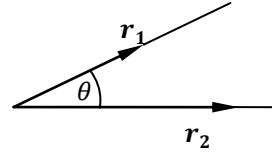
$$16 - 32\lambda + 16\lambda^2 = 12 + 6\lambda^2 \Rightarrow 10\lambda^2 - 32\lambda + 4 = 0$$

Solving, $\lambda = 3.07, 0.13$

ANGLE BETWEEN TWO LINES

Consider the lines;

$$\mathbf{r}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$



The angle between two lines is the angle between the *direction vectors* of the lines.

For $\mathbf{r}_1$, direction vector $\mathbf{d}_1 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and for $\mathbf{r}_2$, direction vector $\mathbf{d}_2 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$$\therefore \mathbf{d}_1 \cdot \mathbf{d}_2 = |\mathbf{d}_1| |\mathbf{d}_2| \cos \theta$$

Examples

1. Find the angle between the pair of lines

a) $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

b) $\mathbf{r} = (1+2\lambda)\mathbf{i} + (3+\lambda)\mathbf{j} + (5+2\lambda)\mathbf{k}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 16 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$

c) $\frac{10-x}{2} = \frac{4-y}{3} = \frac{8-z}{4}$ and $\mathbf{r} = \begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$

Solution

a) $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+3}{3} \Rightarrow \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \therefore \mathbf{d}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\mathbf{r} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{d}_2 = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

From dot product $\mathbf{d}_1 \cdot \mathbf{d}_2 = |\mathbf{d}_1| |\mathbf{d}_2| \cos \theta$

$\mathbf{d}_1 \cdot \mathbf{d}_2 = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (-3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3 - 4 + 3 = -4$

$|\mathbf{d}_1| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, |\mathbf{d}_2| = \sqrt{(-3)^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\therefore \cos \theta = \frac{-4}{\sqrt{14} \times \sqrt{14}} \Rightarrow \theta = 106.6^\circ$

b) $\mathbf{r} = (1+2\lambda)\mathbf{i} + (3+\lambda)\mathbf{j} + (5+2\lambda)\mathbf{k}$

$\Rightarrow \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \therefore \mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 16 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \Rightarrow \mathbf{d}_2 = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = 2\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}$

$\mathbf{d}_1 \cdot \mathbf{d}_2 = (2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}) = 4 + 0 + 10 = 14$

$|\mathbf{d}_1| = \sqrt{2^2 + 1^2 + 2^2} = 3, |\mathbf{d}_2| = \sqrt{2^2 + 0^2 + 5^2} = \sqrt{29}$

$\therefore \cos \theta = \frac{14}{3 \times \sqrt{29}} \Rightarrow \theta = 29.9^\circ$

- c) Left as an exercise

Task

1. Find the angle between the lines:

a) $x-1 = y-3 = z-7$ and $\frac{x-5}{2} = \frac{y+1}{3} = \frac{z}{5}$

ANSWER: $\theta = 120^\circ$

b) $x = 2 - 4t, x = 1 + 5t$

$$y = 1 - 2t, y = -2 - t$$

$$z = 1 + 2t, z = 1 - 2t$$

$$c) \frac{x-3}{2} = \frac{y-3}{5}, z = 0; \frac{x+1}{3} = \frac{y+2}{1}, z = 0 \quad \text{ANSWER: } \theta = 35^\circ$$

$$[\text{HINT: } \mathbf{d}_1 = 2\mathbf{i} + 5\mathbf{j} + 0\mathbf{k}, \mathbf{d}_2 = 3\mathbf{i} + \mathbf{j} + 0\mathbf{k}] \quad \text{ANSWER: } \theta = 50^\circ$$

2. Prove that the vectors $\mathbf{a} = 7\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = 6\mathbf{i} - 16\mathbf{j} - 15\mathbf{k}$ are mutually perpendicular.

3. a) Find the equation of the line through $A(2, 2, 5)$ and $B(1, 2, 3)$

- b) If the line in (a) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ at P , find

i. coordinates of P

ii. angle between the lines (**Uneb 2004**)

$$\text{ANSWER: } \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}, P(3, 2, 7), \theta = 171.9^\circ$$

4. Show that vector $2\mathbf{i} - 5\mathbf{j} + 3.5\mathbf{k}$ is perpendicular to the line $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \lambda(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

(**Uneb 2000**)

5. Find the equation of a line through points $A(2, 2, 5)$, $B(1, 2, 3)$ and $C(0, 2, 1)$

[**HINT:** 1st check whether A, B and C are collinear, if not different equations must be formed]

6. Show that the lines $\mathbf{r}_1 = (1 + 4\lambda)\mathbf{i} + (1 - \lambda)\mathbf{j} + 2\lambda\mathbf{k}$ and

$\mathbf{r}_1 = (5 + 3\mu)\mathbf{i} + 2\mu\mathbf{j} + (2 - 5\mu)\mathbf{k}$ cut at right angles. Find the position vector of point of intersection. **ANSWER: $5\mathbf{i} + 2\mathbf{k}$**

7. The points A, B, C and D have coordinates

$(3, -2, 0), (-1, 2, 4), (0, 1, -1)$ and $(5, -4, 4)$ respectively. Find

a) The point of intersection of the lines AB and CD

b) The angle between the lines in (a) above

$$\text{ANSWER: } (2, -1, 1); 70.5^\circ$$

8. The points A, B and C have p. vs $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} = 6\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ respectively. Find the p. v of a point R on $\overline{BC}$ such that $\overline{AR}$ is perpendicular to $\overline{BC}$. Hence find the perpendicular distance of A from BC . **Answer: $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; $\overline{AR} = \sqrt{21}$**

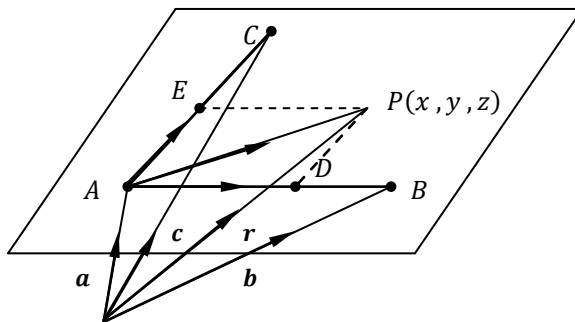
9. The points A, B and C have p. vs $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{i} + 2\mathbf{k}$ and $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ respectively. Find the vector perpendicular to both vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\text{ANSWER: } \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

10. Show that the lines $\mathbf{r}_1 = 2\lambda\mathbf{i} - 3\mathbf{j} + (\lambda - 2)\mathbf{k}$ and $\mathbf{r}_1 = (1 + \mu)\mathbf{i} + (2 - \mu)\mathbf{j} + (2\mu - 5)\mathbf{k}$ do not intersect.

VECTOR EQUATION OF A PLANE

- a) **To find the vector equation of a plane through three non-collinear points A, B and C**



Let the *p.v.s* of A, B, C and P be $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{r}$ respectively.
 $\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP}$. But
 $\overrightarrow{AD} \parallel \overrightarrow{AB}, \overrightarrow{DP} \parallel \overrightarrow{AC}$
 $\therefore \overrightarrow{AP} = \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$
 But $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$
 $\overrightarrow{OA} = \mathbf{r} - \mathbf{a}$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a} \\ \overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} = \mathbf{c} - \mathbf{a} \\ \therefore \mathbf{r} - \mathbf{a} &= \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})\end{aligned}$$

$$\therefore \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

NOTE:

If A is the origin, then the equation of a plane will be given by

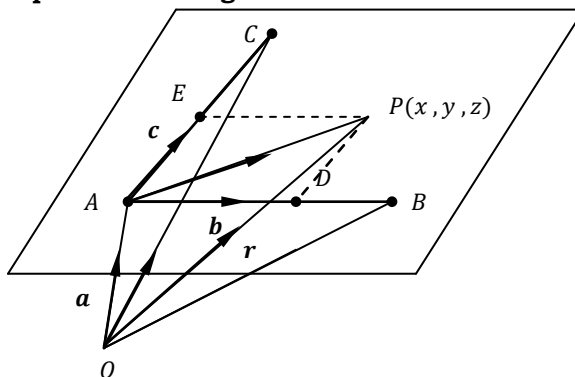
$$\begin{aligned}\overrightarrow{AP} &= \lambda \overrightarrow{AB} + \mu \overrightarrow{AC} \Rightarrow \overrightarrow{OP} = \lambda \overrightarrow{OB} + \mu \overrightarrow{OC} \\ \therefore \mathbf{r} &= \lambda \mathbf{b} + \mu \mathbf{c}\end{aligned}$$

$$\therefore \mathbf{r} = \lambda \mathbf{b} + \mu \mathbf{c}$$

This equation represents an equation of a plane passing through the origin and parallel to *two* given vectors $\mathbf{b}$ and $\mathbf{c}$.

If given two vectors, then the equation of plane containing the two vectors is of the form above.

- b) **To find vector equation of a plane passing through a given point A and parallel to two given vectors $\vec{b}$ and $\vec{c}$.**



Let *p.v* of P be $\mathbf{r}$ and A be $\mathbf{a}$
 From $\overrightarrow{AP} = \overrightarrow{AE} + \overrightarrow{EP}$; But $\overrightarrow{AE} = \lambda \mathbf{c}$,
 $\overrightarrow{EP} = \mu \mathbf{b}$
 $\Rightarrow \overrightarrow{AP} = \lambda \mathbf{c} + \mu \mathbf{b}$

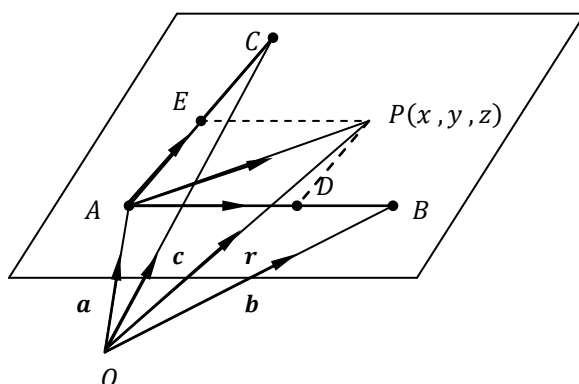
$$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda \mathbf{c} + \mu \mathbf{b}$$

$$\therefore \mathbf{r} = \mathbf{a} + \mu \mathbf{b} + \lambda \mathbf{c}$$

If A is the Origin. i.e. A(0, 0, 0), then the equation of the plane containing *two* vectors $\mathbf{b}$ and $\mathbf{c}$ is

$$\therefore \mathbf{r} = \mu \mathbf{b} + \lambda \mathbf{c}$$

- c) **To find the vector equation of a plane through two given points A and B and parallel to a given vector $\vec{c}$.**



Let the p.v of P, A and B be

$\mathbf{r}, \mathbf{a}$ and $\mathbf{b}$

respectively.

As above, the equation of the plane is

$$\vec{AP} = \lambda \vec{AB} + \mu \vec{c}$$

$$\therefore \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu \mathbf{c}$$

or

$$\boxed{\therefore \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu \mathbf{c}}$$

NOTE:

The above rectangular enclosed equations are called *parametric equations* of a plane.

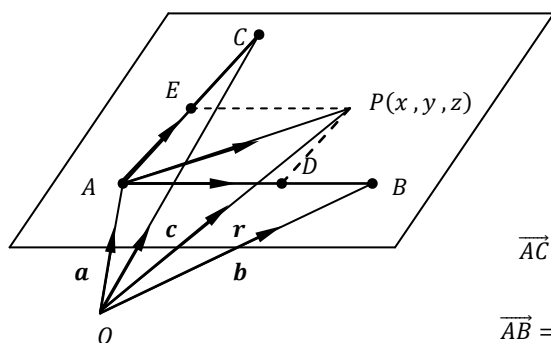
When parameters λ and μ are eliminated from the equation of a plane, the resulting equation of a plane is called *Cartesian equation of a plane* of the form $Ax + By + Cz + D = 0$.

Examples

1. Find the Cartesian equation of a plane
 - a) Passing through the points $A(-1, 1, 2)$, $B(1, -2, 1)$ and $C(2, 2, 4)$
 - b) Containing vectors $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$
 - c) Passing through point $(1, 2, 3)$ and parallel to vectors $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Solution

a)



Let $\vec{OP} = \mathbf{r}$

$$\vec{AP} = \vec{AE} + \vec{EP}$$

But $\vec{AE} \parallel \vec{AC}$ and $\vec{EP} \parallel \vec{AB}$

$$\therefore \vec{AP} = \lambda \vec{AB} + \mu \vec{AC}$$

$$\therefore \vec{AP} = \vec{OP} - \vec{OA} = \mathbf{r} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$$\therefore \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

Expressing the equation in Cartesian form

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$$\Rightarrow x = -1 + 3\lambda + 2\mu \dots\dots\dots (1)$$

$$y = 1 + \lambda - 3\mu \dots\dots\dots (2)$$

$$z = 2 + 2\lambda - \mu \dots\dots\dots (3)$$

Considering (1) and (2), eliminating λ and μ

$$\text{From (1) } x + 1 = 3\lambda + 2\mu \dots\dots\dots (4)$$

$$\text{From (2) } y - 1 = \lambda - 3\mu \dots\dots\dots (5)$$

$$\text{Eqn. (4)} \times 3 - \text{eqn. (3)}$$

$$3y - 3 = 3\lambda - 9\mu$$

$$x + 1 = 3\lambda + 2\mu$$

$$3y - x - 4 = -11\mu \Rightarrow \mu = \frac{x-3y+4}{11} \dots\dots\dots (6)$$

eq. (6) in to eqn. (4)

$$\Rightarrow \lambda = y - 1 + 3\mu = y - 1 + 3 \left[\frac{x-3y+4}{11} \right] = \frac{11y-11+3x-9y+12}{11}$$

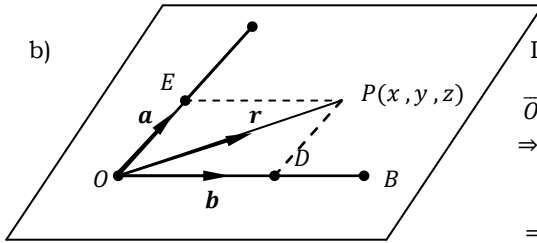
$$\therefore \lambda = \frac{3x+2y+1}{11} \dots\dots\dots (7)$$

eqn. (6), (7) in to (3) to conclude the Cartesian plane equation

$$\Rightarrow z = 2 + 2 \left(\frac{3x+2y+1}{11} \right) - \left(\frac{x-3y+4}{11} \right)$$

$$11z = 22 + 6x + 4y + 2 - x + 3y - 4$$

$$\therefore 5x + 7y - 11z + 20 = 0$$



$$\text{Let } \overrightarrow{OP} = \mathbf{r}, \mathbf{a} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP}, \overrightarrow{OD} \parallel \mathbf{b} \text{ and } \overrightarrow{DP} \parallel \mathbf{a}$$

$$\Rightarrow \mathbf{r} = \lambda \mathbf{b} + \mu \mathbf{a}$$

$$\Rightarrow \mathbf{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

Expressing this equation in Cartesian form

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow x = \lambda \dots\dots\dots (1)$$

$$\Rightarrow y = 3\mu \dots\dots\dots (2)$$

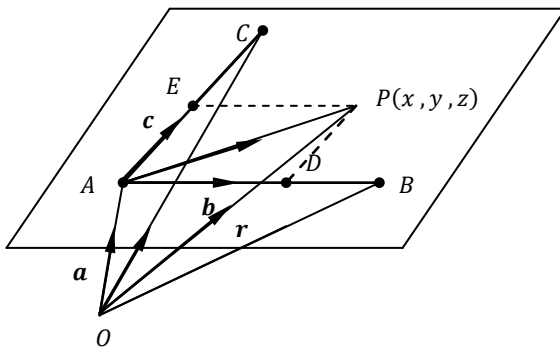
$$\Rightarrow z = 2\lambda \dots\dots\dots (3)$$

eqn (1) in to eqn. (3)

$$\therefore z = 2x \text{ or } 2x - z = 0 \text{ is the required equation of plane.}$$

Note: The eqn. (2) cannot be used in any way to find the required equation of plane.

c)



$$\text{Let } \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

$$\mathbf{r} = \overrightarrow{OP}$$

$$\text{and } \overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} = \lambda \mathbf{b} + \mu \mathbf{c}$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c} \Rightarrow \mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$\Rightarrow x = 1 + \lambda + 2\mu \text{ or } x - 1 = \lambda + 2\mu \dots\dots\dots (1)$$

$$\Rightarrow y = 2 - 2\lambda - 3\mu \text{ or } y - 2 = -2\lambda - 3\mu \dots\dots\dots (2)$$

$$\Rightarrow z = 3 + 3\lambda + \mu \text{ or } z - 3 = 3\lambda + \mu \dots\dots\dots (3)$$

Considering eqn. (1) and eqn. (2)

$$\text{eqn. (1)} \times 2 + \text{eqn. (2)}$$

$$2x - 2 = 2\lambda + 4\mu$$

$$y - 2 = -2\lambda - 3\mu$$

$$2x - 2 + y - 2 = \mu \Rightarrow \mu = 2x + y - 4 \dots\dots\dots (4)$$

eqn. (4) in to eqn. (1)

$$x - 1 = \lambda + 2\mu \Rightarrow \lambda = x - 1 - 2\mu$$

$$\therefore \lambda = x - 1 - 2(2x + y - 4) = -3x - 2y + 7 \dots\dots\dots (5)$$

eqn. (4), (5) in to (3) to conclude equation of plane

$$\Rightarrow z - 3 = 3(-3x - 2y + 7) + 2x + y - 4$$

$$z - 3 = -9x - 6y + 21 + 2x + y - 4 = -7x - 5y + 17$$

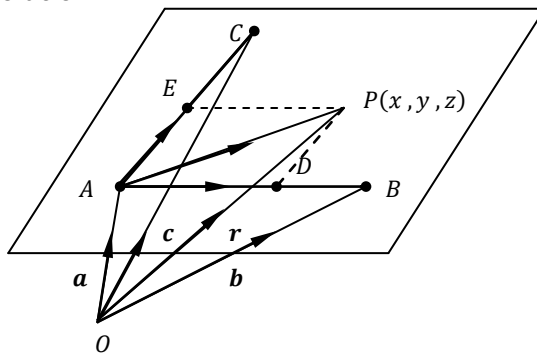
$$\therefore 7x + 5y + z - 20 = 0$$

2. Find the Cartesian equation of a plane passing through

$A(0, 3, -4), B(2, -1, 2)$ and $C(7, 4, -1)$. Show that $Q(10, 13, -10)$ lies on the plane

Uneb (2001)

Solution



Let the p. vs of A,

B, C and P be

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{r}$

respectively.

$$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} \text{ . But}$$

$$\overrightarrow{AD} \parallel \overrightarrow{AB}, \overrightarrow{DP} \parallel \overrightarrow{AC}$$

$$\therefore \overrightarrow{AP} = \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$

$$\text{But } \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$$

$$= \mathbf{r} - \mathbf{a}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a} \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \mathbf{c} - \mathbf{a}$$

$$\therefore \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda \left[\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} \right] + \mu \left[\begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} \right]$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow x = 2\lambda + 7\mu \dots\dots\dots (1)$$

$$\Rightarrow y = 3 - 4\lambda + \mu \text{ or } y - 3 = -4\lambda + \mu \dots\dots\dots (2)$$

$$\Rightarrow z = -4 + 6\lambda + 3\mu \text{ or } z + 4 = 6\lambda + 3\mu \dots\dots\dots (3)$$

Considering eqn. (1) and (2)

eqn. (1) $\times 2$ + eqn. (2)

$$2x = 4\lambda + 14\mu$$

$$y - 3 = -4\lambda + \mu$$

$$2x + y - 3 = 15\mu \Rightarrow \mu = \frac{2x+y-3}{15} \dots\dots\dots (4)$$

eqn. (4) in to eqn. (1)

$$x = 2\lambda + 7\mu \Rightarrow \lambda = \frac{x-7\mu}{2}$$

$$\therefore \lambda = \frac{x-7\left(\frac{2x+y-3}{15}\right)}{2} = \frac{15x-14x-7y+21}{30} = \frac{x-7y+21}{30} \dots\dots\dots (5)$$

eqn. (4), (5) in to (3) to conclude equation of plane

$$\Rightarrow z + 4 = 6\left(\frac{x-7y+21}{30}\right) + 3\left(\frac{2x+y-3}{15}\right)$$

$$\Rightarrow z + 4 = \left(\frac{x-7y+21}{5}\right) + \left(\frac{2x+y-3}{5}\right)$$

$$\Rightarrow 5z + 20 = x - 7y + 21 + 2x + y - 3$$

$$\therefore 3x - 6y - 5z = 2$$

For $Q(10, 13, -10)$ to lie on the plane, it must satisfy the equation

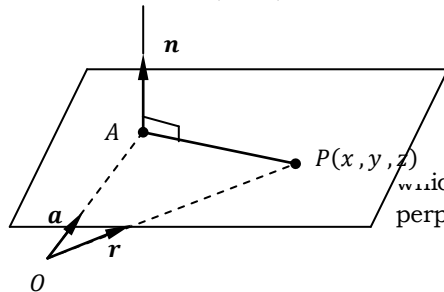
$$\Rightarrow 3(10) - 6(13) - 5(-10) = 2 = \text{RHS}, \text{ hence point lies on the plane.}$$

VECTOR EQUATION OF A PLANE IN NORMAL FORM i.e. Non-parametric/Scalar/Dot-product form

Consider a plane which passes through a point A with p.v $\mathbf{a}$ and which is perpendicular to vector $\mathbf{n}$. Let p.v of point $P(x, y, z)$.

$$\overrightarrow{AP} \perp \mathbf{n}, \therefore \overrightarrow{AP} \cdot \mathbf{n} = 0$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \mathbf{a} \therefore (\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$



$$\Rightarrow \mathbf{r} \cdot \mathbf{n} - \mathbf{a} \cdot \mathbf{n} = 0$$

$$\Rightarrow \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},$$

let $\mathbf{a} \cdot \mathbf{n} = d$ (scalar)

$$\therefore \mathbf{r} \cdot \mathbf{n} = d$$

which is the equation of a plane perpendicular to vector $\mathbf{n}$.

NOTE:

From $\text{vector} = |\text{vector}| \times \text{unit vector} \Rightarrow \mathbf{n} = |\mathbf{n}| \times \hat{\mathbf{n}}, \hat{\mathbf{n}} = \text{unit vector}.$

$\therefore \mathbf{r} \cdot \mathbf{n} = r \cdot \hat{\mathbf{n}} \cdot |\mathbf{n}| = d$ or $\mathbf{r} \cdot \hat{\mathbf{n}} = D$, where $D = \frac{d}{|\mathbf{n}|} = \text{scalar}$.

$\therefore \mathbf{r} \cdot \hat{\mathbf{n}} = D$, also represents equation of a plane perpendicular to $\hat{\mathbf{n}}$.

CARTESIAN EQUATION OF A PLANE

From $\mathbf{r} \cdot \mathbf{n} = d$, $\mathbf{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, let $\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} = d$$

The Cartesian equation of a plane is

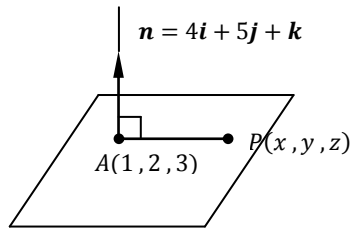
$$\therefore Ax + By + Cz = d$$

Examples:

- Find the equation of a plane through the point $(1, 2, 3)$ and perpendicular to $\mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

(Uneb 2004)

Solution



$$\text{Let } \mathbf{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ let } \mathbf{n} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AP} \perp \mathbf{n}, \therefore \overrightarrow{AP} \cdot \mathbf{n} = 0$$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \mathbf{r} - \mathbf{a} = \mathbf{r} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \left[\mathbf{r} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right] \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} - (4 + 10 + 3) = 0 \text{ or } \mathbf{r} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 17$$

$$\text{or } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 17 \therefore 4x + 5y + z = 17$$

- Show that the equation of the plane through point A with p.v $-2\mathbf{i} + 4\mathbf{k}$ perpendicular to vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ is $x + 3y - 2z + 10 = 0$. (Uneb 2000)

Solution

Using $\mathbf{r} \cdot \mathbf{n} = d$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = d \text{ But } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = d \text{ or } x + 3y - 2z = d$$

But $A(-2, 0, 4)$ lie on plane

$$\Rightarrow (-2) + 3(0) - 2(4) = d \therefore d = -10$$

$$\therefore x + 3y - 2z + 10 = 0$$

- Find the equation of the plane through point A with p.v $5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ perpendicular to vector $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$. (Uneb 1998)

Solution

 Using $\mathbf{r} \cdot \mathbf{n} = d$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = d \text{ But } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

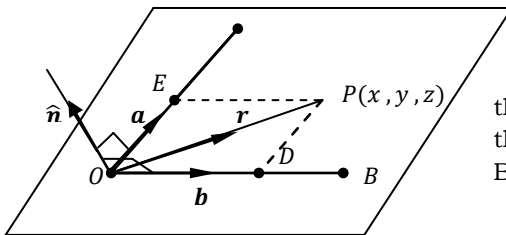
$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = d \text{ or } 3x + 4y - z = d$$

 But $(5, -2, 3)$ lie on plane

$$\Rightarrow 3(5) + 4(-2) - (3) = d \therefore d = 4$$

$$\therefore 3x + 4y - z - 4 = 0$$

4. Given that $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. Determine the equation containing $\mathbf{a}$ and $\mathbf{b}$.

Solution


Since only two vectors are given, and no other point is given, taking the point as the *origin* through which the plane passes, then finding the unit vector normal to the vectors, By cross product,

$$\text{using } \hat{\mathbf{n}} = \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|}$$

$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ -1 & -3 & 2 \\ 1 & -3 & 3 \end{vmatrix} = -3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$$

$$|\mathbf{b} \times \mathbf{a}| = \sqrt{9 + 25 + 36} = \sqrt{70}$$

$$\therefore \hat{\mathbf{n}} = \frac{1}{\sqrt{70}}(3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

$$\text{Using } \mathbf{r} \cdot \hat{\mathbf{n}} = D \Rightarrow \mathbf{r} \cdot \frac{1}{\sqrt{70}}(3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) = D$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \sqrt{70}D = k \text{ (scalar)}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = k, -3x + 5y + 6z = k$$

 But $(0, 0, 0)$ lies on the plane

$$\Rightarrow -3(0) + 5(0) + 6(0) = k$$

$$\therefore -3x + 5y + 6z = 0$$

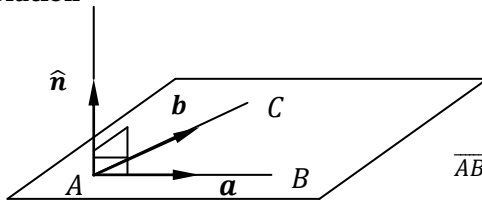
5. Find the Cartesian equation of a plane through;

a) $A(0, 3, -4), B(2, -1, 2)$ and $C(7, 4, -1)$

b) $C(-1, 1, 2), D(1, -2, 1)$ and $E(2, 2, 4)$

Solution

a)



$$\text{Let } \overrightarrow{AB} = \mathbf{a}, \overrightarrow{AC} = \mathbf{b}$$

$$\hat{\mathbf{n}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} \end{aligned}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ -1 & -3 & 2 \\ 1 & -3 & 3 \end{vmatrix} = -18\mathbf{i} + 36\mathbf{j} + 30\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-18)^2 + 36^2 + 30^2} = \sqrt{2520}$$

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{2520}}(-18\mathbf{i} + 36\mathbf{j} + 30\mathbf{k})$$

$$\text{Using } \mathbf{r} \cdot \hat{\mathbf{n}} = D \Rightarrow \mathbf{r} \cdot \frac{1}{\sqrt{2520}}(-18\mathbf{i} + 36\mathbf{j} + 30\mathbf{k}) = D$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} -18 \\ 36 \\ 30 \end{pmatrix} = \sqrt{2520}D = k \text{ (scalar)}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -18 \\ 36 \\ 30 \end{pmatrix} = k, -18x + 36y + 30z = k$$

But A, B and C lie on the plane. Using point $A(0, 3, -4)$

$$\Rightarrow -18(0) + 36(3) + 30(-4) = k \therefore k = -12$$

$$\therefore -18x + 36y + 30z = -12 \dots\dots\dots \div (-6)$$

$$\therefore 3x - 6y - 5z - 2 = 0$$

b) Let $\overrightarrow{CE} = \mathbf{a}, \overrightarrow{CD} = \mathbf{b}$

$$\hat{\mathbf{n}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

$$\mathbf{a} = \overrightarrow{CE} = \overrightarrow{OE} - \overrightarrow{OC}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\mathbf{b} = \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & -1 \end{vmatrix} = 5\mathbf{i} + 7\mathbf{j} - 11\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(5)^2 + 7^2 + (-11)^2} = \sqrt{195}$$

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{195}}(5\mathbf{i} + 7\mathbf{j} - 11\mathbf{k})$$

$$\text{Using } \mathbf{r} \cdot \hat{\mathbf{n}} = D \Rightarrow \mathbf{r} \cdot \frac{1}{\sqrt{195}}(5\mathbf{i} + 7\mathbf{j} - 11\mathbf{k}) = D$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 5 \\ 7 \\ -11 \end{pmatrix} = \sqrt{195}D = k \text{ (scalar)}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \\ -11 \end{pmatrix} = k, 5x + 7y - 11z = k$$

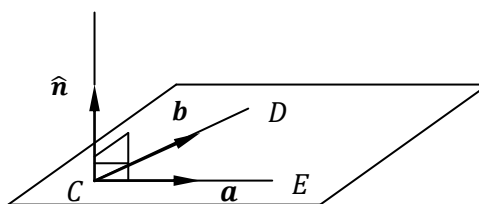
But C, D and E lie on the plane. Using point $C(-1, 1, 2)$

$$\Rightarrow 5(-1) + 7(1) - 11(2) = k \therefore k = -20$$

$$\therefore 5x + 7y - 11z = -20$$

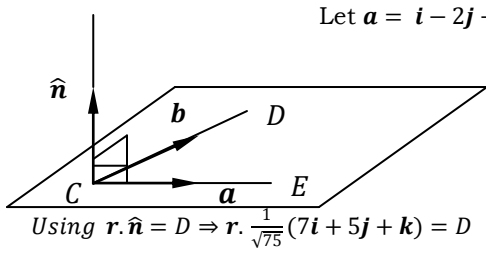
6. Find the equation of a plane passing through point $(1, 2, 3)$ and parallel to vectors

$$\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \text{ and } 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$



Solution

Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $C(1.2, 3)$



The diagram shows a 3D coordinate system with a plane. A normal vector $\hat{n}$ is shown perpendicular to the plane. Two vectors, $\mathbf{a}$ and $\mathbf{b}$, are shown on the plane. A point C is marked on the plane. The plane is labeled with D and E .

$$\hat{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 2 & -3 & 1 \end{vmatrix} = 7\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(7)^2 + (5)^2 + (1)^2} = \sqrt{75}$$

Using $\mathbf{r} \cdot \hat{n} = D \Rightarrow \mathbf{r} \cdot \frac{1}{\sqrt{75}}(7\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = D$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} = \sqrt{75}D = k \text{ (scalar)}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} = k, 7x + 5y + z = k$$

But A, B and C lie on the plane. Using point $C(1.2, 3)$

$$\Rightarrow 7(1) + 5(2) + 3 = k \therefore k = 20$$

$$\therefore 7x + 5y + z = 20$$

Deduction from Cartesian equation of a plane $Ax + By + Cz = d$

From $\mathbf{r} \cdot \mathbf{n} = d$, $\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} = d, Ax + By + Cz = d$$

From the Cartesian equation of a plane, there exists a **NORMAL VECTOR** to plane whose direction ratios are obtained from coefficients of x, y, z in the

equation of plane. i.e. $\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

Example

State the normal vector to the plane;

a) $2x + 3y + 5z = 7$

b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$

c) $\mathbf{r} = (3 + 4\lambda - 3\mu)\mathbf{i} + 2\lambda\mathbf{j} + (3 - 3\mu)\mathbf{k}$

Solution

a) $2x + 3y + 5z = 7$, $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

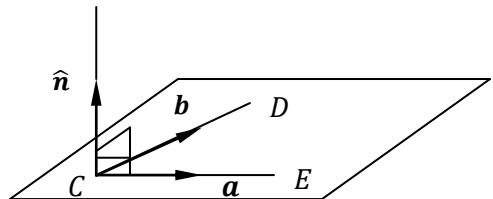
b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$

Let $\mathbf{a} = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$

Using $\hat{n} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 4 & -5 & 6 \\ 7 & 0 & 5 \end{vmatrix} = -25\mathbf{i} + 29\mathbf{j} + 35\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-25)^2 + (29)^2 + (35)^2} = \alpha \text{ (Scalar)}$$



$$\Rightarrow \hat{n} = \frac{1}{\alpha}(-25\mathbf{i} + 29\mathbf{j} + 35\mathbf{k})$$

Direction ratio of normal vector is $-25:29:35$ in simplified form

$$\therefore \mathbf{n} = \begin{pmatrix} -25 \\ 29 \\ 35 \end{pmatrix}$$

$$\text{c) } \mathbf{r} = (3 + 4\lambda - 3\mu)\mathbf{i} + 2\lambda\mathbf{j} + (3 - 3\mu)\mathbf{k}$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$$

$$\text{Let } \mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ -3 & 0 & -3 \end{vmatrix} = -6\mathbf{i} - 12\mathbf{j} + 6\mathbf{k}$$

Normal vector ratio is $-6:-12:6 = 1:2:-1$

$$\therefore \text{Normal vector is } \mathbf{n} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Alternative approach of obtaining normal vector to plane expressed in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

Find in scalar product form the equation of the plane

$$\mathbf{r} = (1 + 3\lambda + 2\mu)\mathbf{i} + (1 + \lambda + 4\mu)\mathbf{j} + (\mu - \lambda)\mathbf{k}$$

Solution

$$\Rightarrow \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \text{ Let } \mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\text{Let the normal vector } \mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

Since $\mathbf{n}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, then by Dot product

$$\mathbf{n} \cdot \mathbf{a} = 0 \Rightarrow (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0$$

$$\therefore 3A + B - C = 0 \dots\dots\dots (1)$$

$$\mathbf{n} \cdot \mathbf{b} = 0 \Rightarrow (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot (2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = 0$$

$$\therefore 2A + 4B + C = 0 \dots\dots\dots (2)$$

Expressing B and C in terms of A only

eqn. (1) + eqn. (2)

$$\Rightarrow 5A + 5B = 0 \therefore A = -B \dots\dots\dots (3)$$

eqn. (3) in to eqn. (1)

$$\Rightarrow 3A - A - C = 0 \therefore C = 2A$$

$$\text{Now } \mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \text{ Direction ratio: } A:B:C$$

$$A:-A:2A = 1:-1:2$$

$$\therefore \mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

$$\text{From } \mathbf{r} \cdot \mathbf{n} = d \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = d. \text{ But } (1, 1, 0) \text{ lies on the plane}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = d \therefore d = 1 - 1 + 0 = 0$$

Equation of plane in dot product form is $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 0$

NOTE:

The above approach can also be used to determine the normal vector to *two* given vectors.

INTERSECTION OF A LINE AND A PLANE

Consider the line $\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and plane in Cartesian form

$$Ax + By + Cz = d$$

From the line, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$$\Rightarrow x = a_1 + \lambda d_1, y = a_2 + \lambda d_2, z = a_3 + \lambda d_3 \dots\dots\dots (1)$$

Substitute eqn. (1) in to eqn. of plane

$$A(a_1 + \lambda d_1) + B(a_2 + \lambda d_2) + C(a_3 + \lambda d_3) = d$$

$$\Rightarrow (Aa_1 + Ba_2 + Ca_3) + \lambda(Ad_1 + Bd_2 + Cd_3) = d$$

From which the value of λ can be obtained. The point of intersection of the line and plane can be obtained from eqn. (1)

$$.i.e. [(a_1 + \lambda d_1), (a_2 + \lambda d_2), (a_3 + \lambda d_3)]$$

Deduction from the rectangular enclosed result

The result above is true for every value of λ hence it is an identity. *i.e.*

$$\begin{aligned} Aa_1 + Ba_2 + Ca_3 &= 0 \\ Ad_1 + Bd_2 + Cd_3 &= 0 \end{aligned}$$

, which are the conditions for a line to lie on a plane.

Remark:

1. The condition $Ad_1 + Bd_2 + Cd_3 = 0$ shows that the line is perpendicular to the normal of the plane, hence parallel to the plane.
2. The condition $Aa_1 + Ba_2 + Ca_3 = 0$ shows that the point (a_1, a_2, a_3) lies on the plane.

Conclusion

For a line to lie in a plane the line must be parallel to the plane and *one* point on the line must also be on plane.

Examples

1. Find the point of intersection of the given line and plane.
 - a) $\mathbf{r} = (2\mathbf{i} - \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j})$, plane: $\mathbf{r} \cdot (5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9$
 - b) $\mathbf{r} = \lambda\mathbf{i} - 2\mathbf{j} + (2\lambda - 1)\mathbf{k}$, plane: $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5$
 - c) $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$, $-4x + 4y - 5z - 3 = 0$

Solution

$$a) \quad \mathbf{r} = (2\mathbf{i} - \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j}) = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}; \quad \mathbf{r} \cdot (5\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 9 \therefore 5x - y + 7z = 9$$

$$\text{From the line, } \Rightarrow x = 2 + \lambda, y = 3\lambda, z = -1 \dots\dots\dots (1)$$

Substitute eqn. (1) in to eqn. of plane

$$5(2 + \lambda) - (3\lambda) + 7(-1) = 9 \Rightarrow 10 + 5\lambda - 3\lambda - 7 = 9$$

$$\therefore \lambda = 3$$

From eqn. (1), point of intersection is;

$$x = 2 + 3 = 5, y = 3(3) = 9, z = -1$$

(5, 9, -1) is the point of intersection of the line and plane.

$$\text{b) } \mathbf{r} = \lambda \mathbf{i} - 2\mathbf{j} + (2\lambda - 1)\mathbf{k} \Rightarrow \mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow x = \lambda, y = -2, z = -1 + 2\lambda \dots\dots\dots (1)$$

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 5 \Rightarrow 2x + y - 3z = 5 \dots\dots\dots (2)$$

eqn. (1) in to eqn. (2)

$$\Rightarrow 2\lambda + (-2) - 3(-1 + 2\lambda) = 5 \therefore 2\lambda - 2 + 3 - 6\lambda = 5$$

$$\therefore \lambda = -1$$

From eqn. (1), point of intersection is;

$$x = -1, y = -2, z = -1 - 2 = -3$$

(-1, -2, -3) is the point of intersection of the line and plane.

$$\text{c) Let } \frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4} = \lambda$$

$$\Rightarrow x = 3 + 2\lambda, y = 4 + 3\lambda, z = 5 + 4\lambda \dots\dots\dots (1)$$

$$\Rightarrow -4x + 4y - 5z - 3 = 0 \dots\dots\dots (2)$$

eqn. (1) in to eqn. (2)

$$\Rightarrow -4(3 + 2\lambda) + 4(4 + 3\lambda) - 5(5 + 4\lambda) - 3 = 0$$

$$\Rightarrow -12 - 8\lambda + 16 + 12\lambda - 25 - 20\lambda - 3 = 0 \text{ or } -16\lambda = 24$$

$$\therefore \lambda = -\frac{3}{2}$$

From eqn. (1), point of intersection is;

$$x = 3 - 3 = 0, y = 4 - \frac{9}{2} = -\frac{1}{2}, z = -1$$

(0, -\frac{1}{2}, -1) is the point of intersection of the line and plane.

2. Prove that the line(s) lie on the given plane(s)

$$\text{a) } \frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}, 4x + 4y - 5z - 3 = 0$$

$$\text{b) } \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } 2x + y + z = 5$$

Solution

$$\text{a) } \frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}, \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \Rightarrow \mathbf{d} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\text{Plane: } 4x + 4y - 5z - 3 = 0 \Rightarrow \mathbf{n} = \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix}$$

For a line to lie in a plane,

- The line is perpendicular to the normal of the plane .i.e. $\mathbf{n} \cdot \mathbf{d} = 0$

- One point on the line lies on the plane

$$\Rightarrow \mathbf{n} \cdot \mathbf{d} = (4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) = 8 + 12 - 20 = 0$$

From the line (3, 4, 5) lies on the line

$\Rightarrow 4(3) + 4(4) - 5(5) - 3 = 0$, which satisfies the plane. Hence point lies on the plane.

$$\text{b) } \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \mathbf{d} = -\mathbf{i} + \mathbf{j}$$

$$2x + 2y + z = 5 \Rightarrow \mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$\Rightarrow \mathbf{n} \cdot \mathbf{d} = (-\mathbf{i} + \mathbf{j} + 0\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = -2 + 2 + 0 = 0$. Hence line is parallel to the plane .i.e. normal $\perp$ plane.

From the line $(2, 0, 1)$ lies on the line

$\Rightarrow 2(2) + 2(0) + 1 = 5$, which satisfies the plane. Hence point lies on the plane.

EQUATION OF A PLANE CONTAINING A GIVEN LINE $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$ AND SATISFYING ANY OTHER GIVEN CONDITION

Let the line be expressed in the Cartesian form $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$ and let the plane be $Ax + By + Cz = d$.

Since the plane is to pass through the point on the line .i.e. (a_1, a_2, a_3) , then

$$Ax + By + Cz = d \dots\dots\dots (1)$$

$$Aa_1 + Ba_2 + Ca_3 = d \dots\dots\dots (2)$$

eqn. (1) – eqn. (2) give the equation of a plane without d

$$\Rightarrow A(x - a_1) + B(y - a_2) + C(z - a_3) = 0 \dots\dots\dots (3)$$

NOTE:

Plane in equations (3) and (1) are the same.

Equation of plane contains the line if;

$$Ad_1 + Bd_2 + Cd_3 = 0 \dots\dots\dots (4)$$

Case I:

From equations (1) and (3)

Let the other condition satisfying the plane be that it also passes through point (α, α, γ)

From eqn. (3),

$$A(\alpha - a_1) + B(\beta - a_2) + C(\gamma - a_3) = 0 \dots\dots\dots (5)$$

Eliminating constants A, B and C from the zeroed equations

$$\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ d_1 & d_2 & d_3 \\ \alpha - a_1 & \beta - a_2 & \gamma - a_3 \end{vmatrix} = 0$$

Case II

Let the given condition be that the required plane is perpendicular to plane $ax + by + cz = k$

Planes $Ax + By + Cz = d$ and $ax + by + cz = k$ are perpendicular if their normal are perpendicular

$$\Rightarrow aA + Bb + Cc = 0 \dots\dots\dots (6)$$

Now eliminating constants A, B and C from the zeroed equations (6) , (3) and (4), the required equation of plane is;

$$\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ d_1 & d_2 & d_3 \\ a & b & c \end{vmatrix} = 0$$

Example

1. Find the equation of a plane containing the line $\mathbf{r} = (2\mathbf{i} + \mathbf{k}) + \lambda(-\mathbf{i} + \mathbf{j})$ and passing through the point with position vector $\mathbf{i} + 3\mathbf{k}$.

Solution

Let the equation of the plane be $Ax + By + Cz = d$ (1)

From the line: $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $(2, 0, 1)$ lies on the plane.

$$\Rightarrow 2A + B(0) + C = d$$

..... (2)

eqn. (1) – eqn. (2) to obtain a zeroed equation

$$\Rightarrow A(x - 2) + B(y - 0) + C(z - 1) = 0 \text{ (3)}$$

Plane (3) passes through $(1, 0, 3)$

$$\Rightarrow A(1 - 2) + B(0 - 0) + C(3 - 1) = 0 \text{ or } -A + 0B + 2C = 0 \text{ (4)}$$

Also eqn. (1) contains line if $\mathbf{n} \cdot \mathbf{d} = 0$

$$\Rightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0 \text{ or } -A + B + 0C = 0 \text{ (5)}$$

Eliminating constants A, B and C from the zeroed equations (3), (4) and (5)

$$\Rightarrow \begin{vmatrix} x-2 & y & z-1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (x-2) \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - y \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} + (z-1) \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} = 0$$

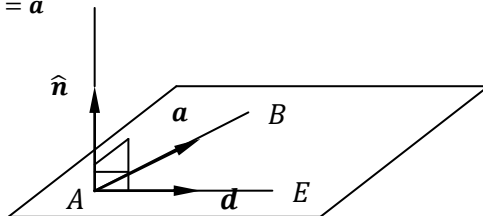
$$\Rightarrow -2(x-2) - y(2) + (z-1)(-1) = 0$$

$$\therefore 2x + 2y + z = 5$$

Alternatively

Let $A(2, 0, 1), B(1, 0, 3)$

$$\overrightarrow{AB} = \mathbf{a}$$



$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

Finding normal vector to plane

$$\hat{\mathbf{n}} = \frac{\mathbf{d} \times \mathbf{a}}{|\mathbf{d} \times \mathbf{a}|}$$

$$\mathbf{d} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{d} \times \mathbf{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\therefore \hat{\mathbf{n}} = \frac{\mathbf{d} \times \mathbf{a}}{|\mathbf{d} \times \mathbf{a}|} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

But normal vector direction ratio is 2:2:1

$$\therefore \mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Using $\mathbf{r} \cdot \mathbf{n} = d$

$$\Rightarrow \mathbf{r} \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = d \text{ or } 2x + 2y + z = d$$

This equation is satisfied by either $A(2, 0, 1)$ or $B(1, 0, 3)$

$$\Rightarrow 4 + 1 = d \therefore d = 5$$

The equation of plane now is $2x + 2y + z = 5$

2. Find the equation of a plane through the line $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ and perpendicular to the plane $x - 2y + 3z = 4$.

Solution

Let the plane be $Ax + By + Cz = d$ (1)

$$\Rightarrow \mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$$

Plane contains the line if the point $(-1, -2, -3)$ on line lies on it

$$\Rightarrow A(-1) + B(-2) + C(-3) = d$$
..... (2)

eqn. (1) – eqn. (2) to obtain a zeroed equation

$$A(x+1) + B(y+2) + C(z+3) = 0$$
..... (3)

Also eqn. (1) contains line if $\mathbf{n} \cdot \mathbf{d} = 0$

$$\Rightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 0 \text{ or } 2A + 3B + 4C = 0$$
..... (4)

For planes to be perpendicular, their normals are at right angles.

From eqn. (1), $\mathbf{n}_1 = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$

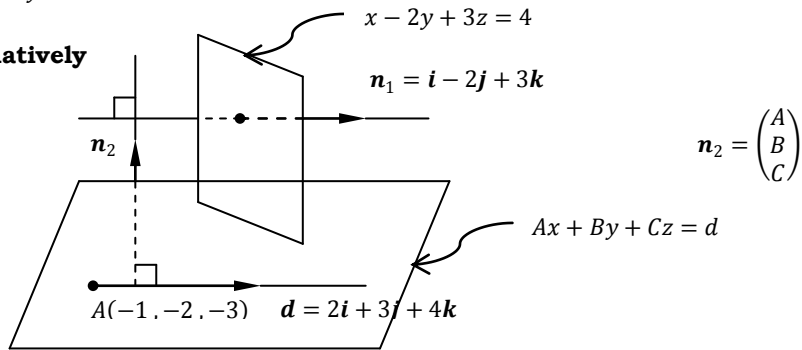
From $x - 2y + 3z = 4$, $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$$\therefore \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \Rightarrow A - 2B + 3C = 0$$
..... (5)

Eliminating constants A, B and C from the zeroed equations (3), (4) and (5)

$$\begin{aligned} \Rightarrow \begin{vmatrix} x+1 & y+2 & z+3 \\ 2 & 3 & 4 \\ 1 & -2 & 3 \end{vmatrix} &= 0 \\ \Rightarrow (x+1) \begin{vmatrix} 3 & 4 \\ -2 & 3 \end{vmatrix} - (y+2) \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} + (z+3) \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} &= 0 \\ \Rightarrow 17(x+1) - 2(y+2) + -7(z+3) &= 0 \\ \therefore 17x - 2y - 7z &= 8 \end{aligned}$$

Alternatively



From the figure, $\mathbf{n}_1 // \mathbf{d}$ and $\mathbf{n}_2$ is perpendicular to the line and $\mathbf{n}_1$

Finding $\mathbf{n}_2$ using cross product

$$\Rightarrow \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & -2 & 3 \end{vmatrix} = 17\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$$

Using $\mathbf{r} \cdot \mathbf{n} = k(\text{scalar})$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 17 \\ -2 \\ -7 \end{pmatrix} = k, 17x - 2y - 7z = k$$

But $(-1, -2, -3)$ lies on the plane

$$\Rightarrow -17 + 4 + 21 = k \therefore k = 8$$

$$\therefore 17x - 2y - 7z = 8 \text{ as before.}$$

3. Find the equation of a plane which passes through the point $(4, 5, -6)$ and is;

- Parallel to the plane $2x + 3y + 5z + 6 = 0$
- Parallel to the line joining points $(5, 2, 3)$ and $(1, 6, 4)$
- Perpendicular to planes $3x + 5y - 6z = 0$ and $7x + y + 2z = 8$

Solution

- a) Let the plane be $Ax + By + Cz = d$ (1)
 Eqn. (1) passes through $(4, 5, -6)$
 $\Rightarrow 4A + 5B - 6C = d$ (2)
 eqn. (1) - eqn. (2)
 $A(x - 4) + B(y - 5) + C(z + 6) = 0$ (3)
 Plane (1) is parallel to $2x + 3y + 5z + 6 = 0$ if their normals are parallel
 $\Rightarrow \mathbf{n}_1 = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \therefore \mathbf{n}_1 = \lambda \mathbf{n}_2$
 $\mathbf{n}_1 = \begin{pmatrix} 2\lambda \\ 3\lambda \\ 5\lambda \end{pmatrix} \Rightarrow \text{Direction ratio of } \mathbf{n}_2 \text{ is } 2\lambda : 3\lambda : 5\lambda = 2 : 3 : 5$
 $\therefore \mathbf{n}_1 = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ (4)
 eqn. (4) in to eqn. (3)
 $2(x - 4) + 3(y - 5) + 5(z + 6) = 0$ or $2x + 3y + 5z + 7 = 0$
- b) Direction vector of line joining $(5, 2, 3)$ and $(1, 6, 4)$ is
 $\begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} = \mathbf{d}$
 Plane (1) or (3) above is parallel to the line (. i.e. Direction vector $\mathbf{d}$) if the normal of plane and $\mathbf{d}$ are parallel
 $\Rightarrow \mathbf{n} = \lambda \mathbf{d} = \begin{pmatrix} -4\lambda \\ 4\lambda \\ \lambda \end{pmatrix}; \begin{pmatrix} -4\lambda \\ 4\lambda \\ \lambda \end{pmatrix} \Rightarrow \text{Direction ratio of } \mathbf{n} \text{ is } -4\lambda : 4\lambda : \lambda = -4 : 4 : 1$
 $\therefore \mathbf{n} = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ (5)
 eqn. (5) in to eqn. (3)
 $-4(x - 4) + 4(y - 5) + 1(z + 6) = 0$ or $4x - 4y - z + 2 = 0$
- c) $\mathbf{n}_1 \perp \mathbf{n}_2, \mathbf{n}_2 \perp \mathbf{n}_3$
 $\Rightarrow (3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}) \cdot (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) = 0$
 $\therefore 3A + 5B - 6C = 0$ (6)
 $\Rightarrow (7\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) = 0$
 $\therefore 7A + B + 2C = 0$ (7)
 Eliminating A, B and C from (3), (6) and (7)
 By cross product,
 $\Rightarrow \begin{vmatrix} x-4 & y-5 & z+6 \\ 3 & 5 & -6 \\ 7 & 1 & 2 \end{vmatrix} = 0$
 $\Rightarrow (x - 4) \begin{vmatrix} 5 & -6 \\ 1 & 2 \end{vmatrix} - (y - 5) \begin{vmatrix} 3 & -6 \\ 7 & 2 \end{vmatrix} + (z + 6) \begin{vmatrix} 3 & 5 \\ 7 & 1 \end{vmatrix} = 0$
 $\Rightarrow 16(x - 4) - 48(y - 5) + -32(z + 6) = 0$
 $\therefore x - 3y - 2z = 1$
4. Find the equation of a plane which passes through points $(6, 2, -4)$ and $(3, 4, 1)$ and is;
- Perpendicular to the plane $x + 5y - 2z = 6$
 - Parallel to the line joining $(1, 0, 3)$ and $(-1, 2, 4)$

Solution

- a) Let plane be: $Ax + By + Cz = d$ (1)
 For $(6, 2, -4)$, $6A + 2B - 4C = d$ (2)
 For $(3, 4, 1)$, $3A + 4C + C = d$ (3)
 $eqn. (1) - eqn. (2)$
 $A(x - 6) + B(y - 2) + C(z + 4) = 0$ (4)
 $eqn. (2) - eqn. (3)$
 $3A + 6B - 5C = 0$ (5)
 For planes to be perpendicular, their normals are also perpendicular
 $\Rightarrow (Ai + Bj + Ck) \cdot (i + 5j - 2k) = 0$
 $\Rightarrow A + 5B - 2C = 0$ (6)
 Eliminating A, B and C from (3), (6) and (7)
 By cross product,
 $\Rightarrow \begin{vmatrix} x-6 & y-2 & z+4 \\ 3 & 6 & -5 \\ 1 & 5 & -2 \end{vmatrix} = 0$
 Simplifying, the plane is
 $\therefore 13x + y + 9z = 44$

- b) Direction vector of the line is;

$$\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \mathbf{d}$$

For a line to be parallel to the plane, the normal vector to plane and direction vector of the line must be perpendicular.

$$\Rightarrow \mathbf{n} \cdot \mathbf{d} = 0 \quad \therefore \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = 0, -2A + 2B + C = 0 \text{ (7)}$$

From (4), (5) and (7), eliminating constants A, B and C

By cross product,

$$\Rightarrow \begin{vmatrix} x-6 & y-2 & z+4 \\ 3 & 6 & -5 \\ -2 & 2 & 1 \end{vmatrix} = 0$$

Simplifying, the plane is

$$\therefore 16x + 7y + 18z = 38$$

NOTE:

The above question can be done using another alternative as in example (2) above.

5. Find the equation of a plane containing the line $\frac{x+1}{-3} = \frac{y-3}{-2} = \frac{z+2}{1}$ and the point $(0, 7, -7)$. Prove that the line $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ also lies on the plane.

Solution

- Let plane be: $Ax + By + Cz = d$ (1)
 $(0, 7, -7)$ lies on the plane
 $\Rightarrow A(0) + B(7) + C(-7) = d$ (2)
 $eqn. (1) - eqn. (2)$
 $A(x - 0) + B(y - 7) + C(z + 7) = 0$ (3)
 $eqn. (1)$ contains line if
 $-3A - 2B + C = 0$ (4)
 The point on line also lies on the plane (3)
 $\Rightarrow A(-1 - 0) + B(3 - 7) + C(-2 + 7) = 0$

$$\therefore -A - 4B + 5C = 0 \dots\dots\dots (3)$$

From (3), (4) and (5), eliminating constants A, B and C

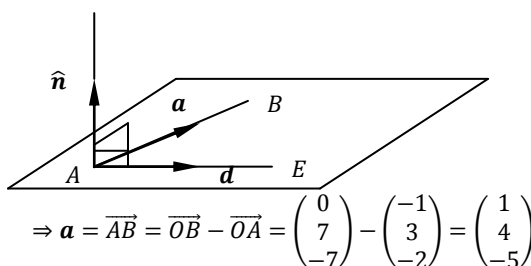
By cross product,

$$\Rightarrow \begin{vmatrix} x & y-2 & z+4 \\ -3 & -2 & 1 \\ -1 & -4 & 5 \end{vmatrix} = 0$$

Simplifying, the plane is

$$\therefore 3x - 7y - 5z + 14 = 0$$

Alternatively



$$\begin{aligned} \text{Let } \overrightarrow{AB} &= \mathbf{a} = \overrightarrow{OB} - \overrightarrow{OA} \\ A(-1, 3, -2), B(0, 7, -7) \\ \text{Let } \mathbf{d} &= -3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \end{aligned}$$

$$\Rightarrow \mathbf{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 7 \\ -7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$$

Finding normal vector to plane

$$\hat{\mathbf{n}} = \frac{\mathbf{d} \times \mathbf{a}}{|\mathbf{d} \times \mathbf{a}|}$$

$$\mathbf{d} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ 1 & 4 & -5 \end{vmatrix} = 6\mathbf{i} - 14\mathbf{j} - 10\mathbf{k} \Rightarrow |\mathbf{d} \times \mathbf{a}| = \sqrt{6^2 + (-14)^2 + (-10)^2} = \sqrt{332}$$

$$\therefore \hat{\mathbf{n}} = \frac{\mathbf{d} \times \mathbf{a}}{|\mathbf{d} \times \mathbf{a}|} = \frac{1}{\sqrt{332}} (6\mathbf{i} - 14\mathbf{j} - 10\mathbf{k})$$

But normal vector direction ratio is $\frac{6}{\sqrt{332}} : -\frac{14}{\sqrt{332}} : -\frac{10}{\sqrt{332}} = 3 : -7 : -5$

$$\therefore \mathbf{n} = 3\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$$

Using $\mathbf{r} \cdot \mathbf{n} = k$

$$\Rightarrow \mathbf{r} \cdot (6\mathbf{i} - 14\mathbf{j} - 10\mathbf{k}) = k \text{ or } 6x - 14y - 10z = k$$

This equation is satisfied by either $A(-1, 3, -2), B(0, 7, -7)$

$$\Rightarrow -3 - 21 + 10 = -14 = k = d \therefore d = -14$$

The equation of plane now is $\therefore 3x - 7y - 5z + 14 = 0$

COPLANAR LINES

Are lines which lie on the same plane. They can meet/do not meet or parallel.

$$\text{Consider } \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \dots\dots\dots (1)$$

$$\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'} \dots\dots\dots (2)$$

$$\text{Let the plane be } Ax + By + Cz = d \dots\dots\dots (3)$$

Eqn. (3) contains (1) if (α, β, γ) lie on the plane

$$\Rightarrow A\alpha + B\beta + C\gamma = d \dots\dots\dots (4)$$

eqn. (3) - eqn. (4)

$$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0 \dots\dots\dots (5)$$

Also Eqn. (3) contains (1) if direction vector of line is perpendicular to plane.

$$\Rightarrow Al + Bm + Cn = d \dots\dots\dots (6)$$

Now eqn. (5) contains (2) if

$$A(\alpha' - \alpha) + B(\beta' - \beta) + C(\gamma' - \gamma) = 0 \dots\dots\dots (7)$$

Also Eqn. (5) contains (2) if direction vector of line is perpendicular to plane
 $\Rightarrow Al' + Bm' + Cn' = d$ (8)

From (6), (7) and (8), eliminating constants A, B and C

By cross product,

$$\therefore \begin{vmatrix} \alpha' - \alpha & \beta' - \beta & \gamma' - \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$
 , which is the condition for two lines to

be coplanar.

From (5), (6) and (8), eliminating constants A, B and C

$$\therefore \begin{vmatrix} x - \alpha & y - \beta & z - \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$
 , which is the equation of plane containing the two lines.

Example

1. Prove that the lines $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ and $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ are coplanar and lie in the plane $x - 2y + z + 7 = 0$

Solution

Equation of a plane containing the first line and parallel to the second line is

$$\begin{vmatrix} x & y-2 & z+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0 \text{ i.e. } x - 2y + z + 7 = 0$$
 , clearly this plane is satisfied by the point (2, 6, 3) which lies on the second line.

2. Show that the lines $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ and $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are coplanar and find the equation of plane containing them.

Solution

Equation of a plane containing the first line and parallel to the second line is

$$\begin{vmatrix} x & y-2 & z+3 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0 \text{ i.e. } 6x - 5y - z = 0$$
 . This plane passes through point

(-1, -1, -1) on the second line so that it contains the second line. Thus the two lines are coplanar and the equation of plane containing them is $6x - 5y - z = 0$.

CONDITION FOR 4-POINTS TO BE COPLANAR

Four points will be coplanar if the plane passing through any 3-points passes through the fourth point.

Example

Show that the points below are coplanar

- a) (0, -1, 0), (2, 1, -1), (1, 1, 1) and (3, 3, 0)
- b) (0, 1, 3), (1, 0, -4), (1, 1, -1) and (1, 2, 2)

Solution

- a) Let A(0, -1, 0), B(2, 1, -1), C(1, 1, 1) and D(3, 3, 0)

Finding the equation of plane through A, B and C

Using $\mathbf{r} \cdot \mathbf{n} = d$

$$\text{Let } \mathbf{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

Ratio of normal direction vector = 4: -3: 2 $\Rightarrow$ Normal vector $\mathbf{n} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = d \text{ or } 4x - 3y + 2z = d$$

But A, B and C lie on this plane

$$\text{For } A(0, -1, 0), 4(0) - 3(-1) + 2(0) = d \therefore d = 3$$

The equation of the plane is $4x - 3y + 2z = 3$

Now for $D(3, 3, 0), 4(3) - 3(3) + 2(0) = 3 = \mathbf{RHS}$. Hence points are coplanar.

- b) Let $A(0, 1, 3), B(1, 0, -4), C(1, 1, -1)$ and $D(1, 2, 2)$

Finding the equation of plane through A, B and C

Using $\mathbf{r} \cdot \mathbf{n} = d$

$$\text{Let } \mathbf{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}$$

$$\mathbf{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 1 & -1 & -7 \\ 1 & 0 & -4 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

Ratio of normal direction vector = 4: -3: 1 $\Rightarrow$ Normal vector $\mathbf{n} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = d \text{ or } 4x - 3y + z = d$$

But A, B and C lie on this plane

$$\text{For } A(0, 1, 3), 4(0) - 3(1) + (3) = d \therefore d = 0$$

The equation of the plane is $4x - 3y + z = 0$

Now for $D(1, 2, 2), 4(1) - 3(2) + (2) = 3 = \mathbf{RHS}$. Hence points are coplanar.

COPLANAR VECTORS

Vectors lying in the same plane are coplanar vectors.

Let vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ lie on the same plane.

For coplanar vectors

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

Examples

Show that the vectors are coplanar

- a) $\mathbf{a} = \mathbf{i} - 4\mathbf{k}, \mathbf{b} = \mathbf{i} - \mathbf{j} - 7\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

- b) $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$

Solution

- a) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 1 & 0 & -4 \\ 1 & -1 & -7 \end{vmatrix} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$c. (\mathbf{a} \times \mathbf{b}) = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot (-4\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = -4 + 6 - 2 = 0$. Thus vectors are coplanar.

b) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 0\mathbf{k}$

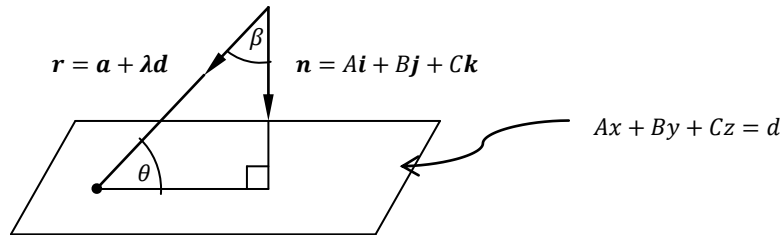
Now $c. (\mathbf{a} \times \mathbf{b}) = (3\mathbf{i} + 4\mathbf{j} + 0\mathbf{k}) \cdot (4\mathbf{i} - 3\mathbf{j} + 0\mathbf{k}) = 12 - 12 + 0 = 0$. Thus vectors are coplanar.

NOTE:

A correct combination of vectors can reduce on the unnecessary calculations to satisfy the condition for coplanar vectors.

ANGLE BETWEEN A LINE AND A PLANE

Consider the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$ and the plane $Ax + By + Cz = d$



Required angle $= \theta = 90 - \beta$

Finding angle β between the line and the normal vector to plane

$$\mathbf{n} \cdot \mathbf{d} = |\mathbf{n}| \cdot |\mathbf{d}| \cos \beta$$

But $\beta = 90 - \theta$

$$\Rightarrow \mathbf{n} \cdot \mathbf{d} = |\mathbf{n}| \cdot |\mathbf{d}| \cos(90 - \theta)$$

$$\Rightarrow \mathbf{n} \cdot \mathbf{d} = |\mathbf{n}| \cdot |\mathbf{d}| \sin \theta \quad [\because \cos(90 - \theta) = \sin \theta]$$

$$\therefore \theta = \sin^{-1} \left(\frac{\mathbf{n} \cdot \mathbf{d}}{|\mathbf{n}| \cdot |\mathbf{d}|} \right)$$

In general the angle between a line and a plane is the complement of the angle between the normal vector of plane and direction vector of the line.

Examples

Find the angle between the following line and plane

a) $\mathbf{r} = (\mathbf{i} - 3\mathbf{j}) + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k})$, plane: $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) = 10$

b) $\mathbf{r} = (2 + \lambda)\mathbf{i} - 3\mathbf{j} + (1 - \lambda)\mathbf{k}$, plane: $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 6$

c) $\frac{x}{4} = \frac{y-1}{-1} = \frac{z+3}{-5}$, $x - 2y + 4z + 3 = 0$

d) $\mathbf{r} = 3\lambda\mathbf{i} + 2\lambda\mathbf{j} - 6\lambda\mathbf{k}$, $\mathbf{r} \cdot (4\mathbf{i} - 3\mathbf{k}) = 20$

Solution

a) $\mathbf{r} = (\mathbf{i} - 3\mathbf{j}) + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \Rightarrow \mathbf{d} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

plane: $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) = 10 \Rightarrow \mathbf{n} = \mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$

Using $\mathbf{n} \cdot \mathbf{d} = |\mathbf{n}| |\mathbf{d}| \sin \theta$

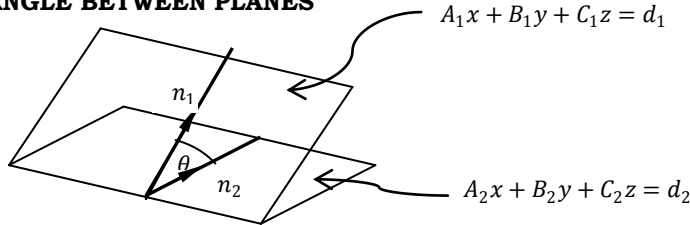
$$\mathbf{n} \cdot \mathbf{d} = (2\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) = 2 + 2 + 7 = 11$$

$$|\mathbf{n}| = \sqrt{1 + 4 + 49} = \sqrt{54}, \quad |\mathbf{d}| = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$\therefore \sin \theta = \frac{\mathbf{n} \cdot \mathbf{d}}{|\mathbf{n}| |\mathbf{d}|} = \frac{11}{\sqrt{54} \times \sqrt{6}} = \frac{11}{18} \Rightarrow \theta = 37.7^\circ$$

- b) $\mathbf{r} = (2 + \lambda)\mathbf{i} - 3\mathbf{j} + (1 - \lambda)\mathbf{k}$, plane: $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 6$
- $$\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \text{plane: } \mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 6 \Rightarrow \mathbf{n} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$$
- $$\mathbf{n} \cdot \mathbf{d} = (4\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 0\mathbf{j} - \mathbf{k}) = 4 + 0 + 1 = 5$$
- $$|\mathbf{n}| = \sqrt{16 + 1 + 1} = \sqrt{18}, |\mathbf{d}| = \sqrt{1 + 0 + 1} = \sqrt{2}$$
- $$\therefore \sin \theta = \frac{\mathbf{n} \cdot \mathbf{d}}{|\mathbf{n}| |\mathbf{d}|} = \frac{5}{\sqrt{18} \times \sqrt{2}} = \frac{5}{6} \Rightarrow \theta = 54.4^\circ$$
- c) $\frac{x}{4} = \frac{y-1}{-1} = \frac{z+3}{-5} \Rightarrow \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix} \therefore \mathbf{d} = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}, x - 2y + 4z + 3 = 0 \Rightarrow \mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$
- $$\mathbf{n} \cdot \mathbf{d} = (4\mathbf{i} - \mathbf{j} - 5\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 4 + 2 - 20 = -14$$
- $$|\mathbf{n}| = \sqrt{16 + 1 + 25} = \sqrt{42}, |\mathbf{d}| = \sqrt{1 + 4 + 16} = \sqrt{21}$$
- $$\therefore \sin \theta = \frac{\mathbf{n} \cdot \mathbf{d}}{|\mathbf{n}| |\mathbf{d}|} = \frac{-14}{\sqrt{21} \times \sqrt{42}} \Rightarrow \theta = -28.1^\circ \therefore \theta = 28.1^\circ \text{ (Numerical value)}$$
- d) Left as an exercise **Answer:** $\sin \theta = \frac{6}{7} \therefore \theta = 59^\circ$

ANGLE BETWEEN PLANES



The angle between planes is the angle between their *normals*.

Using dot-product

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1| |\mathbf{n}_2| \cos \theta$$

$$\therefore \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right)$$

Example

Find the angle between the pair of planes

- a) $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) = 4$ and $\mathbf{r} \cdot (\mathbf{j} + \mathbf{k}) = 1$
 b) $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = 5$ and $\mathbf{r} \cdot \mathbf{j} = 0$
 c) $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 1$ and $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$
 d) $2x + 3y - 5z = 1$ and $x - 2y - z = 0$

Solution

- a) $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j}) = 4 \Rightarrow \mathbf{n}_1 = \mathbf{i} - \mathbf{j} = \mathbf{i} - \mathbf{j} + 0\mathbf{k}$; $\mathbf{r} \cdot (\mathbf{j} + \mathbf{k}) = 1 \Rightarrow \mathbf{n}_2 = \mathbf{j} + \mathbf{k} = 0\mathbf{i} + \mathbf{j} + \mathbf{k}$

Using dot-product

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1| |\mathbf{n}_2| \cos \theta$$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (\mathbf{i} - \mathbf{j} + 0\mathbf{k}) \cdot (0\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0 - 1 + 0 = -1$$

$$|\mathbf{n}_1| = \sqrt{0 + 1 + 1} = \sqrt{2}; |\mathbf{n}_2| = \sqrt{0 + 1 + 1} = \sqrt{2}$$

$$\therefore \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{-1}{\sqrt{2} \times \sqrt{2}} \right) = 120^\circ$$

In terms of acute angles, $\theta = 180 - 120 = 60^\circ$

- b) $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = 4 \Rightarrow \mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$; $\mathbf{r} \cdot \mathbf{j} = 0 \Rightarrow \mathbf{n}_2 = 0\mathbf{i} + \mathbf{j} + 0\mathbf{k}$

Using dot-product

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1| |\mathbf{n}_2| \cos \theta$$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \cdot (0\mathbf{i} + \mathbf{j} + 0\mathbf{k}) = 0 - 2 + 0 = -2$$

$$|\mathbf{n}_1| = \sqrt{1 + 4 + 4} = 3; |\mathbf{n}_2| = \sqrt{0 + 1 + 0} = 1$$

$$\therefore \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{-2}{3} \right) = 131.8^\circ$$

In terms of acute angles, $\theta = 180 - 131.8^\circ = 48.2^\circ$

c) $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 1 \Rightarrow \mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{r} \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0 \Rightarrow \mathbf{n}_2 = \mathbf{i} - \mathbf{j} + \mathbf{k}$

Using dot-product

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2|\cos\theta$$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 1 - 1 + 1 = 1$$

$$|\mathbf{n}_1| = \sqrt{1+1+1} = \sqrt{3}; |\mathbf{n}_2| = \sqrt{1+1+1} = \sqrt{3}$$

$$\therefore \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{1}{\sqrt{3} \times \sqrt{3}} \right) = \cos^{-1} \left(\frac{1}{3} \right) = 70.5^\circ$$

d) $2x + 3y - 5z = 1 \Rightarrow \mathbf{n}_1 = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$; $x - 2y - z = 0 \Rightarrow \mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2|\cos\theta$$

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = (2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 2 - 6 + 5 = 1$$

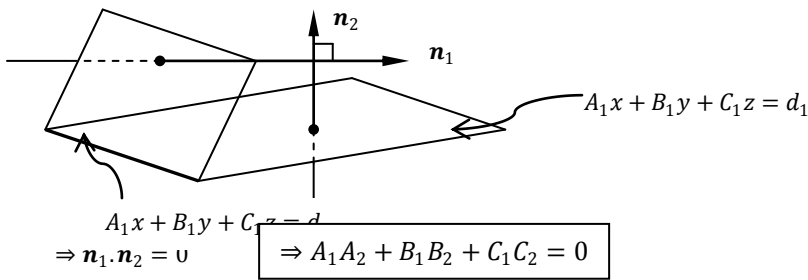
$$|\mathbf{n}_1| = \sqrt{4+9+25} = \sqrt{38}; |\mathbf{n}_2| = \sqrt{1+4+1} = \sqrt{6}$$

$$\therefore \theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right) = \cos^{-1} \left(\frac{1}{\sqrt{38} \times \sqrt{6}} \right) = 86.2^\circ$$

CONDITION FOR PERPENDICULARITY OF 2-PLANES

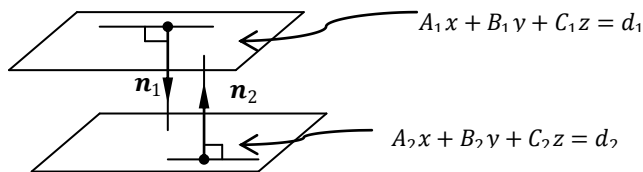
If two planes are perpendicular, then their normals are also perpendicular.

Consider two planes $A_1x + B_1y + C_1z = d_1$ and $A_2x + B_2y + C_2z = d_2$



CONDITION FOR PARALLELISM OF 2-PLANES

If two planes are parallel, then their *normals* are also parallel.



$$\Rightarrow \mathbf{n}_1 = \lambda \mathbf{n}_2$$

$$\Rightarrow A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k} = \lambda(A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k})$$

$$\Rightarrow A_1 = \lambda A_2 \Rightarrow \lambda = \frac{A_1}{A_2}$$

$$\Rightarrow B_1 = \lambda B_2 \Rightarrow \lambda = \frac{B_1}{B_2}$$

$$\Rightarrow C_1 = \lambda C_2 \Rightarrow \lambda = \frac{C_1}{C_2}$$

$$\Rightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \lambda$$

EQUATION OF A PLANE PARALLEL TO A GIVEN PLANE $ax + by + cz = d$

The required plane is generally given as

$$ax + by + cz = k(\text{constant})$$

, where $k = \text{constant}$, that can be obtained from any other condition satisfying the plane.

Example

1. Find the equation of a plane passing through point $(1, 2, 3)$ and parallel to the plane

$$4x - 5y + 6z = 7.$$

Solution

Required plane: $4x - 5y + 6z = k$

But $(1, 2, 3)$ lies on the plane

$$\Rightarrow 4 - 10 + 18 = k \therefore k = 12$$

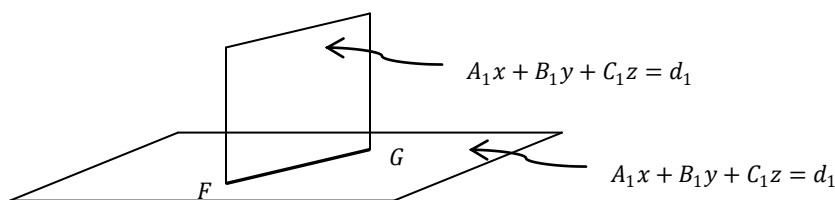
The plane is $4x - 5y + 6z = 12$

2. Find the equation of a plane parallel to plane $2x + 3y - 4z = 1$ and passing through $(1, 2, 3)$. **Answer:** $2x + 3y - 4z + 4 = 0$

INTERSECTION OF PLANES

Intersection of two planes:

Consider two planes $A_1x + B_1y + C_1z = d_1$ and $A_2x + B_2y + C_2z = d_2$



Two planes intersect in a straight *common line*. FG is the common line of intersection

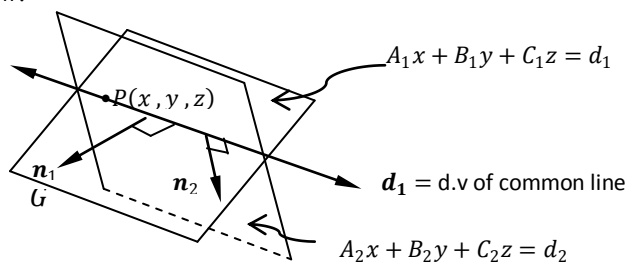
Equation of a plane containing the common line of intersection of two planes:

$$(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0$$

The equation is of the form $U + \lambda V = 0$ where $U = V = 0$ and $\lambda = \text{constant}$ to be obtained from the other condition satisfying the plane.

Equation of the common line to 2-intersecting planes:

This is obtained as below.



Direction vector, d is perpendicular to n_1 and n_2

$$\therefore \mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$$

Using vector equation of a line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$

This line intersects at least one of the xy , xz or the yz - plane. Thus one point on this line has a zero x - or y - or z - coordinate. i.e. Let $x=0$ or $y=0$ or $z=0$
Now solving for x and y if $z=0$ in the equations of planes, yields the coordinate on the common line. Hence, equation of line FG is of the form;

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \lambda(\mathbf{n}_1 \times \mathbf{n}_2) \text{ for } z = 0$$

Example

1. Find the equation of a plane passing through the line of intersection of planes $x + y + z = 6$ and $2x + 3y + 4z = -5$, and is perpendicular to the plane $4x + 5y - 5z = 8$. Find also the equation of line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z = -5$.

Solution

$$x + y + z = 6 \dots\dots\dots (1)$$

$$2x + 3y + 4z = -5 \dots\dots\dots (2)$$

$$\text{Required plane: } (x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$$

$$\Rightarrow x(1 + 2\lambda) + y(1 + 3\lambda) + z(1 + 4\lambda) + (5\lambda - 6) = 0$$

This plane is perpendicular to $4x + 5y - 5z = 8$ if:

$$4(1 + 2\lambda) + 5(1 + 3\lambda) - 5(1 + 4\lambda) = 0$$

$$\Rightarrow 4 + 8\lambda + 5 + 15\lambda - 5 - 20 = 0 \text{ or } 3\lambda + 4 = 0 \therefore \lambda = -\frac{4}{3}$$

$$\text{Equation of plane: } \Rightarrow x\left(1 - \frac{8}{3}\right) + y(1 - 4) + z\left(1 - \frac{16}{3}\right) + \left(-\frac{20}{3} - 6\right) = 0$$

$$-5x - 3y - 13z - 38 = 0 \text{ or } 5x + 3y + 13z + 38 = 0$$

Now finding the equation of the common line:

$$x + y + z = 6, \mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}, \text{ and } 2x + 3y + 4z = -5, \mathbf{n}_2 = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$

Direction vector of line: $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$

$$\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

Finding a point on the required line:

Fixing $z = 0$

$$\Rightarrow x + y + z = 6 \text{ becomes } x + y = 6 \dots\dots\dots (3)$$

$$\Rightarrow 2x + 3y + 4z = -5 \text{ becomes } 2x + 3y = -5 \dots\dots\dots (4)$$

Solving eqns. (3) and (4) simultaneously,

$$x = 23 \text{ and } y = -17$$

$\Rightarrow (23, -17, 0)$ is the point on the required line.

$$\therefore \mathbf{r} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

2. Find the parametric equations of the line which is the intersection of $x + 2y + 3z - 8 = 0$ and $3x + 3y + z + 3 = 0$. Find the equation of the plane containing this line and perpendicular to the plane $2x + 3y - 4z = -2$.

Solution

Fixing $z = 0$ and solving the equations of planes simultaneously,

$$x + 2y + 3z - 8 = 0 \Rightarrow x + 2y = 8 \dots\dots\dots (1)$$

$$3x + 3y + z + 3 = 0 \Rightarrow 3x + 3y = -3 \text{ or } x + y = -1 \dots\dots\dots (2)$$

Solving eqns. (1) and (2) simultaneously,

$$x = 2 \text{ and } y = 3$$

$\Rightarrow (2, 3, 0)$ is the point on the required line.

Direction vectors of the lines are respectively $\mathbf{n}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{n}_2 = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Direction vector of line: $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$

$$\mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 8\mathbf{j} - 9\mathbf{k}$$

The required line is;

$$\therefore \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 8 \\ -9 \end{pmatrix}$$

Required plane:

$$(x + 2y + 3z - 8) + k(3x + 3y + z + 3) = 0$$

$$x(1 + 3k) + y(2 + 3k) + z(3 + k) + 3k - 8 = 0$$

This plane is perpendicular to plane $2x + 3y - 4z = -2$ if:

$$2(1 + 3k) + 3(2 + 3k) - 4(3 + k) = 0 \quad [\because \mathbf{n}_1 \cdot \mathbf{n}_2 = 0]$$

$$\Rightarrow 2 + 6k + 6 - 9k - 12 - 4k = 0 \quad \therefore k = -\frac{4}{7}$$

$\therefore$ Equation of plane is:

$$x\left(1 - \frac{17}{7}\right) + y\left(2 + \frac{12}{7}\right) + z\left(3 - \frac{4}{7}\right) - \frac{12}{7} - 8 = 0$$

$$\therefore -5x + 26y + 17z + 68 = 0$$

Task

- Find the equation of the plane which contains the line of intersection of the planes

$$6x + 4y - 5z = 2 \text{ and } x - 2y + 3z = 0 \text{ and;}$$

- Perpendicular to the plane $3x - 2y + z = 5$
- Parallel to the line with direction cosines proportional to $(1, 3, 2)$.

$$\text{Answer: } 11x + 10y - 13z - 4 = 0, 2x - 20y + 29z + 2 = 0$$

- Show that the lines $\frac{1}{3}(x + 4) = \frac{1}{5}(y + 6) = -\frac{1}{2}(z - 1)$; $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ are coplanar. Find also the coordinates of their point of intersection and the equation of the plane that contains their line.

$$\text{Answer: } (2, 4, -3); 45x - 17y + 25z + 53 = 0$$

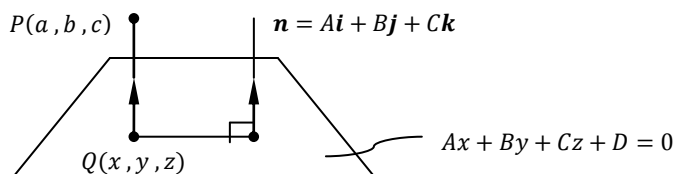
- Show that the lines $\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{2}$, $3x + 2y + z - 2 = 0 = x - 3y + 2z - 3$ are coplanar and find the equation of the plane in which they lie.

[HINT: Required: $(3x + 2y + z - 2) + \lambda(x - 3y + 2z - 3) = 0$]. This plane is parallel to the line if its normal is perpendicular to d.v. of the line. i.e. $3(3 + \lambda) + (2 - 3\lambda) - 2(1 + 2\lambda) = 0 \quad \therefore \lambda = \frac{9}{4} \Rightarrow (-5, -4, 7), 21x - 19y + 22z - 125 = 0$

DISTANCE OF A POINT FROM A PLANE

Consider a point $P(a, b, c)$ and the plane $Ax + By + Cz + D = 0$.

Let $QP = d$



Vector $\overrightarrow{QP}$ is $\perp$ to $\mathbf{n}$

$$\Rightarrow \overrightarrow{QP} = \lambda \mathbf{n}$$

$$\text{But } \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{pmatrix} a-x \\ b-y \\ c-z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a-x \\ b-y \\ c-z \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\Rightarrow a-x = \lambda A \text{ or } x = a - \lambda A$$

$$\Rightarrow b-y = \lambda B \text{ or } y = b - \lambda B$$

$$\Rightarrow c-z = \lambda C \text{ or } z = c - \lambda C$$

$$\therefore Q(x, y, z) \equiv Q(a - \lambda A, b - \lambda B, c - \lambda C) \text{ lies on the plane } Ax + By + Cz + d = 0$$

$$\Rightarrow A(a - \lambda A) + B(b - \lambda B) + C(c - \lambda C) + d = 0$$

$$\text{or } Aa + Bb + Cc + d = \lambda(A^2 + B^2 + C^2) \therefore \lambda = \frac{Aa + Bb + Cc + d}{A^2 + B^2 + C^2} \dots\dots\dots (1)$$

$$\text{But } \overrightarrow{QP} = \text{perpendicular distance} = |\overrightarrow{QP}| = \lambda |\mathbf{n}|$$

$$\therefore \overrightarrow{QP} = \lambda \times \sqrt{A^2 + B^2 + C^2} \dots\dots\dots (2)$$

subst. eqn. (1) in eqn. (2)

$$\Rightarrow \text{Perpendicular distance} = d = \frac{Aa + Bb + Cc + d}{A^2 + B^2 + C^2} \times \sqrt{A^2 + B^2 + C^2}$$

$$\therefore d = \frac{Aa + Bb + Cc + d}{\sqrt{A^2 + B^2 + C^2}} = \text{Perpendicular distance of point } P(a, b, c) \text{ from the plane } Ax + By + Cz + d = 0$$

This distance can sometimes be negative, thus the absolute value (numerical value) is taken to ensure +ve distance.

Let $Ax + By + Cz + d = 0$ be the plane and $P(a, b, c)$ be a point. The distance d from the point is given by;

$$d = \left| \frac{Aa + Bb + Cc + d}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Rule to find perpendicular distance of a point from a plane with equation given in general form $Ax + By + Cz + d = 0$ ($R.H.S = 0$)

“Substitute the coordinates of the point in the equation of plane with $RHS = 0$ and divide the result by

$$\sqrt{(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2 + (\text{coeff. of } z)^2} \text{”}$$

Examples

1. A plane passes through 3 points $A(0, 1, 3)$, $B(1, 0, -4)$ and $C(1, 1, -1)$. Find its equation and length of a perpendicular to it from $(2, 4, -3)$.

Solution

Equation of plane is left for the student as an exercise. **Answer:** $4x - 3y + z = 0$

Using the formula,

$$\text{Perpendicular distance} = \frac{4(2) - 3(4) - 3}{\sqrt{4^2 + (-3)^2 + 1^2}} = -\frac{7}{\sqrt{26}} = \frac{7}{\sqrt{26}} \text{ (numerically)}$$

2. Show that the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$ is $\frac{1}{6}$.

Solution

Find any point on any plane.

Let $x = y = 0$, now finding z

For $2x - 2y + z + 3 = 0 \Rightarrow z = -3$

$\therefore (0, 0, -3)$ lies on this plane.

Now perpendicular distance of $(0, 0, -3)$ from $4x - 4y + 2z + 5 = 0$ is;

$$d = \frac{4(0) - 4(0) + 2(-3) + 5}{\sqrt{4^2 + 4^2 + 2^2}} = -\frac{1}{6} = \frac{1}{6} \text{ (Numerically)}$$

3. Find the equations of the planes parallel to the plane $x - 2y + 2z - 3 = 0$ whose perpendicular distance from $(1, 2, 3)$ is 1 unit.

Solution

Given plane: $x - 2y + 2z - 3 = 0$

Any plane parallel to it is: $x - 2y + 2z + k = 0$ (1)

Since perpendicular distance of eqn. (1) from $(1, 2, 3)$ is 1 unit

$$\Rightarrow \pm \frac{1(1) - 2(2) + 2(3) + k}{\sqrt{1 + 4 + 4}} = 1$$

[use complete formul whenever perpendicular distance is given]

$$\Rightarrow \pm \frac{3+k}{3} = 1 \text{ or } \pm (3+k) = 3 \therefore k = 0, -6$$

From eqn. (1), the required planes are;

$$\therefore x - 2y + 2z = 0, x - 2y + 2z - 6 = 0$$

4. A plane contains the points $A(-4, 9, -9)$ and $B(5, -9, 6)$ and is perpendicular to the line which joins $C(4, -6, k)$ and B . Evaluate k and find the equation of the plane.

Solution

$$\overrightarrow{BC} \parallel \mathbf{n} \Rightarrow \mathbf{n} = \lambda \overrightarrow{BC}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 4 - 5 \\ -6 + 9 \\ k - 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ k - 6 \end{pmatrix}$$

Also $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 + 4 \\ -9 - 9 \\ 6 + 9 \end{pmatrix} = \begin{pmatrix} 9 \\ -18 \\ 15 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{AB} \cdot \overrightarrow{BC} = 0 \therefore -1(9) + 3(-18) + (k - 6)(15) = 0 \dots\dots\dots (\div 3)$$

$$\Rightarrow -51 + 5k = 0 \therefore k = \frac{51}{5}$$

Direction ratio of normal is $-1 : 3 : k - 6, k = \frac{51}{5}$

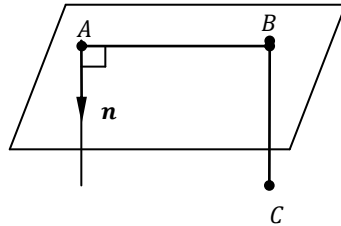
$$\Rightarrow \mathbf{n} \text{ has direction ratio of } -1 : 3 : \left(\frac{51}{5} - 6\right) = -5 : 15 : 21$$

Using $\mathbf{r} \cdot \mathbf{n} = d$

$$\mathbf{r} \cdot \begin{pmatrix} -5 \\ 15 \\ 21 \end{pmatrix} = d, \text{ this plane contains A and B.}$$

$$\text{For } A(-4, 9, -9), d = \begin{pmatrix} -4 \\ 9 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 15 \\ 21 \end{pmatrix} = 34$$

$$\therefore -5x + 15y + 21z - 34 = 0 \text{ or } \mathbf{r} \cdot \begin{pmatrix} -5 \\ 15 \\ 21 \end{pmatrix} = 34$$



COMPLEX NUMBERS

Introduction:

A complex number is that can be expressed in the form:

$$z = (\text{Real part}) + i(\text{Imaginary part}) \text{ .i.e.}$$

$$z = a + ib; a, b \in \mathbb{R}$$

Examples

$$z = 3 + 4i, z = 4i = 0 + 4i, z = 4 = 4 + 0i \text{ where } i = \sqrt{-1}$$

Origin of symbol $i = \sqrt{-1}$

This originates from finding square root of negative numbers.

Consider

$$\text{a) } \sqrt{4}$$

$$\text{b) } \sqrt{-4}$$

Solution

$$\text{a) } \sqrt{4} = \pm 2$$

$$\text{b) } \sqrt{-4} = \sqrt{-1 \times 4} = \sqrt{-1} \times \sqrt{4} \quad [\text{By surds: } \sqrt{ab} = \sqrt{a} \times \sqrt{b}]$$

$$= i \times \pm 2 = \pm 2i \quad [\text{Let } \sqrt{-1} = i \text{ (Mathematical notation)}]$$

Note:

There is no exact square root of a negative number that yields real numbers that are negative or positive.

Algebra of $i = \sqrt{-1}$

1) Simplify

$$\text{a) } i^2$$

$$\text{b) } i^3$$

$$\text{c) } i^5$$

$$\text{d) } i^4$$

Solution

$$\text{a) } i^2 = [i]^2 = [\sqrt{-1}]^2 = [(-1)^{\frac{1}{2}}]^2 = -1$$

$$\therefore i^2 = -1$$

$$\text{b) } i^3 = i^2 \times i = -1 \times i = -i$$

$$\therefore i^3 = -i$$

$$\text{c) } i^5 = i^2 \times i^3 = -1 \times -i = i$$

$$\therefore i^5 = i$$

$$\text{d) } i^4 = (i^2)^2 = (-1)^2 = 1$$

$$\therefore i^4 = 1$$

Note:

$$\text{a. } i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{-1 \times -1} = 1$$

b. The result $i^2 = -1, i^3 = -i$ and $i^4 = 1$ are important in the algebra of $i = \sqrt{-1}$.

2) Simplify

$$\text{a) } i^{20}$$

$$\text{b) } i^{39}$$

$$\text{c) } i^{40}$$

$$\text{d) } i^{15}$$

Solution

$$\text{a) } i^{20} = (i^2)^{10} = (-1)^{10} = 1$$

$$\therefore i^{20} = 1$$

$$\text{b) } i^{39} = (i^3)^{13} = (-i)^{13} = [-1 \times i]^{13} = -[i]^{13}$$

$$= -[i^4]^3 \times i = -[1]^3 \times i = -i$$

$$\therefore i^{39} = -i$$

$$\text{c) } i^{40} = [i^4]^{10} = [1]^{10} = 1$$

$$\therefore i^{40} = 1$$

$$\begin{aligned} \text{d) } i^{15} &= [i^4]^3 \times i^3 = 1 \times -i = -i \\ \therefore i^{15} &= -i \end{aligned}$$

Trial questions

Simplify

$$\text{a) } i^6 \qquad \text{b) } i^8 \qquad \text{c) } i^7 \qquad \text{d) } i^{17}$$

Answer:

$$\begin{aligned} \text{a) } i^6 &= (i^4)(i^2) = 1 \times -1 = -1 \\ \text{b) } i^8 &= [i^4]^2 = 1 \\ \text{c) } i^7 &= (i^4)(i^3) = -i \\ \text{d) } i^{17} &= [i^4]^4 \cdot i = i \end{aligned}$$

OPERATION IN COMPLEX NUMBERS

The following shall be considered:

- i) Addition
- ii) Subtraction
- iii) Multiplication
- iv) Division

Methods used

- Analytical method
- Graphical method

ADDITION OF COMPLEX NUMBERS

A. Analytical method

Given that $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$

Find: i) $z_1 + z_2$ ii) $z_1 + z_2 + z_3$

Solution

$$\begin{aligned} \text{i. } z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ \text{ii. } z_1 + z_2 + z_3 &= (x_1 + iy_1) + (x_2 + iy_2) + (x_3 + iy_3) \\ &= (x_1 + x_2 + x_3) + i(y_1 + y_2 + y_3) \end{aligned}$$

Deduction

When adding complex numbers, add the real parts and imaginary parts separately, and write the result in complex number form.

Examples

$$\begin{aligned} 1) \text{ Given that } z_1 &= 4 + i, z_2 = -3 - i \text{ and } z_3 = 3 + 2i \\ \text{Find i) } z_1 + z_2 \quad \text{ii) } z_1 + z_2 + z_3 \end{aligned}$$

Solution

$$\begin{aligned} \text{i. } z_1 + z_2 &= 4 + i + -3 - i \\ &= (4 - 3) + i(1 - 1) = 1 + 0i = 1 \\ \text{ii. } z_1 + z_2 + z_3 &= 4 + i + -3 - i + 3 + 2i \\ &= (4 - 3 + 3) + i(1 - 1 + 2) = 3 + 2i \\ 2) \text{ Given that } z_1 &= 4, z_2 = 3i \text{ and } z_3 = 3 + 2i \\ \text{Find i) } z_1 + z_2 \quad \text{ii) } z_1 + z_2 + z_3 \end{aligned}$$

Solution

$$\text{i. } z_1 + z_2 = 4 + 3i$$

ii. $z_1 + z_2 + z_3 = 4 + 3i + 3 + 2i = 7 + 5i$

B. Graphical method

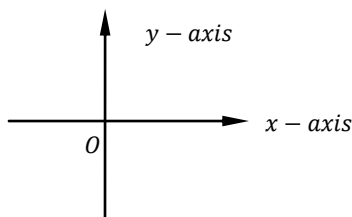
Before we look at this method, let us first look at Argand diagram.

Argand diagram

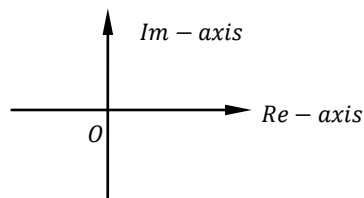
This is a graph having two axes.

i.e. Real (Re) axis as x - axis and Imaginary (Im) axis as y - axis

Cartesian graph



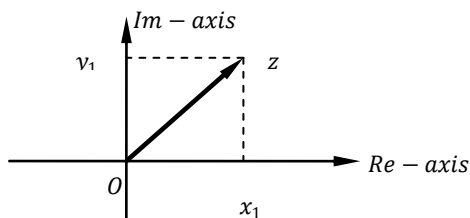
Argand diagram



Representation of a complex number on Argand diagram

Given that $z_1 = x_1 + iy_1$, z has a point (x_1, y_1) in general form. Plotting (x_1, y_1)

From the figure, a complex number is represented using an arrow from the origin to the point representing complex number.



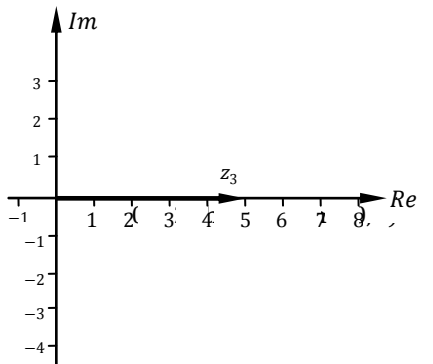
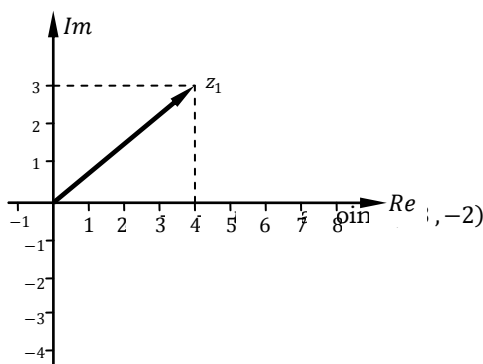
Example

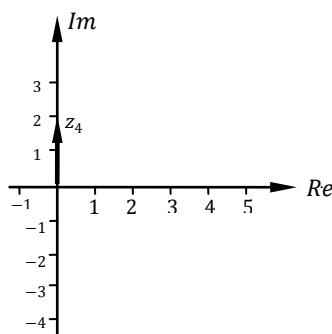
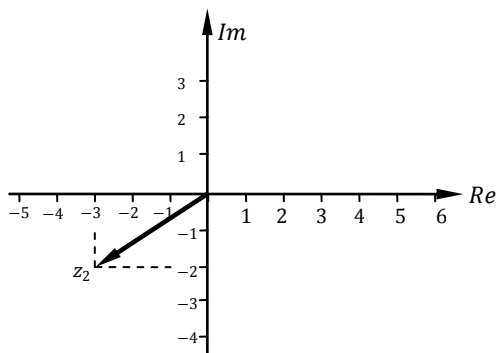
- 1) Given that $z_1 = 4 + 3i$, $z_2 = -3 - 2i$, $z_3 = 5$ and $z_4 = 2i$
Represent the complex numbers on different Argand diagram.

Solution

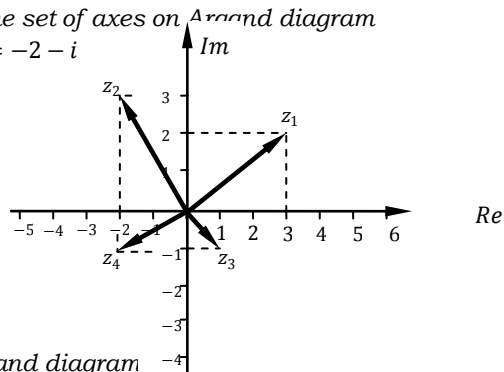
$z_1 = 4 + 3i$ has a point $(4, 3)$

$z_3 = 5 + 0i$ has a point $(5, 0)$





- 2) Represent the complex numbers on the same set of axes on Argand diagram
 $z_1 = 3 + 2i$, $z_2 = -2 + 3i$, $z_3 = 1 - i$ and $z_4 = -2 - i$



Trial questions

Represent the complex numbers on the Argand diagram

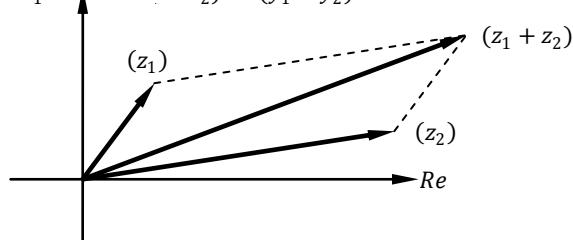
$$z_1 = 3 - \frac{1}{2}i, \quad z_2 = \frac{3}{2} + \frac{1}{2}i, \quad z_3 = 4 - 3i$$

Now adding complex numbers by Graphical method

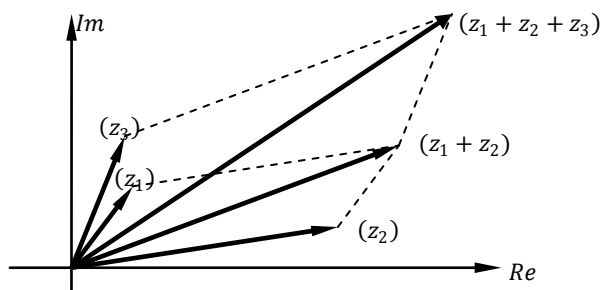
Two complex numbers are added using a parallelogram. Each complex number is represented on the Argand diagram, and the two complex numbers form the adjacent sides of a parallelogram. The addition result is given by the diagonal of a parallelogram.

Consider $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\therefore z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$



Addition of complex numbers by graphical method



Three complex numbers are added as a continuous chain .i.e. $z_1 + z_2 + z_3 = (z_1 + z_2) + z_3$. First add any two .i.e. $z_1 + z_2$ to get the result and this result with z_3 give a diagonal of a parallelogram which is the required result.

Example

Given that $z_1 = 4 + i$, $z_2 = -2 + i$, $z_3 = -3 - 2i$ and $z_4 = 5 + i$

Find:

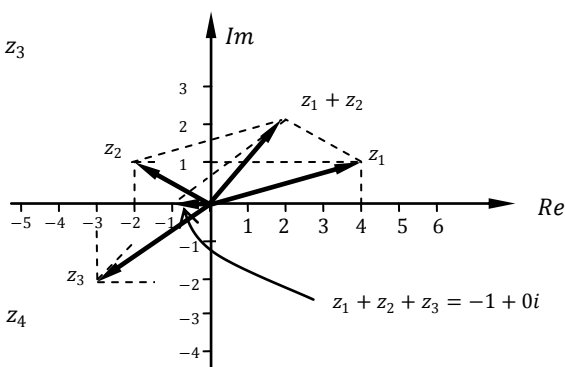
a) $z_1 + z_2 + z_3$

b) $z_2 + z_3 + z_4$

Solution

a) $z_1 + z_2 + z_3 = (z_1 + z_2) + z_3$

Adding $z_1 + z_2$

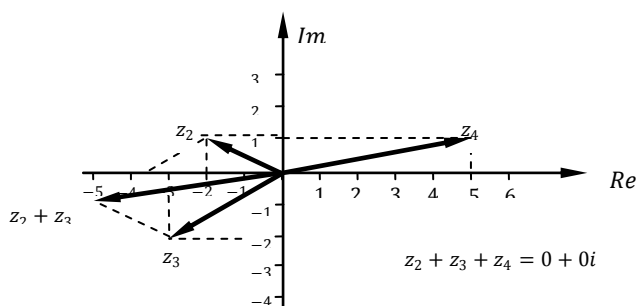


b) $z_2 + z_3 + z_4 = (z_2 + z_3) + z_4$

Adding $z_2 + z_3$

NOTICE:

The diagonal for a parallelogram from z_4 and $z_2 + z_3$ is just a straight line whose midpoint is $(0, 0)$. i.e. Parallelogram cannot be drawn.



Trial questions:

1) Given that $z_1 = 3 + 4i$, $z_2 = -1 + 2i$ and $z_3 = -3i + 2$. Represent the complex numbers clearly on the Argand diagram.

a) $z_1 + z_2$

b) $z_2 + z_3$

c) $z_1 + z_2 + z_3$

2) Given that $z_1 = 4 + 3i$, $z_2 = -4 - 3i$, $z_3 = 5$ and $z_4 = 2i$. using Argand diagram find;

a) $z_1 + z_2$

b) $z_2 + z_3$

c) $z_1 + z_2 + z_3$

Note: $z_3 = 5 = 5 + 0i$, $z_4 = 2i = 0 + 2i$

SUBTRACTION OF COMPLEX NUMBERS

A. Analytical method

Given that $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$

Find: i) $z_1 - z_2$ ii) $z_1 - z_2 - z_3$

Solution

$$\begin{aligned} \text{i. } z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \quad (\text{collecting like terms}) \end{aligned}$$

$$\begin{aligned} \text{ii. } z_1 - z_2 - z_3 &= (x_1 + iy_1) - (x_2 + iy_2) - (x_3 + iy_3) \\ &= (x_1 - x_2 - x_3) + i(y_1 - y_2 - y_3) \quad (\text{collecting like terms}) \end{aligned}$$

Deduction

When subtracting complex numbers, subtract the real parts and imaginary parts separately, and write the result in complex number form.

Examples

1) Given that $z_1 = 3 + 2i$, $z_2 = 4 - 3i$ and $z_3 = 4 + 3i$

Find i) $z_1 + z_2$ ii) $z_1 + z_2 + z_3$

Solution

$$\begin{aligned} \text{i. } z_1 + z_2 &= 4 + i + -3 - i \\ &= (4 - 3) + i(1 - 1) = 1 + 0i = 1 \\ \text{ii. } z_1 + z_2 + z_3 &= 4 + i + -3 - i + 3 + 2i \\ &= (4 - 3 + 3) + i(1 - 1 + 2) = 3 + 2i \end{aligned}$$

2) Given that $z_1 = 3 + 2i$, $z_2 = 4 - 3i$ and $z_3 = 4 + 3i$

Find i) $z_1 - z_2$ ii) $z_1 - z_2 - z_3$

Solution

$$\begin{aligned} \text{i. } z_1 - z_2 &= (3 + 2i) - (4 - 3i) \\ &= (3 - 4) + i(2 - (-3)) \quad (\text{collecting like terms}) = -1 + 5i \\ \text{ii. } z_1 - z_2 - z_3 &= (3 + 2i) - (4 - 3i) - (4 + 3i) \\ &= (3 - 4 - 4) + i(2 - 3 - 3) = -5 + 2i \end{aligned}$$

Task

Given that $z_1 = 4 + 3i$, $z_2 = 3 - i$ and $z_3 = -4 - 3i$. Find:

a) $z_1 - z_2$ **Answer: $1 + 4i$**

b) $z_2 - z_3$ **Answer: $7 + 2i$**

c) $z_1 - z_2 - z_3$ **Answer: $5 + 7i$**

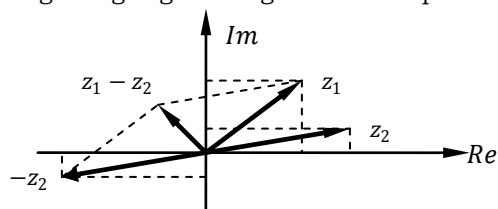
Graphical method

This is done using the Argand diagram just like in addition.

Consider two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. To find $z_1 - z_2$, proceed as follows.

$$z_1 - z_2 = z_1 + (-z_2) \quad [\text{Note this}]$$

Adding using Argand diagram the complex numbers z_1 and $-z_2$

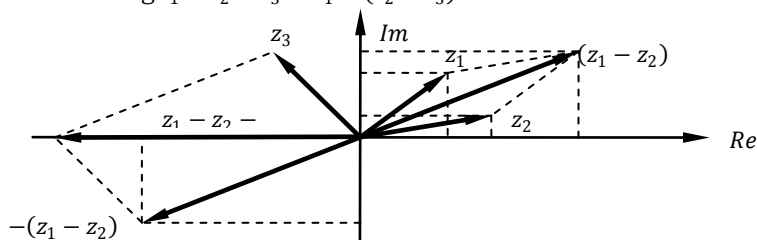


If $z_3 = x_3 + iy_3$, find $z_1 - z_2 - z_3$ if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Solution

$$z_1 - z_2 - z_3 = z_1 - (z_2 + z_3) \text{ or } z_1 - z_2 - z_3 = (z_1 - z_2) - z_3$$

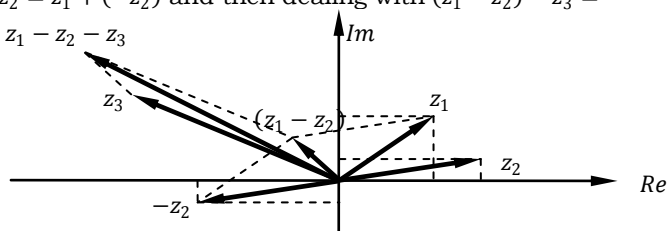
$$\text{Considering } z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$$



$$\text{Considering second alternative } z_1 - z_2 - z_3 = (z_1 - z_2) - z_3$$

$$\text{Dealing with } z_1 - z_2 = z_1 + (-z_2) \text{ and then dealing with } (z_1 - z_2) - z_3 =$$

$$(z_1 - z_2) + (-z_3) \quad z_1 - z_2 - z_3$$



Note:

Two complex numbers are subtracted by addition of the negative of the second complex number being subtracted from the first. i.e. $z_1 - z_2 = z_1 + (-z_2)$. A parallelogram diagonal gives the result, and the answer must be expressed in complex number form.

Example

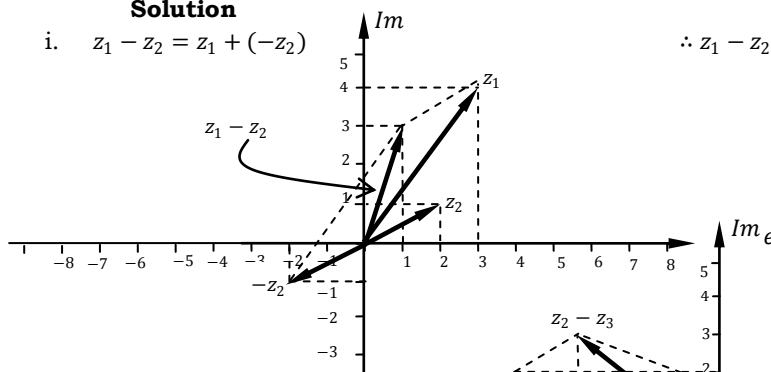
Using the Argand diagram simplify given that $z_1 = 3 + 4i$, $z_2 = 2 + i$ and $z_3 = 6 - 2i$

i) $z_1 - z_2$ ii) $z_2 - z_3$ iii) $z_1 - z_2 - z_3$

Solution

i. $z_1 - z_2 = z_1 + (-z_2)$

$$\therefore z_1 - z_2 = 1 + 3i$$

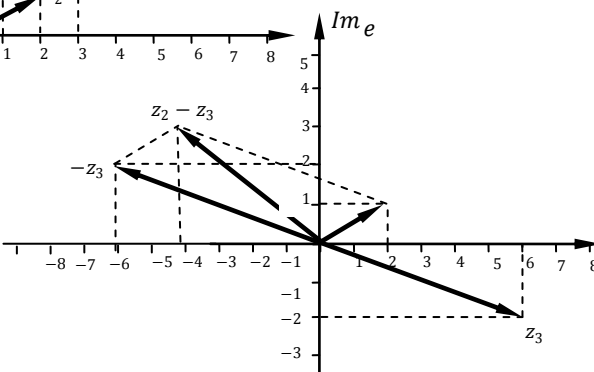


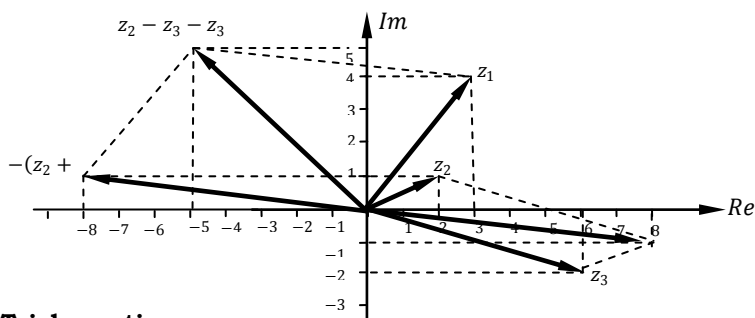
ii. $z_2 - z_3 = z_2 + (-z_3)$

$$\therefore z_2 - z_3 = -4 - 3i$$

iii. $z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$

$$\therefore z_1 - z_2 - z_3 = -5 + 5i$$




Trial questions:

Given that $z_1 = 3 - 2i$, $z_2 = 4$ and $z_3 = 2i$. Using the Argand diagram find;

- i) $z_1 - z_2$ ii) $z_2 - z_3$ iii) $z_1 + z_2 - z_3$ iv) $z_1 - z_2 - z_3$

Answers: i) $-1 - 2i$ (ii) $4 - 2i$ (iii) $7 - 4i$ (iv) $-1 - 4i$

MULTIPLICATION OF COMPLEX NUMBERS
A. Analytical method

Given that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

To find $z_1 \cdot z_2$ proceed as follows:

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1 \cdot x_2 + ix_1 \cdot y_2 + iy_1 \cdot x_2 + i^2 y_1 \cdot y_2 \\ &= (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + y_1 \cdot x_2) \quad [\because i^2 = -1] \end{aligned}$$

Deduction

When multiplying complex numbers, open the brackets just like in expansion of binomials, simplify the expressions and write the result in complex number form.

Examples

Given that $z_1 = 3 + 2i$, $z_2 = -1 + 2i$, $z_3 = 4 - 3i$ and $z_4 = -3 - 4i$. Find:

- i) $z_1 \cdot z_2$ ii) $z_2 \cdot z_3$ iii) $z_3 \cdot z_4$ iv) $z_1 \cdot z_2 \cdot z_3$ v) $z_2 \cdot z_3 \cdot z_4$

Solution

- i. $z_1 \cdot z_2 = (3 + 2i) \cdot (-1 + 2i)$
 $= 3(-1 + 2i) + 2i(-1 + 2i)$
 $= -3 + 6i - 2i + 4i^2 = -3 - 4 + 4i \quad [\because i^2 = -1]$
 $= -7 + 4i$
- ii. $z_2 \cdot z_3 = (-1 + 2i) \cdot (4 - 3i)$
 $= -1(4 - 3i) + 2i(4 - 3i)$
 $= -4 + 3i + 8i - 6i^2 = -4 + 6 + 11i \quad [\because i^2 = -1]$
 $= 2 + 11i$
- iii. $z_3 \cdot z_4 = (4 - 3i) \cdot (-3 - 4i)$
 $= 4(-3 - 4i) - 3i(-3 - 4i)$
 $= -12 - 16i + 9i + 12i^2 = -12 - 12 - 7i \quad [\because i^2 = -1]$
 $= -24 - 7i$
- iv. $z_1 \cdot z_2 \cdot z_3 = [z_1 \cdot z_2] \cdot z_3$. But from (i) above, $z_1 \cdot z_2 = -7 + 4i$
 $= [-7 + 4i] \cdot (4 - 3i)$
 $= -7(4 - 3i) + 4i(4 - 3i)$
 $= -28 + 21i + 16i - 12i^2$
 $= -28 + 12 + 37i$
 $= -16 + 37i$

$$\begin{aligned}
 \text{v. } z_2 \cdot z_3 \cdot z_4 &= [z_2 \cdot z_3] \cdot z_4. \text{ But } z_2 \cdot z_3 = 2 + 11i \\
 &= [2 + 11i] \cdot (-3 - 4i) \\
 &= 2(-3 - 4i) + 11i(-3 - 4i) \\
 &= -6 - 8i - 33i - 44i^2 \\
 &= -6 + 44 - 41i \\
 &= 38 - 41i
 \end{aligned}$$

Note:

Graphical multiplication of complex numbers requires Demoivre's theorem which has not been tackled. Graphical multiplication of complex numbers shall be handled later.

Trial questions

1. Given that $z_1 = 2 + i$, $z_2 = -2 + i$ and $z_3 = 3 - 2i$. Find:
 - i) $z_1 \cdot z_2$
 - ii) $z_2 \cdot z_3$
 - iii) $z_3 \cdot z_4$
 - iv) $z_1 \cdot z_2 \cdot z_3$
2. Given that $z_1 = 1 + i\sqrt{3}$, $z_2 = \sqrt{3} + i$ and $z_3 = 1 + \sqrt{2}i$. Find:
 - i) $z_1 \cdot z_2$
 - ii) $z_2 \cdot z_3$
 - iii) $z_3 \cdot z_4$
 - iv) $z_1 \cdot z_2 \cdot z_3$

Answers:

DIVISION OF COMPLEX NUMBERS

Before we look at this, let us first look at a conjugate of a complex number.

Conjugate of a complex number

If $z = a + bi$, then conjugate of z is $\bar{z} = a - ib$

If $z = a - bi$, then conjugate of z is $\bar{z} = a + ib$

Example

Find $\bar{z}$ if i) $z = 2 + 3i$
 ii) $z = -3 - 2i$

Solution

- i) $z = 2 + 3i$; $\bar{z} = 2 - 3i$
- ii) $z = -3 - 2i$; $\bar{z} = -3 + 2i$

Thus when two complex numbers are to be divided, multiply the numerator and denominator of the quotient being divided by the conjugate of the denominator. In simple terms *rationalize* the denominator.

Consider $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\begin{aligned}
 \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \dots\dots\dots \times \frac{(x_2 - iy_2)}{(x_2 - iy_2)} \\
 &= \frac{(x_1 + iy_1) \cdot (x_2 - iy_2)}{(x_2 + iy_2) \cdot (x_2 - iy_2)} \\
 &= \frac{x_1 \cdot x_2 + ix_1 \cdot y_2 + iy_1 \cdot x_2 + i^2 y_1 \cdot y_2}{x_1 \cdot x_2 - ix_1 \cdot y_2 + iy_1 \cdot x_2 - i^2 y_1 \cdot y_2} \\
 &= \frac{(x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + y_1 \cdot x_2)}{x_2^2 + y_2^2}
 \end{aligned}$$

Note:

$\frac{z_1}{z_2}$, has now a *real* denominator

Example

1. Given that $z_1 = 2 + 3i$, $z_2 = 4 - 3i$ and $z_3 = 1 + i$
 Find i) $\frac{z_1}{z_2}$ ii) $\frac{z_2}{z_3}$

Solution

$$\text{i. } \frac{z_1}{z_2} = \frac{(2+3i)}{(4-3i)} \dots\dots\dots \times \frac{(4+3i)}{(4+3i)}$$

$$\begin{aligned}
 &= \frac{(2+3i)(4+3i)}{(4-3i)(4+3i)} = \frac{8+6i+12i+9i^2}{16+12i-12i-9i^2} \\
 &= \frac{8-9+18i}{16+9} = \frac{-1+18i}{25} \\
 \therefore \frac{z_1}{z_2} &= -\frac{1}{25} + \frac{18}{25}i \\
 \text{ii. } \frac{z_2}{z_3} &= \frac{(4-3i)}{(1+i)} \dots \times \frac{(1-i)}{(1-i)} \\
 &= \frac{(4-3i)(1-i)}{(1+i)(1-i)} = \frac{4-4i-3i+3i^2}{1+i-i-i^2} \\
 &= \frac{4-3-7i}{1+1} = \frac{1-7i}{2} \\
 \therefore \frac{z_2}{z_3} &= \frac{1}{2} - \frac{7}{2}i \\
 2. \text{ Given that } z_1 &= 1+i\sqrt{3}, z_2 = 1-2\sqrt{3}i \text{ and } z_3 = -1+\sqrt{3}i. \text{ Find:} \\
 \text{i) } \frac{z_1}{z_2} \quad \text{ii) } \frac{z_2}{z_3} \quad \text{iii) } \frac{z_3}{z_1 z_2} \quad \text{iv) } \frac{z_2 z_3}{z_1}
 \end{aligned}$$

Solution

$$\begin{aligned}
 \text{i. } \frac{z_1}{z_2} &= \frac{(1+i\sqrt{3})}{(1-2\sqrt{3}i)} \dots \times \frac{(1+2\sqrt{3}i)}{(1+2\sqrt{3}i)} \\
 &= \frac{(1+i\sqrt{3})(1+2\sqrt{3}i)}{(1-2\sqrt{3}i)(1+2\sqrt{3}i)} = \frac{1+2\sqrt{3}i+\sqrt{3}i+2\sqrt{9}i^2}{1+2\sqrt{3}i-2\sqrt{3}i-4\sqrt{9}i^2} \\
 &= \frac{1+3\sqrt{3}i-2(3)}{1+4(3)} = \frac{1-6+3\sqrt{3}i}{1+12} \\
 \therefore \frac{z_1}{z_2} &= -\frac{5}{13} + \frac{3\sqrt{3}}{13}i \\
 \text{ii. } \frac{z_2}{z_3} &= \frac{(1-2\sqrt{3}i)}{(-1+\sqrt{3}i)} \dots \times \frac{(-1-\sqrt{3}i)}{(-1-\sqrt{3}i)} \\
 &= \frac{(1-2\sqrt{3}i)(-1-\sqrt{3}i)}{(-1+\sqrt{3}i)(-1-\sqrt{3}i)} = \frac{-1-\sqrt{3}i+2\sqrt{3}i+2\sqrt{9}i^2}{1+\sqrt{3}i-\sqrt{3}i-\sqrt{9}i^2} \\
 &= \frac{-1-2(3)i+\sqrt{3}i}{1+3} = \frac{-7+\sqrt{3}i}{4} \\
 \therefore \frac{z_2}{z_3} &= -\frac{7}{4} + \frac{\sqrt{3}}{4}i \\
 \text{iii. } \frac{z_3}{z_1 z_2} &= \frac{(-1+\sqrt{3}i)}{(1+i\sqrt{3})(1-2\sqrt{3}i)} = \frac{(-1+\sqrt{3}i)}{1+i\sqrt{3}-2\sqrt{3}i-2\sqrt{9}i^2} \\
 &= \frac{(-1+\sqrt{3}i)}{(7-\sqrt{3}i)} \text{ .Now rationalizing the denominator} \\
 &= \frac{(-1+\sqrt{3}i)}{(7-\sqrt{3}i)} \dots \times \frac{(7+\sqrt{3}i)}{(7+\sqrt{3}i)} \\
 &= \frac{(-1+\sqrt{3}i)(7+\sqrt{3}i)}{(7-\sqrt{3}i)(7+\sqrt{3}i)} = \frac{-7-\sqrt{3}i+7\sqrt{3}i+\sqrt{9}i^2}{49+7\sqrt{3}i-7\sqrt{3}i-\sqrt{9}i^2} \\
 &= \frac{-7-3+6\sqrt{3}i}{49+3} = \frac{-10+6\sqrt{3}i}{52} \\
 &= -\frac{10}{52} + \frac{6\sqrt{3}}{52}i = -\frac{5}{26} + \frac{3\sqrt{3}}{26}i \\
 \therefore \frac{z_3}{z_1 z_2} &= -\frac{5}{26} + \frac{3\sqrt{3}}{26}i \\
 \text{vi. } \frac{z_2 z_3}{z_1} &= \frac{(1-2\sqrt{3}i)(-1+\sqrt{3}i)}{(1+i\sqrt{3})} = \frac{-1+\sqrt{3}i+2\sqrt{3}i-2\sqrt{9}i^2}{(1+i\sqrt{3})} \\
 &= \frac{-1+2(3)+3\sqrt{3}i}{(1+i\sqrt{3})} = \frac{5+3\sqrt{3}i}{(1+i\sqrt{3})} \\
 \text{Now rationalizing the denominator} \\
 \frac{z_2 z_3}{z_1} &= \frac{(5+3\sqrt{3}i)(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{5-5\sqrt{3}i+3\sqrt{3}i-3\sqrt{9}i^2}{1-i\sqrt{3}+i\sqrt{3}-\sqrt{9}i^2} \\
 &= \frac{5+3(3)-i2\sqrt{3}}{1+3} = \frac{14-i2\sqrt{3}}{4}
 \end{aligned}$$

$$= \frac{14}{4} - \frac{2\sqrt{3}}{4}i = \frac{7}{2} - \frac{\sqrt{3}}{2}i \quad \therefore \frac{z_2 z_3}{z_1} = \frac{7}{2} - \frac{\sqrt{3}}{2}i$$

3. Given that $z_1 = 1 + i$, $z_2 = 2 - i$ and $z_3 = 3 + 2i$. Find:

i) $\frac{z_1+z_2}{z_3}$ ii) $\frac{z_1^2}{z_2^2 \cdot z_3}$ iii) $\frac{z_2+z_3}{z_1+z_2}$ iv) $\frac{z_1+z_2+z_3}{z_2-z_3}$ v) $\frac{z_1^3 \cdot z_2}{z_1+z_3}$

Solution

i. $\frac{z_1+z_2}{z_3}$; $z_1 + z_2 = 1 + i + 2 - i = 3$

$\therefore \frac{z_1+z_2}{z_3} = \frac{3}{3+2i}$; rationalizing the denominator

$$\frac{z_1+z_2}{z_3} = \frac{3}{3+2i} = \frac{3(3-2i)}{(3+2i)(3-2i)} \\ = \frac{9-6i}{9-4i^2} = \frac{9-6i}{9+4} \quad \therefore \frac{z_1+z_2}{z_3} = \frac{9}{13} - \frac{6}{13}i$$

ii. $\frac{z_1^2}{z_2^2 \cdot z_3}$;

$z_1^2 = z_1 \cdot z_1 = (1+i)(1+i) = 1+i+i+i^2 = 2i$

$z_2^2 = z_2 \cdot z_2 = (2-i)(2-i) = 4-2i-2i+i^2 = 5-4i$

$\therefore z_2^2 \cdot z_3 = (5-4i)(3+2i) = 15+10i-12i-8i^2 = 23-2i$

$$\therefore \frac{z_1^2}{z_2^2 \cdot z_3} = \frac{2i}{(23-2i)} = \frac{2i(23+2i)}{(23-2i)(23+2i)} \\ = \frac{46i+4i^2}{529+46i-46i-4i^2} = \frac{-4+46i}{529+4} = \frac{-4+46i}{533}$$

$\therefore \frac{z_1^2}{z_2^2 \cdot z_3} = -\frac{4}{533} + \frac{46}{533}i$

iii. $\frac{z_2+z_3}{z_1+z_2} = \frac{2-i+3+2i}{1+i+2-i} = \frac{5+i}{3} \quad \therefore \frac{z_2+z_3}{z_1+z_2} = \frac{5}{3} + \frac{1}{3}i$

iv. $\frac{z_1+z_2+z_3}{z_2-z_3} = \frac{1+i+2-i+3+2i}{2-i-(3+2i)} = \frac{6+2i}{-1-3i}$; rationalizing denominator

$$= \frac{(6+2i)(-1+3i)}{(-1-3i)(-1+3i)} = \frac{-6+18i-2i+6i^2}{1-3i+3i-9i^2} = \frac{-12+16i}{1+9} = -\frac{12}{10} + \frac{16}{10}i$$

$\therefore \frac{z_1+z_2+z_3}{z_2-z_3} = -\frac{6}{5} + \frac{8}{5}i$

v. $\frac{z_1^3 \cdot z_2}{z_1+z_3}$; $z_1^3 = z_1^2 \cdot z_1 = 2i(1+i) = -2+2i$

$\therefore z_1^3 \cdot z_2 = (-2+2i)(2-i) = -4+2i-4i+2i^2 = -2-2i$

$z_1 + z_3 = 1+i+3+2i = 4+3i$

$$\therefore \frac{z_1^3 \cdot z_2}{z_1+z_3} = \frac{(-2-2i)}{(4+3i)} = \frac{(-2-2i)(4-3i)}{(4+3i)(4-3i)} \\ = \frac{-8+6i-8i+6i^2}{16-12i-12i-9i^2} = \frac{-14-2i}{16+9} = -\frac{14}{25} - \frac{2}{9}i$$

$\therefore \frac{z_1^3 \cdot z_2}{z_1+z_3} = -\frac{14}{25} - \frac{2}{9}i$

Trial questions

1) Given that $z = \sqrt{3} + i$; find i) z^2 ii) $\frac{1}{z}$

2) Given that $z = 3 + 4i$, find the value of the expression $z + \frac{25}{z}$

[Hint: 1st find $\frac{25}{z}$ and add the result to z]

3) The total impedance z in an electric circuit with branches z_1 and z_2 is given by: $\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. Given that $z_1 = 3 + i4$ and $z_2 = 5 + i5$, where $i = \sqrt{-1}$, calculate the total impedance in z in the form $a + ib$.

[Hint: $\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} = \frac{z_1+z_2}{z_1 \cdot z_2}$. Now proceed to find $z_1 + z_2$ and $z_1 \cdot z_2$ then z .
Or find $\frac{1}{z_1}$ and $\frac{1}{z_2}$ by rationalisation and then z]

- 4) Given that $z_1 = 3 - i$, $z_2 = 3 + i$. find
- a) $\frac{z_1^2 \cdot z_2^3}{z_1 + z_2}$ b) $\frac{z_2}{z_1}$ c) $\frac{z_2 \cdot z_1}{z_2 - z_1}$

Note:

Graphical division of complex numbers requires Demoivre's theorem. This shall be tackled later.

DIFFERENT FORMS OF COMPLEX NUMBERS

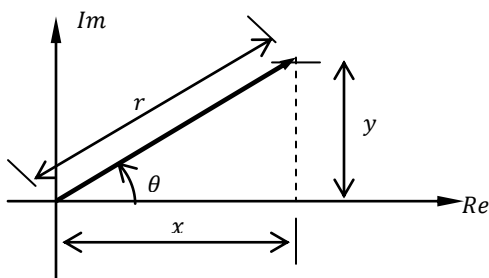
A complex number can mainly be expressed in three forms:

- Standard form .i.e. $z = x + iy$
- Polar form .i.e. $z = r(\cos\theta + i\sin\theta)$
- Exponential form .i.e. $z = re^{i\theta}$

However the most commonly examined at A level are: standard and polar form. Before we look at these in detail, let us first look at modulus and argument of a complex number.

MODULUS AND ARGUMENT OF A COMPLEX NUMBER

Consider the complex number $z = x + iy$ in standard form. Representing z on the Argand diagram



Argument, θ

Argument of a complex number is the angle the complex number makes with *positive Real axis*

Denotion

Argument of z is denoted as $\arg(z)$

From the triangle above,

$$\tan\theta = \frac{y}{x} \therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \arg(z) = \arg(x + iy) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

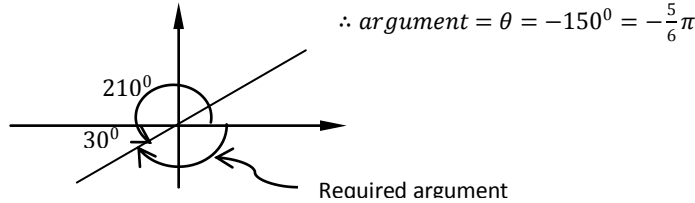
Note:

- When finding $\tan^{-1}\left(\frac{y}{x}\right)$, there is no i on y from $z = x + iy$ which is the commonest mistake students make .i.e. $\tan^{-1}\left(\frac{iy}{x}\right)$, this is not correct.
- The range of θ is $-180^\circ < \theta \leq 180^\circ$ or $-\pi < \theta \leq \pi$ in radians .i.e. *principal argument*
- When θ is being measured, it is done with respect to the *+ve real axis* and θ is *+ve* if measurement is in *anticlockwise direction* where as *-ve* if measured in a *clockwise direction*

- iv. *Measurement of $\theta = \arg(z)$* requires the knowledge of trigonometry about tangents of angles in the four quadrants.

If θ is not in the interval $-\pi < \pi \leq \pi$ then expressing the angle in terms of acute angles must be done to bring the angle in the required interval.

e.g. If $\theta = 210^\circ$ then $\tan 210^\circ = +\tan 30^\circ$



Modulus / Magnitude / length of a complex number

From the right angled triangle above, applying Pythagoras Theorem,

$$x^2 + y^2 = r^2 \therefore r = \sqrt{x^2 + y^2}$$

Denotion

Modulus of z is denoted as $|z| = r$

$$\therefore |z| = r = \sqrt{x^2 + y^2}$$

Note:

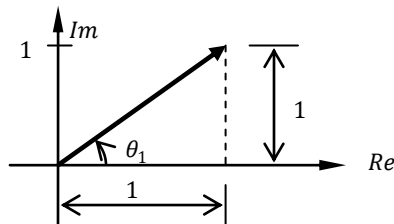
In the formula, there is no use of i on y . i.e. $\therefore |z| = r = \sqrt{x^2 + (iy)^2}$

Examples

1. Given that $z_1 = 1 + i$, $z_2 = -\sqrt{3} + i$, $z_3 = -5 - 12i$ and $z_4 = \sqrt{3} - i$
Find i) $\arg(z_1)$ ii) $\arg(z_2)$ iii) $\arg(z_3)$ iv) $\arg(z_4)$
v) $|z_1|$ vi) $|z_2|$ vii) $|z_3|$ viii) $|z_4|$

Solution

- i. $z_1 = 1 + i$



$$\text{From } |z| = \sqrt{x^2 + y^2}$$

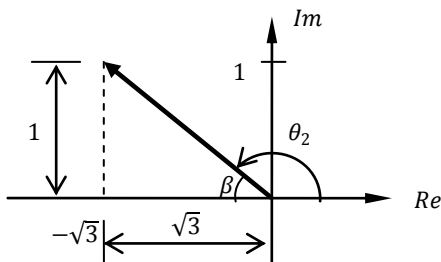
$$\therefore |z_1| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \text{ units}$$

$$\text{From } \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \arg(z_1) = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ = \frac{\pi}{4}$$

$$\therefore |z_1| = \sqrt{2} \text{ units}, \arg(z_1) = 45^\circ = \frac{\pi}{4}$$

- ii. $z_2 = -\sqrt{3} + i$



$$\text{From } |z| = \sqrt{x^2 + y^2}$$

$$\therefore |z_2| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2 \text{ units}$$

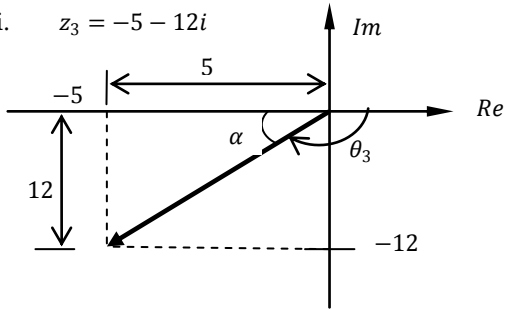
$$\arg(z_2) = 180^\circ - \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{From } \Delta, \beta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

$$\therefore \arg(z_2) = \theta_2 = 180^\circ - 30^\circ = 150^\circ = \frac{5\pi}{6}$$

$$\therefore |z_2| = 2 \text{ units}, \arg(z_2) = 150^\circ = \frac{5\pi}{6}$$

iii. $z_3 = -5 - 12i$



From $|z| = \sqrt{x^2 + y^2}$

$$\therefore |z_3| = \sqrt{(-5)^2 + (-12)^2} = 13 \text{ units}$$

$$\arg(z_3) = \theta_3 = -(180^\circ - \alpha)$$

From Δ , $\alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.4^\circ$

$$\therefore \arg(z_3) = \theta_3 = -(180^\circ - 67.4^\circ) = -112.6^\circ$$

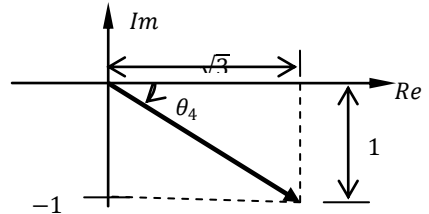
$$\therefore |z_3| = 13 \text{ units}, \arg(z_3) = -112.6^\circ$$

iv. $z_4 = \sqrt{3} - i$, From $|z| = \sqrt{x^2 + y^2}$

$$\therefore |z_4| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2 \text{ units}$$

$$\arg(z_4) = \theta_4 = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ = -\frac{\pi}{6}$$

$$\therefore |z_4| = 2 \text{ units}, \arg(z_4) = 30^\circ = -\frac{\pi}{6}$$



2. Given that $z_1 = 1 + i$, $z_2 = -1 - i$, $z_3 = 2 + 3i$. Find the modulus and argument of:

i) $z_1 \cdot z_2$ ii) $\frac{z_2 \cdot z_3}{z_1}$ iii) $\frac{z_1^2}{z_2 \cdot z_3}$ iv) $\frac{z_2^3}{z_1 \cdot z_3}$ v) z_1^2

Solution

i. $z_1 \cdot z_2 = (1 + i) \cdot (-1 - i) = 1 - i + i - i^2$
 $= 2 + 0i$

$$\therefore |z_1 \cdot z_2| = \sqrt{(2)^2 + 0^2} = 2 \text{ units}$$

$$\arg(z_1 \cdot z_2) = \tan^{-1}\left(\frac{0}{2}\right) = 0^\circ$$

$$\therefore |z_1 \cdot z_2| = 2 \text{ units}; \arg(z_1 \cdot z_2) = 0^\circ$$

ii. $\frac{z_2 \cdot z_3}{z_1} = \frac{(1-i) \cdot (2+3i)}{1+i} = \frac{2+3i-2i-i^2 \cdot 3}{1+i}$
 $= \frac{2+3+i}{1+i} = \frac{5+i}{1+i}$, rationalizing denominator

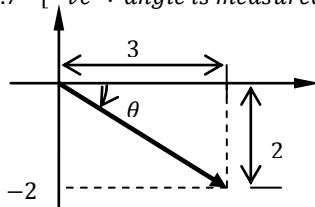
$$\frac{(5+i) \cdot (1-i)}{(1+i) \cdot (1-i)} = \frac{5-5i+i-i^2}{1-i^2} = \frac{5+1-4i}{1+1} = \frac{6-4i}{2} = 3-2i$$

$$\therefore \frac{z_2 \cdot z_3}{z_1} = 3-2i$$

$$\Rightarrow \left| \frac{z_2 \cdot z_3}{z_1} \right| = \sqrt{3^2 + 2^2} = \sqrt{13} \text{ units}$$

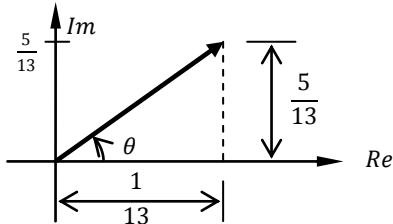
From the figure, $\tan \theta = \frac{2}{3} \therefore \theta = 33.7^\circ$

$$\therefore \arg\left(\frac{z_2 \cdot z_3}{z_1}\right) = -33.7^\circ \text{ [-ve } \because \text{ angle is measured in clockwise direction]}$$



iii. $\frac{z_1^2}{z_2 \cdot z_3} = \frac{(1+i)^2}{(1-i) \cdot (2+3i)} = \frac{1+i+i^2}{2+3i-2i-i^2 \cdot 3} = \frac{2i}{5+i}$
 $\frac{2i}{5+i} = \frac{2i \cdot (5-i)}{(5+i) \cdot (5-i)} = \frac{10i-2i^2}{25-i^2} = \frac{2+10i}{25+1} = \frac{2+10i}{26}$

$$\begin{aligned}\therefore \frac{z_1^2}{z_2 \cdot z_3} &= \frac{1}{13} + \frac{5}{13}i \\ \Rightarrow \left| \frac{z_1^2}{z_2 \cdot z_3} \right| &= \sqrt{\left(\frac{1}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = \sqrt{\left(\frac{1}{13}\right)^2 [1 + 25]} = \frac{\sqrt{26}}{13} \text{ units} \\ \arg\left(\frac{z_1^2}{z_2 \cdot z_3}\right) &= \tan^{-1}\left(\frac{5/13}{1/13}\right) = \tan^{-1}(5) = 78.7^\circ \\ \therefore \left| \frac{z_1^2}{z_2 \cdot z_3} \right| &= \frac{\sqrt{26}}{13} \text{ units}; \arg\left(\frac{z_1^2}{z_2 \cdot z_3}\right) = 78.7^\circ\end{aligned}$$

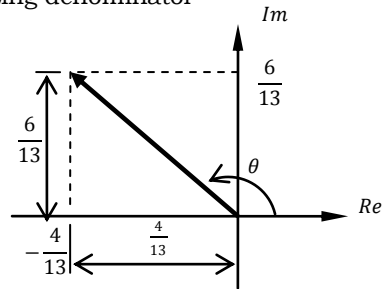


$$\begin{aligned}\text{iv. } \frac{z_2^3}{z_1 \cdot z_3} &= \frac{(1-i)^3}{(1+i)(2+3i)} = \frac{(1-i)^2 \cdot (1-i)}{2+3i+2i+3i^2} = \frac{(1-2i+i^2) \cdot (1-i)}{2-3+5i} \\ &= \frac{(-2i) \cdot (1-i)}{(-1+5i)} = \frac{(-2+2i^2)}{(-1+5i)} = \frac{(-2-2i)}{(-1+5i)}; \text{rationalizing denominator}\end{aligned}$$

$$\begin{aligned}\frac{(-2-2i)}{(-1+5i)} &= \frac{(-2-2i) \cdot (-1-5i)}{(-1+5i) \cdot (-1-5i)} = \frac{2+10i+2i+10i^2}{1+5i-5i-25i^2} \\ \therefore \frac{z_2^3}{z_1 \cdot z_3} &= \frac{2-10+12i}{1+25} = \frac{-8+12i}{26} = -\frac{4}{13} + \frac{6}{13}i \\ \therefore \left| \frac{z_2^3}{z_1 \cdot z_3} \right| &= \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{6}{13}\right)^2} = \sqrt{\left(\frac{1}{13}\right)^2 [16 + 36]} \\ &= \frac{\sqrt{52}}{13} = \frac{\sqrt{4 \times 13}}{13} = \frac{2}{\sqrt{13}} \text{ units}\end{aligned}$$

$$\begin{aligned}\arg\left(\frac{z_2^3}{z_1 \cdot z_3}\right) &= 180^\circ - \tan^{-1}\left(\frac{6/13}{4/13}\right) \\ &= 180^\circ - \tan^{-1}\left(\frac{6}{4}\right) = 180^\circ - 56.3^\circ = 123.7^\circ\end{aligned}$$

$$\therefore \left| \frac{z_2^3}{z_1 \cdot z_3} \right| = \frac{2}{\sqrt{13}} \text{ units}; \arg\left(\frac{z_2^3}{z_1 \cdot z_3}\right) = 123.7^\circ$$



STANDARD FORM OF A COMPLEX NUMBER

A complex number of the form $z = x + iy$ where $x, y \in \mathbb{R}$ is said to be in standard form.

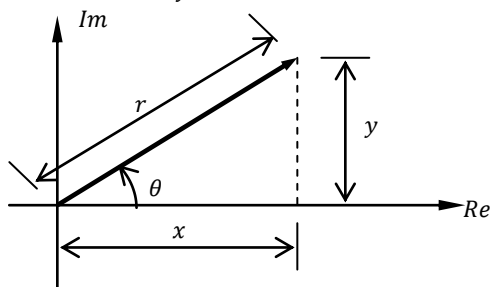
e.g. $z = 2 + 3i, z = 4 - 3i$

POLAR/MODULUS-ARGUMENT FORM OF A COMPLEX NUMBER

This originates from the standard form. A complex number in the form $z = r(\cos\theta + i\sin\theta)$ is in polar form.

Derivation

Consider $z = x + iy$



$$|z| = \sqrt{x^2 + y^2}$$

$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{Now } \cos\theta = \frac{x}{r} \therefore x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \therefore y = r\sin\theta$$

$$\text{From } z = x + iy$$

$$\Rightarrow z = r\cos\theta + r\sin\theta$$

$$\therefore z = r(\cos\theta + i\sin\theta)$$

Note:

Expressing a complex number in polar form requires the knowledge of finding modulus and argument of a complex number.

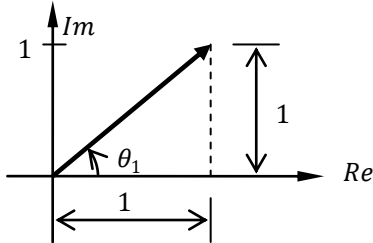
Examples

- 1) Given that $z_1 = 1 + i$, $z_2 = -3i$, $z_3 = -\sqrt{3} - i$ and $z_4 = \sqrt{3} - i$. Express the complex numbers in polar form.

Solution

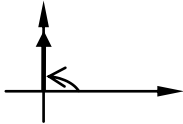
$$z_1 = 1 + i; |z_1| = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ units}; \arg(z_1) = \tan^{-1}(1) = 45^\circ$$

$$\therefore z_1 = \sqrt{2}(\cos(45^\circ) + i\sin(45^\circ))$$



$$z_2 = -3i = 0 + 3i; |z_2| = \sqrt{0^2 + 3^2} = 3 \text{ units}; \arg(z_2) = 90^\circ$$

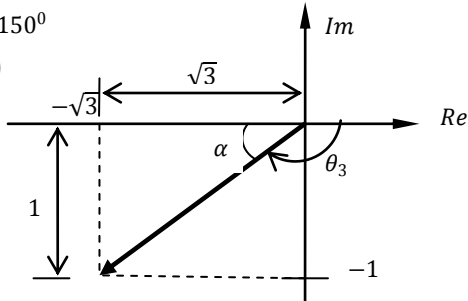
$$\therefore z_2 = 3(\cos(90^\circ) + i\sin(90^\circ))$$



$$z_3 = -\sqrt{3} - i; |z_3| = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \text{ units};$$

$$\arg(z_3) = -\left[180^\circ - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = -150^\circ$$

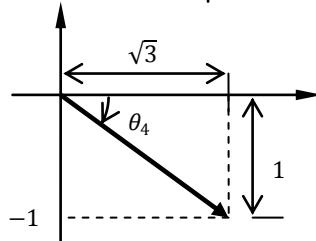
$$\therefore z_3 = 2(\cos(-150^\circ) + i\sin(-150^\circ))$$



$$z_4 = \sqrt{3} - i; |z_4| = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \text{ units};$$

$$\arg(z_4) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^\circ$$

$$\therefore z_4 = 2(\cos(-30^\circ) + i\sin(-30^\circ))$$



- 2) Given that $z_1 = 2 + 3i$, $z_2 = 1 + i$, and $z_3 = 1 - i$. Express the complex numbers in modulus-argument form

i) $\frac{z_1 \cdot z_2}{z_3}$

ii) $\frac{z_2 \cdot z_3}{z_1}$

iii) $z_2 + \frac{z_1 \cdot z_2}{z_3}$

iv) $\frac{z_1 + z_2}{z_2 \cdot z_3}$

v) $\frac{z_1 + z_2}{z_2 - z_3}$

Solution

i. $\frac{z_1 \cdot z_2}{z_3} = \frac{(2+3i) \cdot (1+i)}{1-i} = \frac{2+2i+3i+3i^2}{1-i} = \frac{2-3+5i}{1-i}$

$$\frac{z_1 \cdot z_2}{z_3} = \frac{-1+5i}{1-i}; \text{rationalizing denominator}$$

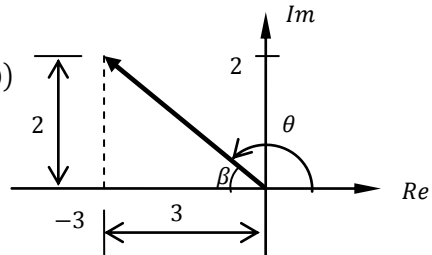
$$\frac{z_1 \cdot z_2}{z_3} = \frac{(-1+5i)(1+i)}{(1-i)(1+i)} = \frac{-1-i+5i+5i^2}{(1)^2-(i)^2} \left[\begin{array}{l} \because (a+b)(a-b) = a^2 - b^2 \\ \text{in the denominator} \end{array} \right]$$

$$= \frac{-1-5+6i}{1+1} = \frac{-6+4i}{2} = -3 + 2i$$

$$\therefore \left| \frac{z_1 \cdot z_2}{z_3} \right| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\arg \left(\frac{z_1 \cdot z_2}{z_3} \right) = 180^\circ - \tan^{-1} \left(\frac{2}{3} \right) = 146.3^\circ$$

$$\therefore \frac{z_1 \cdot z_2}{z_3} = \sqrt{13}(\cos(146.3^\circ) + i\sin(146.3^\circ))$$



ii. $\frac{z_2 \cdot z_3}{z_1} = \frac{(1+i)(1-i)}{2+3i} = \frac{1^2-i^2}{2+3i} = \frac{1+1}{2+3i} = \frac{2}{2+3i}$

$\frac{z_2 \cdot z_3}{z_1} = \frac{2}{(2+3i)}$; rationalizing denominator

$$\frac{z_2 \cdot z_3}{z_1} = \frac{2(2-3i)}{(2+3i)(2-3i)} = \frac{4-6i}{(2)^2-(3i)^2} \left[\begin{array}{l} \because (a+b)(a-b) = a^2 - b^2 \\ \text{in the denominator} \end{array} \right]$$

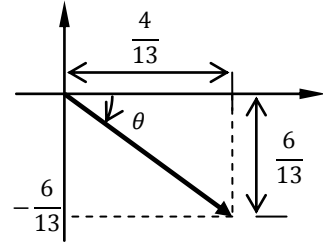
$$= \frac{4-6i}{4+9} = \frac{4}{13} - \frac{6}{13}i$$

$$\therefore \left| \frac{z_2 \cdot z_3}{z_1} \right| = \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{6}{13}\right)^2} = \frac{2}{\sqrt{13}} \text{ units}$$

$$\arg \left(\frac{z_2 \cdot z_3}{z_1} \right) = -\tan^{-1} \left(\frac{6/13}{4/13} \right) = -56.3^\circ$$

$$\therefore \frac{z_2 \cdot z_3}{z_1} = \frac{2}{\sqrt{13}}(\cos(-56.3^\circ) + i\sin(-56.3^\circ))$$

$$= \frac{2}{\sqrt{13}}(\cos(56.3^\circ) - i\sin(56.3^\circ))$$



iii. $z_2 + \frac{z_1 \cdot z_2}{z_3} = (1+i) + \frac{(2+3i)(1+i)}{1-i} = 1+i + \frac{2+2i+3i+3i^2}{1-i}$

$$= 1+i + \frac{2-3+5i}{1-i} = 1+i + \frac{-1+5i}{1-i}$$

$$= 1+i + \frac{(-1+5i)(1+i)}{(1-i)(1+i)} = 1+i + \frac{-1-i+5i+5i^2}{(1)^2-(i)^2}$$

$$= 1+i + \frac{-1-5+6i}{1+1} = 1+i + \frac{-6+4i}{2}$$

$$= 1+i -3+2i$$

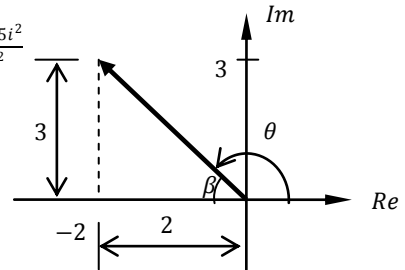
$$\therefore z_2 + \frac{z_1 \cdot z_2}{z_3} = -2 + 3i$$

$$\left| z_2 + \frac{z_1 \cdot z_2}{z_3} \right| = \sqrt{2^2 + 3^2} = \sqrt{13} \text{ units}$$

$$\arg \left(z_2 + \frac{z_1 \cdot z_2}{z_3} \right) = 180^\circ - \tan^{-1} \left(\frac{3}{2} \right)$$

$$= 123.7^\circ$$

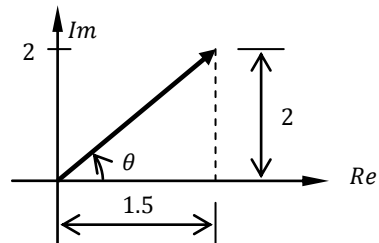
$$\therefore z_2 + \frac{z_1 \cdot z_2}{z_3} = \sqrt{13}(\cos(123.7^\circ) + i\sin(123.7^\circ))$$



iv. $\frac{z_1+z_2}{z_2 \cdot z_3} = \frac{2+3i+1+i}{(1+i)(1-i)} = \frac{3+4i}{(1)^2-(i)^2} = \frac{3+4i}{2} = \frac{3}{2} + 2i$

$$\left| \frac{z_1+z_2}{z_2 \cdot z_3} \right| = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \frac{5}{2} \text{ units}$$

$$\arg \left(\frac{z_1+z_2}{z_2 \cdot z_3} \right) = \tan^{-1} \left(\frac{2}{1.5} \right) = 53.1^\circ$$



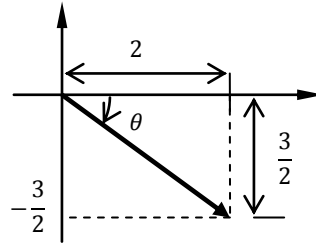
$$\therefore \frac{z_1+z_2}{z_2 \cdot z_3} = \frac{5}{2} (\cos(53.1^\circ) + i \sin(53.1^\circ))$$

$$\begin{aligned} \text{v. } \frac{z_1+z_2}{z_2-z_3} &= \frac{2+3i+1+i}{(1+i)-(1-i)} = \frac{3+4i}{2i} = \frac{3+4i}{0+2i} = \frac{(3+4i)(-2i)}{(2i)(-2i)} \\ &= \frac{-6i-8i^2}{-4i^2} = \frac{8-6i}{4} = 2 - \frac{3}{2}i \end{aligned}$$

$$\therefore \left| \frac{z_1+z_2}{z_2-z_3} \right| = \sqrt{2^2 + \left(\frac{3}{2}\right)^2} = \frac{5}{2} \text{ units}$$

$$\arg\left(\frac{z_1+z_2}{z_2-z_3}\right) = -\tan^{-1}\left(\frac{1.5}{2}\right) = -36.9^\circ$$

$$\begin{aligned} \therefore \frac{z_1+z_2}{z_2-z_3} &= \frac{5}{2} (\cos(-36.9^\circ) + i \sin(-36.9^\circ)) \\ &= \frac{5}{2} (\cos(36.9^\circ) - i \sin(36.9^\circ)) \end{aligned}$$



Trial questions

Given that $z_1 = 4 + 3i$, $z_2 = -4 - 3i$, and $z_3 = 1 - i$. Express the complex numbers in polar form.

$$\begin{array}{lllll} \text{i)} & z_1 \cdot z_2 & \text{ii)} & z_1 \cdot z_2 \cdot z_3 & \text{iii)} & z_3 + \frac{z_1 \cdot z_2}{z_3} & \text{iv)} & \frac{z_1 \cdot z_2^3}{z_3^2} & \text{v)} & \frac{z_1 \cdot z_3}{z_2} + \frac{z_2}{z_2 \cdot z_3} \end{array}$$

$$\text{ANSWER: i) } r = 25, \theta = -106.3^\circ \quad \text{ii) } r = 35.4, \theta = -151.3^\circ \quad \text{iii) } r = 19, \theta = -60.1^\circ \\ \text{iv) } r = 312.5, \theta = 57.5^\circ \quad \text{v) } r = 1.581, \theta = 108.4^\circ$$

MULTIPLICATION OF COMPLEX NUMBERS IN POLAR FORM

Consider two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ which can be expressed in the polar form. i.e.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \text{ respectively.}$$

To find $z_1 \cdot z_2$, proceed as follows;

$$\begin{aligned} z_1 \cdot z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)] \cdot [r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 \cdot r_2 [(\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 \cdot r_2 [\cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2 + i^2 \sin\theta_1 \sin\theta_2] \\ &= r_1 \cdot r_2 [(\cos\theta_1 \cos\theta_2 + i^2 \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)] \\ &= r_1 \cdot r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)] \\ &= r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

In general when multiplying complex numbers in polar form, first multiply their moduli, add their arguments and write the result in polar form. i.e.

$$\text{If } z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$z_n = r_n(\cos\theta_n + i\sin\theta_n)$$

$$\Rightarrow z_1 \cdot z_2 \dots z_n = r_1 \cdot r_2 \dots r_n [\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n)]$$

Deduction

When multiplying complex numbers in polar form,

$$\text{i. } \arg(z_1 \cdot z_2 \dots z_n) = \arg(z_1) + \arg(z_2) + \dots + \arg(z_n)$$

$$\text{ii. } |z_1 \cdot z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|$$

Note

The sense of measurement of the argument *must* be the same. *i.e.* clockwise/anticlockwise or just consider the *principal argument* of complex numbers.

Example

1. Given that $z_1 = 1 + i\sqrt{3}$, $z_2 = \sqrt{3} - i$, $z_3 = -1 - i$ and $z_4 = -1 + i$. Express the complex numbers in polar form. $z_1 \cdot z_2$, $z_3 \cdot z_4$ and $z_1 \cdot z_2 \cdot z_3$

Solution

$$z_1 = 1 + i\sqrt{3}; |z_1| = \sqrt{1^2 + (\sqrt{3})^2} = 2 \text{ units}$$

$$\arg(z_1) = \tan^{-1}(\sqrt{3}) = 60^\circ$$

$$\therefore z_1 = 2(\cos 60^\circ + i\sin 60^\circ)$$

$$z_2 = \sqrt{3} - i; |z_2| = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \text{ units}$$

$$\arg(z_2) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^\circ \text{ [principal argument]}$$

$$\therefore z_2 = 2(\cos(-30^\circ) + i\sin(-30^\circ))$$

$$z_3 = -1 - i; |z_3| = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ units}$$

$$\arg(z_3) = -\left[180^\circ - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = -135^\circ \text{ [principal argument]}$$

$$\therefore z_3 = \sqrt{2}(\cos(-135^\circ) + i\sin(-135^\circ))$$

$$z_4 = -1 + i; |z_4| = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ units}$$

$$\arg(z_4) = 180^\circ - \tan^{-1}(1) = 135^\circ \text{ [principal argument]}$$

$$\therefore z_4 = \sqrt{2}(\cos(135^\circ) + i\sin(135^\circ))$$

$$z_1 \cdot z_2 = [2(\cos 60^\circ + i\sin 60^\circ)] \cdot [2(\cos(-30^\circ) + i\sin(-30^\circ))]$$

$$= 4[\cos(60^\circ + -30^\circ) + i\sin(60^\circ + -30^\circ)]$$

$$= 4(\cos(30^\circ) + i\sin(30^\circ))$$

$$z_3 \cdot z_4 = [\sqrt{2}(\cos(-135^\circ) + i\sin(-135^\circ))] \cdot [\sqrt{2}(\cos(135^\circ) + i\sin(135^\circ))]$$

$$= 2[\cos(-135^\circ + 135^\circ) + i\sin(-135^\circ + 135^\circ)]$$

$$= 2[\cos 0^\circ + i\sin 0^\circ]$$

$$z_1 \cdot z_2 \cdot z_3 = [2(\cos 60^\circ + i\sin 60^\circ)] \cdot [2(\cos(-30^\circ) + i\sin(-30^\circ))] \cdot [\sqrt{2}(\cos(-135^\circ) + i\sin(-135^\circ))]$$

$$= 4\sqrt{2}[\cos(60^\circ + -30^\circ + -135^\circ) + i\sin(60^\circ + -30^\circ + -135^\circ)]$$

$$= 4\sqrt{2}[(\cos(-105^\circ) + i\sin(-105^\circ))]$$

2. Given that $z_1 = 2 + i$, $z_2 = -2 + i$ and $z_3 = 2 - 3i$. Express the complex numbers in polar form. $z_1 \cdot z_2$, $z_3 \cdot z_4$ and $z_1 \cdot z_2 \cdot z_3$

Solution

$$|z_1| = \sqrt{2^2 + 1^2} = \sqrt{5}; \arg(z_1) = 26.6^\circ$$

$$|z_2| = \sqrt{2^2 + 1^2} = \sqrt{5}; \arg(z_2) = 153.4^\circ$$

$$|z_3| = \sqrt{2^2 + 3^2} = \sqrt{13}; \arg(z_3) = -56.3^\circ$$

$$\therefore z_1 \cdot z_2 = [\sqrt{5}[(\cos(26.6^\circ) + i\sin(26.6^\circ))]] \cdot [\sqrt{5}[(\cos(153.4^\circ) + i\sin(153.4^\circ))]]$$

$$= 5[\cos(26.6^\circ + 153.4^\circ) + i\sin(26.6^\circ + 153.4^\circ)]$$

$$= 5(\cos(180^\circ) + i\sin(180^\circ))$$

$$z_3 \cdot z_4 = [\sqrt{5}[(\cos(153.4^\circ) + i\sin(153.4^\circ))]] \cdot [\sqrt{13}[(\cos(-56.3^\circ) + i\sin(-56.3^\circ))]]$$

$$= \sqrt{65}[\cos(-56.3^\circ + 153.4^\circ) + i\sin(-56.3^\circ + 153.4^\circ)]$$

$$\begin{aligned}
 &= \sqrt{65}(\cos(97.1^0) + i\sin(97.1^0)) \\
 z_1 \cdot z_2 \cdot z_3 &= [\sqrt{5}[(\cos(26.6^0) + i\sin(26.6^0))]] \cdot \sqrt{5}[(\cos(153.4^0) + i\sin(153.4^0))] \cdot \sqrt{13}[(\cos(153.4^0) + i\sin(153.4^0))] \\
 &= 5\sqrt{13}[\cos(26.6^0 + 153.4^0 - 56.3^0) + i\sin(26.6^0 + 153.4^0 - 56.3^0)] \\
 &= 5\sqrt{13}[(\cos(123.7^0) + i\sin(123.7^0))]
 \end{aligned}$$

Trial questions

Given that $z_1 = 2$, $z_2 = -1 + i$ and $z_3 = 2i$.

Express z_1 , z_2 , z_3 , $z_1 \cdot z_2$, $z_2 \cdot z_3$ and $z_1 \cdot z_2 \cdot z_3$ in polar form.

ANSWER: $r = 2, \theta = 0$; $r = \sqrt{2}, \theta = 135^0$; $r = 2, \theta = -90^0$; $r = 2\sqrt{2}, \theta = 135^0$; $r = 4\sqrt{2}, \theta = 45^0$

DIVIDING COMPLEX NUMBERS IN POLAR FORM

Consider two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ which can be expressed in the polar form. i.e.

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ respectively.

To find $\frac{z_1}{z_2}$, proceed as follows;

$$\begin{aligned}
 \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}; \text{rationalizing the denominator} \\
 \frac{z_1}{z_2} &= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2) \cdot (\cos\theta_2 - i\sin\theta_2)} \\
 &= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{(\cos\theta_2)^2 - (i\sin\theta_2)^2} \left[\begin{array}{l} \because (a+b)(a-b) = a^2 - b^2 \\ \text{in the denominator} \end{array} \right] \\
 &= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\
 &= \frac{r_1}{r_2} \cdot \frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1} \\
 \therefore \frac{z_1}{z_2} &= \frac{r_1}{r_2} \cdot [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]
 \end{aligned}$$

In general when dividing complex numbers in polar form, first divide their moduli, subtract their arguments and write the result in polar form.

Deduction

When dividing complex numbers in polar form

- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Note

The sense of measurement of the argument *must* be the same. i.e. clockwise/anticlockwise or just consider the *principal argument* of complex numbers.

Example

- Given that $z_1 = 1 + i$, $z_2 = -1 + i$, $z_3 = -i$ and $z_4 = 2$. Express in polar form;
 - $\frac{z_1}{z_2}$
 - $\frac{z_2}{z_3}$
 - $\frac{z_4}{z_2}$

Solution

$$\begin{aligned}
 z_1 &= 1 + i = \sqrt{2}(\cos(45^0) + i\sin(45^0)) \\
 z_2 &= -1 + i = \sqrt{2}(\cos(135^0) + i\sin(135^0)) \\
 z_3 &= -i = 0 - i = 1(\cos(-90^0) + i\sin(-90^0))
 \end{aligned}$$

$$z_4 = 2 = 2 + 0i = 2(\cos(0^0) + i\sin(0^0))$$

[Expressing complex numbers in polar form is left as an exercise]

$$\begin{aligned} \text{i. } \frac{z_1}{z_2} &= \frac{\sqrt{2}(\cos(45^0) + i\sin(45^0))}{\sqrt{2}(\cos(135^0) + i\sin(135^0))} = \cos(45^0 - 135^0) + i\sin(45^0 - 135^0) \\ &\therefore \frac{z_1}{z_2} = [\cos(-90^0) + i\sin(-90^0)] \\ \text{ii. } \frac{z_2}{z_3} &= \frac{\sqrt{2}(\cos(135^0) + i\sin(135^0))}{1(\cos(-90^0) + i\sin(-90^0))} = \sqrt{2}[\cos(135^0 - -90^0) + i\sin(135^0 - -90^0)] \\ &= \sqrt{2}[\cos(225^0) + i\sin(225^0)] \end{aligned}$$

Expressing angles trigonometric ratios in terms *principal arguments*

Angle 225^0 lies in the 3^{rd} quadrant. Thus the angle nearest to +ve real axis is 135^0 . Hence principal argument is -135^0

$$\therefore \frac{z_2}{z_3} = \sqrt{2}[\cos(-135^0) + i\sin(-135^0)]$$

$$\begin{aligned} \text{iii. } \frac{z_4}{z_2} &= \frac{2(\cos(0^0) + i\sin(0^0))}{\sqrt{2}(\cos(135^0) + i\sin(135^0))} = \frac{2}{\sqrt{2}}[\cos(0^0 - 135^0) + i\sin(0^0 - 135^0)] \\ &\therefore \frac{z_4}{z_2} = \frac{2}{\sqrt{2}}[\cos(-135^0) + i\sin(-135^0)] \end{aligned}$$

Summary of rules in complex numbers

Complex number	Modulus rules	Argument rules
$z_1 \cdot z_2$	$ z_1 \cdot z_2 $	$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$
$\frac{z_1}{z_2}$	$\frac{ z_1 }{ z_2 }$	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Note:

- $\arg(z_1) \times \arg(z_2) \neq \arg(z_1 + z_2) \neq \arg(z_1) + \arg(z_2)$
- $\frac{\arg(z_1)}{\arg(z_2)} \neq \arg(z_1) - \arg(z_2)$
- Given that $z_1 = 2 + 3i$, $z_2 = 1 - i$, $z_3 = -\sqrt{3} + i$ and $z_4 = -1 - i\sqrt{3}$. Express the complex numbers in polar form. Hence express the following in polar form:
 - $\frac{z_1 \cdot z_2}{z_3}$
 - $\frac{z_1}{z_2 \cdot z_3}$
 - $\frac{z_1 \cdot z_2}{z_3 \cdot z_4}$

Solution

$$z_1 = 2 + 3i = \sqrt{13}[\cos(56.3^0) + i\sin(56.3^0)]$$

$$z_2 = 1 - i = \sqrt{2}[\cos(-45^0) + i\sin(-45^0)]$$

$$z_3 = -\sqrt{3} + i = 2[\cos(150^0) + i\sin(150^0)]$$

$$z_4 = -1 - i\sqrt{3} = 2[\cos(-120^0) + i\sin(-120^0)]$$

[To express the complex numbers in polar form see notes on polar form of a complex number]

- $$\frac{z_1 \cdot z_2}{z_3} = r(\cos\theta + i\sin\theta)$$

$$r = \left| \frac{z_1 \cdot z_2}{z_3} \right| = \frac{|z_1 \cdot z_2|}{|z_3|} = \frac{|z_1| \cdot |z_2|}{|z_3|} = \frac{\sqrt{13} \times \sqrt{2}}{2} = \frac{\sqrt{26}}{2}$$

$$\theta = \arg\left(\frac{z_1 \cdot z_2}{z_3}\right) = \arg(z_1 \cdot z_2) - \arg(z_3) = \arg(z_1) + \arg(z_2) - \arg(z_3)$$

$$= 56.3^0 + -45^0 - 150^0 = -138.7^0$$

$$\therefore \frac{z_1 \cdot z_2}{z_3} = \frac{\sqrt{26}}{2}[\cos(-138.7^0) + i\sin(-138.7^0)]$$
- $$\frac{z_1}{z_2 \cdot z_3} = r(\cos\theta + i\sin\theta)$$

$$r = \left| \frac{z_1}{z_2 \cdot z_3} \right| = \frac{|z_1|}{|z_2| \cdot |z_3|} = \frac{\sqrt{13}}{\sqrt{2} \times 2} = \frac{1}{2} \sqrt{\left(\frac{13}{2}\right)}$$

$$\theta = \arg\left(\frac{z_1}{z_2 \cdot z_3}\right) = \arg(z_1) - \arg(z_2 \cdot z_3) = \arg(z_1) - [\arg(z_2) + \arg(z_3)]$$

$$= 56.3^\circ - [-45^\circ + 150^\circ] = -48.7^\circ$$

$$\therefore \frac{z_1}{z_2 \cdot z_3} = \frac{1}{2} \sqrt{\left(\frac{13}{2}\right)} [\cos(-48.7^\circ) + i \sin(-48.7^\circ)]$$

c) $\frac{z_1 \cdot z_2}{z_3 \cdot z_4} = r(\cos\theta + i \sin\theta)$

$$r = \left| \frac{z_1 \cdot z_2}{z_3 \cdot z_4} \right| = \frac{|z_1| \cdot |z_2|}{|z_3| \cdot |z_4|} = \frac{\sqrt{13} \times \sqrt{2}}{2 \times 2} = \frac{\sqrt{26}}{4}$$

$$\theta = \arg\left(\frac{z_1 \cdot z_2}{z_3 \cdot z_4}\right) = \arg(z_1) + \arg(z_2) - [\arg(z_3) + \arg(z_4)]$$

$$= 56.3^\circ + -45^\circ - [150^\circ + -120^\circ] = -18.7^\circ$$

$$\therefore \frac{z_1 \cdot z_2}{z_3 \cdot z_4} = \frac{\sqrt{26}}{4} [\cos(-18.7^\circ) + i \sin(-18.7^\circ)]$$

3. Given that $z_1 = 1 + i$, $z_2 = 1 - i$, $z_3 = 2 + 3i$. Express the complex numbers in polar form. Hence express the following in polar form:

i) $\frac{z_2 \cdot z_3}{z_1}$ ii) $\frac{z_1 \cdot z_2}{z_3}$ iii) $z_1 \cdot z_2 + \frac{1}{z_3}$ iv) $\frac{z_1}{z_2} + \frac{z_2 \cdot z_3}{z_1}$

Solution

$$z_1 = 1 + i = \sqrt{2}[\cos(45^\circ) + i \sin(45^\circ)]$$

$$z_2 = 1 - i = \sqrt{2}[\cos(-45^\circ) + i \sin(-45^\circ)]$$

$$z_3 = 2 + 3i = \sqrt{13}[\cos(56.3^\circ) + i \sin(56.3^\circ)]$$

i. $\frac{z_2 \cdot z_3}{z_1} = r(\cos\theta + i \sin\theta)$

$$r = \left| \frac{z_2 \cdot z_3}{z_1} \right| = \frac{|z_2| \cdot |z_3|}{|z_1|} = \frac{\sqrt{2} \times \sqrt{13}}{\sqrt{2}} = \sqrt{13}$$

$$\theta = \arg\left(\frac{z_2 \cdot z_3}{z_1}\right) = \arg(z_2) + \arg(z_3) - \arg(z_1)$$

$$= -45^\circ + 56.3^\circ - 45^\circ = -33.7^\circ$$

$$\therefore \frac{z_2 \cdot z_3}{z_1} = \sqrt{13}[\cos(-33.7^\circ) + i \sin(-33.7^\circ)]$$

ii. $\frac{z_1 \cdot z_2}{z_3} = r(\cos\theta + i \sin\theta)$

$$r = \left| \frac{z_1 \cdot z_2}{z_3} \right| = \frac{|z_1| \cdot |z_2|}{|z_3|} = \frac{\sqrt{2} \times \sqrt{2}}{\sqrt{13}} = \frac{2}{\sqrt{13}}$$

$$\theta = \arg\left(\frac{z_1 \cdot z_2}{z_3}\right) = \arg(z_1) + \arg(z_2) - \arg(z_3)$$

$$= 45^\circ + -45^\circ - 56.3^\circ = -56.3^\circ$$

$$\therefore \frac{z_1 \cdot z_2}{z_3} = \frac{2}{\sqrt{13}}[\cos(-56.3^\circ) + i \sin(-56.3^\circ)]$$

iii. $z_1 \cdot z_2 + \frac{1}{z_3} = r(\cos\theta + i \sin\theta)$

Dealing with $z_1 \cdot z_2$

$$\Rightarrow z_1 \cdot z_2 = (1 + i)(1 - i) = 1^2 - i^2 = 2$$

$$\Rightarrow \frac{1}{z_3} = \frac{1}{2 + 3i} = \frac{1 \cdot (2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{(2 - 3i)}{(2)^2 - (3i)^2} = \frac{2}{13} - \frac{3}{13}i$$

$$\therefore z_1 \cdot z_2 + \frac{1}{z_3} = 2 + \frac{2}{13} - \frac{3}{13}i = \frac{28}{13} - \frac{3}{13}i$$

$$r = \left| z_1 \cdot z_2 + \frac{1}{z_3} \right| = \sqrt{\left(\frac{28}{13}\right)^2 + \left(\frac{3}{13}\right)^2} = \frac{\sqrt{793}}{13}$$

$$\theta = \arg\left(z_1 \cdot z_2 + \frac{1}{z_3}\right) = -\tan^{-1}\left(\frac{3/13}{28/13}\right) = -6.1^\circ$$

$$\begin{aligned}
 \therefore z_1 \cdot z_2 + \frac{1}{z_3} &= \frac{\sqrt{793}}{13} [\cos(-6.1^\circ) + i \sin(-6.1^\circ)] \\
 \text{iv. } \frac{z_1}{z_2} + \frac{z_2 \cdot z_3}{z_1} &= r(\cos\theta + i \sin\theta) \\
 \frac{z_1}{z_2} &= \frac{(1+i)}{(1-i)} = \frac{(1+i) \cdot (1+i)}{(1-i) \cdot (1+i)} = \frac{1+2i+i^2}{1^2-i^2} = \frac{2i}{2} = i \\
 \frac{z_2 \cdot z_3}{z_1} &= \frac{(1-i) \cdot (2+3i)}{(1+i)} = \frac{2+3i-2i-3i^2}{(1+i)} = \frac{2+i+3}{(1+i)} = \frac{(5+i)}{(1+i)} \\
 \frac{(5+i)}{(1+i)} &= \frac{(5+i) \cdot (1-i)}{(1+i) \cdot (1-i)} = \frac{5-5i+i-i^2}{1^2-i^2} = \frac{5-4i+1}{2} = \frac{6-4i}{2} \\
 \therefore \frac{z_2 \cdot z_3}{z_1} &= 3 - 2i \\
 \therefore \frac{z_1}{z_2} + \frac{z_2 \cdot z_3}{z_1} &= i + 3 - 2i = 3 - i \\
 \Rightarrow r &= \left| \frac{z_1}{z_2} + \frac{z_2 \cdot z_3}{z_1} \right| = \sqrt{3^2 + 1^2} = \sqrt{10} \\
 \Rightarrow \theta &= \arg\left(\frac{z_1}{z_2} + \frac{z_2 \cdot z_3}{z_1}\right) = -\tan^{-1}\left(\frac{1}{3}\right) = -18.4^\circ \\
 \therefore \frac{z_1}{z_2} + \frac{z_2 \cdot z_3}{z_1} &= \sqrt{10} [\cos(-18.4^\circ) + i \sin(-18.4^\circ)]
 \end{aligned}$$

Task

Given that $z_1 = 1 + i$, $z_2 = 2i$, $z_3 = -1 + i$. Express the complex numbers in polar form. Hence express the following in polar form:

$$\text{i) } \frac{1}{z_1} \quad \text{ii) } \frac{1}{z_2} \quad \text{iii) } \frac{1}{z_3} \quad \text{iv) } \frac{z_1}{z_2} + z_3 \quad \text{v) } \frac{z_1 \cdot z_2}{z_3} + z_3 \quad \text{vi) } \frac{z_1 \cdot z_2}{z_3} + \frac{z_3}{z_1}$$

$$\text{Answer: i) } r = \frac{1}{\sqrt{2}}, \theta = -45^\circ \quad \text{ii) } r = \frac{1}{2}, \theta = -90^\circ \quad \text{iii) } r = \frac{1}{\sqrt{2}}, \theta = -135^\circ$$

$$\text{iv) } r = \frac{1}{\sqrt{2}}, \theta = -135^\circ \quad \text{v) } r = \sqrt{2}, \theta = 45^\circ \quad \text{vi) } r = 2.236, \theta = 26.5^\circ$$

DEMOIVRE'S THEOREM

This is derived from multiplication of complex numbers in polar form.

Consider $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$

$$z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$$

$$z_3 = r_3(\cos\theta_3 + i \sin\theta_3)$$

$$\Rightarrow z_1 \cdot z_2 \cdot z_3 = r_1 \cdot r_2 \cdot r_3 [\cos(\theta_1 + \theta_2 + \theta_3) + i \sin(\theta_1 + \theta_2 + \theta_3)]$$

$$\text{If } z = z_1 = z_2 = z_3 = r(\cos\theta + i \sin\theta)$$

$$\Rightarrow z \cdot z \cdot z = r \cdot r \cdot r [\cos(\theta + \theta + \theta) + i \sin(\theta + \theta + \theta)]$$

$$\therefore z^3 = r^3 [\cos 3\theta + i \sin 3\theta]$$

$$\therefore z^3 = r^3 [\cos 3\theta + i \sin 3\theta]$$

$$\vdots$$

$$\therefore z^n = r^n [\cos n\theta + i \sin n\theta]$$

$$\Rightarrow [r(\cos\theta + i \sin\theta)]^n = r^n [\cos n\theta + i \sin n\theta]$$

$$\Rightarrow r^n (\cos\theta + i \sin\theta)^n = r^n [\cos n\theta + i \sin n\theta]$$

$$\therefore (\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$$

This is Demoivre's theorem.

Proof

The proof is based on induction.

$$(\cos\theta + i\sin\theta)^n = \cos(n)\theta + i\sin(n)\theta$$

For $n = 1$

$$\mathbf{L.H.S} = (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$$

$$\mathbf{R.H.S} = \cos(1)\theta + i\sin(1)\theta = \cos\theta + i\sin\theta$$

Since $\mathbf{L.H.S} = \mathbf{R.H.S}$, the proof holds for $n = 1$

For $n = k$

$$\Rightarrow (\cos\theta + i\sin\theta)^k = \cos(k)\theta + i\sin(k)\theta \dots\dots\dots (1)$$

Let the proof hold for $n = k$

Now for $n = k + 1$

$$\mathbf{L.H.S} = (\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)^1 \dots\dots\dots (2)$$

$$\begin{aligned} \mathbf{L.H.S} &= (\cos\theta + i\sin\theta)^{k+1} = [\cos(k)\theta + i\sin(k)\theta] \cdot (\cos\theta + i\sin\theta)^1 \\ &= [\cos(k)\theta + i\sin(k)\theta] \cdot (\cos\theta + i\sin\theta) \\ &= \cos(k)\theta\cos\theta + i\cos(k)\theta\sin\theta + i\sin(k)\theta\cos\theta + i^2\sin(k)\theta\sin\theta \\ &= [\cos(k)\theta\cos\theta - \sin(k)\theta\sin\theta] + i[\cos(k)\theta\sin\theta + \sin(k)\theta\cos\theta] \\ &= \cos[k\theta + \theta] + i\sin[k\theta + \theta] \\ &= \cos(k+1)\theta + i\sin(k+1)\theta \end{aligned}$$

$$\mathbf{R.H.S} = \cos(k+1)\theta + i\sin(k+1)\theta$$

Since $\mathbf{L.H.S} = \mathbf{R.H.S}$, the proof holds for $n = k + 1$

Since the proof holds for $n = 1, 2, 3 \dots \dots k, k + 1$, then the proof holds for all values of n .

Deduction from Demoivre's theorem

If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos\theta + i\sin\theta)^n = r^n[\cos n\theta + i\sin n\theta]$

- i. $Arg(z^n) = nArg(z)$
- ii. $|z^n| = |z|^n$

Properties of Argument

$Arg(\lambda z) = Arg(z)$, where $\lambda \geq 0$ and $\lambda \in \mathbb{R}$

Application of Demoivre's Theorem

- Simplifying powers of complex numbers in polar form
- Proof of trigonometric identities
- Simplifying complex number expressions
- Expansions

SIMPLIFYING EXPRESSIONS

1) Simplify

a) $\frac{\cos 3\theta + i\sin 3\theta}{\cos\theta + i\sin\theta}$

b) $\frac{\cos \frac{3}{2}\theta + i\sin \frac{3}{2}\theta}{\cos \frac{1}{2}\theta + i\sin \frac{1}{2}\theta}$

c) $\frac{\cos 2\theta + i\sin 2\theta}{\cos\theta - i\sin\theta}$

d) $\frac{\cos 3\theta - i\sin 3\theta}{\cos n\theta + i\sin n\theta}$

Solution

a) $\frac{\cos 3\theta + i\sin 3\theta}{\cos\theta + i\sin\theta} = \frac{(\cos\theta + i\sin\theta)^3}{(\cos\theta + i\sin\theta)^1} = (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$$\begin{aligned} \text{b)} \quad \frac{\cos \frac{3}{2}\theta + i \sin \frac{3}{2}\theta}{\cos \frac{1}{2}\theta + i \sin \frac{1}{2}\theta} &= \frac{(\cos\theta + i \sin\theta)^{\frac{3}{2}}}{(\cos\theta + i \sin\theta)^{\frac{1}{2}}} = (\cos\theta + i \sin\theta)^{\frac{3}{2} - \frac{1}{2}} = (\cos\theta + i \sin\theta)^1 = \cos\theta + i \sin\theta \\ \text{c)} \quad \frac{\cos 2\theta + i \sin 2\theta}{\cos \theta - i \sin \theta} &= \frac{\cos 2\theta + i \sin 2\theta}{\cos(-1)\theta + i \sin(-1)\theta} = \frac{(\cos\theta + i \sin\theta)^2}{(\cos\theta + i \sin\theta)^{-1}} = (\cos\theta + i \sin\theta)^3 = \cos 3\theta + i \sin 3\theta \\ \text{d)} \quad \frac{\cos 3\theta - i \sin 3\theta}{\cos n\theta + i \sin n\theta} &= \frac{\cos(-3)\theta + i \sin(-3)\theta}{\cos(n)\theta + i \sin(n)\theta} = \frac{(\cos\theta + i \sin\theta)^{-3}}{(\cos\theta + i \sin\theta)^n} = (\cos\theta + i \sin\theta)^{-3-n} \\ &= \cos(3+n)\theta - i \sin(3+n)\theta \end{aligned}$$

2) Express in the form $a + ib$

$$\begin{aligned} \text{a)} \quad &\sqrt{\frac{\cos\theta + i \sin\theta}{\cos 2\theta - i \sin 2\theta}} \\ \text{b)} \quad &\frac{\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)^2}{\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^3 (\cos \pi - i \sin \pi)^{\frac{1}{3}}} \\ \text{c)} \quad &\frac{\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)^2}{(\cos 4\pi - i \sin 4\pi)} \end{aligned}$$

Solution

$$\begin{aligned} \text{a)} \quad &\sqrt{\frac{\cos\theta + i \sin\theta}{\cos 2\theta - i \sin 2\theta}} = \left[\frac{\cos\theta + i \sin\theta}{\cos 2\theta - i \sin 2\theta} \right]^{\frac{1}{2}} = \frac{(\cos\theta + i \sin\theta)^{\frac{1}{2}}}{[(\cos\theta + i \sin\theta)^{-2}]^{\frac{1}{2}}} = \frac{(\cos\theta + i \sin\theta)^{\frac{1}{2}}}{(\cos\theta + i \sin\theta)^{-1}} \\ &= (\cos\theta + i \sin\theta)^{\frac{1}{2} - (-1)} = \cos \frac{3}{2}\theta + i \sin \frac{3}{2}\theta \\ \text{b)} \quad &\frac{\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)^2}{\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^3 (\cos \pi - i \sin \pi)^{\frac{1}{3}}} = \frac{\left[(\cos \pi + i \sin \pi)^{\frac{1}{4}}\right]^2}{\left[(\cos \pi + i \sin \pi)^{\frac{1}{6}}\right]^3 [(\cos \pi + i \sin \pi)^{-1}]^{\frac{1}{3}}} \\ &= \frac{(\cos \pi + i \sin \pi)^{\frac{1}{2}}}{(\cos \pi + i \sin \pi)^{\frac{1}{2}(\cos \pi + i \sin \pi)^{-\frac{1}{3}}}} = (\cos \pi + i \sin \pi)^{\frac{1}{3}} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \\ \text{c)} \quad &\frac{\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)^2}{(\cos 4\pi - i \sin 4\pi)} = \frac{\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \left(\cos \left(-\frac{\pi}{3}\right) + i \sin \left(-\frac{\pi}{3}\right)\right)^2}{(\cos(-4)\pi + i \sin(-4)\pi)} = \frac{(\cos \pi + i \sin \pi)^1 (\cos \pi + i \sin \pi)^{-2}}{(\cos \pi + i \sin \pi)^{-4}} \\ &= (\cos \pi + i \sin \pi)^{\frac{1}{3} - \frac{2}{3} - (-4)} = (\cos \pi + i \sin \pi)^{\frac{11}{3}} = \cos \frac{11}{3}\theta + i \sin \frac{11}{3}\theta \end{aligned}$$

Task

Prove that

$$\begin{aligned} \text{a)} \quad &\frac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}} = 1 \\ \text{b)} \quad &\frac{(\cos 5\theta - i \sin 5\theta)^2 (\cos 7\theta + i \sin 7\theta)^{-3}}{(\cos 4\theta - i \sin 4\theta)^9 (\cos \theta + i \sin \theta)^5} = 1 \\ \text{c)} \quad &\frac{(\cos \alpha + i \sin \alpha)^2}{(\sin \beta + i \cos \beta)^5} = \sin(4\alpha + 5\beta) - i \cos(4\alpha + 5\beta) \\ \text{d)} \quad &\left(\frac{\cos \theta + i \sin \theta}{\sin \theta + i \cos \theta}\right)^4 = \cos 8\theta + i \sin 8\theta \end{aligned}$$

EXPANSION OF $\cos n\theta$ and $\sin n\theta$

From De Moivre's theorem $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$, to expand $\cos n\theta$ and $\sin n\theta$ the **L.H.S** $(\cos\theta + i \sin\theta)^n$ is expanded.

Examples

1. Express $\cos 3\theta$, $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ respectively. Hence find $\tan 3\theta$

Solution

For $n = 3$ in the De Moivre's theorem

$$\Rightarrow (\cos\theta + i \sin\theta)^3 = \cos 3\theta + i \sin 3\theta$$

$$(\cos\theta + i\sin\theta)^3 = 1(\cos\theta)^3(i\sin\theta)^0 + 3(\cos\theta)^2(i\sin\theta)^1 + 3(\cos\theta)^1(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\therefore \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\therefore \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\therefore \tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}, \text{ dividing Numerator and Denominator}$$

by $\cos^3\theta$

$$\Rightarrow \tan 3\theta = \frac{\frac{3\cos^2\theta\sin\theta}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos^3\theta}} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\therefore \tan 3\theta = \frac{3\tan\theta - 4\tan^3\theta}{1 - 3\tan^2\theta}$$

2. Show that;

$$\text{a) } \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

$$\text{b) } \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 6\tan^2\theta + 5\tan^4\theta}$$

Solution

$$\text{a) } \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

$$\text{L.H.S: } \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$

For $n = 4$ in the Demoivre's theorem

$$\Rightarrow (\cos\theta + i\sin\theta)^4 = \cos 4\theta + i \sin 4\theta$$

$$(\cos\theta + i\sin\theta)^4 = 1(\cos\theta)^4(i\sin\theta)^0 + 4(\cos\theta)^3(i\sin\theta)^1 + 6(\cos\theta)^2(i\sin\theta)^2 + 4(\cos\theta)^1(i\sin\theta)^3 + 1(i\sin\theta)^4$$

$$= \cos^4\theta + i4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - i4\cos\theta\sin^3\theta + \sin^4\theta$$

$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

$$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\therefore \sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

$$\therefore \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}, \text{ dividing Numerator and Denominator}$$

by $\cos^4\theta$

$$\Rightarrow \tan 4\theta = \frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}} = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

$$\therefore \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

b) Left as an exercise for the students

3. Prove that;

$$\text{a) } \cos 4\theta = \frac{\tan^4\theta - 6\tan^2\theta + 1}{\tan^4\theta + 2\tan^2\theta + 1}$$

$$\text{b) } \sin 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta + 2\tan^2\theta + 1}$$

Solution

From example (2) above,

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

$$\begin{aligned} \text{a) } \cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{1} \end{aligned}$$

Dividing Numerator and Denominator by $\cos^4\theta$

$$\begin{aligned} \Rightarrow \cos 4\theta &= \frac{\frac{\cos^4\theta}{\cos^4\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}}{\frac{1}{\cos^4\theta}} = \frac{1 - 6\tan^2\theta + \tan^4\theta}{\sec^4\theta} \\ &= \frac{1 - 6\tan^2\theta + \tan^4\theta}{(\sec^2\theta)^2} = \frac{1 - 6\tan^2\theta + \tan^4\theta}{(1 + \tan^2\theta)^2} \end{aligned}$$

$$\therefore \cos 4\theta = \frac{\tan^4\theta - 6\tan^2\theta + 1}{\tan^4\theta + 2\tan^2\theta + 1} \text{ As required}$$

$$\begin{aligned} \text{b) } \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \\ &= \frac{\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{1} \end{aligned}$$

Dividing Numerator and Denominator by $\cos^4\theta$

$$\begin{aligned} \Rightarrow \sin 4\theta &= \frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{1}{\cos^4\theta}} = \frac{4\tan\theta - 4\tan^3\theta}{\sec^4\theta} \\ &= \frac{4\tan\theta - 4\tan^3\theta}{(\sec^2\theta)^2} = \frac{4\tan\theta - 4\tan^3\theta}{(1 + \tan^2\theta)^2} \end{aligned}$$

$$\therefore \sin 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta + 2\tan^2\theta + 1}$$

Task

Use Demoivre's theorem to show that;

$$\text{i. } \cos 2\theta = \frac{1-t^2}{1+t^2}; t = \tan\theta$$

$$\text{ii. } \sin 2\theta = \frac{2t}{1+t^2}; t = \tan\theta$$

EXPANSION OF $\cos^n\theta$ and $\sin^n\theta$

This is based on $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ results.

Consider $z = \cos\theta + i\sin\theta$

$$\Rightarrow z^n = \cos n\theta + i\sin n\theta \dots\dots\dots (1)$$

$$z^{-n} = \frac{1}{z^n} = \cos(-n)\theta + i\sin(-n)\theta = \cos n\theta - i\sin n\theta$$

$$\Rightarrow \frac{1}{z^n} = \cos n\theta - i\sin n\theta \dots\dots\dots (2)$$

Adding the eqns. (1) and (2)

$$z^n + \frac{1}{z^n} = 2\cos n\theta \dots\dots\dots (3)$$

Subtracting the eqns. (1) and (2)

$$z^n - \frac{1}{z^n} = i2\sin n\theta \dots\dots\dots (4)$$

From eqns. (3) and (4), when $n = 1$

$$\Rightarrow z + \frac{1}{z} = 2\cos\theta \dots\dots\dots (5)$$

$$\Rightarrow z - \frac{1}{z} = i2\sin\theta \dots\dots\dots (6)$$

The above equations are important in this kind of expansions.

Example

1. Express

$$\text{a) } \cos^3\theta \text{ in terms of } \cos\theta \text{ and } \cos 3\theta$$

$$\text{b) } \sin^3\theta \text{ in terms of } \sin\theta \text{ and } \sin 3\theta$$

$$\text{c) } \cos^4\theta \text{ in terms of } \cos 4\theta \text{ and } \cos 2\theta$$

$$\text{d) } \sin^4\theta \text{ in terms of } \sin 4\theta \text{ and } \sin 2\theta$$

Solution

- a) To expand $\cos^3\theta, z + \frac{1}{z} = 2\cos\theta$ is used.

Cubing the equation to create $\cos^3\theta$

$$\Rightarrow \left(z + \frac{1}{z}\right)^3 = (2\cos\theta)^3 = 8\cos^3\theta$$

$$z^3 + 3z^2 \cdot \frac{1}{z} + 3z \cdot \frac{1}{z^2} + \frac{1}{z^3} = 8\cos^3\theta$$

$$\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right) = 8\cos^3\theta$$

But from $z^n + \frac{1}{z^n} = 2\cos n\theta$, for $n = 3$,

$$z^3 + \frac{1}{z^3} = 2\cos 3\theta$$

$$\therefore 8\cos^3\theta = 2\cos 3\theta + 3(2\cos\theta)$$

$$= 2\cos 3\theta + 6\cos\theta$$

$$\therefore \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

- b) To expand $\sin^3\theta, z - \frac{1}{z} = i2\sin\theta$ is used.

Cubing the equation to create $\cos^3\theta$

$$\Rightarrow \left(z - \frac{1}{z}\right)^3 = (i2\sin\theta)^3 = -i8\sin^3\theta$$

$$z^3 + 3z^2 \cdot \left(-\frac{1}{z}\right) + 3z \cdot \left(-\frac{1}{z^2}\right) + \left(-\frac{1}{z^3}\right) = -i8\sin^3\theta$$

$$\left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right) = -i8\sin^3\theta$$

But from $z^n - \frac{1}{z^n} = i2\sin n\theta$, for $n = 3$,

$$z^3 - \frac{1}{z^3} = i2\sin 3\theta$$

$$\therefore -i8\sin^3\theta = i2\sin 3\theta - 3(i2\sin\theta)$$

$$= i2\sin 3\theta - i6\sin\theta, \text{ dividing through by } -i8$$

$$\therefore \sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$$

- c) To expand $\cos^4\theta, z + \frac{1}{z} = 2\cos\theta$ is used.

Quarting both sides

$$\Rightarrow \left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$$

$$z^4 + 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} + 4z \cdot \frac{1}{z^3} + \frac{1}{z^4} = 16\cos^4\theta$$

$$\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 = 16\cos^4\theta$$

But from $z^n + \frac{1}{z^n} = 2\cos n\theta$, for $n = 4$ and 2 ,

$$z^4 + \frac{1}{z^4} = 2\cos 4\theta; z^2 + \frac{1}{z^2} = 2\cos 2\theta$$

$$\therefore 16\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$$= 2\cos 4\theta + 8\cos 2\theta + 6, \text{ dividing through by } 16$$

$$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

- d) Left as an exercise for students

Task

- a) Prove that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$
- b) Prove that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

FINDING THE N^{TH} ROOT OF A COMPLEX NUMBER

Consider $z = r(\cos\theta + i \sin\theta)$. To find $\sqrt[n]{z} = z^{\frac{1}{n}}$, proceed as follows.

$$z^{\frac{1}{n}} = [r(\cos\theta + i \sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i \sin\theta)^{\frac{1}{n}}$$

Since the result must also be in polar form, then let

$$r^{\frac{1}{n}}(\cos\theta + i \sin\theta)^{\frac{1}{n}} = R(\cos\phi + i \sin\phi)$$

Recall: From trigonometry,

$$\cos(\theta) = \cos(\theta + 360^0k)$$

$$\sin(\theta) = \sin(\theta + 360^0k), \text{ where } k = 0, 1, 2, 3 \dots$$

$$\therefore r^{\frac{1}{n}}(\cos\theta + i \sin\theta)^{\frac{1}{n}} = R(\cos\phi + i \sin\phi), \text{ becomes;}$$

$$r^{\frac{1}{n}}(\cos(\theta + 360^0k) + i \sin(\theta + 360^0k))^{\frac{1}{n}} = R(\cos\phi + i \sin\phi)$$

$$\Rightarrow r^{\frac{1}{n}} \left[\cos\left(\frac{\theta + 360^0k}{n}\right) + i \sin\left(\frac{\theta + 360^0k}{n}\right) \right] = R(\cos\phi + i \sin\phi)$$

Comparing

$$\Rightarrow R = r^{\frac{1}{n}} \dots \dots \dots (1)$$

$$\Rightarrow \phi = \frac{\theta + 360^0k}{n} \dots \dots \dots (2)$$

where $k = 0, 1, 2, 3 \dots \dots \dots (n-1)$

Taking values of k in to equation (2) yields the argument of different roots to $\sqrt[n]{z}$.

Example

1. Given that $z_1 = 1 + \sqrt{3}i$, $z_2 = 2 + i$, $z_3 = 1 - i$, find;

a) $\sqrt{z_1}$	d) $\sqrt[3]{z_1}$
b) $\sqrt{z_2}$	e) $\sqrt[3]{z_3}$
c) $\sqrt{z_3}$	f) $\sqrt[4]{z_1}$

Solution

Expressing complex numbers in polar form first,

- a) $z_1 = 1 + \sqrt{3}i = 2(\cos(60^0) + i \sin(60^0))$

$$\begin{aligned} \sqrt{z_1} &= [2(\cos(60^0) + i \sin(60^0))]^{\frac{1}{2}} \\ &= \sqrt{2}[(\cos(60^0) + i \sin(60^0))]^{\frac{1}{2}} \\ &= \sqrt{2} \left[\cos\left(\frac{60^0 + 360^0k}{2}\right) + i \sin\left(\frac{60^0 + 360^0k}{2}\right) \right] \end{aligned}$$

Now $k = 0, 1$ [$\because n = 2 \therefore k$ takes on 0 and 1]

For $k = 0$

$$\begin{aligned} \text{Root 1} &= \sqrt{2} \left[\cos\left(\frac{60^0 + 360^0 \times 0}{2}\right) + i \sin\left(\frac{60^0 + 360^0 \times 0}{2}\right) \right] \\ &= 1.225 + 0.707i \end{aligned}$$

For $k = 1$

$$\begin{aligned} \text{Root 2} &= \sqrt{2} \left[\cos\left(\frac{60^0 + 360^0 \times 1}{2}\right) + i \sin\left(\frac{60^0 + 360^0 \times 1}{2}\right) \right] \\ &= -1.225 - 0.707i \end{aligned}$$

$$\therefore \sqrt{z_1} = \pm(1.225 + 0.707i)$$

- b) $z_2 = 2 + i = \sqrt{5}(\cos(26.6^0) + i \sin(26.6^0))$

$$\begin{aligned} \sqrt{z_2} &= [\sqrt{5}(\cos(26.6^0) + i \sin(26.6^0))]^{\frac{1}{2}} \\ &= \sqrt[4]{5} \left[\cos\left(\frac{26.6^0 + 360^0k}{2}\right) + i \sin\left(\frac{26.6^0 + 360^0k}{2}\right) \right] \end{aligned}$$

Now $k = 0, 1$ [$\because n = 2 \therefore k$ takes on 0 and 1]

For $k = 0$

$$\begin{aligned} \text{Root } 1 &= \sqrt[4]{5} \left[\cos\left(\frac{26.6^0 + 360^0 \times 0}{2}\right) + i \sin\left(\frac{26.6^0 + 360^0 \times 0}{2}\right) \right] \\ &= 1.455 + 0.344i \end{aligned}$$

For $k = 1$

$$\begin{aligned} \text{Root } 2 &= \sqrt[4]{5} \left[\cos\left(\frac{26.6^0 + 360^0 \times 1}{2}\right) + i \sin\left(\frac{26.6^0 + 360^0 \times 1}{2}\right) \right] \\ &= -1.455 - 0.344i \end{aligned}$$

$$\therefore \sqrt[4]{z_1} = \pm(1.455 + 0.344i)$$

c) $z_3 = 1 - i = \sqrt{2}(\cos(-45^0) + i \sin(-45^0))$

$$\begin{aligned} \sqrt{z_3} &= z_3^{\frac{1}{2}} = [\sqrt{2}(\cos(-45^0) + i \sin(-45^0))]^{\frac{1}{2}} \\ &= \sqrt[4]{2} \left[\cos\left(\frac{-45^0 + 360^0 k}{2}\right) + i \sin\left(\frac{-45^0 + 360^0 k}{2}\right) \right] \end{aligned}$$

Now $k = 0, 1$ [$\because n = 2 \therefore k$ takes on 0 and 1]

For $k = 0$

$$\begin{aligned} \text{Root } 1 &= \sqrt[4]{2} \left[\cos\left(\frac{-45^0 + 360^0 \times 0}{2}\right) + i \sin\left(\frac{-45^0 + 360^0 \times 0}{2}\right) \right] \\ &= 1.099 - 0.455i \end{aligned}$$

For $k = 1$

$$\begin{aligned} \text{Root } 2 &= \sqrt[4]{2} \left[\cos\left(\frac{-45^0 + 360^0 \times 1}{2}\right) + i \sin\left(\frac{-45^0 + 360^0 \times 1}{2}\right) \right] \\ &= -1.099 + 0.455i \end{aligned}$$

$$\therefore \sqrt{z_3} = \pm(1.099 - 0.455i)$$

d) $\sqrt[3]{z_1} = z_1^{\frac{1}{3}} = [2(\cos(60^0) + i \sin(60^0))]^{\frac{1}{3}}$

$$= \sqrt[3]{2} \left[\cos\left(\frac{60^0 + 360^0 k}{3}\right) + i \sin\left(\frac{60^0 + 360^0 k}{3}\right) \right]$$

Now $k = 0, 1, 2$ [$\because n = 3 \therefore k$ takes on 0, 1 and 2]

For $k = 0$

$$\begin{aligned} \text{Root } 1 &= \sqrt[3]{2} \left[\cos\left(\frac{60^0 + 360^0 \times 0}{3}\right) + i \sin\left(\frac{60^0 + 360^0 \times 0}{3}\right) \right] \\ &= 1.184 + 0.431i \end{aligned}$$

For $k = 1$

$$\begin{aligned} \text{Root } 2 &= \sqrt[3]{2} \left[\cos\left(\frac{60^0 + 360^0 \times 1}{3}\right) + i \sin\left(\frac{60^0 + 360^0 \times 1}{3}\right) \right] \\ &= -0.965 + 0.810i \end{aligned}$$

For $k = 2$

$$\begin{aligned} \text{Root } 3 &= \sqrt[3]{2} \left[\cos\left(\frac{60^0 + 360^0 \times 2}{3}\right) + i \sin\left(\frac{60^0 + 360^0 \times 2}{3}\right) \right] \\ &= -0.219 - 1.241i \end{aligned}$$

$$\therefore \sqrt[3]{z_1} = 1.184 + 0.431i; -0.965 + 0.810i; -0.219 - 1.241i$$

e) $\sqrt[3]{z_3} = z_3^{\frac{1}{3}} = [\sqrt{2}(\cos(-45^0) + i \sin(-45^0))]^{\frac{1}{3}}$

$$= \sqrt[6]{2} \left[\cos\left(\frac{-45^0 + 360^0 k}{3}\right) + i \sin\left(\frac{-45^0 + 360^0 k}{3}\right) \right]$$

Now $k = 0, 1, 2$ [$\because n = 3 \therefore k$ takes on 0, 1 and 2]

For $k = 0$

$$\begin{aligned} \text{Root } 1 &= \sqrt[6]{2} \left[\cos\left(\frac{-45^0 + 360^0 \times 0}{3}\right) + i \sin\left(\frac{-45^0 + 360^0 \times 0}{3}\right) \right] \\ &= 1.084 - 0.291i \end{aligned}$$

For $k = 1$

$$\text{Root } 2 = \sqrt[6]{2} \left[\cos\left(\frac{-45^0 + 360^0 \times 1}{3}\right) + i \sin\left(\frac{-45^0 + 360^0 \times 1}{3}\right) \right]$$

$$= -0.291 + 1.084i$$

For $k = 2$

$$\begin{aligned} \text{Root } 3 &= \sqrt[3]{2} \left[\cos\left(\frac{-45^\circ + 360^\circ \times 2}{3}\right) + i \sin\left(\frac{-45^\circ + 360^\circ \times 2}{3}\right) \right] \\ &= -0.794 - 0.794i \end{aligned}$$

$$\therefore \sqrt[3]{z_1} = 1.084 - 0.291i; -0.291 + 1.084i; -0.794 - 0.794i$$

$$\begin{aligned} \text{f) } \sqrt[4]{z_1} &= \sqrt[4]{z_1} = z_1^{\frac{1}{4}} = \left[2(\cos(60^\circ) + i \sin(60^\circ)) \right]^{\frac{1}{4}} \\ &= \sqrt[4]{2} \left[\cos\left(\frac{60^\circ + 360^\circ k}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ k}{4}\right) \right] \end{aligned}$$

Now $k = 0, 1, 2, 3$ [$\because n = 4 \therefore k$ takes on $0, 1, 2$ and 3]

For $k = 0$

$$\begin{aligned} \text{Root } 1 &= \sqrt[4]{2} \left[\cos\left(\frac{60^\circ + 360^\circ \times 0}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ \times 0}{4}\right) \right] \\ &= 1.149 + 0.308i \end{aligned}$$

For $k = 1$

$$\begin{aligned} \text{Root } 2 &= \sqrt[4]{2} \left[\cos\left(\frac{60^\circ + 360^\circ \times 1}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ \times 1}{4}\right) \right] \\ &= -0.308 + 1.149i \end{aligned}$$

For $k = 2$

$$\begin{aligned} \text{Root } 3 &= \sqrt[4]{2} \left[\cos\left(\frac{60^\circ + 360^\circ \times 2}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ \times 2}{4}\right) \right] \\ &= -1.149 - 0.308i \end{aligned}$$

$$\begin{aligned} \text{Root } 3 &= \sqrt[4]{2} \left[\cos\left(\frac{60^\circ + 360^\circ \times 2}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ \times 2}{4}\right) \right] \\ &= 0.308 - 1.149i \end{aligned}$$

$$\therefore \sqrt[4]{z_1} = 1.149 + 0.308i; -0.308 + 1.149i; -1.149 - 0.308i; 0.308 - 1.149i$$

Task

Find the fourth root of:

- $5 + 12i$
- $-4 + 3i$
- $4 - 3i$

ANSWER:

EXPRESSING POWERED COMPLEX NUMBER IN POLAR FORM

The rules of modulus and argument of complex numbers are employed.

Example

- Express the following in polar form:

$$\text{a. } \frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}$$

$$\text{c. } \frac{1+2i}{1-(1-i)^2}$$

$$\text{b. } \frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}$$

$$\text{d. } \frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}$$

Solution

$$\text{a. let } z = \frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3} = r(\cos\theta + i \sin\theta)$$

$$\begin{aligned} r = |z| &= \left| \frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3} \right| = \frac{|(1+i)^2||1+i\sqrt{3}|}{|(1-i\sqrt{3})^3|} \\ &= \frac{|1+i|^2|1+i\sqrt{3}|}{|1-i\sqrt{3}|^3} = \frac{(\sqrt{1+1})^2(\sqrt{1+3})}{(\sqrt{1+3})^3} = \frac{2 \times 2}{2^3} = \frac{1}{2} \end{aligned}$$

$$\theta = \arg(z) = \arg\left(\frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}\right)$$

$$\begin{aligned}
 &= \arg(1+i)^2 \cdot (1+i\sqrt{3}) - \arg(1-i\sqrt{3})^3 \\
 &\left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \right] \\
 &= 2\arg(1+i) + \arg(1+i\sqrt{3}) - 3\arg(1-i\sqrt{3}) \\
 &\left[\because \arg(z^n) = n\arg(z) \right] \\
 &= 2 \times 45^\circ + 60^\circ - 3 \times (-60^\circ) = 330^\circ \text{ or } -30^\circ \text{ [Principal angle]} \\
 \therefore z &= \frac{1}{2}(\cos(-30^\circ) + i \sin(-30^\circ))
 \end{aligned}$$

b. let $z = \frac{(1-i)^2(3+4i)^3}{(-5+12i)^2} = r(\cos\theta + i \sin\theta)$

$$\begin{aligned}
 r = |z| &= \left| \frac{(1-i)^2(3+4i)^3}{(-5+12i)^2} \right| = \frac{|(1-i)^2||3+4i|^3}{|(-5+12i)^2|} \\
 &= \frac{|1-i|^2|3+4i|^3}{|-5+12i|^2} = \frac{(\sqrt{1+1})^2(\sqrt{9+16})^3}{(\sqrt{25+144})^2} = \frac{2 \times 5^3}{13^2} = \frac{250}{169} \\
 \theta &= \arg(z) = \arg\left(\frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}\right) \\
 &= \arg(1-i)^2 \cdot (3+4i)^3 - \arg(-5+12i)^2 \\
 &= 2\arg(1-i) + 3\arg(3+4i) - 2\arg(-5+12i) \\
 &= 2 \times -45^\circ + 3 \times 53.1^\circ - 2 \times 112.6^\circ = -155.9^\circ \text{ [Principal angle]} \\
 \therefore z &= \frac{250}{169}(\cos(-155.9^\circ) + i \sin(-155.9^\circ))
 \end{aligned}$$

c. let $z = \frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-2i+i^2)} = \frac{1+2i}{1+2i} = 1+0i = r(\cos\theta + i \sin\theta)$

$$\begin{aligned}
 r = |z| &= |1+0i| = \sqrt{1+0} = 1 \\
 \theta &= \arg(z) = 0 \\
 \therefore z &= 1(\cos 0 + i \sin 0)
 \end{aligned}$$

d. let $z = \frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}} = r(\cos\theta + i \sin\theta)$

$$\begin{aligned}
 r = |z| &= \left| \frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}} \right| = \frac{|(1+i\sqrt{3})^{12}|}{|(\sqrt{3}-i)^{11}|} = \frac{|1+i\sqrt{3}|^{12}}{|\sqrt{3}-i|^{11}} \\
 &= \frac{(\sqrt{1+3})^{12}}{(\sqrt{3+1})^{11}} = \frac{2^{12}}{2^{11}} = 2 \\
 \theta &= \arg(z) = \arg\left(\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}\right) \\
 &= \arg(1+i\sqrt{3})^{12} - \arg(\sqrt{3}-i)^{11} \\
 &= 12\arg(1+i\sqrt{3}) - 11\arg(\sqrt{3}-i) \\
 &= 12 \times 60^\circ - 11 \times -30^\circ = 1050^\circ \text{ or } -30^\circ \text{ [Principal angle]} \\
 \therefore z &= 2(\cos(-30^\circ) + i \sin(-30^\circ))
 \end{aligned}$$

STATING THE PRINCIPAL ANGLE OF A COMPLEX NUMBER

The interval of a principal angle is $-\pi < \theta \leq \pi$ or $-180^\circ < \theta \leq 180^\circ$

The knowledge of trigonometry of angles in quadrants is required.

Definition

The *principal angle* is the angle whose complex number (number) is nearest to the *positive Real axis*.

Example

- 1) Given that $z_1 = \sqrt{3} - i$, $z_2 = -1 + i$ and $z_3 = 1 - i\sqrt{3}$. Find the principal argument of;
- a) z_1^5

b) z_2^7 c) z_3^8

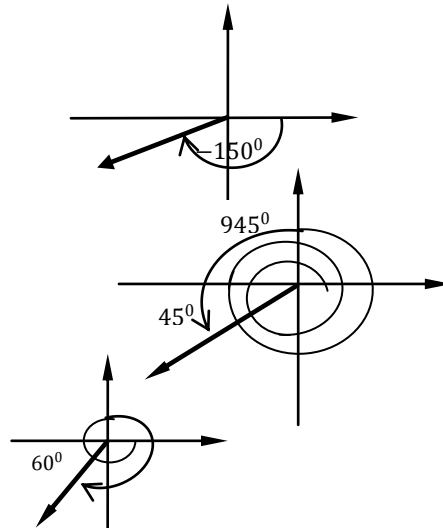
Solution

a) $\arg(z_1^5) = 5\arg(z_1)$
 $= 5 \times -30^\circ = -150^\circ$

b) $\arg(z_2^7) = 7\arg(z_2)$
 $= 7 \times 135^\circ = 945^\circ$

But $\tan 945^\circ = -\tan 135^\circ$
 $\therefore \arg(z_2^7) = -135^\circ$

c) $\arg(z_3^8) = 8\arg(z_3)$
 But $\arg(z_3) = -60^\circ$
 $\therefore \arg(z_3^8) = -60^\circ \times 8 = -480^\circ$
 $\therefore \arg(z_3^8) = -120^\circ$



- 2) Given that $z_1 = 1 + i$, $z_2 = -1 + i$ and $z_3 = -1 - i$. Find the principal argument of;

c) z_1^5

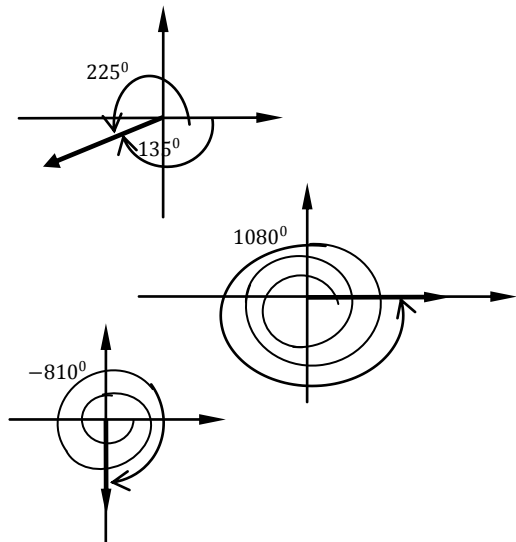
d) z_2^8 c) z_3^6

Solution

a) $\arg(z_1^5) = 5\arg(z_1)$
 $= 5 \times 45^\circ = 225^\circ$
 $\therefore \arg(z_1^5) = -135^\circ$

b) $\arg(z_2^8) = 8\arg(z_2)$
 But $\arg(z_2) = 135^\circ$
 $\therefore \arg(z_2^8) = 135^\circ \times 8 = 1080^\circ$
 $\therefore \arg(z_2^8) = 0^\circ$

c) $\arg(z_3^6) = 6\arg(z_3)$
 But $\arg(z_3) = -135^\circ$
 $\therefore \arg(z_3^6) = -135^\circ \times 6 = -810^\circ$
 $\therefore \arg(z_3^6) = -90^\circ$



Task

Given that $z_1 = 5 + 12i$, $z_2 = -2 + i$ and $z_3 = 8 - i4$. Find the principal argument of;

a) z_1^3

b) z_2^5 c) z_3^8

Answer: (a) -157.9° (b) -47.2° (c) -65°

FURTHER EXAMPLES ON PROOFS BY DEMOIVRE'S THEOREM

The knowledge of trigonometry is applicable.

Example

1. If $z_1 = \cos \alpha + i \sin \alpha$, $z_2 = \cos \beta + i \sin \beta$, show that $\frac{1}{2i} \left[\frac{z_1}{z_2} - \frac{z_2}{z_1} \right] = \sin(\alpha - \beta)$

Solution

$$\begin{aligned}
 \frac{z_1}{z_2} &= \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \cos(\alpha - \beta) + i\sin(\alpha - \beta) \\
 \frac{z_2}{z_1} &= \frac{\cos\beta + i\sin\beta}{\cos\alpha + i\sin\alpha} = \cos(\beta - \alpha) + i\sin(\beta - \alpha) \\
 \therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} &= \cos(\alpha - \beta) + i\sin(\alpha - \beta) - [\cos(\beta - \alpha) + i\sin(\beta - \alpha)] \\
 &= \cos(\alpha - \beta) + i\sin(\alpha - \beta) - \cos(\beta - \alpha) - i\sin(\beta - \alpha) \\
 &= \cos(\alpha - \beta) + i\sin(\alpha - \beta) - \cos[-(\alpha - \beta)] - i\sin[-(\alpha - \beta)] \\
 &= \cos(\alpha - \beta) + i\sin(\alpha - \beta) - \cos(\alpha - \beta) + i\sin(\alpha - \beta) \\
 \therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} &= 2i\sin(\alpha - \beta) \\
 \therefore \frac{1}{2i} \left[\frac{z_1}{z_2} - \frac{z_2}{z_1} \right] &= \frac{1}{2i} \times 2i\sin(\alpha - \beta) = \sin(\alpha - \beta)
 \end{aligned}$$

2. If $z = \cos\theta + i\sin\theta$, prove that $\frac{2}{1+z} = 1 - i\tan\frac{\theta}{2}$ and $\frac{1+z}{1-z} = i\cot\frac{\theta}{2}$

Solution

$$\begin{aligned}
 \frac{2}{1+z} &= \frac{2}{1+\cos\theta + i\sin\theta} = \frac{2}{2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\
 &= \frac{1}{\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})} = \frac{1}{\cos\frac{\theta}{2}} \cdot \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1} \\
 &= \frac{1}{\cos\frac{\theta}{2}} \cdot \cos\left(-\frac{\theta}{2}\right) + i\sin\left(-\frac{\theta}{2}\right) = \frac{1}{\cos\frac{\theta}{2}} \cdot \left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right] \\
 \therefore \frac{2}{1+z} &= 1 - i\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = 1 - i\tan\frac{\theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1+z}{1-z} &= \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = \frac{(1+\cos\theta) + i\sin\theta}{(1-\cos\theta) - i\sin\theta} \\
 &= \frac{2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\cos\frac{\theta}{2}[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}]}{2\sin\frac{\theta}{2}[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}]}
 \end{aligned}$$

Writing $\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}$ in the form $\cos\varphi + i\sin\varphi$

Recall: $\cos\left(90^\circ - \frac{\theta}{2}\right) = \sin\frac{\theta}{2}$ and $\sin\left(90^\circ - \frac{\theta}{2}\right) = \cos\frac{\theta}{2}$

$$\begin{aligned}
 &= \cot\frac{\theta}{2} \cdot \frac{[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}]}{[\cos(90^\circ - \frac{\theta}{2}) - i\sin(90^\circ - \frac{\theta}{2})]} \\
 &= \cot\frac{\theta}{2} \cdot \frac{[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}]}{[\cos\{-(90^\circ - \frac{\theta}{2})\} - i\sin\{-(90^\circ - \frac{\theta}{2})\}]} \\
 &= \cot\frac{\theta}{2} \cdot \left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right] \cdot \left[\cos\left(\frac{\theta}{2} - 90^\circ\right) + i\sin\left(\frac{\theta}{2} - 90^\circ\right)\right]^{-1} \\
 &= \cot\frac{\theta}{2} \cdot \left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right] \cdot \left[\cos\left\{-\left(\frac{\theta}{2} - 90^\circ\right)\right\} + i\sin\left\{-\left(\frac{\theta}{2} - 90^\circ\right)\right\}\right] \\
 &= \cot\frac{\theta}{2} \cdot \left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right] \cdot \left[\cos\left(90^\circ - \frac{\theta}{2}\right) + i\sin\left(90^\circ - \frac{\theta}{2}\right)\right] \\
 &= \cot\frac{\theta}{2} \cdot \left[\cos\left(\frac{\theta}{2} + 90^\circ - \frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2} + 90^\circ - \frac{\theta}{2}\right)\right] \\
 &= \cot\frac{\theta}{2} \cdot [\cos(90^\circ) + i\sin(90^\circ)]
 \end{aligned}$$

$$\therefore \frac{1+z}{1-z} = i\cot\frac{\theta}{2}$$

3. Prove that $(\sin\theta + i\cos\theta)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$

Solution

$$(\sin\theta + i\cos\theta)^n = ?$$

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \text{ and } \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \sin\theta + i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$$

$$\begin{aligned}\therefore (\sin\theta + i \cos\theta)^n &= \left[\cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right) \right]^n \\ &= \cos\left\{n\left(\frac{\pi}{2} - \theta\right)\right\} + i \sin\left\{n\left(\frac{\pi}{2} - \theta\right)\right\} \\ \therefore (\sin\theta + i \cos\theta)^n &= \cos\left(\frac{n\pi}{2} - n\theta\right) + i \sin\left(\frac{n\pi}{2} - n\theta\right)\end{aligned}$$

4. (a) Find the modulus and argument of $(a + ib)^2$ where a and b are real numbers. Hence show that $\tan^{-1}\left(\frac{2ab}{a^2 - b^2}\right) = 2 \tan^{-1}\left(\frac{b}{a}\right)$
- (b) Two complex numbers z_1 and z_2 are such that $|z_1 - z_2| = |z_1 + z_2|$. Show that $|\theta_1 - \theta_2| = \frac{\pi}{2}$, where θ_1 and θ_2 are arguments of z_1 and z_2 respectively.

Solution

- a) let $z = (a + ib)^2$

$$\begin{aligned}|z| &= |a + ib|^2 = [\sqrt{a^2 + b^2}]^2 \\ \therefore |z| &= a^2 + b^2\end{aligned}$$

$$\arg(z) = \arg(a + ib)^2 = 2\arg(a + ib) = 2 \tan^{-1}\left(\frac{b}{a}\right)$$

$$\text{Also } z = (a + ib)^2 = a^2 + i2ab + i^2b^2$$

$$= a^2 - b^2 + i2ab$$

$$\Rightarrow \arg(z) = \tan^{-1}\left(\frac{2ab}{a^2 - b^2}\right) = 2 \tan^{-1}\left(\frac{b}{a}\right)$$

- b) $|\theta_1 - \theta_2| = \frac{\pi}{2}$; $|z_1 - z_2| = |z_1 + z_2|$

$$\text{Let } z_1 = r_1(\cos\theta_1 + i \sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$$

$$\begin{aligned}z_1 - z_2 &= r_1(\cos\theta_1 + i \sin\theta_1) - r_2(\cos\theta_2 + i \sin\theta_2) \\ &= r_1\cos\theta_1 + i r_1\sin\theta_1 - r_2\cos\theta_2 - r_2i \sin\theta_2 \\ &= (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)\end{aligned}$$

$$\begin{aligned}z_1 + z_2 &= r_1(\cos\theta_1 + i \sin\theta_1) + r_2(\cos\theta_2 + i \sin\theta_2) \\ &= r_1\cos\theta_1 + i r_1\sin\theta_1 + r_2\cos\theta_2 + r_2i \sin\theta_2 \\ &= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)\end{aligned}$$

$$\begin{aligned}|z_1 - z_2| &= \sqrt{[(r_1\cos\theta_1 - r_2\cos\theta_2)]^2 + [(r_1\sin\theta_1 - r_2\sin\theta_2)]^2} \\ &= \sqrt{r_1^2\cos^2\theta_1 - 2r_1\cos\theta_1r_2\cos\theta_2 + r_2^2\cos^2\theta_2 \\ &\quad + r_1^2\sin^2\theta_1 - 2r_1\sin\theta_1r_2\sin\theta_2 + r_2^2\sin^2\theta_2} \\ &= \sqrt{r_1^2(\cos^2\theta_1 + \sin^2\theta_1) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &\quad + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)}\end{aligned}$$

$$\therefore |z_1 - z_2| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

$$\begin{aligned}|z_1 + z_2| &= \sqrt{[(r_1\cos\theta_1 + r_2\cos\theta_2)]^2 + [(r_1\sin\theta_1 + r_2\sin\theta_2)]^2} \\ &= \sqrt{r_1^2\cos^2\theta_1 + 2r_1\cos\theta_1r_2\cos\theta_2 + r_2^2\cos^2\theta_2 \\ &\quad + r_1^2\sin^2\theta_1 + 2r_1\sin\theta_1r_2\sin\theta_2 + r_2^2\sin^2\theta_2} \\ &= \sqrt{r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &\quad + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)}\end{aligned}$$

$$\therefore |z_1 + z_2| = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}$$

Now $|z_1 - z_2| = |z_1 + z_2|$; squaring both sides;

$$r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$$

Simplifying

$$\begin{aligned} -\cos(\theta_1 - \theta_2) &= \cos(\theta_1 - \theta_2) \text{ or } 2\cos(\theta_1 - \theta_2) = 0 \\ \cos(\theta_1 - \theta_2) &= 0 \Rightarrow \theta_1 - \theta_2 = \cos^{-1}(0) = \frac{\pi}{2} \\ \therefore \theta_1 - \theta_2 &= \frac{\pi}{2} \text{ or } |\theta_1 - \theta_2| = \frac{\pi}{2} \end{aligned}$$

Task

1. If $z = 1 + \cos 2\theta + i \sin 2\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Prove that $|z| = 2\cos \theta$ and $\arg(z) = \theta$
2. If $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \lambda + i \sin \lambda$ and $x + y + z = 0$, prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$
3. If $\arg\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = \frac{\pi}{2}$, show that $|z_1| = |z_2|$

SOLVING EQUATIONS IN COMPLEX NUMBERS

a) Equations involving equality of complex numbers

Equality of complex numbers occurs if:

- i. If $a + ib = 0$, then $a = 0$, $b = 0$
- ii. If $a + ib = c + id$, then $a = c$, $b = d$

Example

- 1) Solve the simultaneous equations;

$$z_1 + z_2 = 8, 4z_1 - i3z_2 = 26 + 8i$$

Solution

$$\text{Let } z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$$

$$\Rightarrow z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = 8 + 0i$$

$$\therefore x_1 + x_2 = 8 \dots\dots\dots (1)$$

$$y_1 + y_2 = 0 \dots\dots\dots (2)$$

$$4z_1 - i3z_2 = 26 + 8i$$

$$\Rightarrow 4(x_1 + iy_1) - i3(x_2 + iy_2) = 26 + i8$$

$$4x_1 + i4y_1 - i3x_2 - i^2 3y_2 = 26 + i8$$

$$(4x_1 + 3y_2) + i(4y_1 - 3x_2) = 26 + i8$$

$$4x_1 + 3y_2 = 26 \dots\dots\dots (3)$$

$$4y_1 - 3x_2 = 8 \dots\dots\dots (4)$$

From eqn. (1)

$$x_2 = 8 - x_1 \dots\dots\dots (5)$$

eqn. (5) in to (4)

$$4y_1 - 3(8 - x_1) = 8 \Rightarrow 4y_1 + 3x_1 - 24 = 8$$

$$4y_1 + 3x_1 = 32 \dots\dots\dots (6)$$

From eqn. (2)

$$y_2 = -y_1 \dots\dots\dots (7)$$

eqn. (7) in to (3)

$$4x_1 - 3y_1 = 26 \dots\dots\dots (8)$$

Solving eqn. (6) and (8)

$$x_1 = 8, y_1 = 2$$

From eqn. (2) and eqn. (5)

$$y_2 = -2, x_2 = 0$$

$$\therefore z_1 = 8 + 2i, z_2 = 0 - 2i$$

Alternatively

$$z_1 + z_2 = 8 \dots\dots\dots (1)$$

$$4z_1 - i3z_2 = 26 + 8i \dots\dots\dots (2)$$

$$\text{eqn. (1)} \times 4 - \text{eqn. (2)}$$

$$\begin{array}{r} 4z_1 + 4z_2 = 32 + 0i \\ - \quad 4z_1 - i3z_2 = 26 + 8i \\ \hline (4 + 3i)z_2 = 6 - 8i \\ \therefore z_2 = \frac{6-8i}{4+3i} = \frac{(6-8i)(4-3i)}{(4+3i)(4-3i)} \\ = \frac{24-18i-32i+i^2 24}{16-(3i)^2} = \frac{0-50i}{16+9} \end{array}$$

$$\therefore z_2 = 0 - 2i$$

From eqn. (1)

$$z_1 = 8 - z_2 = 8 - (-2i) = 8 + 2i$$

$$\therefore z_1 = 8 + 2i$$

- 2) Given that $(3 + 2z^*)z = 5 + 2z$, where z^* is the conjugate of z . find z .

Solution

$$3z + 2zz^* = 5 + 2z$$

$$\Rightarrow 2zz^* = 5 - z$$

$$\text{Let } z = x + iy \Rightarrow z^* = x - iy$$

$$\therefore 2(x + iy)(x - iy) = 5 - (x + iy)$$

$$\Rightarrow 2(x^2 - (iy)^2) = (5 - x) - iy$$

$$\Rightarrow 2x^2 + 2y^2 = (5 - x) - iy$$

Equating Real parts and imaginary parts

$$\therefore 2x^2 + 2y^2 = 5 - x \dots\dots\dots (1)$$

$$\therefore 0 = -y \Rightarrow y = 0$$

For $y = 0$, $x = ?$

$$\Rightarrow 2x^2 = 5 - x \text{ or } 2x^2 + x - 5 = 0$$

$$\therefore x = 1.351, -1.851$$

$$\therefore z = 1.351 + 0i; -1.851 + 0i$$

- 3) Given that z is a complex number such that $z = \frac{p}{2-i} + \frac{q}{1+3i}$, where p and q are real.

Given that $\arg(z) = \frac{\pi}{2}$ and $|z| = 7$, find p and q .

Solution

$$\begin{aligned} z &= \frac{p}{2-i} + \frac{q}{1+3i} = z = \frac{p(2+i)}{(2-i)(2+i)} + \frac{q(1-3i)}{(1+3i)(1-3i)} \\ &= \frac{2p+2i}{4-i^2} + \frac{q-3qi}{1-i^2 3^2} = \frac{2p}{5} + \frac{2}{5}i + \frac{q}{10} - \frac{3q}{10}i \\ \therefore z &= \left(\frac{2p}{5} + \frac{q}{10}\right) + i\left(\frac{2}{5} - \frac{3q}{10}\right) \end{aligned}$$

$$\arg(z) = \tan^{-1}\left(\frac{\left(\frac{2}{5} - \frac{3q}{10}\right)}{\left(\frac{2p}{5} + \frac{q}{10}\right)}\right) = \tan^{-1}\left(\frac{2p-3q}{4p+q}\right) = \frac{\pi}{2}$$

$$\therefore \frac{2p-3q}{4p+q} = \tan \frac{\pi}{2} = \infty$$

$$\Rightarrow 4p + q = 0 \dots\dots\dots (1)$$

$$|z| = 7 = \sqrt{\left(\frac{2p}{5} + \frac{q}{10}\right)^2 + \left(\frac{2}{5} - \frac{3q}{10}\right)^2}$$

Squaring both sides

$$\Rightarrow 49 = \frac{(4p+q)^2 + (2p-3q)^2}{100}$$

$$\therefore 4900 = (4p + q)^2 + (2p - 3q)^2 \dots\dots\dots (2)$$

From eqn. (1)

$$q = -4p \dots\dots\dots (3)$$

eqn. (3) in to (2)

$$\Rightarrow 4900 = (4p - 4p)^2 + (2p - 3(-4p))^2$$

$$\Rightarrow 4900 = (14p)^2$$

Finding square root on both sides

$$\pm 70 = 14p \therefore p = \pm 5$$

For $p = \pm 5, q = ?$

From eqn. (1)

$$q = -4p = \pm 5 \times (-4) = \pm 20$$

$$\therefore p = 5, q = -20; p = -5, q = 20$$

- 4) Given that $u = -10 + i9$ as a complex number,

Find the complex number v which satisfies the equation $uv = -11 + i28$.

Hence verify that $|u + v| = 8\sqrt{2}$.

Solution

$$uv = -11 + i28 \Rightarrow v = \frac{-11+i28}{u} = \frac{-11+i28}{-10+i9}$$

$$= \frac{(-11+i28)(-10-i9)}{(-10+i9)(-10-i9)}$$

$$= \frac{110+i99-i280-i^2252}{(-10)^2-(i9)^2} = \frac{110-i181+252}{100+81}$$

$$= \frac{362}{181} - i\frac{181}{181} = 2 - i \therefore v = 2 - i$$

$$|u + v| = |-1 + i9 + 2 - i| = 8\sqrt{2}$$

$$= |-8 + 8i| = \sqrt{8^2 + 8^2} = 8\sqrt{2}$$

- 5) Find z if $\arg(z + 1) = \frac{\pi}{4}$ and $\arg(z - 1) = \frac{2\pi}{3}$

Solution

Let $z = x + iy, z + 1 = x + 1 + iy$

$$\Rightarrow \arg(z + 1) = \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan \frac{\pi}{4} = \frac{y}{x+1} = 1$$

$$\therefore y = x + 1 \dots\dots\dots (1)$$

$$z - 1 = x - 1 + iy$$

$$\Rightarrow \arg(z - 1) = \tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan \frac{2\pi}{3} = \frac{y}{x-1} = -\sqrt{3}$$

$$\therefore -\sqrt{3}x + \sqrt{3} = y \dots\dots\dots (2)$$

eqn. (1) = eqn. (2)

$$x + 1 = -\sqrt{3}x + \sqrt{3}$$

$$x(1 + \sqrt{3}) + 1 - \sqrt{3} = 0 \therefore x = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$x = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{3-2\sqrt{3}+1}{3-1} = 1 - \sqrt{3} \therefore x = 1 - \sqrt{3}$$

From eqn. (1)

$$y = x + 1 = 1 - \sqrt{3} + 1 = 2 - \sqrt{3}$$

$$\therefore z = (1 - \sqrt{3}) + i(2 - \sqrt{3})$$

- 6) If $(1 + i3)z_1 = 5(1 + i)$, find z_1

Solution

$$\begin{aligned}(1 + i3)z_1 &= 5(1 + i) \Rightarrow z_1 = \frac{5(1+i)}{(1+i3)} \\ z_1 &= \frac{(5+5i)}{(1+i3)} = \frac{(5+5i)(1-i3)}{(1+i3)(1-i3)} = \frac{5-i15+i5-i^215}{1^2-(i3)^2} \\ &= \frac{5+15-10i}{1+9} = \frac{20-i10}{10} = 2 - i\end{aligned}$$

$$\therefore z_1 = 2 - i$$

- 7) Given that x and y are real, find the values of x and y which satisfy the equation

$$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$$

Solution

$$\begin{aligned}\frac{2y+4i}{2x+y} - \frac{y}{x-i} &= 0 \\ \frac{2y}{2x+y} + \frac{4}{2x+y}i - \frac{y(x+i)}{(x-i)(x+i)} &= 0 \\ \frac{2y}{2x+y} + \frac{4}{2x+y}i - \frac{xy}{x^2+1} - \frac{y}{x^2+1}i &= 0\end{aligned}$$

For equality of complex numbers

$$\text{For real part: } \frac{2y}{2x+y} - \frac{xy}{x^2+1} = 0 \dots\dots\dots (1)$$

$$\text{For imaginary part: } \frac{4}{2x+y} - \frac{y}{x^2+1} = 0 \dots\dots\dots (2)$$

From eqn. (2)

$$\frac{4}{2x+y} = \frac{y}{x^2+1}, \text{ substituting in eqn. (1)}$$

$$\Rightarrow \frac{2y}{2x+y} - \frac{4}{2x+y} \cdot x = 0$$

$$\therefore \frac{2y-4x}{2x+y} = 0 \Rightarrow 2y - 4x = 0$$

$$\therefore y = 2x \dots\dots\dots (3)$$

Eqn. (3) in to eqn. (1)

$$\Rightarrow \frac{2(2x)}{2x+2x} - \frac{x(2x)}{x^2+1} = 0 \therefore 1 - \frac{2x^2}{x^2+1} = 0$$

$$\Rightarrow x^2 - 2x^2 + 1 = 0 \text{ or } x^2 = 1$$

$$\therefore x = \pm 1$$

From eqn. (3)

$$y = 2(\pm 1) = \pm 2$$

$$\therefore x = 1, y = 2; x = -1, y = -2$$

- 8) Find the values of x and y in the equation

$$\frac{x}{2+i3} - \frac{y}{3-i2} = \frac{6+2i}{1+i8} \quad (\text{Uneb 2006})$$

Solution

Expressing complex numbers in the form $a + ib$

$$\begin{aligned}\frac{x}{2+i3} - \frac{y}{3-i2} &= \frac{6+2i}{1+i8} \\ \frac{x(2-i3)}{(2+i3)(2-i3)} - \frac{y(3+i2)}{(3-i2)(3+i2)} &= \frac{(6+2i)(1-i8)}{(1+i8)(1-i8)} \\ \frac{2x-i3x}{4+9} - \frac{3y+i2y}{9+4} &= \frac{6-i48+i2-i^216}{1+64} \\ \frac{2x-i3x}{13} - \frac{3y+i2y}{13} &= \frac{6+16-i46}{65} \\ \Rightarrow \frac{2x}{13} - i\frac{3x}{13} - \frac{3y}{13} - i\frac{2y}{13} &= \frac{22}{65} - i\frac{46}{65}\end{aligned}$$

Equating real parts and imaginary parts

$$\text{For real part: } \Rightarrow \frac{2x}{13} - \frac{3y}{13} = \frac{22}{65} \dots \dots \dots \times 65$$

$$\therefore 10x - 15y = 22 \dots \dots \dots (1)$$

$$\text{For imaginary part: } -\frac{3x}{13} - \frac{2y}{13} = -\frac{46}{65} \dots \dots \dots \times 65$$

$$\therefore 15x + 10y = 46 \dots \dots \dots (2)$$

Solving eqn. (1) and eqn. (2) simultaneously
 $x = 2.8, y = 0.4$

Alternatively

Making the equation linear

$$\frac{x}{2+i3} - \frac{y}{3-i2} = \frac{6+2i}{1+i8}$$

$$\Rightarrow \frac{x(3-i2)-y(2+i3)}{(2+i3)(3-i2)} = \frac{6+2i}{1+i8}$$

$$\Rightarrow \frac{3x-i2x-2y-i3y}{6-i4+i9-i^26} = \frac{6+2i}{1+i8}$$

$$\Rightarrow \frac{(3x-2y)+i(-2x-3y)}{12+i5} = \frac{(6+2i)}{(1+i8)} \dots \dots \dots \times (12+i5)$$

$$\Rightarrow (3x-2y) + i(-2x-3y) = \frac{(6+2i)(12+i5)}{(1+i8)}$$

$$= \frac{72+i30+i24+i^210}{1+i8}$$

$$\frac{72+i30+i24+i^210}{1+i8} = \frac{62+i54}{1+i8} = \frac{(62+i54)(1-i8)}{(1+i8)(1-i8)}$$

$$= \frac{62+i54-i496-i^2432}{1+64}$$

$$= \frac{494}{65} - i \frac{442}{65} = \frac{38}{5} - i \frac{34}{5}$$

$$\therefore (3x-2y) = \frac{38}{5} \text{ or } 15x-10y = 38 \dots \dots \dots (1)$$

$$(-2x-3y) = -\frac{38}{5} \text{ or } 10x+15y = 34 \dots \dots \dots (2)$$

Solving eqn. (1) and eqn. (2) simultaneously
 $\therefore x = 2.8, y = 0.4$

Task

Given that x and y are real, find the values of x and y which satisfy the equation

a) $\frac{x+4i}{2x+y} - \frac{x+i}{y} = 0$

b) $\frac{3z}{1-i} + \frac{3z}{i} = \frac{4}{3-i}$

Answer: (b) $x = 0.27, y = 0.53$

ROOTS OF A GIVEN EQUATION

It can be proved that if $A + iB = x$ is a root of a quadratic equation $ax^2 + bx + c = 0$, then its conjugate is also a root.

Formulation of a quadratic equation with two given roots

The general formula used is:

$$x^2 - (\text{sum of the roots})x + \text{product of roots} = 0$$

Example

1) Find the equation whose one root is;

- (i) $2 - 3i$ (ii) $1 + i$ (iii) $3 + 4i$

Solution

i) Since $x = 2 - 3i$ is a root, then the other root $\bar{x} = 2 + 3i$ is also a root.

Sum of the roots: $2 - 3i + 2 + 3i = 4$

Product of roots: $(2 - 3i)(2 + 3i) = (2)^2 - (i3)^2 = 4 + 9 = 13$

$$\therefore x^2 - 4x + 13 = 0$$

ii) $x = 1 + i \Rightarrow x = 1 - i$ is also a root.

Sum of the roots: $1 + i + 1 - i = 2$

Product of roots: $(1 + i)(1 - i) = (1)^2 - (i)^2 = 1 + 1 = 2$

$$\therefore x^2 - 2x + 2 = 0$$

iii) $x = 3 + 4i \Rightarrow x = 3 - 4i$ is also a root.

Sum of the roots: $3 + 4i + 3 - 4i = 6$

Product of roots: $(3 + 4i)(3 - 4i) = (3)^2 - (i4)^2 = 9 + 16 = 25$

$$\therefore x^2 - 6x + 25 = 0$$

2) Show that $z = 1$ is the root of the equation $z^3 - 5z^2 + 7z - 5 = 0$. Hence solve the equation. (Unib 2003)

Solution

$$\text{Let } f(z) = z^3 - 5z^2 + 7z - 5 = 0$$

For $z = 1$

$\Rightarrow f(1) = (1)^3 - 5(1)^2 + 7(1) - 5 = 0$. Hence $z = 1$ is a root. Thus $(z - 1)$ is a root of $f(z)$. Factorizing $f(z)$

$$\begin{array}{r} z^2 - 4z + 5 \\ z - 1 \overline{) z^3 - 5z^2 + 7z - 5} \\ \underline{z^3 - z^2} \\ -4z^2 + 9z - 5 \\ \underline{-4z^2 + 4z} \\ 5z - 5 \\ \underline{5z - 5} \\ 0 \end{array}$$

$$\therefore f(z) = (z - 1)(z^2 - 4z + 5) = 0$$

$$z^2 - 4z + 5 = 0$$

$$z = \frac{4 \pm \sqrt{16 - 4(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$\therefore z = 2 \pm i$$

3) Show that $z = 2 + 3i$ is a root of the equation $z^4 - 5z^3 + 18z^2 - 17z + 13 = 0$

Solution

$$z = 2 + 3i$$

$$z^2 = (2 + 3i)^2 = 4 + 12i + i^2 9 = -5 + 12i$$

$$z^3 = (-5 + 12i)(2 + 3i) = -10 - 15i + 24i + i^2 36$$

$$= -10 + 9i - 36 = -46 + 9i$$

$$z^4 = (-5 + 12i)^2 = 25 - 120i + i^2 144 = -119 - 120i$$

$$\text{Let } f(z) = z^4 - 5z^3 + 18z^2 - 17z + 13 = 0$$

$$f(z) = -119 - 120i - 5(-46 + 9i) + 18(-5 + 12i) - 17(2 + 3i) + 13$$

$$= (-119 + 230 - 90 - 34 + 13) + i(-120 - 45 + 216 - 51) = 0$$

Hence $z = 2 + 3i$ is a root of the equation.

Since $2 + 3i$ is a root of the equation, also $2 - 3i$ is a root.

Quadratic factor in $f(z) = z^4 - 5z^3 + 18z^2 - 17z + 13$:

$$\text{Sum of roots: } 2 + 3i + 2 - 3i = 4$$

Product of roots: $(2 + 3i)(2 - 3i) = (2)^2 - (3i)^2 = 4 + 9 = 13$

$\therefore z^2 - 4z + 13$ is a quadratic factor of $f(z)$

Factorizing $f(z)$

$$\begin{array}{r}
 z^2 - 4z + 13 \overline{) z^4 - 5z^3 + 18z^2 - 17z + 13} \\
 \underline{z^4 - 4z^3 + 13z^2} \\
 -z^3 + 5z^2 - 17z + 13 \\
 \underline{-z^3 + 4z^2 - 3z} \\
 z^2 - 4z + 13 \\
 \underline{z^2 - 4z + 13} \\
 0
 \end{array}$$

$$\therefore f(z) = (z^2 - 4z + 13)(z^2 - z + 1) = 0$$

$$z^2 - z + 1 = 0$$

$$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2} \quad \therefore z = \frac{1 \pm i\sqrt{3}}{2}$$

- 4) Show that $z - 1 - i$ is a factor of the expression $z^4 - 6z^3 + 23z^2 - 34z + 26$. Hence solve the equation $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$

Solution

If $z - 1 - i$ is a factor then the root of the equation $f(z) = 0$ must be $z - 1 - i = 0$ or

$z = 1 + i$. Thus showing that $z = 1 + i$ is a root of $f(z) = 0$.

Let $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$

$$z = 1 + i$$

$$z^2 = (1 + i)^2 = 1 + 2i + i^2 = 2i$$

$$z^3 = (2i)(1 + i) = 2i + i^2 2 = -2 + 2i$$

$$z^4 = (2i)^2 = -4$$

$$\text{From } f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$$

$$f(z) = -4 - 6(-2 + 2i) + 23(2i) - 34(1 + i) + 26$$

$$= (-4 + 12 - 34 + 26) + i(-12 + 46 - 34) = 0$$

Hence $z = 1 + i$ is a root of the equation hence $z - 1 - i$ is a factor of the expression

$$z^4 - 6z^3 + 23z^2 - 34z + 26$$

Since $1 + i$ is a root of the equation, also $1 - i$ is a root.

Quadratic factor in $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$:

Sum of roots: $1 + i + 1 - i = 2$

Product of roots: $(1 + i)(1 - i) = (1)^2 - (i)^2 = 1 + 1 = 2$

$\therefore z^2 - 2z + 2$ is a quadratic factor of $f(z)$

Factorizing $f(z)$

$$\begin{array}{r}
 z^2 - 2z + 2 \overline{) z^4 - 6z^3 + 23z^2 - 34z + 26} \\
 \underline{z^4 - 2z^3 + 2z^2} \\
 -4z^3 + 21z^2 - 34z + 26 \\
 \underline{-4z^3 + 8z^2 - 8z} \\
 13z^2 - 26z + 26 \\
 \underline{13z^2 - 26z + 26} \\
 0
 \end{array}$$

$$\therefore f(z) = (z^2 - 2z + 2)(z^2 - 4z + 13) = 0$$

$$z^2 - 4z + 13 = 0$$

$$z = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\therefore z = 2 \pm 3i$$

- 5) Solve the equation $z^4 - 6z^2 + 25 = 0$

Solution

$z^4 - 6z^2 + 25 = 0$ is an equation reducible to quadratic of the form $au^2 +$

$$bu + c = 0$$

$$\text{Let } z^2 = u$$

$$\Rightarrow u^2 - 6u + 25 = 0$$

$$\therefore u = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

$$\text{But } z^2 = u \therefore z = \sqrt{u}$$

$$\text{For } u = 3 \pm 4i; \text{ let } z = a + bi$$

$$\Rightarrow (a + bi)^2 = 3 + 4i$$

$$\Rightarrow a^2 + i2ab - b^2 = 3 + 4i$$

$$\therefore a^2 - b^2 = 3 \dots\dots\dots (1)$$

$$\Rightarrow 2ab = \pm 4 \text{ or } ab = \pm 2$$

$$\therefore a = \pm \frac{2}{b} \dots\dots\dots (2)$$

eqn. (2) in to eqn. (1)

$$\Rightarrow \left(\frac{2}{b}\right)^2 - b^2 = 3 \Rightarrow \frac{4}{b^2} - b^2 = 3$$

$$\text{Let } m = b^2$$

$$\Rightarrow \frac{4}{m} - m = 3 \therefore m^2 + 3m - 4 = 0$$

Solving the equation yields $m = -4, 1$

$$\Rightarrow b = \sqrt{m}. \text{ Now for } m = -4, b = \sqrt{-4} = \pm 2i$$

$$\text{For } m = 1, b = \sqrt{1} = \pm 1$$

From eqn. (3)

$$a = \pm \frac{2}{b} \Rightarrow a = \pm \frac{2}{b}. \text{ For } b = \pm 2i \Rightarrow a = \pm \frac{2}{\pm 2i} = \pm i$$

$$\text{Also for } b = \pm 1 \Rightarrow a = \pm \frac{2}{\pm 1} = \pm 2$$

$$\therefore b = \pm 1, a = \pm 2; b = \pm 2i, a = \pm i$$

$$\Rightarrow z = \pm(2 + i)$$

Task

1. Show that $z = 1$ is a root of the equation $z^4 - 3z^3 + 4z^2 - 3z + 1 = 0$. Hence find the other roots. **Answer:** $1, 1, \frac{1}{2}(1 \pm i\sqrt{3})$
2. Given that that the complex number z and its conjugate $\bar{z}$ satisfy the equation: $z\bar{z} + 3\bar{z} = 34 - 12i$. Find the values of z . (**Uneb 2010**)
3. Given the complex number such that $z = \frac{p}{2-i} + \frac{q}{1+3i}$ where p and q are real. If $\arg(z) = \frac{\pi}{2}$ and $|z| = 7$, find the values of p and q
4. Find the real values of a and b such that $(a + ib)^2 = i$, hence or otherwise solve the equation $z^2 + 2z + 1 - i = 0$, where $z = a + ib$.
5. Solve for z in the form $a + ib$ if $\frac{2z-3}{1-4i} = \frac{3}{1+i}$

Equations involving n^{th} root of a complex number

Example

Solve the following equations

- i) $z^3 + 1 = 0$
- ii) $z^3 - 8 = 0$
- iii) $z^4 - 8 - i 8\sqrt{3} = 0$
- iv) $z^3 - 5 + i 12 = 0$
- v) $z^3 - \left(\frac{2-i}{2+i}\right) = 0$

Solution

- i) $z^3 + 1 = 0 \Rightarrow z^3 = -1 + 0i \therefore z = \sqrt[3]{-1 + 0i}$
 Expressing z in polar form
 $\Rightarrow z = 1 (\cos(-180^\circ) + i \sin(-180^\circ))$
 $\sqrt[3]{z} = [\cos(-180^\circ) + i \sin(-180^\circ)]^{\frac{1}{3}}$
 $= \cos\left(\frac{-180^\circ + 360^\circ k}{3}\right) + i \sin\left(\frac{-180^\circ + 360^\circ k}{3}\right), \text{ where } k = 0, 1 \text{ and } 2$
 For $k = 0, \sqrt[3]{z} = \cos\left(\frac{-180^\circ + 360^\circ \times 0}{3}\right) + i \sin\left(\frac{-180^\circ + 360^\circ \times 0}{3}\right)$
 $= \frac{1}{2} - \frac{i\sqrt{3}}{2}$
 For $k = 1, \sqrt[3]{z} = \cos\left(\frac{-180^\circ + 360^\circ \times 1}{3}\right) + i \sin\left(\frac{-180^\circ + 360^\circ \times 1}{3}\right)$
 $= \frac{1}{2} + \frac{i\sqrt{3}}{2}$
 For $k = 2, \sqrt[3]{z} = \cos\left(\frac{-180^\circ + 360^\circ \times 2}{3}\right) + i \sin\left(\frac{-180^\circ + 360^\circ \times 2}{3}\right)$
 $= -1 + 0i$
 $\therefore \sqrt[3]{z} = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}, -1$
- ii) $z^3 - 8 = 0 \Rightarrow z^3 = 8 + 0i \therefore z = \sqrt[3]{8 + 0i}$
 Expressing z in polar form
 $\Rightarrow z = 8 (\cos(0^\circ) + i \sin(0^\circ))$
 $\sqrt[3]{z} = [8 (\cos(0^\circ) + i \sin(0^\circ))]^{\frac{1}{3}}$
 $= 2 \left[\cos\left(\frac{0^\circ + 360^\circ k}{3}\right) + i \sin\left(\frac{0^\circ + 360^\circ k}{3}\right) \right], \text{ where } k = 0, 1 \text{ and } 2$
 For $k = 0, \sqrt[3]{z} = 2 \left[\cos\left(\frac{0^\circ + 360^\circ \times 0}{3}\right) + i \sin\left(\frac{0^\circ + 360^\circ \times 0}{3}\right) \right]$
 $= 2 + 0i$
 For $k = 1, \sqrt[3]{z} = 2 \left[\cos\left(\frac{0^\circ + 360^\circ \times 1}{3}\right) + i \sin\left(\frac{0^\circ + 360^\circ \times 1}{3}\right) \right]$
 $= -1 + i\sqrt{3}$
 For $k = 2, \sqrt[3]{z} = 2 \left[\cos\left(\frac{0^\circ + 360^\circ \times 2}{3}\right) + i \sin\left(\frac{0^\circ + 360^\circ \times 2}{3}\right) \right]$
 $= -1 - i\sqrt{3}$
 $\therefore \sqrt[3]{z} = 2 + 0i, -1 \pm i\sqrt{3}$
- iii) $z^4 - 8 - i 8\sqrt{3} = 0 \Rightarrow z^4 = -8 + i 8\sqrt{3} = z_1 \text{ (say)}$
 $\therefore z = \sqrt[4]{z_1}$
 Expressing z_1 in polar form
 $|z_1| = \sqrt{(8)^2 + (8\sqrt{3})^2} = 8\sqrt{1+3} = 16$
 $\arg(z_1) = \arg[8(-1 + i\sqrt{3})]$

$$\begin{aligned}
 &= \arg(-1 + i\sqrt{3}) \quad [\because \arg(\lambda z) = \arg(z) \text{ for } \lambda > 0] \\
 \therefore \arg(z_1) &= 120^\circ \\
 \Rightarrow z_1 &= 16(\cos(120^\circ) + i \sin(120^\circ)) \\
 \Rightarrow z &= \sqrt[4]{z_1} = \sqrt[4]{[16(\cos(120^\circ) + i \sin(120^\circ))]} \\
 &= 2[\cos(120^\circ) + i \sin(120^\circ)]^{\frac{1}{4}} \\
 &= 2 \left[\cos\left(\frac{120^\circ + 360^\circ k}{4}\right) + i \sin\left(\frac{120^\circ + 360^\circ k}{4}\right) \right], \text{ where } k = 0, 1, 2 \text{ and } 3 \\
 \text{For } k = 0, \sqrt[4]{z_1} &= 2 \left[\cos\left(\frac{120^\circ + 360^\circ \times 0}{4}\right) + i \sin\left(\frac{120^\circ + 360^\circ \times 0}{4}\right) \right] \\
 &= \sqrt{3} + i \\
 \text{For } k = 1, \sqrt[4]{z} &= 2 \left[\cos\left(\frac{120^\circ + 360^\circ \times 1}{4}\right) + i \sin\left(\frac{120^\circ + 360^\circ \times 1}{4}\right) \right] \\
 &= -1 + i\sqrt{3} \\
 \text{For } k = 2, \sqrt[4]{z} &= 2 \left[\cos\left(\frac{120^\circ + 360^\circ \times 2}{4}\right) + i \sin\left(\frac{120^\circ + 360^\circ \times 2}{4}\right) \right] \\
 &= -\sqrt{3} - i \\
 \text{For } k = 3, \sqrt[4]{z} &= 2 \left[\cos\left(\frac{120^\circ + 360^\circ \times 3}{4}\right) + i \sin\left(\frac{120^\circ + 360^\circ \times 3}{4}\right) \right] \\
 &= 1 - i\sqrt{3} \\
 \therefore z &= \pm(\sqrt{3} + i), \pm(1 - i\sqrt{3})
 \end{aligned}$$

iv) Left as an exercise **Answer:** $1[102^\circ 18', 1[222^\circ 18', 1[342^\circ 18', 0.953 - i 0.304$

LOCUS PROBLEMS IN COMPLEX NUMBERS

Consider a complex number $z = x + iy$. If z varies when subjected to some given condition, the corresponding set of points in the Argand diagram is called the *locus* of a point $P(x, y)$ representing z .

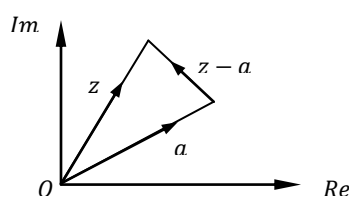
Loci are defined by:

- Distances
- Arguments/ angles

In general, equations of loci in Argand diagram involve moduli and arguments of complex variables.

Locus from Distances:

Consider the figure below.



$$z = x + iy, P(x, y)$$

$$a = a_1 + i a_2$$

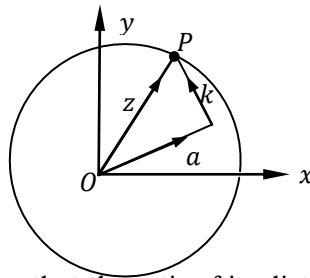
$$z - a = \overrightarrow{AP}$$

Finding magnitude of $\overrightarrow{AP}$

$$\Rightarrow |\overrightarrow{AP}| = |z - a| = k \text{ (+ve constant)}$$

Thus Distance of P from A is a constant. Hence

$|z - a| = k$ is the equation of a circle with centre a at the point A and radius k


NOTE

If a point $P(x, y)$ move so that the ratio of its distance from two fixed points A and B is constant, then the locus is a circle. This locus is referred to as *Apollonius circle* and is represented in the Argand diagram by the equation of the form

$$|z - a| = k|z - b| \text{ or } \left| \frac{z-a}{z-b} \right| = k \text{ for } k \neq 1$$

Example

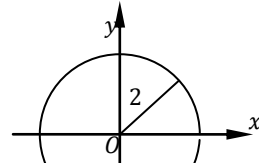
- 1) Sketch the loci defined by the equations:

- i) $|z| = 2$
- ii) $|z - 3 + 2i| = 5$
- iii) $|z - 4 + 3i| = 4$
- iv) $|z + i| = 3$

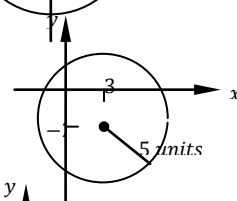
Solution

These are straight forward equations. Re-writing in the form $|z - a| = k$

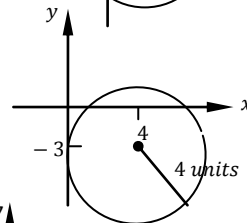
- i) $|z| = 2 \Rightarrow |z - (0 + 0i)| = 2$
 $\therefore$ Centre of circle: $(0, 0)$; Radius: 2
 $|x + iy| = 2 \Rightarrow x^2 + y^2 = 2^2$



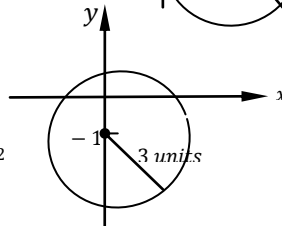
- ii) $|z - 3 + 2i| = 5 \Rightarrow |z - (3 - 2i)| = 5$
 Centre $(3, -2)$: Radius : 5
 $|(x - 3) + i(y + 2)| = 5 \Rightarrow (x - 3)^2 + (y + 2)^2 = 5^2$



- iii) $|z - 4 + 3i| = 4 \Rightarrow |z - (4 - 3i)| = 4$
 Centre $(4, -3)$: Radius : 4
 $|(x - 4) + i(y + 3)| = 5 \Rightarrow (x - 4)^2 + (y + 3)^2 = 4^2$



- iv) $|z + i| = 3 \Rightarrow |z - (0 - i)| = 3$
 Centre $(0, -1)$: Radius : 3
 $|(x - 0) + i(y + 1)| = 3 \Rightarrow x^2 + (y + 2)^2 = 3^2$



- 2) A point P represents a complex number $z = x + iy$ on an Argand diagram. Describe the locus of P if;
- a) $2|z + 1| = |z - 2|$

b) $|z + 4i| = 3|z - 4|$

c) $\left| \frac{z+i}{z-5-2i} \right| = 2$

d) $\left| \frac{z-1}{z+1-i} \right| = \frac{2}{3}$

e) $\left| \frac{z}{z+6} \right| = 5$

Solution

a) $2|z + 1| = |z - 2| \Rightarrow \frac{|z-2|}{|z+1|} = 2$

 Let $z = x + yi$

$$\frac{|z-2|}{|z+1|} = \frac{|(x-2)+iy|}{|(x+1)+iy|} = 2$$

$$\Rightarrow \frac{\sqrt{(x-2)^2+(y)^2}}{\sqrt{(x+1)^2+(y)^2}} = 2, \text{ squaring both sides}$$

$$\Rightarrow \frac{(x-2)^2+(y)^2}{(x+1)^2+(y)^2} = 4 \therefore \frac{x^2-4x+4+y^2}{x^2+2x+1+y^2} = 4$$

$$\Rightarrow x^2 - 4x + 4 + y^2 = 4[x^2 + 2x + 1 + y^2]$$

$$\Rightarrow 3x^2 + 12x + 3y^2 = 0 \therefore x^2 + y^2 + 4x = 0$$

 Circle with centre $(-2, 0)$, Radius = $\sqrt{4 + 0 - 0} = 2 \text{ units}$

b) $|z + 4i| = 3|z - 4| \Rightarrow \frac{|z+4i|}{|z-4|} = 3$

$$\frac{|z+4i|}{|z-4|} = \frac{|x+i(y+4)|}{|(x-4)+iy|} = 3$$

$$\Rightarrow \frac{\sqrt{x^2+(y+4)^2}}{\sqrt{(x-4)^2+y^2}} = 3, \text{ squaring both sides}$$

$$\Rightarrow \frac{x^2+(y+4)^2}{(x-4)^2+y^2} = 9 \therefore \frac{x^2+y^2+8y+16}{x^2-8x+16+y^2} = 9$$

$$\Rightarrow x^2 + y^2 + 8y + 16 = 9[x^2 - 8x + 16 + y^2]$$

$$\Rightarrow 8x^2 - 72x - 8y + 128 + 8y^2 = 0 \therefore x^2 + y^2 - 9x - y + 16 = 0$$

 Circle with centre $\left(\frac{9}{2}, \frac{1}{2}\right)$, Radius = $\sqrt{\frac{81}{4} + \frac{1}{4} - 16} = \frac{\sqrt{21}}{2} \text{ units}$

c) $\left| \frac{z+i}{z-5-2i} \right| = 2 \Rightarrow \frac{|z+i|}{|z-5-2i|} = 2$

$$\frac{|z+i|}{|z-5-2i|} = \frac{|x+i(y+1)|}{|(x-5)+i(y-2)|} = 2$$

$$\Rightarrow \frac{\sqrt{x^2+(y+1)^2}}{\sqrt{(x-5)^2+(y-2)^2}} = 2, \text{ squaring both sides}$$

$$\Rightarrow \frac{x^2+(y+1)^2}{(x-5)^2+(y-2)^2} = 4 \therefore \frac{x^2+y^2+2y+1}{x^2-10x+25+y^2-4y+4} = 4$$

$$\Rightarrow x^2 + y^2 + 2y + 1 = 4[x^2 + y^2 - 10x - 4y + 29]$$

$$\Rightarrow 3x^2 + 3y^2 - 40x - 18y + 115 = 0 \therefore x^2 + y^2 - \frac{40}{3}x - 6y + \frac{115}{3} = 0$$

 Circle with centre $\left(\frac{20}{3}, 3\right)$, Radius = $\sqrt{\frac{400}{9} + 9 - \frac{115}{3}} = \frac{\sqrt{136}}{3} = \frac{2}{3}\sqrt{34} \text{ units}$

d) $\left| \frac{z-1}{z+1-i} \right| = \frac{2}{3} \Rightarrow \frac{|z-1|}{|z+1-i|} = \frac{2}{3}$

$$\frac{|z-1|}{|z+1-i|} = \frac{|(x-1)+iy|}{|(x+1)+i(y-1)|} = \frac{2}{3}$$

$$\Rightarrow \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+(y-1)^2}} = \frac{2}{3}, \text{ squaring both sides}$$

$$\Rightarrow \frac{(x-1)^2+y^2}{(x+1)^2+(y-1)^2} = \frac{4}{9} \therefore \frac{x^2+y^2-2x+1}{x^2+2x+1+y^2-2y+1} = \frac{4}{9}$$

$$\Rightarrow 9[x^2 + y^2 - 2x + 1] = 4[x^2 + 2x + 1 + y^2 - 2y + 1]$$

$$\Rightarrow 9x^2 + 9y^2 - 18x + 9 = 4x^2 + 8x + 8 + 4y^2 - 8y$$

$$\therefore 5x^2 + 5y^2 - 26x + 8y + 1 = 0$$

Circle with centre $\left(\frac{13}{5}, -\frac{4}{5}\right)$, Radius = $\sqrt{\frac{169}{25} + \frac{16}{25} - 1} = \frac{\sqrt{184}}{5} = \frac{2}{5}\sqrt{46}$ units

Task

If $z = x + iy$, determine the Cartesian equation of loci of the point z which moves in the Argand diagram so that;

- i) $|z + 2i|^2 + |z - 2i|^2 = 40$
- ii) $|z + 2i|^2 - |z - 2i|^2 = 24$
- iii) $|z + ki|^2 + |z - ki|^2 = 30k^2$

ANSWER: (i) $x^2 + y^2 = 16$ (ii) $y = 3$ (iii) $x^2 + y^2 = 4k^2$

NOTE:

For $\left|\frac{z-a}{z-b}\right| = k$ and $k = 1$, this is a case where locus is a straight line

Example

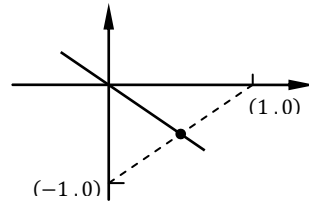
Sketch the locus of the point $P(x, y)$ representing the complex number $z = x + iy$ given that

- i) $|z - 1| = |z + i|$
- ii) $\left|\frac{z+i}{z-5-2i}\right| = 1$

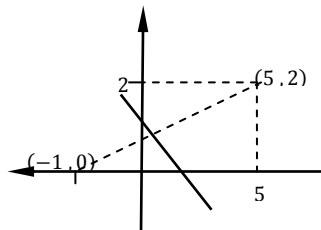
Write also the Cartesian equation of the locus.

Solution

- i) $|z - 1| = |z + i|$; $z = x + iy$
 $|(x - 1) + iy| = |x + i(y + 1)|$
 $\Rightarrow \sqrt{(x - 1)^2 + y^2} = \sqrt{x^2 + (y + 1)^2}$;
 squaring both sides
 $\Rightarrow (x - 1)^2 + y^2 = x^2 + (y + 1)^2$
 $\Rightarrow x^2 + y^2 - 2x + 1 = x^2 + y^2 + 2y + 1$
 $\Rightarrow -2x = 2y \therefore y = -x$ (Line with slope = -1)



- ii) $\left|\frac{z+i}{z-5-2i}\right| = 1 \Rightarrow |z + i| = |z - 5 - 2i|$
 $|x + i(y + 1)| = |(x - 5) + i(y - 2)|$
 $\Rightarrow \sqrt{x^2 + (y + 1)^2} = \sqrt{(x - 5)^2 + (y - 2)^2}$; squaring both sides
 $\Rightarrow x^2 + (y + 1)^2 = (x - 5)^2 + (y - 2)^2$
 $\Rightarrow x^2 + y^2 + 2y + 1 = x^2 + y^2 - 10x - 4y + 25 + 4$
 $\Rightarrow 6y = -10x + 28 \therefore y = -\frac{5}{3}x + \frac{14}{3}$ (Line with slope = $-\frac{5}{3}$)



LOCUS PROBLEMS FROM EQUALITY OF COMPLEX NUMBERS

1. Given that $\operatorname{Re}\left(\frac{z-2i}{z+4}\right) = 0$, prove that in the Argand diagram, the locus of z is a circle. Find the centre and write as a complex number. Show that the radius is $\sqrt{5}$

Solution

$$\frac{z-2i}{z+4} = \frac{x+iy-2i}{x+iy+4} = \frac{x+i(y-2)}{(x+4)+iy}; \text{ rationalizing the denominator,}$$

$$\Rightarrow \frac{[x+i(y-2)][(x+4)-iy]}{[(x+4)+iy][(x+4)-iy]} = \frac{x^2+4x-ixy+i(x+4)(y-2)-i^2(y^2-2y)}{(x+4)^2-i^2y^2}$$

$$\Rightarrow \frac{z-2i}{z+4} = \frac{x^2+4x-ixy+i(x+4)(y-2)+(y^2-2y)}{(x+4)^2+y^2}$$

$$\therefore \frac{z-2i}{z+4} = \frac{(x^2+4x+y^2-2y)}{(x+4)^2+y^2} + i \frac{(x+4)(y-2)-xy}{(x+4)^2+y^2}$$

For Real part = 0

$$\Rightarrow \frac{(x^2+4x+y^2-2y)}{(x+4)^2+y^2} = 0$$

$\therefore x^2 + 4x + y^2 - 2y = 0$, which is the required locus of a circle.

$$\text{Centre: } \left(-\frac{4}{2}, \frac{-2}{2}\right) = (-2, 1)$$

$$\therefore \text{Centre: } -2 + i$$

$$\text{Radius: } \sqrt{2^2 + 1^2 - 0} = \sqrt{5}$$

2. If $z = x + iy$ where x and y are real, show that

i) when $\frac{z+i}{z+2}$ is purely imaginary the locus of the point $P(x, y)$ is a circle of radius $\frac{1}{2}\sqrt{5}$

ii) When $\frac{z-2i}{2z-i}$ is purely imaginary the locus of a point $P(x, y)$ in the Argand diagram is a circle. Write its centre as a complex number.

iii) When $\frac{z-i}{z+2}$ is purely real the locus of the point $P(x, y)$ is a straight line.

Solution

- i) For purely imaginary complex number, Real part = 0

$$\frac{z+i}{z+2} = \frac{x+iy+i}{x+iy+2} = \frac{x+i(y+1)}{(x+2)+iy}; \text{ rationalizing the denominator,}$$

$$\Rightarrow \frac{[x+i(y+1)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]} = \frac{x^2+2x-ixy+i(x+2)(y+1)-i^2(y^2+y)}{(x+2)^2-i^2y^2}$$

$$\Rightarrow \frac{z+i}{z+2} = \frac{x^2+2x-ixy+i(x+2)(y+1)+(y^2+y)}{(x+2)^2+y^2}$$

$$\therefore \frac{z+i}{z+2} = \frac{(x^2+2x+y^2+y)}{(x+2)^2+y^2} + i \frac{(x+2)(y+1)-xy}{(x+2)^2+y^2}$$

For Real part = 0

$$\Rightarrow \frac{(x^2+2x+y^2+y)}{(x+2)^2+y^2} = 0$$

$\therefore x^2 + y^2 + 2x + y = 0$, which is the required locus of a circle.

$$\therefore \text{Centre: } \left(-1, -\frac{1}{2}\right)$$

$$\therefore \text{Centre: } 1 - \frac{1}{2}i$$

$$\text{Radius: } \sqrt{1^2 + \left(\frac{1}{2}\right)^2 - 0} = \frac{1}{2}\sqrt{5}$$

- ii) $\frac{z-2i}{2z-i} = \frac{x+iy-2i}{2x+i2y-i} = \frac{x+i(y-2)}{2x+i(2y-1)}$; rationalizing the denominator,

$$\Rightarrow \frac{z-2i}{2z-i} = \frac{[x+i(y-2)][2x-i(2y-1)]}{[2x+i(2y-1)][2x-i(2y-1)]} = \frac{2x^2-i(2xy-x)+i(2xy-4x)-i^2(y-2)(2y-1)}{(2x)^2-i^2(2y-1)^2}$$

$$\Rightarrow \frac{z-2i}{2z-i} = \frac{2x^2-i(2xy-x)+i(2xy-4x)+2y^2-5y+2}{4x^2+(2y-1)^2}$$

$$\therefore \frac{z-2i}{2z-i} = \frac{(2x^2+2y^2-5y+2)}{4x^2+(2y-1)^2} + i \frac{-3x}{4x^2+(2y-1)^2}$$

For Real part = 0

$$\Rightarrow \frac{(2x^2+2y^2-5y+2)}{4x^2+(2y-1)^2} = 0$$

$\therefore 2x^2 + 2y^2 - 5y + 2 = 0$ or $x^2 + y^2 - \frac{5}{2}y + 1 = 0$, which is the required locus of a circle.

$$\therefore \text{Centre: } \left(0, \frac{5}{4}\right)$$

$$\therefore \text{Centre: } 0 + \frac{5}{4}i$$

iii) $\frac{z-i}{z+2}$ is purely Real if Imaginary part = 0

$$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{(x+2)+iy}; \text{ rationalizing the denominator,}$$

$$\Rightarrow \frac{[x+i(y-1)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]} = \frac{x^2+2x-ixy+i(x+2)(y-1)-i^2(y^2-y)}{(x+2)^2-i^2y^2}$$

$$\Rightarrow \frac{z+i}{z+2} = \frac{x^2+2x-ixy+i(x+2)(y-1)+(y^2-y)}{(x+2)^2+y^2}$$

$$\therefore \frac{z+i}{z+2} = \frac{(x^2+2x+y^2-y)}{(x+2)^2+y^2} + i \frac{(x+2)(y-1)-xy}{(x+2)^2+y^2}$$

For Imaginary part = 0

$$\Rightarrow \frac{(x+2)(y-1)-xy}{(x+2)^2+y^2} = 0$$

$$\therefore (x+2)(y-1) - xy = 0$$

$$\Rightarrow xy - x + 2y - 2 - xy = 0$$

$$\therefore 2y = x + 2, \text{ which is a straight line.}$$

Task

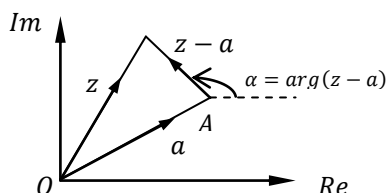
- 1) If $z = \frac{3}{3+\cos\theta + i\sin\theta}$, Prove that the locus of z is a circle $x^2 + y^2 = 4x - 3$
- 2) If n is a variable and $z = 4n + i3(1-n)$, show that the locus of z is a straight line.

LOCUS PROBLEMS FROM ARGUMENT OF COMPLEX NUMBER

Two major cases shall be considered.

Case I: $\arg(z-a) = \alpha$

P is a point representing complex number $z = x + iy$. A is a complex number $a = c + id$. $\overline{AP}$ is a straight line



Deduction

$\arg(z-a) = \alpha$, is the equation of a half line with end point A inclined at an angle α to the +ve Real axis.

Example

1. Sketch the loci defined by the equations

i) $\arg(z - 1 - 2i) = \frac{\pi}{4}$

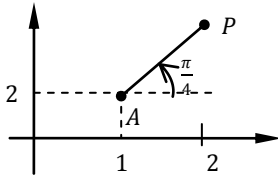
ii) $\arg(z + 2) = -\frac{2\pi}{3}$

iii) $\arg(z + 1 - 3i) = \frac{\pi}{6}$

iv) $\arg(z + 1 - i) = -\frac{\pi}{4}$

Solution

- i)
- $\arg(z - 1 - 2i) = \frac{\pi}{4}$
- , can be written as
- $\arg[z - (1 + 2i)] = \frac{\pi}{4} \equiv \arg(z - a) = \alpha$

 This is a locus of half line with end point $(1, 2)$ inclined at an angle $\frac{\pi}{4}$

Note:

Direction arrow of angle measurement is in anticlockwise since

$$\arg(z - 1 - 2i) = \frac{\pi}{4} = +ve$$

Finding Cartesian equation of line AP

 Let $z = x + iy$

$$\Rightarrow \arg(z - 1 - 2i) = \arg((x - 1) + (y - 2)i) = \frac{\pi}{4}$$

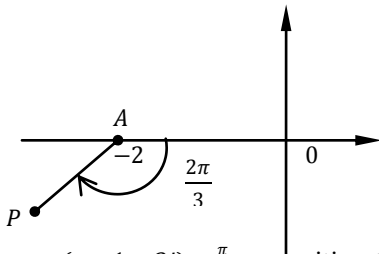
$$\Rightarrow \tan^{-1}\left(\frac{y-2}{x-1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{y-2}{x-1} = \tan \frac{\pi}{4} = 1$$

$$\therefore y - 2 = x - 1 \text{ or } y = x + 1$$

- ii)
- $\arg(z + 2) = -\frac{2\pi}{3}$
- , re-writing the equation

$$\Rightarrow \arg(z + 2) = \arg(z - (-2 + 0i)) = -\frac{2\pi}{3} \equiv \arg(z - a) = \alpha$$



Finding Cartesian equation of locus

 Let $z = x + iy$

$$\Rightarrow \arg(z + 2) = \arg((x + 2) + yi) = -\frac{2\pi}{3}$$

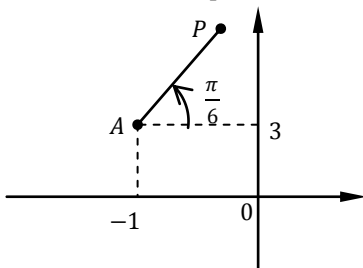
$$\Rightarrow \tan^{-1}\left(\frac{y}{x+2}\right) = -\frac{2\pi}{3}$$

$$\Rightarrow \frac{y}{x+2} = \tan\left(-\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$\therefore y = -\sqrt{3}(x + 2)$$

- iii)
- $\arg(z + 1 - 3i) = \frac{\pi}{6}$
- , re-writing the equation

$$\Rightarrow \arg(z + 1 - 3i) = \arg(z - (-1 + 3i)) = \frac{\pi}{6} \equiv \arg(z - a) = \alpha$$

 Half line with end point $A(-1, 3)$


Finding Cartesian equation of locus

 Let $z = x + iy$

$$\Rightarrow \arg(z + 1 - 3i) = \arg((x + 1) + (y - 3)i) = \frac{\pi}{6}$$

$$\Rightarrow \tan^{-1}\left(\frac{y-3}{x+1}\right) = \frac{\pi}{6}$$

$$\Rightarrow \frac{y-3}{x+1} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y - 3 = \frac{1}{\sqrt{3}}(x + 1)$$

$$\therefore y = \frac{1}{\sqrt{3}}x + \left(\frac{1}{\sqrt{3}} + 3\right)$$

iv) Left as an exercise for the student

Case II: $\arg\left(\frac{z-a}{z-b}\right) = \gamma$

Let $\arg(z-a) = \alpha$, $\arg(z-b) = \beta$ and $\arg\left(\frac{z-a}{z-b}\right) = \gamma$

where $\gamma = \alpha - \beta$ ($\pm 2\pi$ if necessary)

NOTE:

γ can either be positive or negative depending on the turn of the half lines from the end points at a and b . *Anti clockwise turn* is $\overrightarrow{BP}$ to $\overrightarrow{AP}$ and γ is $+ve$ and for *clockwise turn* i.e. $\overrightarrow{AP}$ to $\overrightarrow{BP}$. Two cases arise here

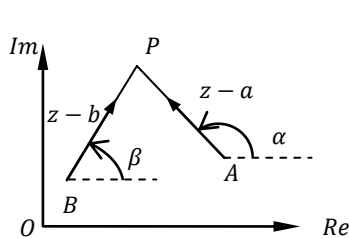


Fig $\arg\left(\frac{z-a}{z-b}\right) > 0$ i.e. $\alpha > \beta$

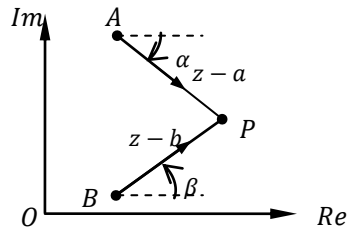


Fig $\arg\left(\frac{z-a}{z-b}\right) < 0$ i.e. $\alpha < \beta$

In general, the equation $\arg\left(\frac{z-a}{z-b}\right) = \lambda$, where $\lambda = \text{constant}$ represents a circular arc with end points A and B .

Example

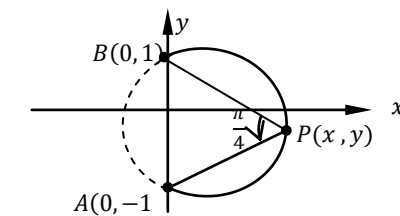
1. The point $P(x, y)$ represents $z = x + iy$ as the complex number on the Argand diagram. Describe and sketch the locus of P if;

- $\arg\left(\frac{z+i}{z-i}\right) = \frac{\pi}{4}$
- $\arg\left(\frac{z-3}{z-1}\right) = \frac{\pi}{4}$
- $\arg\left(\frac{z+2}{z-i}\right) = \frac{\pi}{3}$
- $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$
- $\arg\left(\frac{z-3-i}{z+5-3i}\right) = \frac{\pi}{3}$

Solution

a) $\arg\left(\frac{z+i}{z-i}\right) = \frac{\pi}{4}$; re-writing the equation in the form $\arg\left(\frac{z-a}{z-b}\right) = \lambda$

$\arg\left(\frac{z+i}{z-i}\right) = \frac{\pi}{4} = \arg\left[\frac{z-(0-i)}{z-(0+i)}\right] = \frac{\pi}{4}$, circular arc with end points $A(0, -1)$ and $B(0, 1)$



$$\therefore \arg\left(\frac{z+i}{z-i}\right) = \tan^{-1} \left[\frac{2x}{\frac{x^2+(y-1)^2}{x^2+(y+1)^2}} \right] = \frac{\pi}{4}$$

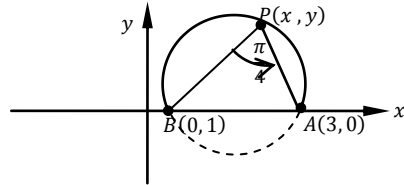
Cartesian equation of locus:

$$\begin{aligned} \frac{z+i}{z-i} &= \frac{x+i(y+1)}{x+i(y-1)} = \frac{[x+i(y+1)][x-i(y-1)]}{[x+i(y-1)][x-i(y-1)]} \\ &= \frac{x^2-i(xy-x)+i(xy+x)-i^2(y+1)(y-1)}{x^2-i^2(y-1)^2} \\ &= \frac{(x^2+y^2-1)+i(2x)}{x^2+(y-1)^2} \end{aligned}$$

$$\Rightarrow \frac{2x}{x^2+y^2-1} = \tan \frac{\pi}{4} = 1$$

$$\therefore x^2 + y^2 - 2x - 1 = 0, \text{ centre } (1, 0)$$

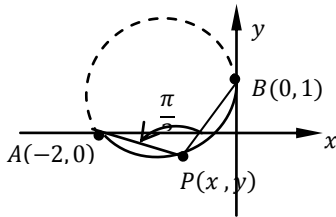
b) $\arg\left(\frac{z-3}{z-1}\right) = \arg\left[\frac{z-(3+0i)}{z-(1+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $A(3, 0)$ and $B(0, 1)$



Cartesian equation of locus:

$$\begin{aligned} \frac{z-3}{z-1} &= \frac{(x-3)+iy}{(x-1)+iy} = \frac{[(x-3)+iy][(x-1)-iy]}{[(x-1)+iy][(x-1)-iy]} \\ &= \frac{(x-3)(x-1)-i(xy-3y)+i(xy-y)-i^2y^2}{(x-1)^2+(iy)^2} \\ &= \frac{x^2-4x+3+i(3y-y)+y^2}{x^2-i^2(y-1)^2} \\ &= \frac{(x^2+y^2-4x+3)+i(2y)}{x^2+(y-1)^2} \\ \therefore \arg\left(\frac{z-3}{z-1}\right) &= \tan^{-1} \left[\frac{2y/(x^2+(y-1)^2)}{(x^2+y^2-4x+3)/(x^2+(y-1)^2)} \right] = \frac{\pi}{4} \\ \therefore x^2 + y^2 - 4x - 2y + 3 &= 0, \text{ centre } (2, 1) \end{aligned}$$

c) $\arg\left(\frac{z+2}{z-i}\right) = \arg\left[\frac{z-(-2+0i)}{z-(0+i)}\right] = \frac{\pi}{3}$, circular arc with end points $A(-2, 0)$ and $B(0, 1)$



Cartesian equation of locus:

$$\begin{aligned} \frac{z+2}{z-i} &= \frac{(x+2)+iy}{x+i(y-1)} = \frac{[(x+2)+iy][x-i(y-1)]}{[x+i(y-1)][x-i(y-1)]} \\ \Rightarrow \frac{z+2}{z-i} &= \frac{x^2+2x-i(y-1)(x+2)+ixy-i^2(y^2-y)}{x^2-i^2(y-1)^2} \\ &= \frac{x^2+y^2+2x-y+i(xy+2y-x-2+xy)}{x^2+(y-1)^2} \\ &= \frac{x^2+y^2+2x-y+i(-x+2y-2)}{x^2+(y-1)^2} \end{aligned}$$

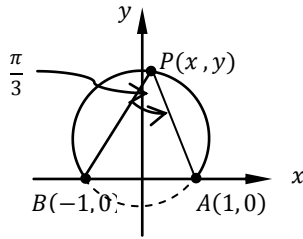
$$\begin{aligned} \therefore \arg\left(\frac{z+2}{z-i}\right) &= \tan^{-1} \left[\frac{(-x+2y-2)/(x^2+(y-1)^2)}{(x^2+y^2+2x-y)/(x^2+(y-1)^2)} \right] = \frac{\pi}{3} \\ \Rightarrow \frac{-x+2y-2}{x^2+y^2+2x-y} &= \tan \frac{\pi}{3} = \sqrt{3} \\ \Rightarrow -x+2y-2 &= \sqrt{3}(x^2+y^2+2x-y) \\ \Rightarrow x^2+y^2+2x-y &= -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y - \frac{2}{\sqrt{3}} \\ \therefore x^2+y^2 + \left(2+\frac{1}{\sqrt{3}}\right)x + \left(-\frac{2}{\sqrt{3}}-1\right)y + \frac{2}{\sqrt{3}} &= 0, \text{ centre } (-0.7, -0.1) \end{aligned}$$

d) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \arg\left[\frac{z-(1+0i)}{z-(-1+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $A(1, 0)$ and $B(-1, 0)$

Note:

Since $\overrightarrow{BP}$ is rotating through $\frac{\pi}{4}$ (Anticlock wise) to $\overrightarrow{AP}$, arrow direction is as shown.

Cartesian equation of locus:



$$\begin{aligned}\frac{z-1}{z+1} &= \frac{(x-1)+iy}{(x+1)+iy} = \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]} \\ \Rightarrow \frac{z-1}{z+1} &= \frac{x^2-1-i(xy-x)+i(xy+y)-i^2y^2}{(x-1)^2+y^2} \\ \Rightarrow \frac{z-1}{z+1} &= \frac{x^2+y^2-1+i(xy+y-xy+y)}{(x-1)^2+y^2} \\ &= \frac{(x^2+y^2-1)+i(2y)}{(x-1)^2+y^2}\end{aligned}$$

$$\begin{aligned}\therefore \arg\left(\frac{z-1}{z+1}\right) &= \tan^{-1}\left[\frac{2y}{(x^2+y^2-1)}\right] = \tan\frac{\pi}{4} = 1 \\ \therefore x^2+y^2-2y-1 &= 0, \text{ centre } (0, 1)\end{aligned}$$

- e) $\arg\left(\frac{z-3-i}{z+5-3i}\right) = \frac{\pi}{3} \Rightarrow \arg\left[\frac{z-(3+i)}{z-(-5+3i)}\right] = \frac{\pi}{3}$, circular arc with end points $A(3, 1)$ and $B(-5, 3)$

Cartesian equation of locus:

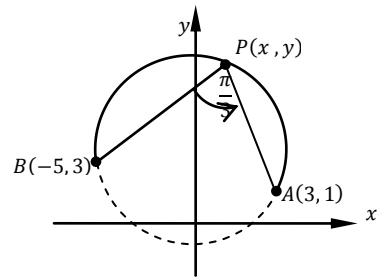
$$\begin{aligned}\frac{z-3-i}{z+5-3i} &= \frac{(x-3)+i(y-1)}{(x+5)+i(y-3)} = \frac{[(x-3)+i(y-1)][(x+5)-i(y-3)]}{[(x+5)+i(y-3)][(x+5)-i(y-3)]} \\ &= \frac{(x-3)(x+5)-i(x-3)(y-3)+i(y-1)(x+5)-i^2(y-3)(y-1)}{(x+5)^2+(y-3)^2} \\ &= \frac{x^2+2x-15+i(xy+5y-x-5-xy+3y+3x-9)+y^2-4y+3}{(x+5)^2+(y-3)^2} \\ &= \frac{(x^2+y^2+2x-4y-12)+i(2x+8y-14)}{(x+5)^2+(y-3)^2}\end{aligned}$$

$$\therefore \arg\left(\frac{z-3-i}{z+5-3i}\right) = \tan^{-1}\left[\frac{2x+8y-14}{x^2+y^2+2x-4y-12}\right] = \frac{\pi}{3}$$

$$\therefore \frac{2x+8y-14}{x^2+y^2+2x-4y-12} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\therefore x^2+y^2+2x-4y-12 = \frac{2}{\sqrt{3}}x + \frac{8}{\sqrt{3}}y - \frac{14}{\sqrt{3}}$$

$$\therefore x^2+y^2 + \left(2 - \frac{2}{\sqrt{3}}\right)x + \left(-4 - \frac{8}{\sqrt{3}}\right)y + \left(\frac{14}{\sqrt{3}} - 12\right) = 0, \text{ centre } (-0.4, 4.1)$$



Task

Sketch the locus of the following

- $\arg\left(\frac{z+1}{z-1}\right) = -\frac{2\pi}{3}$
 - $\arg\left(\frac{z-4}{z-2i}\right) = -\frac{\pi}{2}$
 - $\arg\left(\frac{z-2i}{z-1}\right) = \frac{\pi}{3}$
2. Represent each of the following loci on the Argand diagram
- $\arg(z-1) = \arg(z+1)$
 - $\arg(z) = \arg(z-1-i)$
 - $\arg(z-2) = \pi + \arg(z)$
 - $\arg(z-1) = \pi + \arg(z-i)$
 - $\arg(z) = \arg(z-4+2i) + \frac{\pi}{2}$
 - $\arg(w-5+3i) = \arg(w+2+3i) + \frac{\pi}{2}$

Solution

This example can be transformed in to forms as in example (1) above

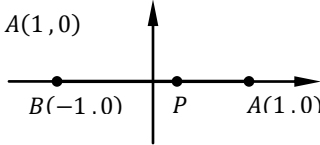
a) $\arg(z-1) = \arg(z+1) \Rightarrow \arg(z-1) - \arg(z+1) = 0$

$$\therefore \arg\left(\frac{z-1}{z+1}\right) = 0 \quad \left[\because \arg(z_1) - \arg(z_2) = \arg\left(\frac{z_1}{z_2}\right) \right]$$

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$$

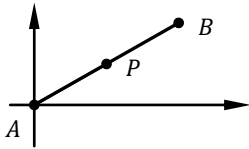
$$= \frac{x^2-1+i(-xy+y+xy+y)+y^2}{(x+1)^2+y^2} = \frac{x^2+y^2-1-i2y}{(x+1)^2+y^2}$$

$$\Rightarrow \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = 0 \therefore y = 0, \text{ is the locus of half straight line with end points } B(-1, 0) \text{ and } A(1, 0)$$



b) $\arg(z) = \arg(z-1-i) \Rightarrow \arg(z) - \arg(z-1-i) = 0$

$$\therefore \arg\left(\frac{z}{z-1-i}\right) = \arg\left[\frac{z-(0+0i)}{z-(1+i)}\right] = 0$$



Since the angle $\overrightarrow{BP}$ is to turn to $\overrightarrow{AP}$ is zero, this is a straight line joining A and B (circular arc in form of a straight line).

Cartesian equation of locus:

$$\frac{z}{z-1-i} = \frac{x+iy}{(x-1)+i(y-1)} = \frac{[x+iy][(x-1)-i(y-1)]}{[(x-1)+i(y-1)][(x-1)-i(y-1)]}$$

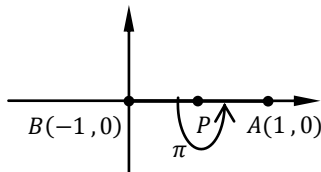
$$\therefore \frac{z}{z-1-i} = \frac{x^2-x+i(-xy+x+xy-y)+y^2-y}{(x-1)^2+(y-1)^2} = \frac{x^2+y^2-x-y+i(x-y)}{(x-1)^2+(y-1)^2}$$

$$\therefore \arg\left(\frac{z}{z-1-i}\right) = \tan^{-1}\left(\frac{x-y}{x^2+y^2-x-y}\right) = 0$$

$$\therefore x = y, \text{ is the locus of half straight line with end points } B(1, 1) \text{ and } A(0, 0)$$

c) $\arg(z-2) = \pi + \arg(z) \Rightarrow \arg(z-2) - \arg(z) = \pi$

$$\therefore \arg\left(\frac{z-2}{z}\right) = \arg\left[\frac{z-(2+0i)}{z-(0+0i)}\right] = \pi, \text{ circular arc in form of a straight line with end points } B(0, 0) \text{ and } A(2, 0)$$



Cartesian equation of locus:

$$\frac{z-2}{z} = \frac{(x-2)+iy}{(x+iy)} = \frac{[(x-2)+iy][x-iy]}{[x+iy][x-iy]}$$

$$= \frac{x^2-2x+i(-xy-2y+xy)+y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2-2x+i2y}{x^2+y^2}$$

$$\therefore \arg\left(\frac{z-2}{z}\right) = \tan^{-1}\left(\frac{2y}{x^2+y^2-2x}\right) = \pi$$

$$\Rightarrow \frac{2y}{x^2+y^2-2x} = \tan\pi = 0 \therefore y = 0$$

d) Left as an exercise for the students.

e) $\arg(z) = \arg(z-4+2i) + \frac{\pi}{2} \Rightarrow \arg(z) - \arg(z-4+2i) = \frac{\pi}{2}$

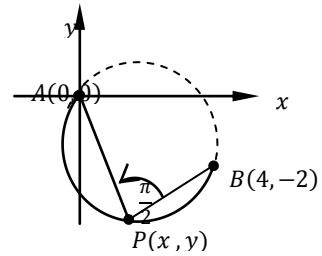
$$\therefore \arg\left(\frac{z}{z-4+2i}\right) = \arg\left[\frac{z-(0+0i)}{z-(4-2i)}\right] = \frac{\pi}{2}, \text{ circular arc with end points } B(4, -2) \text{ and } A(0, 0)$$

Cartesian equation of locus:

$$\frac{z}{z-4+2i} = \frac{(x^2+y^2-4x+2y)+i(-2x-4y)}{(x-4)^2+(y+2)^2} \quad \left[\begin{array}{l} \text{show this as a} \\ \text{student} \end{array} \right]$$

$$\therefore \arg\left(\frac{z}{z-4+2i}\right) = \tan^{-1}\left(\frac{-2x-4y}{x^2+y^2-4x+2y}\right) = \frac{\pi}{2}$$

$$\therefore x^2+y^2-4x+2y=0, \text{ since no value for } \tan\frac{\pi}{2}$$



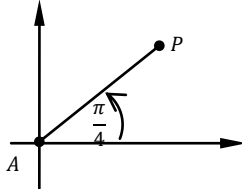
Further examples on locus involving arguments

- 1) Given that $z = x + iy$, find and sketch the locus $\arg(z) + \arg(\lambda z) = \frac{\pi}{2}$ if;
- $\lambda > 0$
 - $\lambda < 0$, where λ is a real number.

Solution

It can be shown that $\lambda = \lambda + 0i$

- a) $\arg(\lambda z) = \arg(\lambda) + \arg(z)$
 $\therefore \arg(\lambda z) = 0 + \arg(z) = \arg(z)$
 $\Rightarrow \arg(\lambda z) = \arg(z)$
 $\therefore \arg(z) + \arg(z) = \frac{\pi}{2} \Rightarrow 2\arg(z) = \frac{\pi}{2}$
 $\therefore \arg(z) = \frac{\pi}{4}$
 $\therefore \arg(z) = \arg[z - (0 + 0i)] = \frac{\pi}{4}$, which is half line with one end point as $(0, 0)$

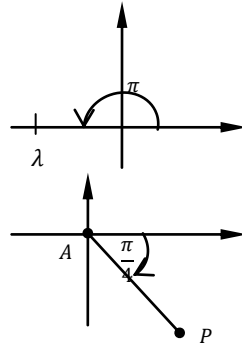


- b) $\arg(\lambda z) = \arg(\lambda) + \arg(z)$, for $\lambda < 0$. i.e. λ is -ve
 $\arg(\lambda) = \pi$ (The principal argument)
 $\Rightarrow \arg(\lambda z) = \pi + \arg(z)$

$$\therefore \arg(z) + \arg(\lambda z) = \arg(z) + \pi + \arg(z) = \frac{\pi}{2}$$

$$\Rightarrow 2\arg(z) = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

$$\therefore \arg(z) = -\frac{\pi}{4}$$



- 2) If $\arg(2z) + \frac{\pi}{4} = \arg(z + i)$, show that the locus of z is a circle.

Solution

$$\arg(2z) + \frac{\pi}{4} = \arg(z + i) \Rightarrow \arg(z + i) - \arg(2z) = \frac{\pi}{4}$$

$$\Rightarrow \arg(z + i) - \arg(z) = \frac{\pi}{4}$$

$$\Rightarrow \arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$$

$$\frac{z+i}{z} = \arg\left[\frac{z-(0-i)}{z-(0+0i)}\right] = \frac{\pi}{4}, \text{ circular arc with end points } B(0, 0) \text{ and } A(0, -1)$$

Cartesian equation of locus:

$$\frac{z+i}{z} = \frac{x+i(y+1)}{(x+iy)} = \frac{[x+i(y+1)][x-iy]}{[x+iy][x-iy]}$$

$$\therefore \frac{z+i}{z} = \frac{x^2-ixy+i(xy+x)+y^2+y}{x^2+y^2} = \frac{x^2+y^2+y+ix}{x^2+y^2}$$

$$\therefore \arg\left(\frac{z+i}{z}\right) = \tan^{-1}\left(\frac{x}{x^2+y^2+y}\right) = \frac{\pi}{4}$$

$$\therefore x^2 + y^2 - x + y = 0, \text{ which is a circle of centre } \left(\frac{1}{2}, -\frac{1}{2}\right) \text{ and radius } \frac{\sqrt{2}}{2} \text{ units}$$

Locus problems involving inequalities

This involves finding the equation of the locus, representing it on the Argand diagram and shading the unwanted region.

The knowledge portrayed in the previous examples shall be applied here. Locus problems may involve arguments, modulus of inequality complex number.

Example

1. Shade on separate Argand diagrams the regions in which;

- i) $|z - 4 + 4i| < 4$
- ii) $|z - 3| > |z - 5|$
- iii) $\left|\frac{z+i}{z-5-2i}\right| \geq 2$
- iv) $\left|\frac{z+2}{z-i}\right| > 3$
- v) $3 < |z| < 5$
- vi) $4 < |z + a| \leq 5$, where $a = 1 + i$

Solution

- i) Recall: $|z - 4 + 4i| = 4$ represents a circle of centre $(4, -4)$ and radius 4 units.

Let $z = x + iy$

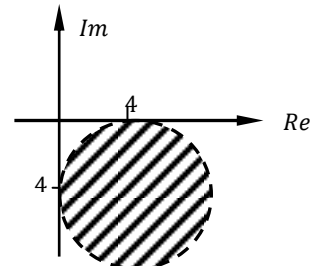
$$|z - 4 + 4i| \Rightarrow |(x - 4) + i(y + 4)| < 4$$

$$\therefore (x - 4)^2 + (y + 4)^2 < 4^2$$

Using the centre coordinate to find the wanted region to be shaded,

$$\therefore (4 - 4)^2 + (-4 + 4)^2 < 4^2, \text{ which is true.}$$

Hence shading the outside region of a circle,



- ii) $|z - 3| > |z - 5|$

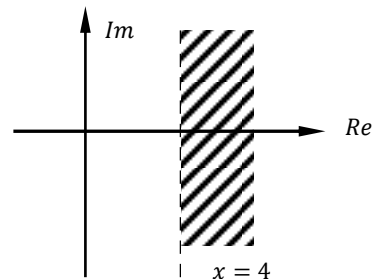
Finding the Cartesian inequality of locus

$$|(x - 3) + iy| > |(x - 5) + iy|, \text{ squaring both sides}$$

$$\Rightarrow (x - 3)^2 + y^2 > (x - 5)^2 + y^2$$

$$\Rightarrow x^2 - 6x + 9 > x^2 - 10x + 25$$

$$\Rightarrow 4x > 16 \therefore x > 4$$



- iii) $\left|\frac{z+i}{z-5-2i}\right| \geq 2$

Finding the Cartesian inequality of locus

$$\Rightarrow |z + i| \geq 2|z - 5 - 2i|$$

$$\therefore |x + i(y + 1)| \geq 2|(x - 5) + i(y - 2)|, \text{ squaring both sides}$$

$$\Rightarrow x^2 + (y + 1)^2 \geq 4[(x - 5)^2 + (y - 2)^2]$$

$$\Rightarrow x^2 + y^2 + 2y + 1 \geq 4x^2 - 40x + 100 + 4y^2 - 16y + 16$$

$$\Rightarrow 3x^2 + 3y^2 - 40x - 18y + 115 \leq 0$$

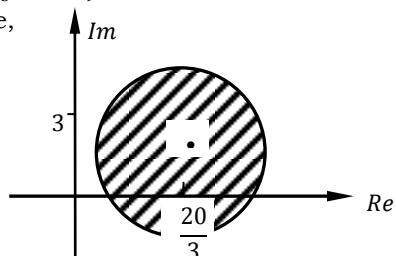
$$\therefore x^2 + y^2 - \frac{40}{3}x - 6y + \frac{115}{3} \leq 0, \text{ which is a circle of centre } \left(\frac{20}{3}, 3\right) \text{ and}$$

$$\text{radius } \sqrt{\left(\frac{20}{3}\right)^2 + 3^2 - \frac{115}{3}} = \frac{2}{3}\sqrt{34}$$

Using the centre coordinate to find the wanted region to be shaded,

$$\therefore \left(\frac{20}{3}\right)^2 + (3)^2 - \frac{40}{3}\left(\frac{20}{3}\right) - 6(3) + \frac{115}{3} = -\frac{274}{9} \leq 0, \text{ which satisfies the inequality.}$$

Shading the inside of the circle,



iv) $\left|\frac{z+2}{z-i}\right| > 3$

Finding the Cartesian inequality of locus

$$\Rightarrow |z+2| > 3|z-i|$$

$$\therefore |(x+2) + iy| > 3|x + i(y-1)|, \text{ squaring both sides}$$

$$\Rightarrow (x+2)^2 + y^2 > 9[x^2 + (y-1)^2]$$

$$\Rightarrow x^2 + 4x + 4 + y^2 > 9x^2 + 9y^2 - 18y + 9$$

$$\Rightarrow 8x^2 + 8y^2 - 4x - 18y + 5 > 0$$

$$\therefore x^2 + y^2 - \frac{1}{2}x - \frac{9}{4}y + \frac{5}{8} > 0, \text{ which is a circle of centre } \left(\frac{1}{4}, \frac{9}{8}\right) \text{ and}$$

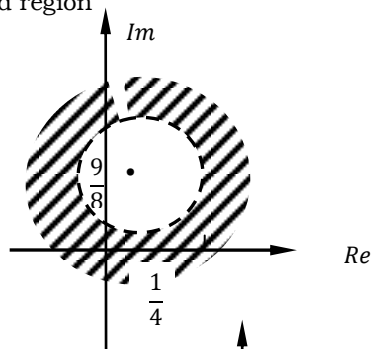
$$\text{radius } \sqrt{\left(-\frac{1}{4}\right)^2 + \left(-\frac{9}{8}\right)^2 - \frac{5}{8}} = \frac{3}{8}\sqrt{5}$$

Using the centre coordinate to find the wanted region to be shaded,

$$\therefore \left(\frac{1}{4}\right)^2 + \left(\frac{9}{8}\right)^2 - \frac{1}{8} - \frac{81}{32} + \frac{5}{8} = -\frac{45}{64} \leq 0,$$

, which does not satisfy the inequality.

Shading the outside of the circle,



v) $3 < |z| < 5$, for $3 < |z|$, $3^2 < x^2 + y^2$

$$\text{For } |z| < 5, x^2 + y^2 \leq 5^2$$

Using the centre coordinate to find the wanted

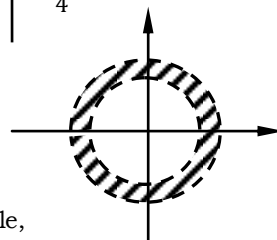
region to be shaded,

$$\text{For } 9 < x^2 + y^2 \Rightarrow 3^2 < 0^2 + 0^2, \text{ this is not true.}$$

Shading the outside of circle,

$$\text{Now for } |z| < 5, x^2 + y^2 < 5^2$$

$$\Rightarrow 0^2 + 0^2 < 5^2, \text{ this is true. Shading the inside of circle,}$$



vi) $4 < |z+a| \leq 5$, $a = 1+i$

$$\Rightarrow 4^2 < (x+1)^2 + (y+1)^2 \leq 5^2$$

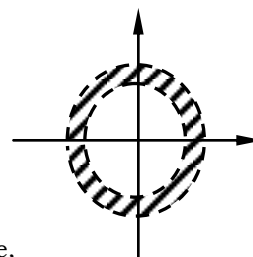
Using the centre $(-1, -1)$ to find the wanted region to be shaded,

$$\text{For } 4^2 < (x+1)^2 + (y+1)^2 \Rightarrow 4^2 < 0^2 + 0^2$$

, this is not true shading the outside of circle,

$$\text{Now for } (x+1)^2 + (y+1)^2 \leq 5^2$$

$$\Rightarrow 0^2 + 0^2 \geq 5^2, \text{ this is true. Shading the inside of circle,}$$



NOTE:

When testing region that is to be shaded, always use the centre to represent the inner region of the locus.

2. Shade the region represented by;

- $\arg(z + a) \leq \frac{\pi}{4}, a = 1 + i$
- $\arg(z - 1 + i) > \frac{\pi}{6}$
- $\arg(z + 2 - i) < -\frac{2\pi}{3}$
- $\arg(z - i) \geq -\frac{\pi}{3}$
- $-\frac{\pi}{2} \leq \arg(z + i) \leq \frac{2\pi}{3}$
- $0 < \arg(z - a) \leq \frac{\pi}{2}, \text{ where } a = 1 + i$
- $-\frac{\pi}{6} < \arg(z + a) < \frac{\pi}{4}; a = 2 + i$

Solution

- a) $\arg(z + a) \leq \frac{\pi}{4}, a = 1 + i$

Recall: $\arg(z - a) = \alpha$, represents a half a line with one end as a and making an angle of α with +ve real axis.

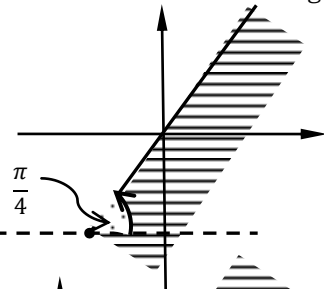
Cartesian inequality:

$$\arg(z + 1 + i) \leq \frac{\pi}{4}$$

$$\arg[(x + 1) + i(y + 1)] \leq \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{y+1}{x+1}\right) \leq \frac{\pi}{4} \Rightarrow y + 1 \leq (x + 1)\tan\frac{\pi}{4}$$

$\therefore y + 1 \leq x + 1$ or $y \leq x$, shading below the line,



- b) $\arg(z - 1 + i) > \frac{\pi}{6} \Rightarrow \arg[z - (1 - i)] > \frac{\pi}{6}$

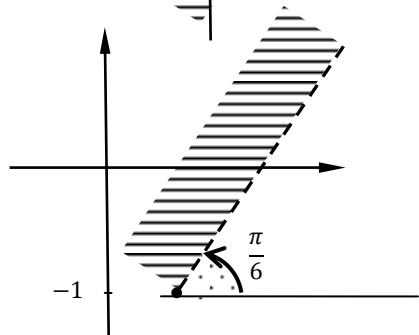
Cartesian inequality:

$$\arg[(x - 1) + i(y + 1)] > \frac{\pi}{6}$$

$$\Rightarrow \tan^{-1}\left(\frac{y+1}{x-1}\right) > \frac{\pi}{6} \Rightarrow y + 1 > (x - 1)\tan\frac{\pi}{6}$$

$$\therefore y + 1 > (x - 1)\frac{1}{\sqrt{3}} \text{ or } y > \frac{1}{\sqrt{3}}x + \left(-1 - \frac{1}{\sqrt{3}}\right)$$

, shading the above region of the line,



- c) $\arg(z + 2 - i) < -\frac{2\pi}{3} \Rightarrow \arg[z - (-2 + i)] < -\frac{2\pi}{3}$

Cartesian inequality:

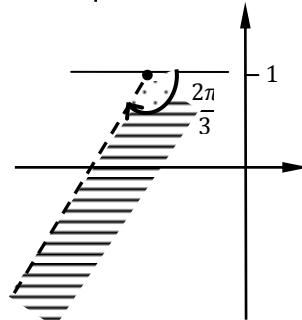
$$\arg[(x + 2) + i(y - 1)] < -\frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{y-1}{x+2}\right) < -\frac{2\pi}{3}$$

$$\Rightarrow y - 1 < (x + 2)\tan\left(-\frac{2\pi}{3}\right)$$

$$\therefore y - 1 < (x + 2)\sqrt{3} \text{ or } y < \sqrt{3}x + (1 + 2\sqrt{3})$$

, shading the below region of the line,



- d) left as an exercise

e) $-\frac{\pi}{2} \leq \arg(z + i) \leq \frac{2\pi}{3}$

$$\Rightarrow -\frac{\pi}{2} \leq \arg(z + i); \arg(z + i) \leq \frac{2\pi}{3}$$

Representing these on the Argand diagram,

$$\text{For } -\frac{\pi}{2} \leq \arg(z + i) \Rightarrow -\frac{\pi}{2} \leq \arg[z - (0 - i)]$$

Cartesian inequality:

$$-\frac{\pi}{2} \leq \arg[x + i(y + 1)]$$

$$\Rightarrow -\frac{\pi}{2} \leq \tan^{-1}\left(\frac{y+1}{x}\right) \text{ or } \tan\left(-\frac{\pi}{2}\right) \leq \frac{y+1}{x}$$

$\therefore 0 \leq x$ or $x \geq 0$, shading the **R.H.S** of the line,

$$\text{For } \arg(z + i) \leq \frac{2\pi}{3} \Rightarrow \arg[z - (0 - i)] \leq \frac{2\pi}{3}$$

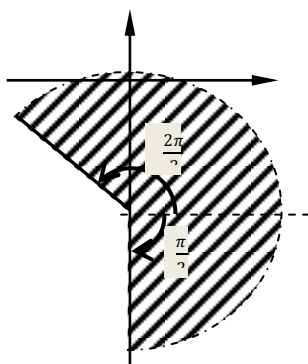
Cartesian inequality:

$$\arg[x + i(y + 1)] \leq \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{y+1}{x}\right) \leq \frac{2\pi}{3} \text{ or } \frac{y+1}{x} \leq \tan\left(\frac{2\pi}{3}\right)$$

$$\therefore \frac{y+1}{x} \leq -\sqrt{3} \text{ or } y + 1 \leq -\sqrt{3}x$$

$\therefore y \leq -\sqrt{3}x - 1$, shading the below region of the line,



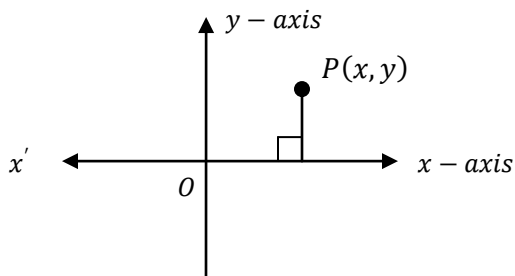
f) Left as an exercise

g) Left as an exercise

COORDINATE GEOMETRY THE STRAIGHT LINE

1) Position of a point in a plane

Consider the figure below



To locate a point $P(x, y)$ distances OM and MP are used.

$OM = x$, $MP = y$. $\overline{MP}$ and $\overline{OM}$ are perpendicular lines, to x and y axes respectively.

Distances $\overline{MP}$ and $\overline{OM}$ are **RECTANGULAR COORDINATES**

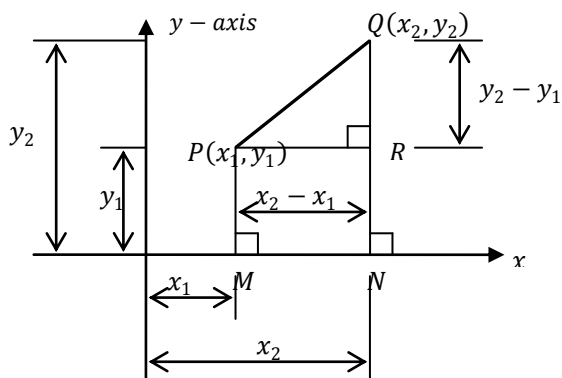
Ox and Oy are x -axis and y -axis respectively.

The x -coordinate of P is called **ABSCISSA** and the y -coordinate of P is called **ORDINATE**. O is the **origin**.

2) To find the distance between two given coordinates.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points in $x-y$ plane.

From the figure, distances are indicated w.r.t coordinates.



From ΔPQR ,
 $\overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2$

$$\therefore \overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

NOTE

If $P(0, 0)$ is the origin, then

$$\overline{PQ} = \sqrt{x^2 + y^2}$$

Example

1) Find the distance between the points

- $A(-6, 5)$ and $B(-1, -7)$
- $A(4, 4)$ and $B(1, -1)$
- $C(-1, -3)$ and $D(4, -7)$

Solution

- a) $A(-6, 5)$, $B(-1, -7) \Rightarrow (x_1, y_1) = (-6, 5)$ and $(x_2, y_2) = (-1, -7)$

$$\overline{AB} = \sqrt{(-1 - (-6))^2 + (-7 - 5)^2} = \sqrt{25 + 144} = 13 \text{ units}$$

- b) $A(4, 4)$, $B(1, -1) \Rightarrow (x_1, y_1) = (4, 4)$ and $(x_2, y_2) = (1, -1)$

$$\overline{AB} = \sqrt{(1 - 4)^2 + (-1 - 4)^2} = \sqrt{9 + 25} = \sqrt{34} \text{ units}$$

- c) $C(-1, -3)$, $D(4, -7) \Rightarrow (x_1, y_1) = (-1, -3)$ and $(x_2, y_2) = (4, -7)$

$$\overline{CD} = \sqrt{(4 - (-1))^2 + (-7 - (-3))^2} = \sqrt{25 + 16} = \sqrt{41} \text{ units}$$

Mathematics for an A-Level Student Volume II

- 2) Given that $A(a, 3)$, $B(1, -9)$ and $\overline{AB} = 13$ units, find the possible values of a .

Solution

$$\overline{AB} = 13 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \text{ but } A(a, 3) \equiv A(x_1, y_1), B(1, -9) \equiv B(x_2, y_2)$$

$$\Rightarrow 13 = \sqrt{(1 - a)^2 + (-9 - 3)^2}, \text{ squaring both sides}$$

$$169 = (1 - a)^2 + (-9 - 3)^2 \Rightarrow 25 = (1 - a)^2, \text{ finding the square root on both sides gives}$$

$$\Rightarrow \pm 5 = 1 - a \therefore a = 1 \pm 5 \text{ or } a = 6, -4$$

NOTE

Modulus /magnitude also represent length/distance between two given points.

Task

Given that $B(-4, 2b)$ and $D(-1, 1)$ are end points of a line segment, find the possible values of b if

a) $\overline{BD} = 5$ **Ans:** 2.5, -1.5

b) $\overline{BD} = \sqrt{58}$ **Ans:** 4, -3

3) Ratio formula (Internal and External division)

To find the coordinates of the point that divides the line joining the given points in the given ratio, say $m:n$

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points and $P(x, y)$, the point which divides AB in the ratio $m:n$.

The ratio of division is $\overline{AP} : \overline{PB} = m : n$

$$\Rightarrow \frac{\overline{AP}}{\overline{PB}} = \frac{m}{n}$$

By similar Δs .i.e. ΔPCB

$$\Rightarrow \frac{\overline{AP}}{\overline{PB}} = \frac{\overline{AE}}{\overline{PC}} = \frac{\overline{PE}}{\overline{BC}} = \frac{m}{n}$$

$$\therefore \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} \dots \dots \dots (1)$$

Since x and y are unknowns in (1) then

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x} \Rightarrow mx_2 - mx = nx - nx_1$$

$$\Rightarrow mx_2 + nx_1 = mx + nx$$

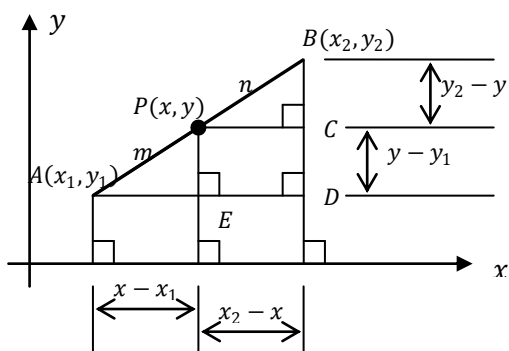
$$\therefore x(m + n) = mx_2 + nx_1 \Rightarrow x = \frac{mx_2 + nx_1}{m + n}$$

$$\text{Also } \frac{m}{n} = \frac{y - y_1}{y_2 - y} \Rightarrow my_2 - my = ny - ny_1$$

$$\Rightarrow my_2 + ny_1 = my + ny$$

$$\therefore y(m + n) = my_2 + ny_1 \Rightarrow y = \frac{my_2 + ny_1}{m + n}$$

$$\therefore P(x, y) = P \left[\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right]$$



, which is sometimes referred to as **SECTIONAL FORMULA**

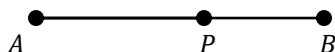
NOTE

- 1) The ratio of division can be stated in two ways.

a) *Internal division*

The ratio is $\overline{AP}:\overline{PB} = m:n$ (i.e. both m and n are +ve values).

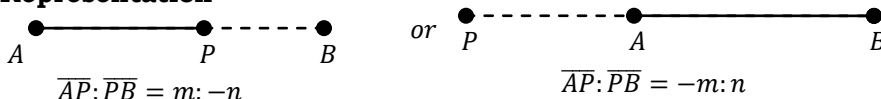
Representation



b) *External division*

The ratio is $\overline{AP}:\overline{PB} = m:-n$ (i.e. both m and n are of opposite signs).

Representation



Ratio formula of external division becomes

$$\therefore P(x, y) = P\left[\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right]$$

- 2) If P is the *midpoint* of AB , then $m = n$. Hence $P\left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right]$
- 3) Sometimes the ratio of division i.e. $\overline{AP}:\overline{PB} = m:n$ can be expressed in the form $\overline{AP}:\overline{PB} = k:1$
- Thus the ratio formula for the coordinates of a point P dividing $\overline{AB}$ in the ratio $k:1$ is $P\left[\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}\right]$
- 4) If the coordinates of the dividing point are given and the ratio is required, then take the ratio as $\overline{AP}:\overline{PB} = k:1$, for simplicity.

Examples

- 1) Find the coordinates of the point that divides the line $A(-7, 2)$ and $B(4, -4)$ in the ratio
- a) $-1:3$ b) $1:2$ c) $4:-3$ d) $4:1$

Solution

Given: $A(-7, 2) \equiv A(x_1, y_1)$; $B(4, -4) \equiv B(x_2, y_2)$

Let $P(x, y)$ be the point of division

- a) Ratio $\overline{AP}:\overline{PB} = -1:3 \equiv m:n$ [External division]

Using the general ratio formula

$$P\left[\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right] \equiv P\left[\frac{-1 \times 4 + 3 \times -7}{-1 + 3}, \frac{-1 \times -4 + 3 \times 2}{-1 + 3}\right] = P\left(-\frac{25}{2}, 5\right)$$

- b) Ratio $\overline{AP}:\overline{PB} = 1:2 \equiv m:n$ [Internal division]

$$\therefore P\left[\frac{1 \times 4 + 2 \times -7}{1 + 2}, \frac{1 \times -4 + 2 \times 2}{1 + 2}\right] = P\left(\frac{5}{3}, 0\right)$$

- c) $\overline{AP}:\overline{PB} = 4:-3 \equiv m:n$ [External division]

$$\therefore P\left[\frac{4 \times 4 + 3 \times -7}{4 - 3}, \frac{4 \times -4 + 3 \times 2}{4 - 3}\right] = P(37, -10)$$

- d) Ratio $\overline{AP}:\overline{PB} = 4:1 \equiv m:n$

$$\therefore P\left[\frac{4 \times 4 + 1 \times -7}{4 + 1}, \frac{4 \times -4 + 1 \times 2}{4 + 1}\right] = P\left(\frac{9}{5}, \frac{14}{5}\right)$$

- 2) Find the coordinates of the point which divides the line joining points $(2, -4)$ and $(7, 1)$

- a) Internally in the ratio 2:3
- b) Externally in the ratio 1:3

Solution

Let $A(2, -4) \equiv A(x_1, y_1)$ and $B(7, 1) \equiv B(x_2, y_2)$

Let $P(x, y)$ be the required point

From the ratio formula $P\left[\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right]$

- a) For internal division, $\overline{AP}:\overline{PB} = 2:3$ [Both ratio values are +ve]

$$\therefore P\left[\frac{2 \times 7 + 3 \times 2}{2+3}, \frac{2 \times 1 + 3 \times -4}{2+3}\right] = P(4, -2)$$

- b) For external division, $\overline{AP}:\overline{PB} = 1:-3$ [One of the ratio value is -ve]

$$\therefore P\left[\frac{1 \times 7 + (-3) \times 2}{1-3}, \frac{1 \times 1 + (-3) \times -4}{1-3}\right] = P\left(-\frac{1}{2}, -\frac{13}{2}\right)$$

- 3) Find the ratio in which the given points divide the given lines

- i) Point $(1, 3)$, line through $(0, 1)$ and $(-4, -7)$
- ii) Point $(1, -4)$, line through $(0, -5)$ and $(4, -1)$
- iii) Point $(1, -1)$, line through $(-1, -7)$ and $(-2, -10)$

Solution

- i) Let the required ratio be $\overline{AP}:\overline{PB} = k:1$

$$\therefore P\left[\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right] = P(1, 3), \text{ using the } x\text{-coordinate,}$$

$$\Rightarrow \frac{kx_2+x_1}{k+1} = 1 \Rightarrow kx_2 + x_1 = k + 1,$$

$$\text{Now } A(0, 1) \equiv A(x_1, y_1) \text{ and } B(-4, -7) \equiv B(x_2, y_2) \Rightarrow x_1 = 0, x_2 = -4$$

$$\Rightarrow -4k + 0 = k + 1 \therefore k = -\frac{1}{5}$$

$$\text{Ratio is } k:1 = -\frac{1}{5}:1 \text{ or } -1:5 \quad [\text{External division}]$$

Note: If the y -coordinate was used in finding k -value, $k = -\frac{1}{5}$ i.e.

$$\frac{ky_2+y_1}{k+1} = 3 \Rightarrow \frac{-7k+1}{k+1} = 3 \therefore k = -\frac{1}{5}$$

- ii) $\overline{AP}:\overline{PB} = k:1$, $A(0, -5) \equiv A(x_1, y_1)$, $B(4, -1) \equiv B(x_2, y_2)$

$$\therefore P\left[\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right] = P(1, -4) \Rightarrow \frac{kx_2+x_1}{k+1} = 1 \text{ or } \frac{ky_2+y_1}{k+1} = -4$$

$$\text{Using the } x\text{-coordinate, } \frac{k(4)+0}{k+1} = 1 \Rightarrow 4k = k + 1 \therefore k = \frac{1}{3}$$

$$\text{Ratio is } k:1 = \frac{1}{3}:1 \text{ or } 1:3 \quad [\text{Internal division}]$$

- iii) $\overline{AP}:\overline{PB} = k:1$, $A(-1, -7) \equiv A(x_1, y_1)$, $B(-2, -10) \equiv B(x_2, y_2)$

$$\therefore P\left[\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right] = P(1, -1) \Rightarrow \frac{kx_2+x_1}{k+1} = 1 \text{ or } \frac{ky_2+y_1}{k+1} = -1$$

$$\text{Using the } x\text{-coordinate, } \frac{k(-2)-1}{k+1} = 1 \Rightarrow -2k - 1 = k + 1 \therefore k = -\frac{2}{3}$$

$$\text{Ratio is } k:1 = -\frac{2}{3}:1 \text{ or } -2:3 \quad [\text{External division}]$$

- 4) Given that the three vertices of a rectangle are $A(2, -2)$, $B(8, 4)$, $C(5, 7)$. Find the fourth vertex.

Solution

Let $D(x, y)$, diagonals $\overline{AC}$ and $\overline{BD}$ meet at

$M = \text{midpoint of diagonals}$

$$\therefore \text{midpoint of } \overline{AC} = M\left(\frac{2+5}{2}, \frac{7+(-2)}{2}\right) = M\left(\frac{7}{2}, \frac{5}{2}\right)$$

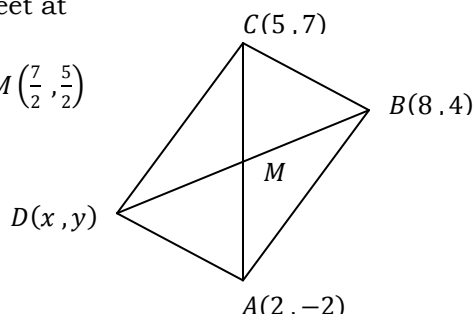
$$\text{midpoint of } \overline{BD} = M\left(\frac{x+8}{2}, \frac{y+4}{2}\right)$$

Equating x and y coordinates of M

$$\Rightarrow \frac{7}{2} = \frac{x+8}{2} \therefore x = -1$$

$$\Rightarrow \frac{5}{2} = \frac{y+4}{2} \therefore y = 1$$

$\therefore (-1, 1)$, is the required vertex.



- 5) If the three vertices of the triangle are points $A(2, 3)$, $B(-1, -3)$ and $C(3, 2)$. Find the coordinates of the fourth vertex if $ABCD$ are vertices of a parallelogram.

Solution

Diagonals $\overline{BC}$ and $\overline{AD}$ meet

at $M = \text{mid point}$

For diagonal $\overline{BC}$:

$$M\left(\frac{-1+3}{2}, \frac{-3+2}{2}\right) = M\left(\frac{2}{2}, -\frac{1}{2}\right)$$

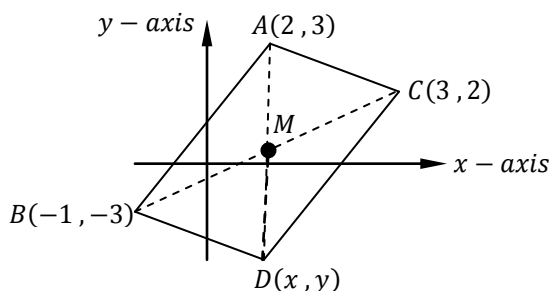
For diagonal $\overline{AD}$:

$$M\left(\frac{2+x}{2}, \frac{3+y}{2}\right) = M\left(\frac{2}{2}, -\frac{1}{2}\right)$$

$$\text{But } \frac{2}{2} = \frac{2+x}{2} \Rightarrow x = 0$$

$$\text{Also } -\frac{1}{2} = \frac{3+y}{2} \Rightarrow y = -4$$

$\therefore D(0, -4)$



NOTE:

Numbers of this kind can equally be done using distances between two points. i.e. $\overline{AB} = \overline{CD}$ and $\overline{BD} = \overline{AC}$

$$\Rightarrow \sqrt{(-1-2)^2 + (-3-3)^2} = \sqrt{(x-3)^2 + (y-2)^2}$$

$$\text{squaring, } 9 + 36 = (x-3)^2 + (y-2)^2$$

$$\Rightarrow 45 = x^2 - 6x + 9 + y^2 - 4y + 4 \therefore x^2 + y^2 - 6x - 4y = 32 \dots\dots\dots (1)$$

$$\Rightarrow \sqrt{(x+1)^2 + (y+3)^2} = \sqrt{(3-2)^2 + (2-3)^2}$$

$$\text{squaring, } 1 + 1 = (x+1)^2 + (y+3)^2 \therefore x^2 + y^2 + 2x + 6y = -8 \dots\dots\dots (2)$$

Solving (1) and (2) simultaneously, (1) - (2)

$$\Rightarrow -8x - 10y = 40 \text{ or } 4x + 5y = -20 \therefore x = \frac{-20-5y}{4} \dots\dots\dots (3)$$

eqn.(3) in to (2)

$$\Rightarrow \left[-\left(\frac{20+5y}{4}\right)\right]^2 + y^2 + 2\left[\frac{-20-5y}{4}\right] + 6y + 8 = 0, \text{ Simplifying and collecting like terms yields}$$

$41y^2 + 256y + 368 = 0$, solving the quadratic equation by quadratic formula gives

$$\therefore y = -\frac{92}{41}, -4$$

Now for $y = -4, x = ?$

From eqn. (3), $x = \frac{-20+20}{4} = 0 \therefore (0, -4)$ is the 4th vertex.

Also for $y = -\frac{92}{41}, x = \frac{-20+5 \times 92/41}{4} = -\frac{90}{41} \Rightarrow \left(-\frac{90}{41}, -\frac{92}{41}\right)$ is also a vertex. But plotting the coordinates of a parallelogram $\left(-\frac{90}{41}, -\frac{92}{41}\right)$ does not give a shape of parallelogram. Hence $(0, -4)$ is the required vertex.

- 6) ABCD is quadrilateral with $A(2, -2), B(5, -1), C(6, 2)$ and $D(3, 1)$. show that the quadrilateral is a rhombus.

Solution

For a rhombus

- All sides are equal
- Diagonals meet at 90°
- Opposite angles are $\neq 90^\circ$ but obtuse

$$\overline{AB} = \sqrt{(5-2)^2 + (-1-(-2))^2} = \sqrt{10}$$

$$\overline{BC} = \sqrt{(6-5)^2 + (2-(-1))^2} = \sqrt{10}$$

$$\overline{CD} = \sqrt{(3-6)^2 + (1-2)^2} = \sqrt{10}$$

$$\overline{AD} = \sqrt{(2-3)^2 + (-2-1)^2} = \sqrt{10}$$

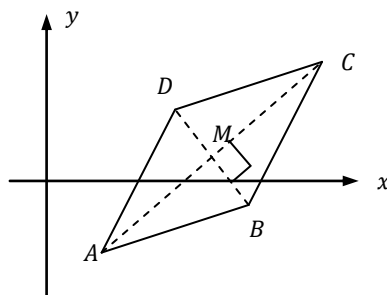
Hence all sides are equal.

Now finding coordinates of M, the point of intersection of diagonals.

$$\text{For } \overline{AC}: M\left[\frac{2+6}{2}, \frac{-2+2}{2}\right] = M(4, 0)$$

Now considering $\triangle AMB$, applying Pythagoras theorem

$$\Rightarrow \overline{AM}^2 + \overline{MB}^2 = \overline{AB}^2 \Rightarrow \left[\sqrt{(4-2)^2 + (0+2)^2}\right]^2 + \left[\sqrt{(4-5)^2 + (0+1)^2}\right]^2 = \left[\sqrt{10}\right]^2 = \overline{AB}^2. \text{ Hence this is a rhombus.}$$



NOTE:

This question can still be handled by vectors.

COORDINATES OF THE CENTROID OF A TRIANGLE WITH GIVEN VERTICES.

Consider $\triangle ABC$ D, E and F

are midpoints of lines

$\overline{AB}, \overline{BC}$ and $\overline{AC}$ respectively.

Lines joining vertices to opposite

midpoints are *medians*, and

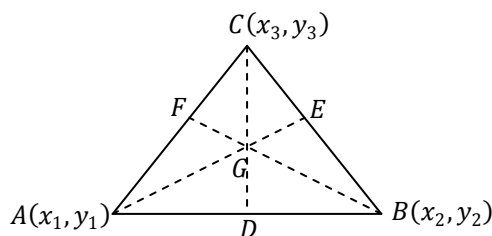
their intersection is the

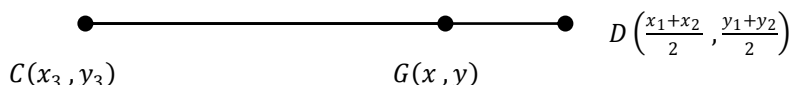
CENTROID.(G)

.i.e. Medians are concurrent at G.

Taking line $\overline{CD}$, G divides CD in the ratio $2 : 1$. i.e. $\overline{CG} : \overline{GD} = 2 : 1$

[This is a standard ratio, and is stated from the vertex
to the base containing the midpoint]





From the ratio theorem

$$G \left[\frac{2 \times \left(\frac{x_1+x_2}{2} \right) + 1 \times x_3}{2+1}, \frac{2 \times \left(\frac{y_1+y_2}{2} \right) + 1 \times y_3}{2+1} \right] = G \left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right]$$

$$\therefore \text{Centroid is } \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right)$$

Example

1. Find the centroid of the triangles below

- a) $A(2, 1), B(1, 2)$ and $C(3, 6)$
- b) $A(0, 1), B(3, 2)$ and $C(-3, -9)$

Solution

Let $G(x, y)$ be the required centroid

But $G \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right)$

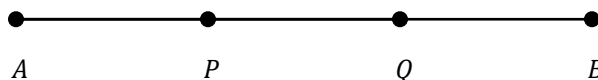
- a) $G \left(\frac{2+1+3}{3}, \frac{1+2+6}{3} \right) = G(2, 3)$
- b) $G \left(\frac{0+3-3}{3}, \frac{1+2-9}{3} \right) = G(0, -2)$

Task

- 1) Find the coordinates of one vertex of triangle having two of its vertices as $(5, 4), (6, 9)$ and its centroid as $(5, 6)$ **Ans:** $(4, 5)$
- 2) Find the ratio in which the join of
 - a) $(5, 8)$ and $(-9, -2)$ is divided by the point $(-2, 3)$ **Ans:** $1:1$
 - b) $(1, 4)$ and $(3, -2)$ is divided by the point $(-3, 16)$ **Ans:** $2:3$ (externally)
 - c) $(2, 3)$ and $(4, 1)$ is divided by the x-axis. **Ans:** $3:1$ (externally)

[HINT: Find the point of division by solving line joining points and x – axis]
- 3) The line joining points $(1, -2)$ and $(-3, 4)$ is trisected. Find the coordinates of point of trisection.

[HINT: Trisection means dividing line in to 3 – equal parts]. i. e.



[P and Q are needed. considering P; Ratio of division is $1:2 = \overline{AP}:\overline{PB}$
Now considering Q; Ratio of division is $\overline{AQ}:\overline{QB} = 2:1$]

Answer: $\left(-\frac{1}{3}, 0 \right), \left(-\frac{5}{3}, 2 \right)$

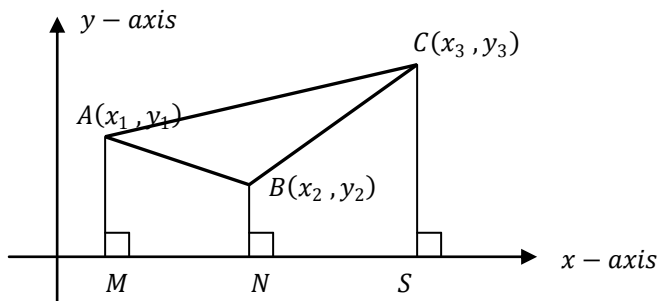
- 4) The three angular points of a parallelogram are; $(-2, -1), (1, 0)$ and $(4, 3)$. Find the coordinates of the fourth vertex. **Ans:** $(1, 2)$

- 5) The points $A(-2, 3), B(1, -2), C(8, 1)$ are vertices of a parallelogram. Find the coordinates of the fourth vertex. **Answer: (5, 0)**
- 6) $ABCD$ with $A(2, -2), B(5, -1), C(6, 2)$ and $D(x, y)$ is a rhombus. Find D .
Ans: (3, 1)

AREA OF A TRIANGLE

To find area of a triangle given its three vertices

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be three vertices of $\triangle ABC$.



Area of $\triangle ABC$ = Area of $ACMS$ - (Area of $AMBN$ + Area of $NSBC$)

Area of trapezium $ACMS = \frac{1}{2}(x_3 - x_1)(y_1 + y_3)$

Area of trapezium $AMNB = \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$

Area of trapezium $NBCS = \frac{1}{2}(x_3 - x_2)(y_2 + y_3)$

$$\begin{aligned} \text{Area of } AMBN + \text{Area of } NSBC &= \frac{1}{2}(x_2 - x_1)(y_1 + y_2) + \frac{1}{2}(x_3 - x_2)(y_2 + y_3) \\ &= \frac{1}{2}[x_2(y_1 + y_2) - x_1(y_1 + y_2) + x_3(y_2 + y_3) - x_2(y_2 + y_3)] \\ &= \frac{1}{2}[-x_1(y_1 + y_2) + x_2(y_1 - y_3) + x_3(y_2 + y_3)] \end{aligned}$$

$$\therefore \text{Area of } \triangle ABC = \frac{1}{2}(x_3 - x_1)(y_1 + y_3) - \frac{1}{2}[-x_1(y_1 + y_2) + x_2(y_1 - y_3) + x_3(y_2 + y_3)]$$

$$\begin{aligned} &= \frac{1}{2}[x_3(y_1 + y_3) - x_1(y_1 + y_3) + x_1(y_1 + y_2) - x_2(y_1 - y_3) - x_3(y_2 + y_3)] \\ &= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \\ &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \times \text{determinant of } \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \end{aligned}$$

$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note

During the derivation of the formula, the brackets containing the y 's have been maintained through out the derivation, while those of x 's are expanded.

Examples

Find the area of the triangle whose vertices are

- a) $(4, 6), (0, 4)$ and $(6, 2)$
- b) $(-1, 2), (5, 6)$ and $(5, -4)$
- c) $(4, 0), (-6, 0)$ and $(2, 6)$

Solution

$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\begin{aligned} \text{a) Area} &= \frac{1}{2} \begin{vmatrix} 4 & 6 & 1 \\ 0 & 4 & 1 \\ 6 & 2 & 1 \end{vmatrix} = \frac{1}{2} \left[4 \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} + 6 \begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix} \right] \\ &= \frac{1}{2} [4(4 - 2) - 0 + 6(6 - 4)] = \frac{1}{2} (8 + 12) = 10 \text{ sq. units} \end{aligned}$$

$$\begin{aligned} \text{b) Area} &= \frac{1}{2} \begin{vmatrix} -1 & 2 & 1 \\ 5 & 6 & 1 \\ 5 & -4 & 1 \end{vmatrix} = \frac{1}{2} \left[-1 \begin{vmatrix} 6 & 1 \\ -4 & 1 \end{vmatrix} - 5 \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + 5 \begin{vmatrix} 2 & 1 \\ 6 & 1 \end{vmatrix} \right] \\ &= \frac{1}{2} [-6(6 + 4) - 5(2 + 4) + 5(2 - 6)] = \frac{1}{2} \times (-60) = 30 \text{ sq. units (numerically)} \end{aligned}$$

$$\begin{aligned} \text{c) Area} &= \frac{1}{2} \begin{vmatrix} 4 & 0 & 1 \\ -6 & 0 & 1 \\ 2 & 6 & 1 \end{vmatrix} = \frac{1}{2} \left[4 \begin{vmatrix} 0 & 1 \\ 6 & 1 \end{vmatrix} - 0 \begin{vmatrix} -6 & 1 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} -6 & 1 \\ 2 & 1 \end{vmatrix} \right] \\ &= \frac{1}{2} [4(0 - 6) + 6(0 - 6) + 2(0 - 0)] = \frac{1}{2} \times (-60) = 30 \text{ sq. units (numerically)} \end{aligned}$$

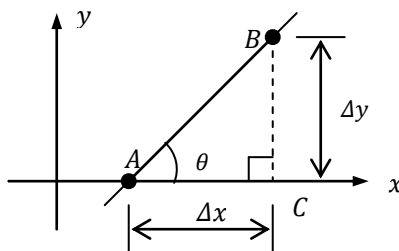
GRADIENT/SLOPE OF A LINE

Consider the figure below

$$\text{Slope, } m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

$$\text{But } \tan \theta = \frac{BC}{AC} = \frac{\Delta y}{\Delta x}$$

$\therefore m = \tan \theta$



Thus slope of a line is the *tangent* of the angle which the line makes with the *+ve x - axis*.

NOTE

- 1) The measurement of the angle can either be *+ve. i.e.* anticlockwise direction measurement or *-ve. i.e.* clockwise direction measurement *w.r.t.* *+ve x - axis*.
- 2) If $0^\circ < \theta < 90^\circ$, the slope of line is *+ve* and if $90^\circ < \theta < 180^\circ$, the slope of line is *-ve*.
If $\theta = 0^\circ$, $m = \tan 0 = 0$, line is parallel to *x - axis*

If $\theta = 180^\circ$, $m = \tan 180 = 0$, line is oppositely directed to the +ve sense of x - axis

If $\theta = 90^\circ$, $m = \tan 90 = \infty$, line is parallel to the y - axis. i. e vertical

Example

State the slope of the following lines

a) $y = x + 1$

b) $y = -\sqrt{3}x + 1$

c) $2y + 3x - 1 = 0$

d) $4y + 3x + 1 = 0$

e) Hence state the angle the lines make with the x - axis

Solution

a) $y = x + 1$, slope, $m = 1 \therefore \tan \theta = 1 \Rightarrow \theta = 45^\circ$

b) $y = -\sqrt{3}x + 1$, slope, $m = -\sqrt{3} \therefore \tan \theta = -\sqrt{3} \Rightarrow \theta = -60^\circ$ or 120°

c) $2y + 3x - 1 = 0 \Rightarrow y = -\frac{3}{2}x + \frac{1}{2}$, $m = -\frac{3}{2} \therefore \tan \theta = -\frac{3}{2} \Rightarrow \theta = -56.3^\circ$ or 123.7°

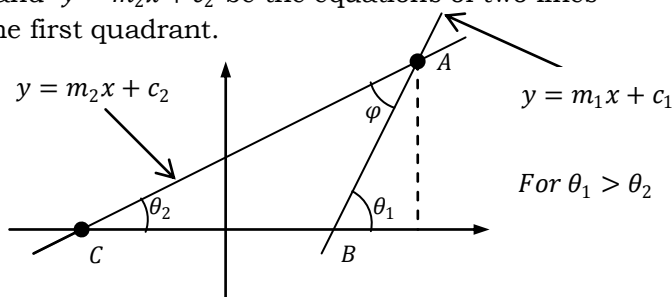
d) Left as an exercise. **Ans:** -36.9° or 143.1°

NOTE

1. The knowledge of trigonometry of finding tangents of angles is applicable
2. Before stating the gradient of a line, make sure that it is expressed in the form $Y = Mx + C$

ANGLE BETWEEN TWO LINES

Let $y = m_1x + c_1$ and $y = m_2x + c_2$ be the equations of two lines intersecting in the first quadrant.



Let ϕ = angle between lines, θ_1 and θ_2 be angles the lines make with x - axis

From the figure

$$\phi + \theta_2 = \theta_1 \therefore \phi = \theta_1 - \theta_2$$

Introducing tangent on both sides

$$\tan \phi = \tan(\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}, \text{ but } m_1 = \tan \theta_1, m_2 = \tan \theta_2$$

$$\therefore \tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$\therefore \phi = \tan^{-1} \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right)$

, which gives angle between two lines.

NOTE

1. From the figure above, θ_1 and θ_2 both are *acute* and $\theta_1 > \theta_2$ so that $\tan \theta_1 > \tan \theta_2 \therefore \varphi$ is *acute*. however if $m_2 > m_1$, then φ will be *obtuse*. ($90^\circ < \varphi < 180^\circ$)
2. If the lines are *parallel*, then $\varphi = 0$ or 180°
 $\therefore \tan 0 = \frac{m_1 - m_2}{1 + m_1 m_2} = 0 \Rightarrow m_1 - m_2 = 0$

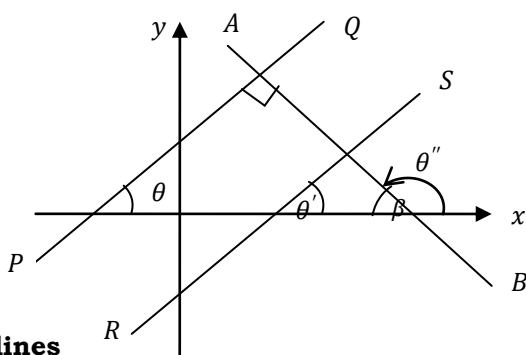
$\therefore$ For parallel lines, $m_1 = m_2$

3. If the lines are *perpendicular*, then $\varphi = 90^\circ$
 $\therefore \tan 90 = \frac{m_1 - m_2}{1 + m_1 m_2} = \infty \Rightarrow m_1 m_2 + 1 = 0$

$\therefore$ For perpendicular lines, $m_1 m_2 = -1$

Alternative derivation of condition for parallelism and perpendicularity of lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$.

Consider the figure below



Parallel lines

If PQ and RS are parallel, then

$$\theta = \theta' \text{ or } \tan \theta = \tan \theta' \Rightarrow m_1 = m_2$$

$\therefore$ If PQ is the line $y = m_1 x + c_1$ and RS is $y = m_2 x + c_2$, then the lines will be parallel if

$m_1 = m_2$

Perpendicular line

If PQ is perpendicular to AB, then $\beta + \theta'' = 180^\circ$

$$\text{Also } \theta + \beta = 90^\circ \Rightarrow \beta = 90 - \theta$$

$$\therefore 90 - \theta + \theta'' = 180 \Rightarrow \theta'' = 90 + \theta$$

From trigonometry, $\tan \theta'' = \tan(90 + \theta) = -\cot \theta$

$$\therefore \tan \theta'' = -\frac{1}{\tan \theta} \Rightarrow m_2 = -\frac{1}{m_1}$$

$\therefore m_1 m_2 = -1$

, which is the condition for perpendicular line

Examples

Find the angle between two straight lines whose equations are

- $y = 2x + 7$ and $y = \frac{1}{3}x - 4$
- $2y + 3x + 1 = 0$ and $y = x + 3$
- $2x - 3y + 1 = 0$ and $2x - 3y + 2 = 0$
- $2x + 3y - 2 = 0$ and $-3x + 2y + 3 = 0$

Solution

a) $y = 2x + 7$; $m_1 = 2$ and for $y = \frac{1}{3}x - 4$; $m_2 = \frac{1}{3}$

Using $\therefore \tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2}$, $m_1 > m_2$

$$\Rightarrow \tan \phi = \frac{2 - \frac{1}{3}}{1 + \frac{2}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1 \therefore \phi = \tan^{-1}(1) = 45^\circ$$

b) $2y + 3x + 1 = 0 \Rightarrow y = -\frac{3}{2}x - \frac{1}{2}$; $m_1 = -\frac{3}{2}$ and for $y = x + 3$; $m_2 = 1$

$$m_2 > m_1 \therefore \tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2} \Rightarrow \phi = \tan^{-1}\left(\frac{1 - (-\frac{3}{2})}{1 + (-\frac{3}{2})}\right) = -78.69^\circ, 101.31^\circ$$

c) $2x + 3y - 2 \Rightarrow y = \frac{2}{3}x + \frac{1}{3}$; $m_1 = \frac{2}{3}$ and

$$-3x + 2y + 3 = 0 \Rightarrow y = \frac{3}{2}x - \frac{3}{2}; m_2 = \frac{3}{2}$$

$$\therefore \phi = \tan^{-1}\left(\frac{\frac{2}{3} - \frac{3}{2}}{1 + \frac{2}{3} \times \frac{3}{2}}\right) = 0. \text{ Hence lines are parallel}$$

d) $2x + 3y - 2 \Rightarrow y = -\frac{2}{3}x + \frac{2}{3}$; $m_2 = -\frac{2}{3}$ and

$$-3x + 2y + 3 = 0 \Rightarrow y = \frac{3}{2}x - \frac{3}{2}; m_1 = \frac{3}{2}$$

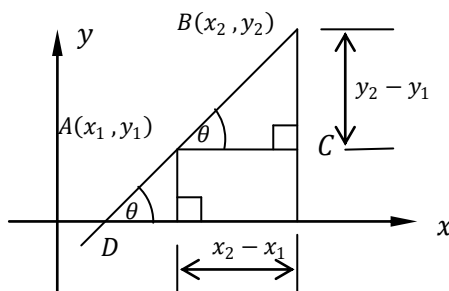
$$\therefore \tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{3}{2} - (-\frac{2}{3})}{1 + \frac{3}{2} \times (-\frac{2}{3})} = +\infty \therefore \phi = 90^\circ. \text{ Hence lines are perpendicular.}$$

SLOPE OF THE LINE JOINING POINTS $A(x_1, y_1)$ AND $B(x_2, y_2)$

Consider the figure below

$$\angle B\hat{D}X = \angle B\hat{A}C = \theta$$

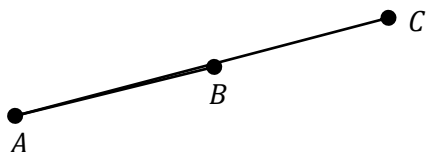
$$\text{Slope of } \overline{AB} = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$$



$$\therefore \text{Slope} = m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Diff. in Ordinates}}{\text{Diff. in Absiscea}}$$

COLLINEARITY OF THREE POINTS

Consider points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ lying on the same line.



$$\text{Grad } \overline{AB} = \text{Grad } \overline{BC} = \text{Grad } \overline{AC}$$

$$\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$$

This is the condition for three collinear points.

NOTE

Also three points will be collinear if the area of triangle formed by joining

the 3 points is **ZERO**. i.e. $\therefore \text{Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Examples

1) Show that the points are collinear

- a. $(2, 5)$, $(3, 7)$ and $(-1, -1)$
- b. $(2, -3)$, $(4, 1)$ and $(-2, -7)$
- c. $(0, -4)$, $(2, 2)$ and $(-1, -7)$

Solution

a. Let $A(2, 5)$, $B(3, 7)$ and $C(-1, -1)$

$$\text{Grad } \overline{AB} = \text{Grad } \overline{BC} \Rightarrow \text{Grad } \overline{AB} = \frac{7-5}{3-2} = 2; \text{Grad } \overline{BC} = \frac{-1-7}{-1-3} = 2$$

Since $\text{Grad } \overline{AB} = \text{Grad } \overline{BC} = 2$, points lie on a straight line.

Alternatively

$$\text{For points to be on a straight line, then } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow \frac{1}{2} \begin{vmatrix} 2 & 5 & 1 \\ 3 & 7 & 1 \\ -1 & -1 & 1 \end{vmatrix} &= \frac{1}{2} \left[2 \begin{vmatrix} 7 & 1 \\ -1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 5 & 1 \\ 7 & 1 \end{vmatrix} \right] \\ &= \frac{1}{2} [2(7 + 1) - 3(5 + 1) - (5 - 7)] = 0. \text{ Hence} \end{aligned}$$

points are collinear

b. Let $A(2, -3)$, $B(4, 1)$ and $C(-2, -7)$

$$\text{For A, B and C to be collinear, } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow \frac{1}{2} \begin{vmatrix} 2 & -3 & 1 \\ 4 & 1 & 1 \\ -2 & -7 & 1 \end{vmatrix} &= \frac{1}{2} \left[2 \begin{vmatrix} 1 & 1 \\ -7 & 1 \end{vmatrix} - 4 \begin{vmatrix} -3 & 1 \\ -7 & 1 \end{vmatrix} - 2 \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} \right] \\ &= \frac{1}{2} [2(8) - 4(4) + 8] = 0 \therefore \text{Points are collinear.} \end{aligned}$$

NOTE

3 –Collinear points form 3 –parallel lines. Hence the condition for parallel lines is the same as that of collinear points.

c. Left as an exercise for the reader.

VARIOUS FORMS OF EQUATION OF A STRAIGHT LINE

A. **To find equation of a line given any two coordinate points on it.**

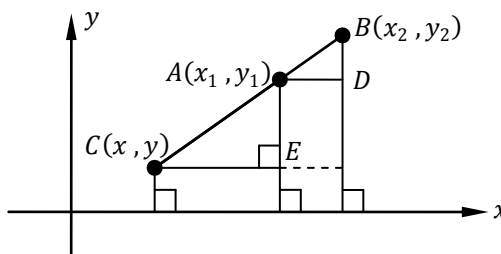
i.e. Two point form of equation of a line

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points.

Let $C(x, y)$ be a general point on line.

By similar Δs , ΔACE and ΔBAD are similar.

$$\begin{aligned} \therefore \frac{CE}{AD} &= \frac{AE}{BD} \\ \Rightarrow \frac{x_1 - x}{x_2 - x_1} &= \frac{y_1 - y}{y_2 - y_1} \Rightarrow \frac{x_1 - x}{x_2 - x_1} = \frac{y_1 - y}{y_2 - y_1} \\ \Rightarrow \frac{y_1 - y}{x_1 - x} &= \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$



Sometimes the equation is written as

$$\frac{y - y_1}{x - x_1} = m \text{ or } y - y_1 = m(x - x_1) \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

$\therefore y - y_1 = m(x - x_1) \text{ or } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m$

, which is the equation of a line.

From this equation other forms can be deduced.

B. **To find the equation of a line passing through the given point with a given slope, m**

Let $P(x_1, y_1)$ be the given point.

But $m = (\text{known})$

$\Rightarrow y - y_1 = m(x - x_1)$

, which is the equation of the line.

NOTE

1) If the line cuts the y – axis at a distance c from the origin i.e. given point is $(0, c)$, then the equation will be

$$y - c = m(x - 0) \Rightarrow \boxed{y = mx + c}$$

2) If the line passes through the origin, $(0, 0)$ its equation will be given by $y - 0 = m(x - 0) \Rightarrow$

$y = mx$

3) If the line passes through the point (x_1, y_1) with a zero gradient i.e. $m = 0$, its equation will be given by

$$y - y_1 = 0(x - x_1) \Rightarrow \boxed{y = y_1 \text{ (Horizontal)}}$$

- 4) If the line passes through the point (x_1, y_1) with an *infinite* gradient
i.e. $m = \infty$, its equation will be given by

$$x = x_1 \text{ (Vertical)}$$

Example

- 1) Find the equation of a line passing through point(s)
- $(2, 3)$ and $(-3, -5)$
 - $(-2, 1)$ and having the gradient -3
 - $(-4, 1)$ and making an angle of 135° with x -axis
 - $(-4, -5)$ and $(-5, 1)$

Solution

- a. Equation of line passing through 2-points is $y - y_1 = m(x - x_1)$,

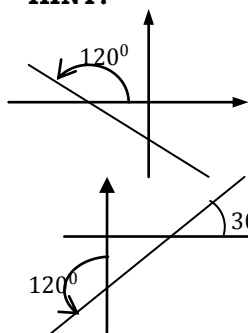
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
- b. Let $(2, 3) \equiv (x_1, y_1)$ and $(-3, -5) \equiv (x_2, y_2)$
- c. $\Rightarrow m = \frac{-5-3}{-3-2} = \frac{8}{5} \therefore y - 3 = \frac{8}{5}(x - 2)$ or $8x - 5y - 1 = 0$
- d. Equation of a line passing through a known point and having a known gradient is
 $y - y_1 = m(x - x_1); m = -3, (x_1, y_1) \equiv (-2, 1)$
 $y - 1 = -3(x - -2) \therefore 3x + y + 5 = 0$
- c. Given: $(-4, 1)$ lies on line, angle made by line and x -axis is $\theta = 135^\circ$
 $\therefore m = \tan \theta = \tan 135^\circ = -1$
 $\therefore y - y_1 = m(x - x_1)$, must be used.
 $\Rightarrow y - 1 = -1(x + 4) \therefore x + y + 5 = 0$
- d. Left as an exercise **Answer: $6x + y + 29 = 0$**

Task

Find the equation of a line passing through $(0, -3)$ and making an angle of

- 120° with x -axis
- 120° with y -axis

HINT:



$$\theta = +120^\circ; \text{hence slope is -ve. i.e. } m = \tan 120^\circ = -\sqrt{3}$$

$$\Rightarrow y + 3 = -\sqrt{3}(x - 0) \therefore y + \sqrt{3}x + 3 = 0$$

$$\theta = +120^\circ; \text{measured w.r.t } y\text{-axis.}$$

$$\text{slope is +ve. i.e. } m = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y + 3 = \frac{1}{\sqrt{3}}(x - 0) \therefore y - \frac{1}{\sqrt{3}}x + 3 = 0$$

C. To find of a line passing through a given point (x_1, y_1) and parallel to a given line $ax + by + c = 0$

From $ax + by + c = 0$ or $y = -\frac{a}{b}x - \frac{c}{b} \Rightarrow \text{slope}, m = -\frac{a}{b}$

Required line:

$\Rightarrow y - y_1 = m'(x - x_1)$, but for parallel lines $m = m'$ (slopes are equal)

$\Rightarrow y - y_1 = -\frac{a}{b}(x - x_1) \Rightarrow by - by_1 = -ax + ax_1$

$\therefore ax + by = ax_1 + by_1$, but $ax_1 + by_1$ can be evaluated to give a constant, say d

$\Rightarrow ax + by = d$, which is the required line.

In general, the equation of a line passing through a given point (x_1, y_1) and parallel to the given line $ax + by + c = 0$ is

$ax + by + k = 0$

NOTE

The two lines always differ in only constants.

Example

Find the equation of a line passing through

a. Point $(4, 1)$ and parallel to $2x + y + 1 = 0$

b. Point $(-1, 2)$ and parallel to $y = 2x - 5$

c. Point $(4, 7)$ and parallel to $2y = 4x - 50$

Solution

a. The line parallel to $2x + y + 1 = 0$ is $2x + y + k = 0$

b. But $(4, 1)$ lies on this required line

c. $\Rightarrow 2(4) + 1 + k = 0 \therefore k = -9$

d. $\therefore 2x + y - 9 = 0$, is the line.

e. The line parallel to the line $y = 2x - 5$ or $2x - y - 5 = 0$ is $2x - y + k = 0$

This line passes through the point $(-1, 2)$

$\Rightarrow 2(-1) - 2 + k = 0 \therefore k = 4$

$\therefore 2x - y + 4 = 0$

Note

The **R.H.S** of the given parallel line to required must be equal to zero.

f. Line parallel to $2y = 4x - 5$ or $4x - 2y - 5 = 0$ is $4x - 2y + k = 0$

This line passes through $(4, 7)$

$\Rightarrow 4(4) - 2(7) + k = 0 \therefore k = -2$

$\therefore 4x - 2y - 2 = 0$

D. To find equation of a line passing through a point (x_1, y_1) and perpendicular to the given line $ax + by + c = 0$

From $ax + by + c = 0$ or $y = -\frac{a}{b}x - \frac{c}{b}$, slope of this line is $m_1 = -\frac{a}{b}$

But for perpendicular lines, $m_1 \times m_2 = -1$; m_2 is the slope of the required line $\therefore m_2 = \frac{b}{a}$

Required line: $y - y_1 = m_2(x - x_1)$

$$\Rightarrow y - y_1 = \frac{b}{a}(x - x_1) \text{ or } ay - ay_1 = bx - bx_1 \Rightarrow bx - ay = bx_1 - ay_1$$

$$\Rightarrow bx - ay = d$$

Generally the line parallel to $ax + by + c = 0$ is $bx - ay + k = 0$

Example

Find the equation of a line passing through

- (2, 3) and perpendicular to $x + 2y + 5 = 0$ (**Unneb 2011**)
- (-3, 1) and perpendicular to $2x + 3y - 4 = 0$
- (4, 5) and perpendicular to $2y = 4x - 5$
- (4, 6) and perpendicular to $x + y - 5 = 0$

Solution

a) $x + 2y + 5 = 0$ or $y = -\frac{1}{2}x - \frac{5}{2} \therefore m_1 = -\frac{1}{2}$

Let m_2 be slope of the required line

From $m_1 \times m_2 = -1$; $m_2 = 2$ for perpendicular lines

Required line:

$$y - y_1 = m_2(x - x_1) \Rightarrow y - 3 = 2(x - 2) = 2x - 4$$

$$\therefore 2x - y - 1 = 0$$

b) $2x + 3y - 4 = 0$ or $y = -\frac{2}{3}x + \frac{4}{3} \therefore m_1 = -\frac{2}{3}$

For perpendicular lines $m_1 \times m_2 = -1$; $m_2 = \frac{3}{2}$

Required line:

$$y - y_1 = m_2(x - x_1) \Rightarrow y - 1 = \frac{3}{2}(x - -3) \Rightarrow 2y - 2 = 3x + 9$$

$$\therefore 3x - 2y + 11 = 0$$

c) $2y = 4x - 5$ or $y = 2x - \frac{5}{2} \therefore m_1 = 2$

From $m_1 \times m_2 = -1$; $m_2 = -\frac{1}{2}$

Required line:

$$y - y_1 = m_2(x - x_1) \Rightarrow y - 5 = -\frac{1}{2}(x - 4) \Rightarrow 2y - 10 = -x + 4$$

$$\therefore x + 2y - 14 = 0$$

d) Left as an exercise to the reader **Ans:** $y = x + 2$

E. To find the equation of a line in terms of intercepts which it makes on the axes. (Intercept form)

Consider the figure below.

Let the line AB cut the x -axis at A and y -axis at B.

Let $\overline{OA} = a$ units and $\overline{OB} = b$ units

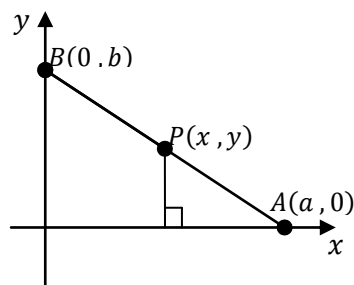
$\overline{OA}$ and $\overline{OB}$ are intercepts on

x -axis and y -axis respectively.

Let $P(x, y)$ be any point on the line

By similar triangles,

$\triangle PAC$ and $\triangle OAB$ are similar



$$\Rightarrow \frac{\overline{PC}}{\overline{OB}} = \frac{\overline{AC}}{\overline{OA}} \Rightarrow \frac{y}{b} = \frac{a-x}{a} = 1 - \frac{x}{a}$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1 \text{ or } \frac{x}{x\text{-intercept}} + \frac{y}{y\text{-intercept}} = 1$$

Example

- 1) Find the equation of a line that cuts off intercepts of +2 and +3 on the +ve x and y – axis respectively

Solution

Required line: $\frac{x}{a} + \frac{y}{b} = 1$, $a = x - \text{intercept}$, $b = y - \text{intercept}$

$$\therefore \frac{x}{2} + \frac{y}{3} = 1$$

- 2) Find the equation of a line that cuts off intercepts of –3 and –4 on x and y – axis respectively.

Solution

Required line: $\frac{x}{x\text{-intercept}} + \frac{y}{y\text{-intercept}} = 1$

$$\therefore \frac{x}{-3} + \frac{y}{-4} = 1 \text{ or } \frac{x}{3} + \frac{y}{4} = -1$$

- 3) Find the equation of the line cutting off an intercept of a on x – axis and making an angle of θ with x – axis in the first quadrant. Take $\theta < 90^\circ$

Solution

Given $\overline{OA} = a$ and $\angle OAB = \theta$

Required $B(0, b)$ and the line

From trigonometry

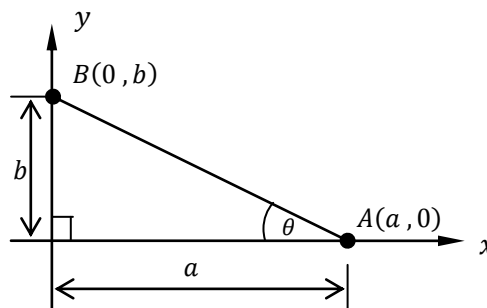
$$\tan \theta = \frac{b}{a} \therefore b = a \tan \theta$$

$$\therefore B(0, a \tan \theta)$$

Required line:

$$\frac{x}{x\text{-intercept}} + \frac{y}{y\text{-intercept}} = 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{a \tan \theta} = 1 \text{ or } x + y \cot \theta = a$$



- 4) Find the equation of a line cutting off +ve intercepts on x-and y – axes, making an angle of $\theta < 90^\circ$ with x – axis and distance between intercepts is 13 units

Solution

Now expressing intercepts in terms of 13 and θ

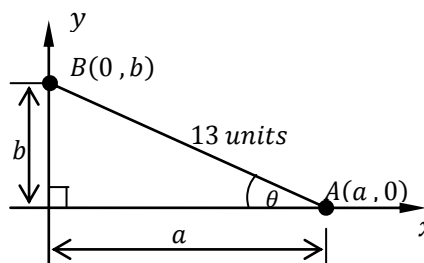
$$\cos \theta = \frac{a}{13} \therefore a = 13 \cos \theta$$

$$\sin \theta = \frac{b}{13} \therefore b = 13 \sin \theta$$

$$\therefore A(13 \cos \theta, 0) \text{ and } B(0, 13 \sin \theta)$$

Required line:

$$\frac{x}{x\text{-intercept}} + \frac{y}{y\text{-intercept}} = 1$$



$$\Rightarrow \frac{x}{13\cos\theta} + \frac{y}{13\sin\theta} = 1 \text{ or } x\sec\theta + y\csc\theta = 13$$

Task

The points A and B lie on the positive sides of x and y – axis respectively. If the length of AB is 15 units and $\angle OAB = \theta$, where O is the origin, find the equation of line AB (Leave θ in your answer) (Uneb 2010)

- F. To find the equation of a line in terms of length of perpendicular, d upon it from origin and the angle θ the perpendicular makes with the x – axis**

Consider the figure below

From the figure, $\angle O\hat{B}A = \alpha$, $\angle B\hat{O}C = 90 - \theta$

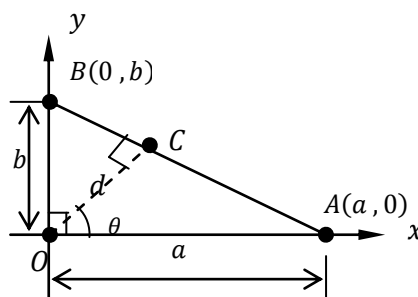
And $\angle A\hat{O}C = \theta$

$$\text{From } \triangle OAC, \cos\theta = \frac{d}{a} \therefore a = \frac{d}{\cos\theta}$$

$$\text{From } \triangle OBC, \sin\theta = \frac{d}{b} \therefore b = \frac{d}{\sin\theta}$$

$$\text{From intercept equation } \frac{x}{\frac{d}{\cos\theta}} + \frac{y}{\frac{d}{\sin\theta}} = 1$$

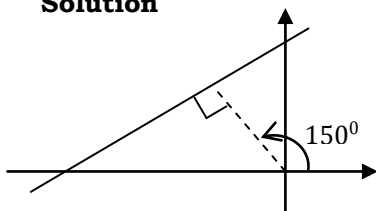
$$\text{Required line is } \boxed{x\cos\theta + y\sin\theta - d = 0}$$



Examples

- 1. Find the equation of a line whose perpendicular distance from the origin is 8 units and angle made by the perpendicular with x – axis is 150°**

Solution



$$\text{Using } x\cos\theta + y\sin\theta - d = 0$$

$$\theta = 150^\circ, d = 8$$

$$\Rightarrow x\cos 150^\circ + y\sin 150^\circ - 8 = 0$$

$$\Rightarrow x[-\cos 30^\circ] + y[\sin 30^\circ] = 8$$

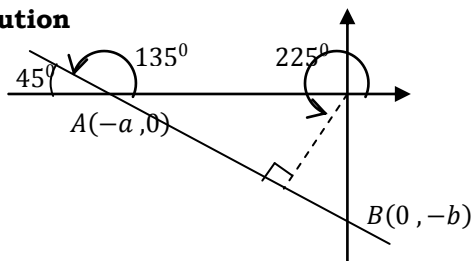
$$\Rightarrow -x\sqrt{3} + \frac{y}{2} = 8 \text{ or } 2\sqrt{3}x - y + 16 = 0$$

Note

Numbers involving angles and distances need a clear sketch for illustration.

- 2. Find the equation of a line cutting off negative intercepts on x and y – axis and its perpendicular distance from the origin is 16 units if the line makes an angle of 135° with x – axis**

Solution



From interpretation of the diagram, angle made by the perpendicular with +ve x – axis is 225°

Required line:

$$x\cos\theta + y\sin\theta - d = 0$$

$$\begin{aligned} \Rightarrow x \cos 225^\circ + y \sin 225^\circ - 16 &= 0 \\ \Rightarrow x[-\cos 45^\circ] + y[-\sin 45^\circ] - 16 &= 0 \\ \therefore -\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} &= 16 \text{ or } x + y + 16\sqrt{2} = 0 \end{aligned}$$

- 3.** Find the equation of a line which passes through $(4, \sqrt{3})$ and whose perpendicular distance from the origin on to the line makes an angle of 60° with the x -axis

Solution

Required line:

$$x \cos 60^\circ + y \sin 60^\circ - d = 0 \text{ This line passes through } (4, \sqrt{3})$$

$$\Rightarrow \frac{4}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = d \therefore d = \frac{7}{2}$$

$$\therefore \frac{x}{2} + y \frac{\sqrt{3}}{2} = \frac{7}{2} \text{ or } x + \sqrt{3}y - 7 = 0$$

- G. To find equation of a line passing through the point of intersection of the two given lines and satisfying any other given condition**

Let the given lines be

$$U = a_1x + b_1y + c_1 = 0 \text{ .i.e. } U = 0$$

$$V = a_2x + b_2y + c_2 = 0 \text{ .i.e. } V = 0$$

Required line is

$$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$$

Where λ = arbitrary constant, and can be obtained from any other given condition.

Example

- 1.** Find the equation of a straight line passing through the point of intersection of the pair of lines
- $3x - 4y + 5 = 0$ and $5x + 2y - 6 = 0$ and is parallel to the line $2x - y + 5 = 0$
 - $x - 2y + 3 = 0$ and $2x + 4y - 9 = 0$ and containing the point $(1, 1)$
 - $2x - 5y + 4 = 0$ and $5x - 6y - 3 = 0$ and passing through the point $(-5, -2)$
 - $5x + 2y - 4 = 0$ and $3x + y - 1 = 0$ and perpendicular to the line $4x - 15y + 6 = 0$

e) Solution

- a) Equation of a line passing through point of intersection of the other two lines is $(3x - 4y + 5) + \lambda(5x + 2y - 6) = 0$

$$\Rightarrow (3 + 5\lambda)x + (2\lambda - 4)y + (5 - 6\lambda) = 0 \dots\dots\dots (1)$$

eqn. (1) is parallel to $2x - y + 5 = 0$ when both have the same gradient

$$\Rightarrow -\frac{(3+5\lambda)}{(2\lambda-4)} = 2 \text{ or } 3 + 5\lambda = -2(2\lambda - 4) = 8 - 4\lambda$$

$$\therefore \lambda = \frac{5}{9} \dots\dots\dots (2)$$

eqn. (2) in to eqn. (1)

$$\Rightarrow \left(3 + \frac{25}{9}\right)x + \left(\frac{10}{9} - 4\right)y + \left(5 - \frac{30}{9}\right) = 0 \dots\dots\dots (\times 9)$$

$$\therefore 52x - 26y + 15 = 0 \text{ is the line required.}$$

Alternatively

Solving the equations simultaneously for point of intersection

$$3x - 4y + 5 = 0 \dots\dots\dots(\times 1) \Rightarrow 3x - 4y + 5 = 0$$

$$5x + 2y - 6 = 0 \dots\dots\dots(\times 2) \Rightarrow 10x + 4y - 12 = 0$$

Adding the equations to eliminate y

$$13x = 7 \Rightarrow x = \frac{7}{13}$$

$$\text{From } 3x - 4y + 5 = 0, \text{ and } x = \frac{7}{13} \Rightarrow 4y = 3 \times \frac{7}{13} + 5 \therefore y = \frac{43}{26}$$

$\therefore \left(\frac{7}{13}, \frac{43}{26}\right)$, is the point of intersection

Now finding equation of a line passing through $\left(\frac{7}{13}, \frac{43}{26}\right)$ and parallel to

$$2x - y + 5 = 0 \text{ is } 2x - y + k = 0; \left(\frac{7}{13}, \frac{43}{26}\right) \text{ lies on this line}$$

$$\Rightarrow \frac{14}{13} - \frac{43}{26} + k = 0 \therefore k = \frac{15}{26}$$

$$\therefore 2x - y + \frac{15}{26} = 0 \Rightarrow 52x - 26y + 15 = 0$$

b) Required line:

$$(x - 2y + 3) + \lambda(2x + 4y - 9) = 0, (1, 1) \text{ lies on this line}$$

$$\Rightarrow (1 - 2 + 3) + \lambda(2 + 4 - 9) = 0 \therefore \lambda = \frac{2}{3}$$

$$\Rightarrow (x - 2y + 3) + \frac{2}{3}(2x + 4y - 9) = 0$$

$$\Rightarrow 3x - 6y + 9 + 4x + 8y - 18 = 0$$

$$\therefore 7x + 2y - 9 = 0 \text{ is the required line}$$

c) Left as an exercise **Answer: $x - 2y + 1 = 0$**

d) Required line:

$$(5x + 2y - 4) + \lambda(3x + y - 1) = 0$$

$$\Rightarrow (5 + 3\lambda)x + (2 + \lambda)y + (-4 - \lambda) = 0 \dots\dots\dots (1)$$

$$\text{slope } m_1 = -\frac{(5+3\lambda)}{(2+\lambda)}$$

$$\text{eqn. (1) is perpendicular to } 4x - 15y + 6 = 0$$

$$\text{Slope } m_2 = \frac{4}{15}$$

$$\text{But for perpendicular lines } m_1 \times m_2 = -1$$

$$\Rightarrow -\frac{(5+3\lambda)}{(2+\lambda)} \times \frac{4}{15} = -1 \Rightarrow 20 + 12\lambda = 30 + 13\lambda \therefore \lambda = -\frac{10}{3} \dots\dots\dots (2)$$

$$\text{eqn. (2) in to eqn. (1)}$$

$$\Rightarrow \left(5 - \frac{30}{3}\right)x + \left(2 - \frac{10}{3}\right)y + \left(-4 + \frac{10}{3}\right) = 0 \Rightarrow -5x - \frac{4}{3}y - \frac{2}{3} = 0 \dots\dots (\times -3)$$

$$15x + 4y + 2 = 0$$

Task

Do (b), (c) and (d) using the 2nd alternative. i.e. 1st get point of intersection, and apply the suitable form of equation of line

Note

When finding point of intersection of two lines, solve the equations simultaneously.

PERPENDICULAR DISTANCE OF A GIVEN POINT FROM A GIVEN LINE

Let the given line be $ax + by + k = 0$, which can be expressed as $y = mx + c$ or $mx - y + c = 0$, and the point $A(p, q)$

Consider the figures below

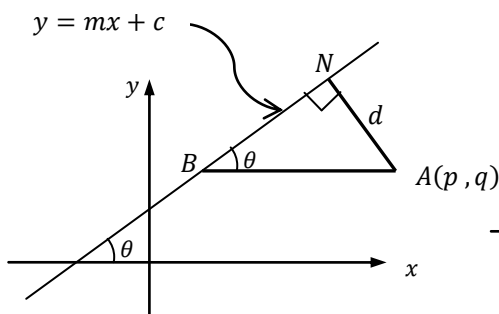


Fig.1. A and Origin on same side

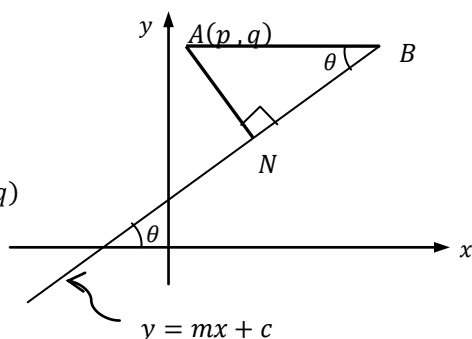


Fig.1. A and Origin on opposite side

$A(p, q)$ is at a distance, d from the line

$\overline{AB}$ is horizontal . i. e. parallel to x - axis

Finding coordinates

At B, y - coordinate = $q \therefore$ from $y = mx + c$, $x = \frac{q-c}{m}$

$$\therefore B\left(\frac{q-c}{m}, q\right)$$

$$\text{From fig.1; } \overline{AB} = p - \left(\frac{q-c}{m}\right) = \frac{mp-q+c}{m}$$

$$\text{From fig.1; } \overline{AB} = \frac{q-c}{m} - p = -\left[p - \left(\frac{q-c}{m}\right)\right] = -\left[\frac{mp-q+c}{m}\right]$$

$\Rightarrow$ For any point A,

$$\overline{AB} = \pm \left[\frac{mp-q+c}{m}\right]$$

From trigonometry,

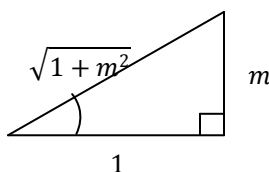
$d = \overline{AB}\sin\theta$, but $m = \tan\theta$

$$\therefore \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

$\therefore$ perpendicular distance, $d = \overline{AB}\sin\theta$

$$= \pm \left[\frac{mp-q+c}{m}\right] \cdot \frac{m}{\sqrt{1+m^2}}$$

$$= \pm \left[\frac{mp-q+c}{\sqrt{1+m^2}}\right]$$



$\therefore$ Perpendicular distance of a point (p, q) from line

$$y = mx + c \text{ or } mx - y + c = 0 \text{ is } \pm \left[\frac{mp-q+c}{\sqrt{1+m^2}}\right]$$

Mathematics for an A-Level Student Volume II

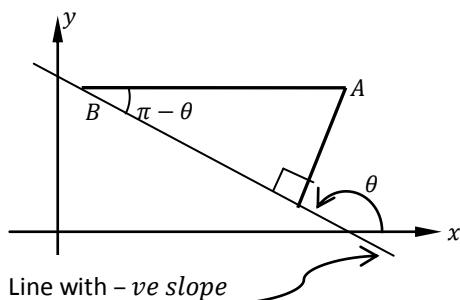
Line is $ax + by + k = 0 \Rightarrow y = -\frac{a}{b}x - \frac{k}{b}$, where $m = -\frac{a}{b}$, $c = -\frac{k}{b}$ in the equation $y = mx + c$, where $m = -\frac{a}{b}$, $c = -\frac{k}{b}$ in the equation $y = mx + c$

$$\therefore \text{Perpendicular distance} = \pm \left(\frac{-\frac{a}{b}p - q - \frac{k}{b}}{\sqrt{\left(-\frac{a}{b}\right)^2 + 1}} \right) = \pm \left[\frac{ap + bq + k}{\sqrt{a^2 + b^2}} \right]$$

$\therefore$ Perpendicular Distance of (p, q) from the line $ax + by + k = 0$
is $\pm \left[\frac{ap + bq + k}{\sqrt{a^2 + b^2}} \right]$

Note

- 1) When finding perpendicular distance of a point (p, q) from the line $ax + by + k = 0$ (**R.H.S = 0**)
*“Substitute the point in the line when **R.H.S = 0** and divide the result by the square root of the sum of squares of coefficient of x and y in the equation”*
- 2) The above result is also valid for lines with negative gradient since
 $\overline{AN} = d = \overline{AB} \sin(\pi - \theta) = \overline{AB} \sin \theta$



- 3) The sign ($\pm$) before the expression is adjusted with the sign of k or with sign of b if $k = 0$ in the equation $ax + by + k = 0$, so the perpendicular distance, d is *+ve* when the *point* and the *origin* are on *opposite sides* of the line, and *-ve* if they are on the *same side* of line.

Example

1. Find the perpendicular distance of the given point from the stated lines in each of the following
 - a) Point $(-5, -6)$, line: $3x + 4y + 12 = 0$
 - b) Point $(2, -5)$, line: $12x - 5y + 7 = 0$
 - c) Point $(-5, 6)$, line: $12x + 5y - 9 = 0$

Hence state whether the point and the origin are on the same side of the line.

Solution

- a) $(-5, -6) \equiv (p, q)$; $3x + 4y + 12 = 0 \equiv ax + by + k = 0$

$$\begin{aligned}\therefore \text{perpendicular distance} &= \pm \left[\frac{ap+bq+k}{\sqrt{a^2+b^2}} \right] \\ &= \pm \left[\frac{3 \times (-5) + 4 \times (-6) + 12}{\sqrt{3^2+4^2}} \right]\end{aligned}$$

Adjusting the sign on $k = 12$ before the expression

$$\therefore \text{perpendicular distance} = - \left[\frac{3 \times (-5) + 4 \times (-6) + 12}{\sqrt{3^2+4^2}} \right] = \frac{27}{5} \text{ units}$$

Since perpendicular distance is +ve, then the point and the origin are on the opposite sides of the line.

b) $(2, -5) \equiv (p, q); 12x - 5y + 7 = 0 \equiv ax + by + k = 0$

$$\begin{aligned}\therefore \text{perpendicular distance} &= \pm \left[\frac{12 \times (3) + (-5) \times (-5) + 7}{\sqrt{12^2 + (-5)^2}} \right] \\ &= - \left[\frac{36 + 25 + 7}{13} \right] \quad \left\{ \begin{array}{l} \text{sign on} \\ k = 7 \text{ is +ve, hence taking -ve sign before expression} \end{array} \right\} \\ &= - \frac{68}{13}, \text{ hence distance is } \frac{68}{13} \text{ units (taking numerical value)}\end{aligned}$$

Since the perpendicular distance is negative, then the origin and the point are on the same side of the line.

c) $(-5, 6) \equiv (p, q); 12x + 5y - 9 = 0 \equiv ax + by + k = 0$

$$\begin{aligned}\therefore \text{perpendicular distance} &= \pm \left[\frac{12 \times (-5) + (5) \times (6) - 9}{\sqrt{12^2 + 5^2}} \right] \\ &= + \left[\frac{-60 + 30 - 9}{13} \right]\end{aligned}$$

{+ve sign before expression is taken $\therefore k = -7$ }

$\therefore$ perpendicular distance = 3 units (taking numerical value)

Since perpendicular distance is -ve, the origin and the point are on the same side of the line.

2. If p and q be lengths of the perpendiculars from the origin upon the straight lines $4x + 3y - 5\cos\alpha = 0$ and $3x - 4y - \frac{5}{2}\sin\alpha = 0$ respectively, prove that $p^2 + 4q^2 = 1$

Solution

$$p = \pm \left[\frac{4 \times 0 + 3 \times 0 - 5\cos\alpha}{\sqrt{4^2 + 3^2}} \right] = \pm \cos\alpha \Rightarrow p = -5\cos\alpha \dots\dots\dots (1)$$

$$q = \pm \left[\frac{3 \times 0 - 4 \times 0 - \frac{5}{2}\sin\alpha}{\sqrt{3^2 + 4^2}} \right] = \pm \frac{\sin\alpha}{2} \Rightarrow q = -\frac{\sin\alpha}{2} \dots\dots\dots (2)$$

eqn. (1)² + (2)²

$$\Rightarrow p^2 + 4q^2 = \sin^2\alpha + \cos^2\alpha = 1 \therefore p^2 + 4q^2 = 1$$

3. Given that U and V are lengths of perpendicular from the point $(1, -1)$ upon the straight lines $4x + 3y - 5\cos\theta = 0$ and $3x - 4y - 5\sin\theta = 0$. Show that the equation connecting perpendicular distances U and V is an equation of a circle. Hence state its centre and radius

Solution

$$U = \pm \left[\frac{4 \times 1 + 3 \times (-1) - 5\cos\theta}{\sqrt{4^2 + 3^2}} \right] = \pm \frac{1 - 5\cos\theta}{5}$$

$$\therefore U = - \left(\frac{1 - 5\cos\theta}{5} \right) \text{ or } U + \frac{1}{5} = \cos\theta \dots\dots\dots (1)$$

$$V = \pm \left[\frac{3 \times 1 - 4 \times (-1) - 5 \sin \theta}{\sqrt{3^2 + 4^2}} \right] = \pm \frac{7 - 5 \sin \theta}{5}$$

$$\therefore V = -\left(\frac{7}{5} - \sin \theta\right) \text{ or } V + \frac{7}{5} = \sin \theta \dots\dots\dots(2)$$

$$\text{From } \cos^2 \theta + \sin^2 \theta = 1$$

$$\Rightarrow \left(U + \frac{1}{5}\right)^2 + \left(V + \frac{7}{5}\right)^2 = 1, \text{ which is the equation of a circle of centre } \left(-\frac{1}{5}, -\frac{7}{5}\right) \text{ and radius 1 unit}$$

NOTE

This example illustrates the relevance of perpendicular distance of a point from a line in determining the path traced by the point *w.r.t.* the line *i.e.* locus of a point.

EQUATIONS OF BISECTORS OF THE ANGLES BETWEEN TWO STRAIGHT LINES

Let AB and CD be two given lines whose equations are:

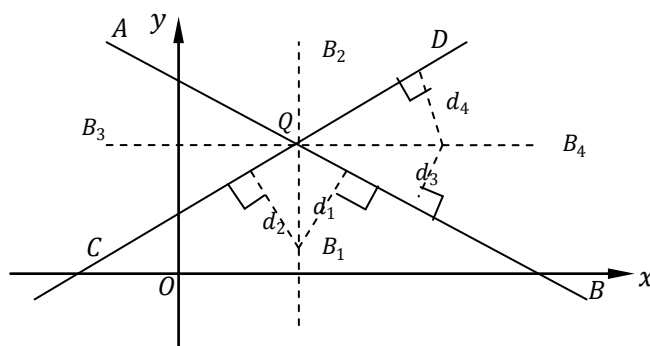
$$a_1x + b_1y + c_1 = 0, \text{ and}$$

$$a_2x + b_2y + c_2 = 0 \text{ respectively}$$

Let Q be the point of intersection of the lines.

Consider the figure below

$\overline{B_1B_2}, \overline{B_3B_4}$ are angle bisectors



From the figure, B_1 is the point on the angle bisector having coordinates $B_1(x_1, y_1)$ and $B_2(x_2, y_2)$

d_1 and d_2 are perpendicular distances of B_1 from lines AB and CD respectively, and d_3 and d_4 are perpendicular distances of B_3 from AB and CD respectively.

$$\Rightarrow d_1 = \pm \frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} \text{ and } d_2 = \pm \frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}$$

$$\Rightarrow d_3 = \pm \frac{a_1x_2 + b_1y_2 + c_1}{\sqrt{a_1^2 + b_1^2}} \text{ and } d_4 = \pm \frac{a_2x_2 + b_2y_2 + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Assuming c_1 and c_2 are of the same sign, then $d_1 = d_2$ and $d_3 = d_4$

NOTE:

From the figure B_1 and the origin are on the same side of the lines, and B_4 and origin are on the same side of CD but opposite side of AB

$\therefore d_1 = -ve$, and $d_2 = -ve$

$$\Rightarrow -\left(\frac{a_1x_1+b_1y_1+c_1}{\sqrt{a_1^2+b_1^2}}\right) = -\left(\frac{a_2x_1+b_2y_1+c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

$\therefore d_3 = +ve$ and $d_4 = -ve$

$$+\left(\frac{a_1x_2+b_1y_2+c_1}{\sqrt{a_1^2+b_1^2}}\right) = -\left(\frac{a_2x_2+b_2y_2+c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

$\Rightarrow$ The point $B_1(x_1, y_1)$ lies on the line whose equation is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

$\Rightarrow$ The point $B_4(x_2, y_2)$ lies on the line whose equation is

$$+\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Combining the equations

Angle bisector equations of angles made by the lines

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Note

The upper sign (+ve) gives the equation of the bisector of the angle containing the origin when c_1 and c_2 are of the same sign. *i.e.* Always re-arrange the equations to make c_1 and c_2 of the same sign before using them in the formula

Example

1) Find the equations of angle bisectors between the following lines

- a) $7x - 24y + 55 = 0$ and $4x - 3y - 65 = 0$
- b) $3x + 4y - 5 = 0$ and $6x - 8y + 7 = 0$
- c) $\sqrt{3}x + y - 7 = 0$ and $x - \sqrt{3}y + 1 = 0$
- d) $3x + 4y - 2 = 0$ and $5x - 12y + 2 = 0$

Solution

a) $7x - 24y + 55 = 0$ and $4x - 3y - 65 = 0$ since the constants are not of the same sign, re-arranging the equations for same constants

$7x - 24y + 55 = 0$ and $-4x + 3y + 65 = 0$ (Re-arranged)

$\Rightarrow$ Equation of the bisector containing the origin is

$$-\left[\frac{7x-24y+55}{-\sqrt{7^2+24^2}}\right] = -\left[\frac{-4x+3y+65}{-\sqrt{4^2+3^2}}\right] \Rightarrow \frac{7x-24y+55}{25} = \frac{-4x+3y+65}{5}$$

$\therefore 9x - 13y - 90 = 0$ is the required line

And equation of bisector of the angle which does not contain the origin is

$$+ \left[\frac{7x-24y+55}{-\sqrt{7^2+24^2}} \right] = - \left[\frac{-4x+3y+65}{-\sqrt{4^2+3^2}} \right] \Rightarrow \frac{7x-24y+55}{-25} = \frac{-4x+3y+65}{5}$$

$\therefore 13x + 9y - 380 = 0$ is the required line

- b) $3x + 4y - 5 = 0$ and $6x - 8y + 7 = 0$ since the constants are not of the same sign,

Re-arranging the equations for same constants

$$3x + 4y - 5 = 0 \text{ and } -6x + 8y - 7 = 0 \text{ (Re-arranged)}$$

Equation of bisector containing the origin is

$$- \left[\frac{3x+4y-5}{+\sqrt{3^2+4^2}} \right] = - \left[\frac{-6x+8y-7}{+\sqrt{6^2+8^2}} \right] \Rightarrow \frac{3x+4y-5}{5} = \frac{-6x+8y-7}{10}$$

$$\therefore 16y - 17 = 0$$

- c) $\sqrt{3}x + y - 7 = 0$ and $x - \sqrt{3}y + 1 = 0$ re-arranging the equations for same constants

$$\sqrt{3}x + y - 7 = 0 \text{ and } -x + \sqrt{3}y - 1 = 0$$

Equation of bisector containing the origin is

$$- \left[\frac{\sqrt{3}x+y-7}{+\sqrt{3+1}} \right] = - \left[\frac{-x+\sqrt{3}y-1}{+\sqrt{3+1}} \right] \Rightarrow \frac{\sqrt{3}x+y-7}{2} = \frac{x-\sqrt{3}y+1}{2}$$

$$\therefore (\sqrt{3}-1)x + (1+\sqrt{3})y - 8 = 0$$

- d) $3x + 4y - 2 = 0$ and $5x - 12y + 2 = 0$ or $-5x + 12y - 2 = 0$

Equation of bisector containing the origin is

$$- \left(\frac{3x+4y-2}{+\sqrt{3^2+4^2}} \right) = - \left(\frac{-5x+12y-2}{+\sqrt{5^2+12^2}} \right) \Rightarrow \frac{3x+4y-2}{5} = \frac{5x+12y-2}{13}$$

$$\therefore 64x - 8y - 16 = 0 \text{ or } 8x - y - 2 = 0$$

Equation of bisector of angle not containing origin

$$+ \left(\frac{3x+4y-2}{+\sqrt{3^2+4^2}} \right) = - \left(\frac{-5x+12y-2}{+\sqrt{5^2+12^2}} \right) \Rightarrow \frac{3x+4y-2}{5} = \frac{5x-12y+2}{13}$$

$$\therefore 14x + 112y - 36 = 0 \text{ or } 7x + 56y - 18 = 0$$

NOTE

Referring to the last example, when finding equation of bisector containing the origin, the distances must be *-ve*, hence *-ve* sign before the bracketed terms, and the inside sign in the bracket must be opposite to that of the constant in the equation, and when finding the equation of bisector not containing the origin the equated distances must be opposite in sign. *i.e.* *+ve* and then *-ve* to the second.

TRIGONOMETRY

Introduction:

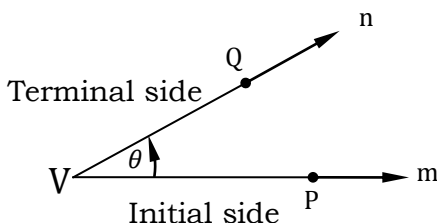
This deals with the study of relationship between the sides and angles of a triangle.

Angles and their measure

Angles

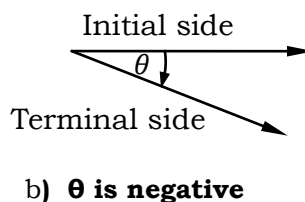
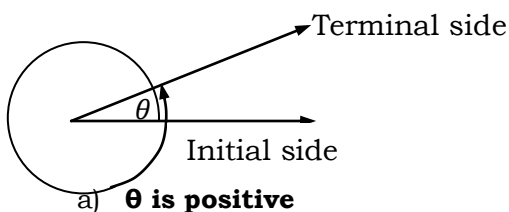
The **angle** is formed by rotating (in planes) a ray m , called the *initial side* of the angle, around the end point until it coincides with a ray n , called the *terminal side* of the angle.

The common end point V of m and n is called the *vertex*.



A counter clockwise produces a *positive angle* and a clockwise rotation produces a *negative angle*.

Consider the fig. below



Co terminal angles:

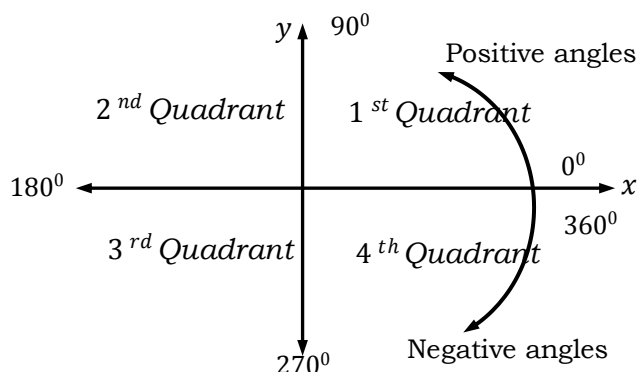
Any angles having the same terminal and initial sides are said to be *co terminal*.

The six ratios are called *trigonometrical ratios*, and each has a name.

Angle measured with respect to x- axis

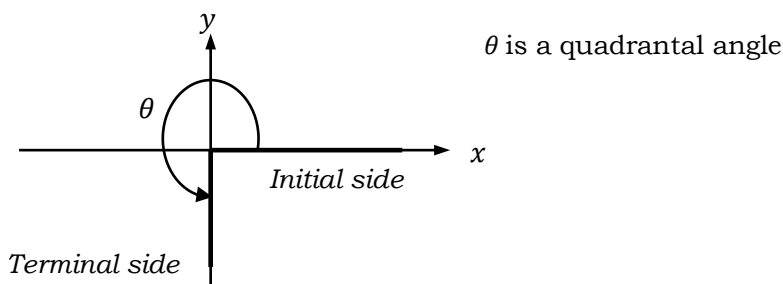
Any angle in a rectangular coordinate system ($x - y$ axes) is said to be in *standard position* If its vertex is at the *origin* and the initial side is along the positive $x -$ axis.

Consider the fig. below



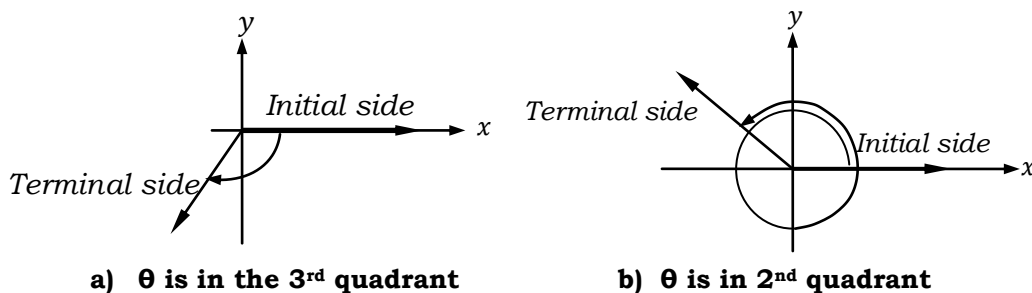
Quadrantal angle

If the terminal side of an angle in standard position lies along the coordinate axes, the angle is said to be a *Quadrantal angle*



Angle in a quadrant

If the terminal side does not lie along a coordinate axis, the angle is referred to in terms of the quadrant in which the terminal side lies.

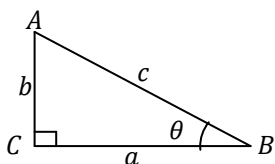


The Trigonometrical functions of acute angles:

In many applications of trigonometry, you are given the measure of *two sides* of *right angled triangle* or the measure of an *acute angle* and *one side*, and you are asked to find the measure of the remaining sides and acute angles, this is called *solving a triangle*.

Solving a triangle is accomplished by using special *trigonometrical ratios* associated with each acute angle θ consider the fig. below,

Mathematics for an A-Level Student Volume II



There are altogether six ratios between the sides of the ΔABC viz:

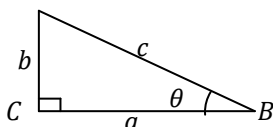
$$\frac{b}{c}, \frac{c}{b}, \frac{a}{b}, \frac{b}{a}, \frac{a}{c}, \frac{c}{a}$$

The six ratios are called *trigonometric ratios* and each has a name

i.e. *Sine (sin)*, *Cosine (cos)*, *tangent (tan)*, *Cosecant (cosec)*, *Secant (sec)* and *cotangent (cot)*.

Definitions of the ratios:

Consider the right angled ΔABC



$$\cos \theta = \frac{a}{c}$$

$$\operatorname{cosec} \theta = \frac{c}{b}$$

$$\sin \theta = \frac{b}{c}$$

$$\sec \theta = \frac{c}{a}$$

$$\tan \theta = \frac{b}{a}$$

$$\cot \theta = \frac{a}{b}$$

Side a = side adjacent to angle θ

Side b = side opposite to the angle θ

Side c = opposite side to the right angle or hypotenuse

Using these definitions, then the alternative form;

$$\cos \theta = \frac{\text{Adj}}{\text{Hyp}}$$

$$\sin \theta = \frac{\text{Opp}}{\text{Hyp}}$$

$$\tan \theta = \frac{\text{Opp}}{\text{Adj}}$$

$$\operatorname{cosec} \theta = \frac{\text{Hyp}}{\text{Opp}}$$

$$\sec \theta = \frac{\text{Hyp}}{\text{Adj}}$$

$$\cot \theta = \frac{\text{Adj}}{\text{Opp}}$$

Relation between the six ratios

From the above definitions, three of these ratios are the reciprocals of the other viz:

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}, \sec \theta = \frac{1}{\cos \theta} \text{ and } \cot \theta = \frac{1}{\tan \theta}$$

NOTE:

$$\tan \theta = \frac{b}{a} = \left(\frac{b}{c}\right) / \left(\frac{a}{c}\right) = \frac{\sin \theta}{\cos \theta} \text{ and hence } \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}$$

NOTE:

1. When using reciprocal relations many students tend to associate *cosecant* with *cosine* and *secant* with *sine*, just the opposite is the case. Cosecant goes with sine and secant with cosine.
2. In remembrance of reciprocal relations, in each of the three pairs of reciprocals, there is one ratio with the prefix “CO” thus the reciprocal of sin is cosec, of sec is cos, of tan is cot.

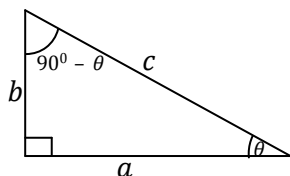
Application of reciprocal trig Ratios:

- Proof of identities
- Solution to trig equations
- Elimination of parameters
- Simplifying trig. Expressions
- Integration

Related examples shall be tackled later.

Complementary Angles and Co-functions;

Consider a right angled triangle below.



$$\sin \theta = \frac{b}{c}, \quad \cos (90 - \theta) = \frac{a}{c}, \quad \tan \theta = \frac{b}{a} = \frac{1}{a/b} = \frac{1}{\tan (90 - \theta)}$$

$$\therefore \tan \theta = \cot (90 - \theta)$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{a/c} = \frac{1}{\sin (90 - \theta)} \therefore \sec \theta = \operatorname{cosec} (90 - \theta)$$

$$\therefore \sin \theta = \cos (90 - \theta), \sec \theta = \operatorname{cosec} (90 - \theta), \text{ and } \tan \theta = \cot (90 - \theta)$$

The trigonometrical ratios cosine, cotangent and cosecant contain the prefix “CO”. so some times are referred to as the *co functions* of sine, tangent and secant respectively.

Application of complementary Angles and co functions:

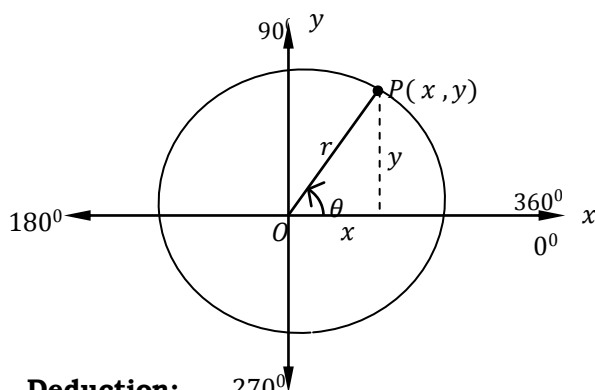
- Proof of identities
- Solution to trig equations
- Elimination of parameters
- Simplifying trig. Expressions
- Integration

Related examples shall be tackled later.

General angle (angles of any magnitude)

Consider an angle θ as an arbitrary in standard position in rectangular coordinate system and $P(x, y)$ a point r units from the origin on the terminal side of θ . Six ratios involving r and coordinates of P can be derived, taking $\overline{OP} = r$ (always +ve),

ANGLES IN 1st QUADRANT, $0^\circ \leq \theta \leq 90^\circ$ (Acute angle case)

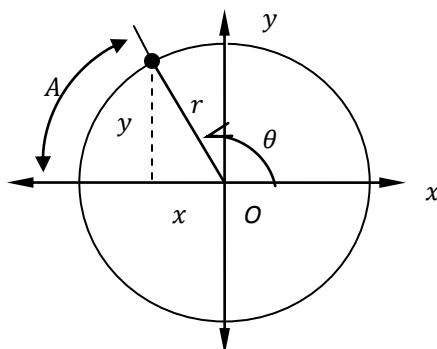


$$\begin{aligned}\sin \theta &= \frac{+y}{+r} = +\frac{y}{r} \\ \cos \theta &= \frac{+x}{+r} = +\frac{x}{r} \\ \tan \theta &= \frac{+y}{+x} = +\frac{y}{x} \\ \cot \theta &= \frac{+x}{+y} = +\frac{x}{y} \\ \sec \theta &= \frac{+r}{+x} = +\frac{r}{x} \\ \operatorname{cosec} \theta &= \frac{+r}{+y} = +\frac{r}{y}\end{aligned}$$

Deduction: 270°

All trig. Ratios of angles in 1st quadrant are positive

ANGLES IN 2nd QUADRANT, $90^\circ \leq \theta \leq 180^\circ$ (Obtuse angle case)



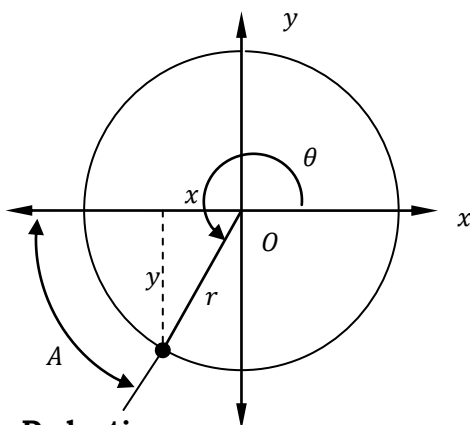
$$A = 180 - \theta$$

$$\begin{aligned}\sin \theta &= \frac{+y}{+r} = +\frac{y}{r} & \cos \theta &= \frac{-x}{+r} = -\frac{x}{r} \\ \tan \theta &= \frac{+y}{-x} = -\frac{y}{x} & \cot \theta &= \frac{-x}{+y} = -\frac{x}{y} \\ \sec \theta &= \frac{+r}{-x} = -\frac{r}{x} & \operatorname{cosec} \theta &= \frac{+r}{+y} = +\frac{r}{y}\end{aligned}$$

Deduction:

Only sines and cosecants of angles in 2nd quadrant are positive, the other trig. Ratios are negative.

ANGLES IN 3rd QUADRANT, $180^\circ \leq \theta \leq 270^\circ$



$$A = \theta - 180^\circ$$

$$\sin \theta = \frac{-y}{+r} = -\frac{y}{r}$$

$$\cos \theta = \frac{-x}{+r} = -\frac{x}{r}$$

$$\cot \theta = \frac{-x}{-y} = +\frac{x}{y}$$

$$\operatorname{cosec} \theta = \frac{+r}{-y} = -\frac{r}{y}$$

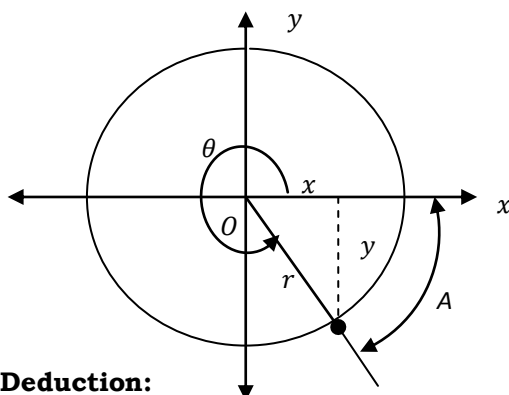
$$\sec \theta = \frac{+r}{-x} = -\frac{r}{x}$$

$$\tan \theta = \frac{-y}{-x} = +\frac{y}{x}$$

Deduction:

Only tangents and cotangent of angles in 3rd quadrant are positive, the other trig ratios are negative.

ANGLES IN 4th QUADRANT, $270^\circ \leq \theta \leq 360^\circ$ (Reflex angle case)



$$A = 360^\circ - \theta$$

$$\sin \theta = \frac{-y}{+r} = -\frac{y}{r}$$

$$\cos \theta = \frac{+x}{+r} = +\frac{x}{r}$$

$$\cot \theta = \frac{+x}{-y} = -\frac{x}{y}$$

$$\sec \theta = \frac{+r}{+x} = +\frac{r}{x}$$

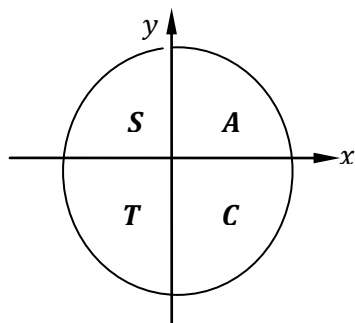
$$\operatorname{cosec} \theta = \frac{+r}{-y} = -\frac{r}{y}$$

$$\tan \theta = \frac{-y}{+x} = -\frac{y}{x}$$

Deduction:

Only cosines and secants of angles in 4th quadrant are positive, the other trig. Ratios are negative

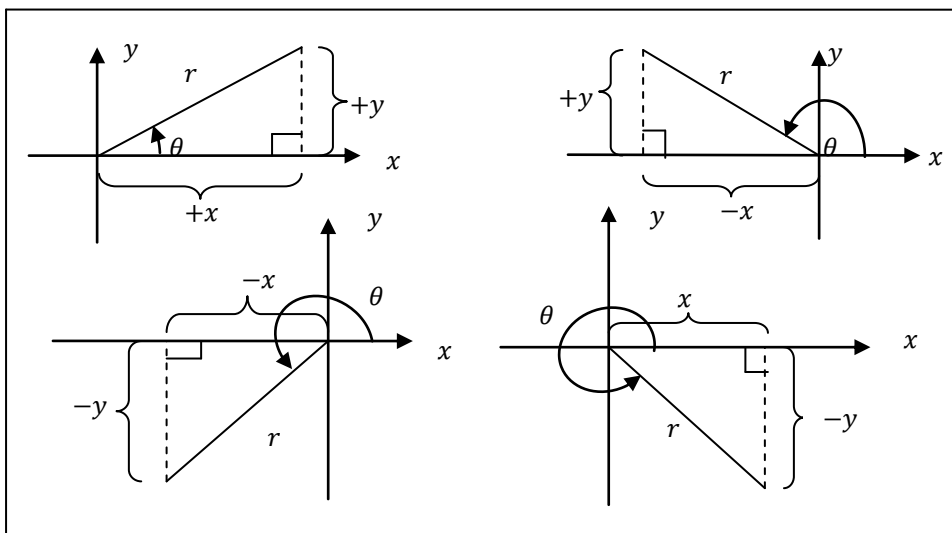
Summary of the results:



The abbreviated letters S, T and C denote the Sines, Tangents and Cosines of angles being positive. Even their reciprocal ratios are positive. A denotes ALL trig ratios of angles in that quadrant are positive; the rest undefined letters are negative for the ratios.

NOTE:

1. The positive/negative sign on x and y are assumed as the sign conventions of rectangular coordinates axes.
2. The triangle being formed with its base on the horizontal axis (x- axis) is called the *reference triangle*.
3. The acute angle (always taken positive) between the terminal side of θ and the horizontal axis (x – axis) is called the *reference angle*.
4. $r = \sqrt{x^2 + y^2} > 0$.
5. A student should be very careful to verify each item on the reference triangle below in each quadrant.



Application of the above results:

- Evaluating trig. functions
- Expressing trig ratios of angles in terms of acute angled trig. Ratios
- Proof of identities
- Evaluation of inverse of trig functions in terms of acute angles.

Evaluation of trigonometric functions:

1. Given that $\cos \theta = -\frac{8}{17}$ and θ is obtuse,

Find (i) $\sin \theta$ (ii) $\tan \theta$ (iii) $\cot \theta$ (iv) $\operatorname{cosec} \theta$ (v) $\sec \theta$.

Solution

Given $\cos \theta = -\frac{8}{17}$, $90^\circ < \theta < 180^\circ$,

$$\cos \theta = \frac{\text{Adj}}{\text{Hyp}} = \frac{-8}{17}$$

By Pythagoras theorem, $b^2 + 8^2 = 17^2$

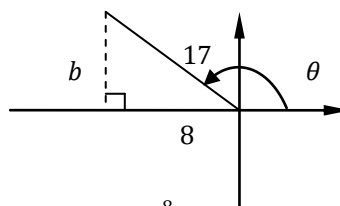
$$\therefore b = 15$$

i) $\sin \theta = +\frac{15}{17}$

ii) $\tan \theta = \frac{15}{-8}$

iv) $\sec \theta = \frac{+17}{15}$

v) $\sec \theta = \frac{-17}{8}$



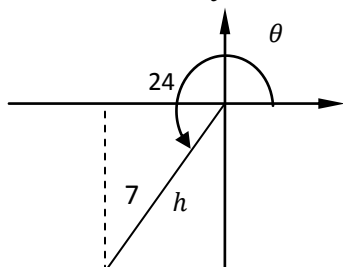
iii) $\cot \theta = \frac{-8}{15}$

2. If $\tan \theta = \frac{7}{24}$ and θ is reflex, find without using tables/calculator the values;

- i) $\sec \theta$ ii) $\sin \theta$ iii) $\cot \theta$ iv) $\operatorname{cosec} \theta$ v) $\cos \theta$

Solution

Given θ is reflex, then $180^\circ < \theta < 270^\circ$



$$\text{Since } \tan \theta = \frac{\text{Opp}}{\text{Adj}} = \frac{7}{24}$$

By Pythagoras theorem,
 $7^2 + 24^2 = h^2$, $h = 25$

i) $\sec \theta = \frac{1}{\cos \theta} = \frac{1}{-24/25} = -\frac{25}{24}$

ii) $\sin \theta = \frac{-7}{25}$

iii) $\cot \theta = \frac{1}{\tan \theta} = \frac{1}{7/24} = \frac{24}{7}$

iv) $\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{1}{-7/25} = -\frac{25}{7}$

v) $\cos \theta = -\frac{24}{25}$

Task:

If $\sin \theta = \frac{-4}{5}$ for $270^\circ < \theta < 360^\circ$, find

- i) $\operatorname{cosec} \theta$ ii) $\sec \theta$ iii) $\cot \theta$ iv) $\cos \theta$
 v) $\tan \theta$ vi) $3\cos \theta - 5\tan \theta$ vi) $4\tan \theta - 3\operatorname{cosec} \theta$.

Evaluating inverse of trig Functions in terms of acute angles:

Find the exact values of two angles less 360° .

i) $\tan^{-1}(-0.36397)$

iv) $\operatorname{cosec}^{-1}(-2.7290)$

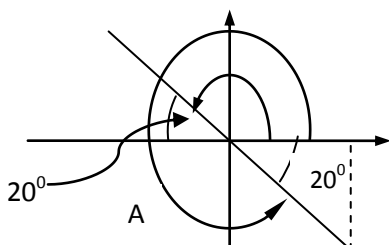
ii) $\cot^{-1}(1.73205)$

v) $\sec^{-1}(-1.0670)$

iii) $\sin^{-1}(0.866)$

Solution

i) let $A = \tan^{-1}(-0.36397) \Rightarrow \tan A = -0.36397$ (–ve)



Now finding $\tan^{-1}(-0.36397) = 20^\circ$
 (Reference angle)

$$\therefore A = 180 - 20 = 160^\circ$$

$$A = 360 - 20 = 340^\circ$$

$$\therefore \tan^{-1}(-0.36397) = 160^\circ, 340^\circ$$

ii) Let $B = \cot^{-1}(1.73205)$

$$\cot B = 1.73205, \text{ but } \cot B = \frac{1}{\tan \theta}$$

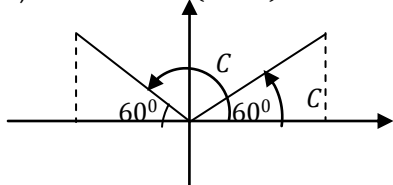
$$\therefore \tan \theta = \frac{1}{1.73205} = 0.5774 (+ve)$$

$$\tan^{-1} 0.5774 = 30^{\circ} \text{ (Reference angle)}$$

$$\therefore B = 30^{\circ}, 180 + 30^{\circ} = 210^{\circ}$$

$$\therefore \cot^{-1}(1.73205) = 30^{\circ}, 210^{\circ}$$

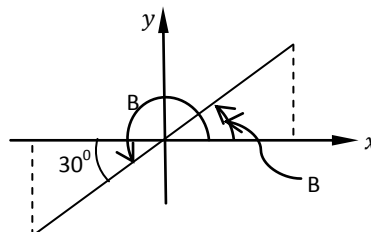
iii) Let $C = \sin^{-1}(0.866) \Rightarrow \sin C = 0.866 (+ve)$



$$\sin^{-1}(0.866) = 60^{\circ} \text{ (reference angle)}$$

$$\therefore C = 180 - 60^{\circ} = 120^{\circ}, 60^{\circ}$$

$$\therefore \sin^{-1}(0.866) = 60^{\circ}, 120^{\circ}$$



iv) Let $D = \operatorname{cosec}^{-1}(-2.7290)$

$$\Rightarrow \operatorname{cosec} D = -2.7290$$

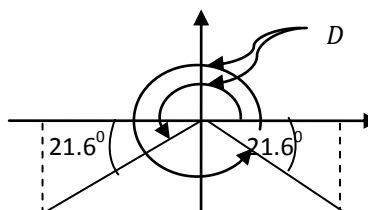
$$\text{, but } \operatorname{cosec} D = \frac{1}{\sin D} = -2.7290$$

$$\therefore \sin D = -0.3664 (-ve)$$

$$\sin^{-1}(+0.3664) = 21.5^{\circ} \text{ (reference angle)}$$

$$\therefore D = 180 + 21.5 = 201.5^{\circ} \text{ or } 360 - 21.5 = 338.5^{\circ}$$

$$\therefore \operatorname{cosec}^{-1}(-2.7290) = 201.6^{\circ}, 338.5^{\circ}$$



Task:

Find two values of ϕ less than 360° which satisfy the following trig functions.

i) $\cot \phi = -0.06315$ [$\cot^{-1}(-0.06315)$] (Ans: $93.6^{\circ}, 273.6^{\circ}$)

ii) $\sec \phi = -1.0670$ [$\sec^{-1}(-1.0670)$] (Ans: $159.6^{\circ}, 200.4^{\circ}$)

iii) $\operatorname{cosec} \phi = -1.4142$ [$\operatorname{cosec}^{-1}(-1.4142)$] (Ans: $225^{\circ}, 315^{\circ}$)

Expressing trig ratios in terms of trig ratios of acute angles:

Express in terms of trig ratios of acute angles,

$\sin 170^{\circ}$ iv) $\sec(-172^{\circ})$

i) $\tan 200^{\circ}$ v) $\operatorname{cosec}(-590^{\circ})$

ii) $\cot 200^{\circ}$ vi) $\cot(-860^{\circ})$

Solution:

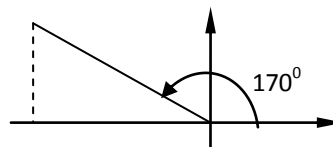
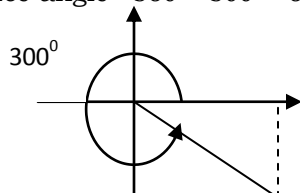
i) $\sin 170^{\circ}$, $+170^{\circ}$ means counter clockwise direction

$$\text{Reference angle} = 180 - 170 = 10^{\circ}$$

$$\therefore \sin 170^{\circ} = +\sin 10^{\circ}$$

ii) $\tan 300^{\circ}$

$$\text{Reference angle} = 360 - 300 = 60^{\circ}$$



$$\tan 300^{\circ} = -\tan 60^{\circ}$$

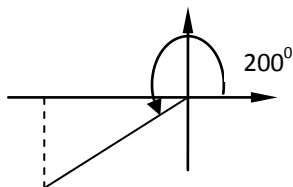
(-ve sign means quadrant in which angle lies)

iii) $\cot 200^\circ = \frac{1}{\tan 200^\circ}$

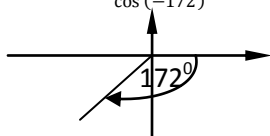
Reference angle = $200 - 180 = 20^\circ$

$\tan 200^\circ = +\tan 20^\circ$

$\therefore \cot 200^\circ = \frac{1}{+\tan 20^\circ} = +\cot 20^\circ$



iv) $\sec(-172^\circ) = \frac{1}{\cos(-172^\circ)}$, -172° means clockwise measurement



Reference angle = $180 - 172 = 8^\circ$

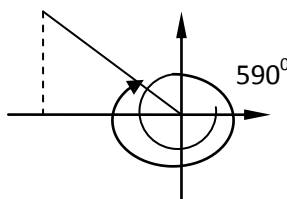
$\therefore \cos(-172^\circ) = -\cos 8^\circ$

$\therefore \sec(-172^\circ) = \frac{1}{-\cos 8^\circ} = -\sec 8^\circ$

v) $\operatorname{cosec}(-590^\circ) = \frac{1}{\sin(-590^\circ)}$

but $\sin(-590^\circ) = -\sin 50^\circ$

$\operatorname{cosec}(-590^\circ) = \frac{1}{-\sin 50^\circ} = -\operatorname{cosec} 50^\circ$



Note:

$590^\circ = 360^\circ + 230^\circ$, $230^\circ = 180^\circ + 50^\circ$

The final reference angle is 50° .

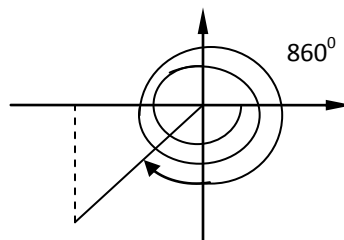
vi) $\cot(-860^\circ) = \frac{1}{\tan(-860^\circ)}$

$860 = 2 \times 360 + 140$, $140 = 180 - 40$

$\therefore$ Reference angle = 40°

$\tan(-860^\circ) = +\tan 40^\circ$

$\cot(-860^\circ) = \frac{1}{\tan(-860^\circ)} = \frac{1}{+\tan 40^\circ} = +\cot 40^\circ$



Task:

Express in terms of trigonometric ratios of acute angles

a) $\tan(-500^\circ)$

c) $\operatorname{cosec}(+950^\circ)$

b) $\cot(-773^\circ)$

Ans: a) $+\tan 40^\circ$

b) $-\cot 55^\circ$

c) $-\operatorname{cosec} 50^\circ$

Exact values for trigonometrical ratios of special angles

Special angles take on 30° and 45° or $\frac{\pi}{6}$ and $\frac{\pi}{4}$ radians multiples.

The trig functions of these angles can easily be evaluated exactly with the use of tables or calculator.

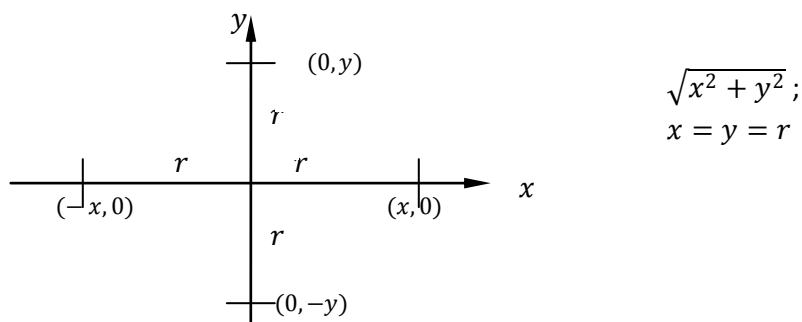
Quadrantal angles:

Are angles with their terminal side lying along the coordinate axes.

These angles are multiples of 90° or $\frac{\pi}{2}$ radians. They are the easiest angles to work with.

Mathematics for an A-Level Student Volume II

Consider a point (x, y) on the coordinate axes r -units from the *origin*.



a) **The function of zero degrees.** ($y = 0, x = 0$)

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{r} = 0, \quad \tan 0^\circ = \frac{y}{x} = \frac{0}{r} = 0$$

$$\cos 0^\circ = \frac{+x}{r} = \frac{r}{r} = +1 \quad \text{And} \quad \sec 0^\circ = \frac{r}{+x} = \frac{r}{r} = +1$$

Cotangent and cosecant of 0° do not exist
since a calculation of them involves a division
by a zero

b) **The function of 90° ,** ($x = 0, y = r$)

$$\sin 90^\circ = \frac{+y}{r} = \frac{+r}{+r} = 1, \quad \cot 90^\circ = \frac{+x}{+y} = \frac{0}{r} = 0$$

$$\cos 90^\circ = \frac{+x}{r} = \frac{0}{r} = 0 \quad \text{And} \quad \operatorname{cosec} 90^\circ = \frac{r}{+y} = \frac{r}{r} = +1$$

Tangent and secant of 90° do not exist since a
calculation of them involves a division by a

c) **The function of 180° ,** ($x = -r, y = 0$)

$$\sin 180^\circ = \frac{y}{r} = \frac{0}{+r} = 0, \quad \tan 180^\circ = \frac{y}{-x} = \frac{0}{-r} = 0$$

$$\cos 180^\circ = \frac{-x}{r} = \frac{-r}{r} = -1 \quad \text{And} \quad \sec 180^\circ = \frac{r}{-x} = \frac{r}{-r} = -1$$

Cotangent and cosecant of 180° do not exist since
a calculation of them involves a division by a zero

d) The function of 270° and 360° , ($x = -r$, $y = 0$)

It is left for the student to verify the following.

$$\sin 270^\circ = -1$$

$$\sin 360^\circ = 0$$

$$\cos 270^\circ = 0$$

$$\cos 360^\circ = 1$$

$$\tan 270^\circ \text{ does not exist}$$

$$\tan 360^\circ = 0$$

$$\cot 270^\circ = 0$$

$$\cot 360^\circ \text{ does not exist}$$

$$\sec 270^\circ \text{ does not exist}$$

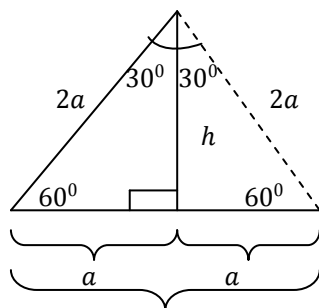
$$\sec 360^\circ = 1$$

$$\operatorname{cosec} 270^\circ = -1$$

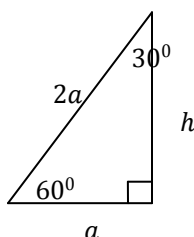
$$\operatorname{cosec} 360^\circ \text{ does not exist}$$

e) Evaluation for Angles with 30° , 60° or 45°

Consider the equilateral triangle of side $2a$.



$2a$

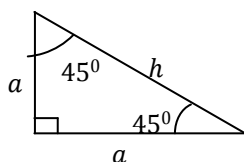


By Pythagoras theorem

$$h^2 = (2a)^2 - a^2$$

$$h = \sqrt{3a^2} = a\sqrt{3}$$

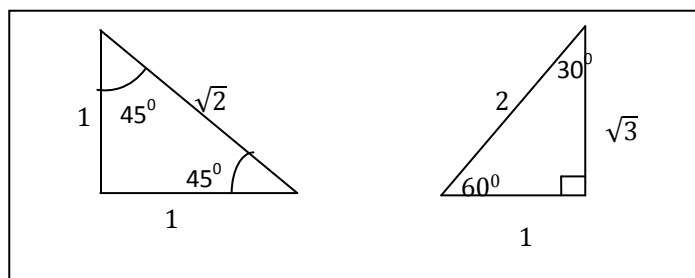
Consider also an isosceles right angled triangle of side a .



$$h^2 = a^2 + a^2$$

$$\therefore h = \sqrt{2a^2} = a\sqrt{2}$$

Taking $a = 1$ for both reference triangles, then



Application of the special angles reference triangle

- Evaluation of exact angle trigonometric ratios. e.g. $\sin 30^\circ, \sec(-240^\circ)$ etc.
- Solving trigonometric equations.

Evaluation of exact angle trig ratios

Find the exact value of;

i) $\sec(-240^\circ)$	iii) $\cot(315^\circ)$
ii) $\operatorname{cosec}(600^\circ)$	iv) $\tan(-765^\circ)$

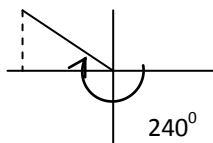
Solution

i) $\sec(-240^\circ) = \frac{1}{\cos(-240^\circ)}$

Reference angle = $240 - 180 = 60^\circ$

$\cos(-240^\circ) = -\cos 60^\circ = -1/2$

$\therefore \sec(-240^\circ) = \frac{1}{-1/2} = -2$

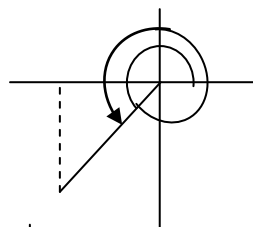


ii) $\operatorname{cosec}(600^\circ) = \frac{1}{\sin(600^\circ)}$

Reference angle = $(360 + 240) - 540^\circ$

$= (360 + 180 + 60) - 540 = 60^\circ$

$\operatorname{cosec}(600^\circ) = \frac{1}{-\sin 60^\circ} = \frac{1}{-\sqrt{3}/2} = \frac{-2}{\sqrt{3}}$

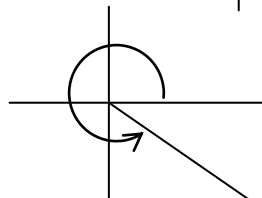


iii) $\cot(315^\circ) = \frac{1}{\tan(315^\circ)}$

Reference angle = $360 - 315 = 45^\circ$

$\tan(315^\circ) = -\tan(45^\circ) = \frac{-1}{\sqrt{2}}$

$\therefore \cot(315^\circ) = \frac{1}{-1/\sqrt{2}} = -\sqrt{2}$



iv) Left as an exercise to the student. **Ans: -1**

Solving trigonometric equations

Solve for $0^\circ < \theta < 360^\circ$ the equations;

i) $\cot \theta = -\sqrt{3}$ iii) $\sin \theta = \frac{\sqrt{3}}{2}$

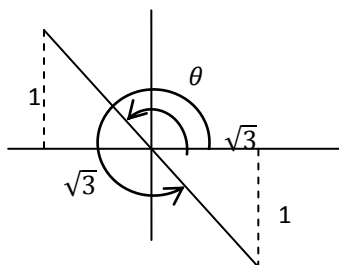
ii) $\operatorname{cosec} \theta = -2$

Solution

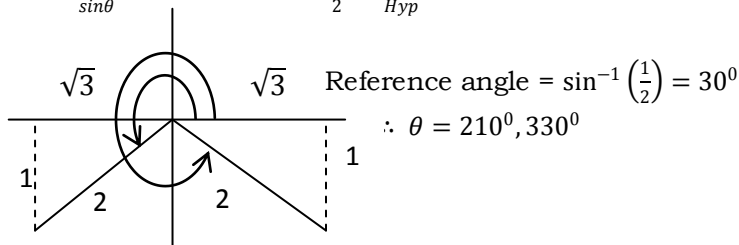
i) $\cot \theta = -\sqrt{3} \Rightarrow \frac{1}{\tan \theta} = -\sqrt{3} \therefore \tan \theta = \frac{-1}{\sqrt{3}} = \frac{\text{Opp}}{\text{Adj}}$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ = \text{reference angle}$

$\therefore \theta = 150^\circ, 300^\circ$



ii) $\operatorname{cosec} \theta = -2 \Rightarrow \frac{1}{\sin \theta} = -2 \therefore \sin \theta = \frac{-1}{2} = \frac{\text{Opp}}{\text{Hyp}}$



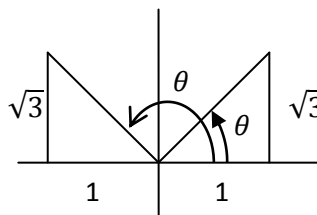
iii) $\sin \theta = \frac{\sqrt{3}}{2}$

$\theta = 180 - \text{Reference angle}$,

θ is the reference angle

Reference angle = $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$

$\theta = 120^\circ, 60^\circ$



TRIGONOMETRIC IDENTITIES.

Solving trigonometric equations using algebraic approach

Suggested approach for solving Trig equations algebraically.

1. Regard one particular trig function as function and solve for it,
 - a) Consider using algebraic manipulation such as factoring, combining or separating functions
 - b) Consider using identities
2. After solving for trig function solve for it.

Example:

Exact solutions using factoring

Solve for $-180^\circ < \theta < +180^\circ$ the equations;

i) $2\cos^2\theta - \cos\theta = 0$

iv) $2\cos^2\theta + 3\cos\theta + 1 = 0$

ii) $\tan^2\theta + \tan\theta = 0$

v) $4\sin^2\theta - \sin\theta - 1 = 0$

iii) $4\cos^3\theta = \cos\theta$

Solution:

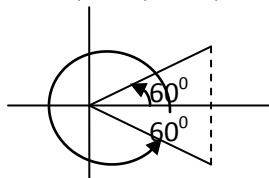
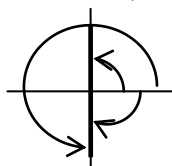
Step 1: Solve for $\cos x$: $2\cos^2\theta - \cos\theta = 0$, factor out $\cos x$

$\cos x (\cos x - 1) = 0$ [a.b = 0 if a = 0, b = 0]

$\therefore \cos x = 0, 2\cos x - 1 = 0$ or $\cos x = 1/2$

Step 2: Solve each equation for the required interval.

$\cos x = 0$, $x = \cos^{-1}(0) = 90^\circ, 270^\circ, -90^\circ, -270^\circ$



For $\cos x = 1/2$,
 $\cos^{-1}(1/2) = 60^\circ$
 $\therefore x = \pm 60^\circ, \pm 90^\circ$

ii) $\tan^2 \theta + \tan \theta = 0$ [$\because \tan \theta = x, x^2 + x - 0$]

$\tan \theta (\tan \theta + 1) = 0$ [$x(x + 1) = 0$]

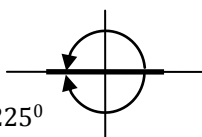
$\therefore \tan \theta = 0, \tan \theta + 1 = 0$ or $\tan \theta = -1$

For $\tan \theta = 0$

$\theta = 0^\circ, \pm 180^\circ$

$\theta = 135^\circ, 315^\circ, -45^\circ, -225^\circ$

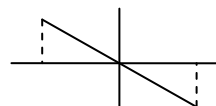
$\therefore \theta = 0^\circ, \pm 180^\circ, 135^\circ$



For $\tan \theta = -1$

$\Rightarrow \tan^{-1}(1) = 45^\circ$

$\theta = 135^\circ, 315^\circ, -45^\circ, -225^\circ$



iii) $4\cos^3 \theta = \cos \theta \Rightarrow 4\cos^3 \theta - \cos \theta = 0$

$\cos \theta [4\cos^2 \theta - 1] = 0 \Rightarrow \cos \theta = 0, 4\cos^2 \theta - 1 = 0$ or $\cos \theta = \pm \frac{1}{2}$

From $\cos \theta = 0$

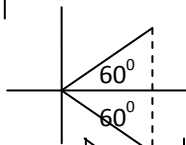
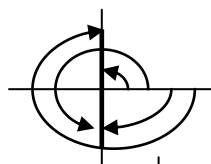
$\theta = \pm 90^\circ, \pm 270^\circ$

From $\cos \theta = \pm \frac{1}{2}$

Considering

$\cos \theta = +\frac{1}{2}, \cos^{-1}(1/2) = 60^\circ$

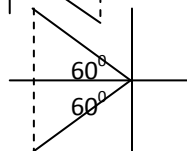
$\therefore \theta = \pm 60^\circ, \pm 300^\circ$



Considering $\cos \theta = -\frac{1}{2}, \cos^{-1}(\frac{1}{2}) = 60^\circ$

$\therefore \theta = \pm 120^\circ, \pm 240^\circ$

$\therefore \theta = \pm 60^\circ, \pm 120^\circ$



iv) $2\cos^2 \theta + 3\cos \theta + 1 = 0$ [let $\cos \theta = x, 2x^2 + 3x + 1 = 0$]

$2\cos^2 \theta + 2\cos \theta + \cos \theta + 1 = 0$ [$2x^2 + 2x + x + 1 = 0$]

$2\cos \theta (\cos \theta + 1) + 1(\cos \theta + 1) = 0$ [$2x(x + 1) + 1(x + 1) = 0$]

$(\cos \theta + 1)(2\cos \theta + 1) = 0$ [$(2x + 1)(x + 1) = 0$]

$\therefore \cos \theta + 1 = 0$ or $\cos \theta = -1$

$2\cos \theta + 1 = 0$ or $\cos \theta = -1/2$

Considering $\cos \theta = -1$

$\cos^{-1}(1) = 180^\circ$

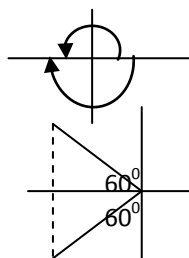
$\therefore \theta = \pm 180^\circ$

Considering $\cos \theta = -1/2$

$\cos^{-1}(1/2) = 60^\circ$

$\therefore \theta = \pm 120^\circ, \pm 240^\circ$

$\therefore \theta = \pm 120^\circ, \pm 180^\circ$



a) $4\sin^2 \theta = 1$

[let $x = \sin \theta, 4x^2 = 1$]

$4\sin^2 \theta - 1 = 0$

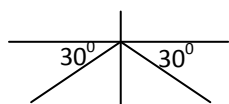
[$4x^2 - 1 = 0$]

$(2\sin \theta + 1)(2\sin \theta - 1) = 0$ [$(2x + 1)(2x - 1) = 0$]

$\therefore 2\sin \theta + 1 = 0$ or $\sin \theta = -1/2$

$2\sin \theta - 1 = 0$ or $\sin \theta = 1/2$

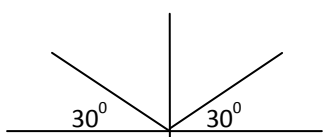
Dealing with $\sin \theta = -1/2$



$$\sin^{-1}(1/2) = 30^\circ$$

$$\theta = 210^\circ, 330^\circ, -30^\circ, -150^\circ$$

Dealing with $\sin \theta = 1/2$



$$\sin^{-1}(1/2) = 30^\circ$$

$$\theta = 30^\circ, 150^\circ$$

$$\therefore \theta = \pm 30^\circ, \pm 150^\circ$$

v) $2\sin^2\theta - \sin\theta - 1 = 0$

$$2\sin^2\theta - 2\sin\theta - 1 = 0$$

$$2\sin\theta(\sin\theta - 1) + 1(\sin\theta - 1) = 0$$

$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$

$$\therefore \sin\theta - 1 = 0 \text{ or } \sin\theta = 1$$

$$2\sin\theta + 1 = 0 \text{ or } \sin\theta = -1/2$$

Dealing with $\sin \theta = 1$

$$+90^\circ, -270^\circ$$

Dealing with $\sin \theta = -1/2$

$$\sin^{-1}(1/2) = 30^\circ$$

$$\therefore \theta = 210^\circ, 330^\circ, -30^\circ, -150^\circ$$

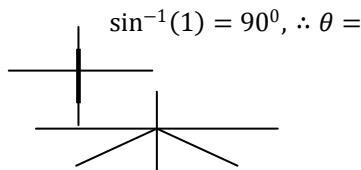
$$\Rightarrow \therefore \theta = -30^\circ, -150^\circ, +90^\circ$$

$$[\text{let } x = \sin\theta, 2x^2 - x - 1 = 0]$$

$$[2x^2 - 2x + x - 1 = 0]$$

$$[2x(x - 1) + 1(x - 1) = 0]$$

$$[(x - 1)(2x + 1) = 0]$$



$$\sin^{-1}(1) = 90^\circ, \therefore \theta =$$

Example

Approximate solutions using substitution

Solve for $0^\circ < \theta < 360^\circ$ the equations

i) $5\sin(2\theta - 5) = -3.045$

ii) $\sin(2\theta + 30^\circ) = 0.8$

iii) $\tan(3\theta - 45^\circ) = \frac{1}{2}$

iv) $\sec(2\theta + 30^\circ) = 3$

v) $\cot\left(\frac{1}{2}\theta + 45^\circ\right) = \sqrt{3}$

Solution

i) $5\sin(2\theta - 5) = -3.045$, let $u = 2\theta - 5 \Rightarrow 5\sin u = -3.045$

$$\Rightarrow \sin u = \frac{-3.045}{5} = -0.609$$

$$\sin^{-1}(0.609) = 37.5^\circ$$

$$u = 217.5^\circ, 322.5^\circ + 360^\circ k, k = 0, 1, 2, 3, \dots$$

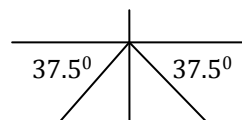
[$\therefore 360k$ is added to get other angles giving the same result]

$$\Rightarrow u = 2\theta - 5 = 217.5^\circ, 322.5^\circ, 577.5^\circ, 682.5^\circ$$

$$2\theta = (217.5^\circ, 322.5^\circ, 577.5^\circ, 682.5^\circ) + 5^\circ$$

$$= 222.5^\circ, 327.5^\circ, 582.5^\circ, 687.5^\circ$$

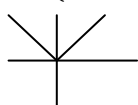
$$\therefore \theta = 111.25^\circ, 163.75^\circ, 291.25^\circ, 343.75^\circ$$



NOTE:

If $\sin \theta = a$, then $\theta = \sin^{-1}(a) \pm 360^\circ k$; $k = 0, 1, 2, 3, \dots$, and $\sin^{-1}(a) < 90^\circ$

ii) $\sin(2\theta + 30^\circ) = 0.8$, let $u = 2\theta + 30^\circ \Rightarrow \sin u = 0.8$



$$\sin^{-1}(0.8) = 53.1^\circ$$

$$u = 53.1^\circ, 126.7^\circ + 360^\circ k$$

$$\therefore 2\theta + 30^\circ = 53.1^\circ, 126.7^\circ, 413.1^\circ, 486.7^\circ, 773.1^\circ$$

$$2\theta = 23.1^\circ, 96.7^\circ, 383.1^\circ, 456.7^\circ, 743.1^\circ (\text{Ignored})$$

$$\theta = 11.55^\circ, 48.36^\circ, 191.55^\circ, 228.35^\circ$$

iii) left as an exercise **Ans:** $\theta = 35.79^\circ, 81.86^\circ, 215.79^\circ, 203.86^\circ, 263.86^\circ, 323.86^\circ$

iv) $\sec(2\theta + 30^\circ) = 3 \Rightarrow \frac{1}{\cos(2\theta + 30^\circ)} = 3, \therefore \cos(2\theta + 30^\circ) = 1/3$

$$\text{let } u = 2\theta + 30^\circ \Rightarrow \cos u = 1/3$$

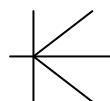
$$\cos^{-1}(1/3) = 70.5^\circ$$

$$u = 70.5^\circ, 289.5^\circ + 360^\circ k$$

$$\therefore 2\theta + 30^\circ = 70.5^\circ, 289.5^\circ, 430.5^\circ, 649.5^\circ$$

$$2\theta = 40.5^\circ, 259.5^\circ, 400.5^\circ, 619.5^\circ$$

$$\therefore \theta = 20.25^\circ, 129.75^\circ, 200.25^\circ, 309.75^\circ$$



NOTE:

If $\cos \theta = b$, $\theta = \cos^{-1}(b) \pm 360^\circ k$ and $\cos^{-1}(b) < 90^\circ$

v) $\cot\left(\frac{1}{2}\theta + 45^\circ\right) = \sqrt{3} \Rightarrow \frac{1}{\tan\left(\frac{1}{2}\theta + 45^\circ\right)} = \sqrt{3}$

$$\Rightarrow \tan\left(\frac{1}{2}\theta + 45^\circ\right) = \frac{1}{\sqrt{3}}, \text{ let } u = \frac{1}{2}\theta + 45^\circ$$

$$\Rightarrow \tan u = 1/\sqrt{3}$$

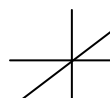
$$\tan^{-1}(1/\sqrt{3}) = 30^\circ$$

$$\therefore u = 30^\circ, 210^\circ + 180^\circ k; k = 0, 1, 2, 3, \dots$$

$$\therefore \frac{1}{2}\theta + 45^\circ = 30^\circ, 210^\circ, 390^\circ, 570^\circ, 750^\circ$$

$$\frac{1}{2}\theta = -15^\circ, 165^\circ, 345^\circ, 525^\circ, 705^\circ \therefore \theta = (-15^\circ, 165^\circ, 345^\circ, 525^\circ, 705^\circ) \times 2$$

$$\therefore \theta = 330^\circ$$



NOTE:

If $\tan \theta = a$, then $\theta = \tan^{-1}(a) \pm 180^\circ k$ i.e cumulate the angle $\tan^{-1}(a)$ with $\pm 180^\circ$

Approximate solutions to trig equations using a quadratic formula

Solve for $-180^\circ \leq \theta \leq +180^\circ$ the equation

i) $2\cos^2\theta - 4\cos\theta + 1 = 0$

iv) $2\sec^2\theta - \sec\theta - 2 = 0$

ii) $3\tan^2\theta + 5\tan\theta - 1 = 0$

iii) $3\operatorname{cosec}^2\theta + \operatorname{cosec}\theta - 8 = 0$

Solution

i) $2\cos^2\theta - 4\cos\theta + 1 = 0$ [let $u = \cos\theta, 2u^2 - 4u + 1 = 0$]

$$\cos \theta = u = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{16-8}}{4} \Rightarrow \cos \theta = \frac{4+\sqrt{8}}{4} \text{ or } \frac{4-\sqrt{8}}{8}$$

$$\therefore \cos \theta = 1.707107, \cos \theta = 0.292893$$

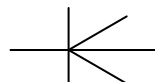
Dealing with $\cos \theta = 1.707107$

No values of θ since $-1 \leq \theta \leq +1$

Dealing with $\cos \theta = 0.292893$, $\cos^{-1}(0.292893) = 72.93^\circ$

$$\therefore \theta = \pm 72.97^\circ, 287.03^\circ$$

$$\therefore \theta = \pm 72.97^\circ$$



ii) $3\tan^2\theta + 5\tan\theta - 1 = 0$ [let $u = \tan\theta$, $3u^2 + 5u - 1 = 0$]

$$\tan \theta = \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-1)}}{2(3)} = \frac{-5 \pm \sqrt{37}}{6}$$

$$\therefore \tan \theta = \frac{-5+\sqrt{37}}{6} = 0.18046 \text{ or } \frac{-5-\sqrt{37}}{6} = -1.84713$$

Dealing with $\tan \theta = 0.18046$

$$\tan^{-1}(0.18046) = 10.2^\circ$$

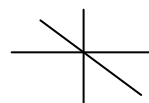
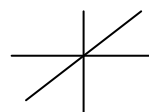
$$\therefore \theta = 10.2^\circ, 190.2^\circ, -169.8^\circ$$

Dealing with $\tan \theta = -1.84713$

$$\tan^{-1}(1.84713) = 61.6^\circ$$

$$\Rightarrow \theta = -61.6^\circ, -241.6^\circ, 118.4^\circ, 298.4^\circ$$

$$\therefore \theta = 10.2^\circ, 118.4^\circ, -61.6^\circ, -169.8^\circ$$



iii) $3\operatorname{cosec}^2\theta + \operatorname{cosec}\theta - 8 = 0$ [let $u = \operatorname{cosec}\theta$, $3u^2 + u - 8 = 0$]

$$\operatorname{cosec}\theta = \frac{-1 \pm \sqrt{(1)^2 - 4(3)(-8)}}{2(3)} = \frac{-1 \pm \sqrt{97}}{6}$$

$$\therefore \operatorname{cosec}\theta = \frac{-1+\sqrt{97}}{6} \text{ or } \frac{-1-\sqrt{97}}{6} \Rightarrow \operatorname{cosec}\theta = 1.4748 \text{ or } -1.80814$$

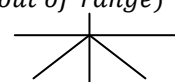
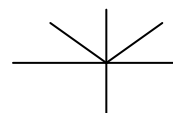
$$\text{But } \operatorname{cosec}\theta = \frac{1}{\sin \theta} \Rightarrow \frac{1}{\sin \theta} = 1.4748 \Rightarrow \sin \theta = 0.6781 \text{ also}$$

$$\sin \theta = \frac{1}{-1.80814} = -0.5531$$

Dealing with $\sin \theta = -0.6781$, $\sin^{-1}(0.6781) = 42.7^\circ$

$$\therefore \theta = 42.7^\circ, 137.3^\circ, (\text{no need for negative angles since are out of range})$$

Dealing with $\sin \theta = -0.5531$, $\sin^{-1}(0.5531) = 33.6^\circ$



$$\therefore \theta = -33.6^\circ, -146.4^\circ, (\text{no need for positive angles since are out of range})$$

$$\therefore \theta = 42.7^\circ, 137.3^\circ, -33.6^\circ, -146.4^\circ$$

iv) Left as an exercise. **Ans: $\pm 38.9^\circ$**

Solve $8\cos^4x - 10\cos^2x + 3 = 0$ for x in the range

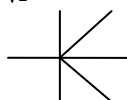
$$0^\circ \leq x \leq 180^\circ \quad (\text{Uneb2012})$$

Solution

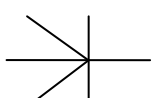
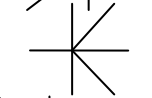
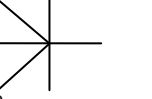
$$\text{Let } u = \cos^2x \Rightarrow 8u^2 - 10u + 3 = 0 \text{ Solving } u = \frac{3}{4}, \frac{1}{2} \Rightarrow \cos^2x = \frac{3}{4}, \frac{1}{2}$$

$$\therefore \cos x = \pm \frac{\sqrt{3}}{2} \text{ or } \cos x = \pm \frac{1}{\sqrt{2}}$$

$$\text{For } \cos x = +\frac{\sqrt{3}}{2}$$



$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$$

For $\cos x = -\frac{\sqrt{3}}{2}$		$\Rightarrow x = 30^0, 330^0$ $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^0$ $\Rightarrow x = 150^0, 210^0$
For $\cos x = +\frac{1}{\sqrt{2}}$		$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$ $\Rightarrow x = 45^0, 315^0$
For $\cos x = -\frac{1}{\sqrt{2}}$		$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$ $\Rightarrow x = 135^0, 225^0$

$\therefore x = 30^0, 45^0, 135^0, 150^0$ Reject $x = 210^0, 225^0, 330^0$

Equations involving creation of tangent ratio

Solve for $0^0 < \theta < 360^0$ the equations

a) $\cos \theta + \sin \theta = 0$	d) $3\sin^2 \theta - \sin \theta \cos \theta - 2\cos^2 \theta = 0$
b) $2\sin \theta - 4\cos \theta = 0$	$2\cos^2 \theta - 8\sin \theta \cos \theta + 3\sin^2 \theta = 0$
c) $4\cos \theta - 3\sin \theta = 0$	

Solution

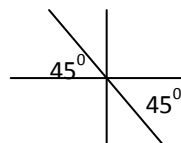
Equation of the form $a\cos(m\theta) + b\sin(m\theta) = 0$ where a , m , and b are constants require creation of *tangent*, dividing through by $\cos(m\theta)$.

- a) $\cos \theta + \sin \theta = 0$ (dividing by $\cos \theta$)

$$\frac{\cos \theta}{\cos \theta} + \frac{\sin \theta}{\cos \theta} = \frac{0}{\cos \theta} \Rightarrow 1 + \tan \theta = 0 \therefore \tan \theta = -1$$

$$\tan^{-1}(1) = 45^0$$

$$\therefore \theta = 135^0, 315^0$$

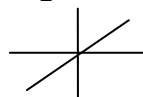


- b) $2\sin \theta - 4\cos \theta = 0$ (dividing by $\cos \theta$)

$$\frac{2\sin \theta}{\cos \theta} - \frac{4\cos \theta}{\cos \theta} = \frac{0}{\cos \theta} \Rightarrow 2\tan \theta - 4 = 0 \text{ or } \tan \theta = 2$$

$$\tan^{-1}(2) = 63.4^0$$

$$\therefore \theta = 63.4^0, 243.4^0$$



- c) Left as an exercise **Ans: $53.1^0, 233.1^0$**

- d) $3\sin^2 \theta - \sin \theta \cos \theta - 2\cos^2 \theta = 0$ is of the form

$a \sin^2(m\theta) + b \sin(m\theta)\cos(m\theta) + c \cos^2 \theta = d$

Or $a \sin^2 \theta + b \sin(2m\theta) + c \cos^2(m\theta) = d$ is solved by division by $\cos^2(m\theta)$

Dividing through by $\cos^2 \theta$ yields $\frac{3\sin^2 \theta}{\cos^2 \theta} - \frac{\sin \theta \cos \theta}{\cos^2 \theta} - \frac{2\cos^2 \theta}{\cos^2 \theta} = 0$

$$\Rightarrow 3\tan^2 \theta - \tan \theta - 2 = 0 \Rightarrow \tan \theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-2)}}{2(3)}$$

$$\therefore \tan \theta = \frac{1 \pm \sqrt{25}}{6} = \frac{1 \pm 5}{6} \Rightarrow \tan \theta = \frac{1+5}{6} = 1 \text{ of } \frac{1-5}{6} = -2/3$$

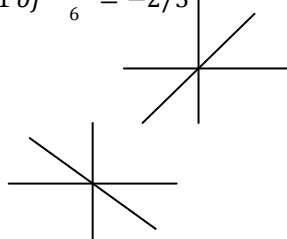
Dealing with $\tan \theta = 1$

$$\tan^{-1}(1) = 45^0$$

$$\therefore \theta = 45^0, 225^0$$

Dealing with $\tan \theta = -2/3$

$$\tan^{-1}(2/3) = 23.69^0$$



$$\Rightarrow \theta = 146.31^\circ, 326.31^\circ$$

$$\therefore \theta = 45^\circ, 225^\circ, 146.31^\circ, 326.31^\circ$$

e) $2\cos^2\theta - 8\sin\theta\cos\theta + 3\sin^2\theta = 0$, dividing through by $\cos^2\theta$ yields

$$2 - 8\tan\theta + 3\tan^2\theta = 0$$

Solving the equation yields $\tan\theta = 2.3874, 0.2792$ (CAL)

Dealing with $\tan\theta = 2.3874$

$$\tan^{-1}(2.3874) = 67.3^\circ$$

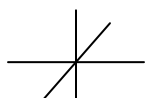
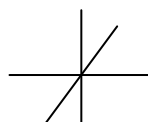
$$\therefore \theta = 67.3^\circ, 247.3^\circ$$

Dealing with $\tan\theta = 0.2792$

$$\tan^{-1}(0.2792) = 15.6^\circ$$

$$\therefore \theta = 15.6^\circ, 195.6^\circ$$

$$\therefore \theta = 67.3^\circ, 247.3^\circ, 15.6^\circ, 195.6^\circ$$



Task:

Solve for $0^\circ < \theta < 360^\circ$ the equations

i) $4\cos 2\theta - 3\sin 2\theta = 0$

ii) $3\sin^2\theta - \sin 2\theta \cos 2\theta - 4\cos^2 2\theta = 0$

iii) $10\sin^2 x + 10\sin x \cos x - \cos^2 x = 0$

Ans: (i) $26.6^\circ, -153.4^\circ$ (ii) $26.6^\circ, 116.6^\circ, -63.5^\circ$ (iii) $14^\circ, -166^\circ, -56.3^\circ, 123.7^\circ$

Evaluating trig equations involving reciprocal of trig ratios, and solving using factorizing Or quadratic formula

Solve for $0^\circ < \theta < 360^\circ$ the equations

a) $3\cos\theta + 2\sec\theta + 7 = 0$

d) $\sec\theta = 2\cos\theta$

b) $5\sin\theta + 6\operatorname{cosec}\theta = 17$

e) $\tan\theta + 2\cot\theta = 3$

c) $\cot\theta = 5\cos\theta$

Solution

a) $3\cos\theta + 2\sec\theta + 7 = 0 \Rightarrow 3\cos\theta + \frac{2}{\cos\theta} + 7 = 0$ (multiplying through by $\cos\theta$)

$$3(\cos\theta)^2 + 7\cos\theta + 2 = 0 \quad [\text{let } u = \cos\theta, 3u^2 + 7u + 2 = 0]$$

$$3u^2 + 7u + 2 = 0 \Rightarrow u = -\frac{1}{3} \text{ or } -2 \text{ (CAL)}$$

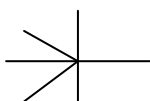
$$\therefore \cos\theta = -\frac{1}{3}, \cos\theta = -2$$

For $\cos\theta = -2$, No values since $-1 \leq \theta \leq +1$

For $\cos\theta = -\frac{1}{3}$

$$\cos^{-1}\left(\frac{1}{3}\right) = 70.5^\circ$$

$$\therefore \theta = \pm 109.5^\circ$$



b) $5\sin\theta + 6\operatorname{cosec}\theta = 17 \Rightarrow 5\sin\theta + \frac{6}{\sin\theta} = 17$

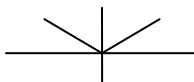
$$5(\sin\theta)^2 - 17\sin\theta + 6 = 0 \quad [\text{let } u = \sin\theta, 5u^2 - 17u + 6 = 0]$$

Solving $u = 3, 0.4 \Rightarrow \sin\theta = 3$ and $\sin\theta = 0.4$

For $\sin\theta = 3$, no values of θ since $-1 \leq \theta \leq +1$

For $\sin \theta = 0.4$

$$\sin^{-1}(0.4) = 23.6^\circ$$

$$\therefore \theta = 23.6^\circ, 156.4^\circ$$


c) $\cot \theta = 5 \cos \theta \Rightarrow \frac{\cos \theta}{\sin \theta} = 5 \cos \theta \Rightarrow \cos \theta = 5 \cos \theta \sin \theta$

$$\cos \theta - 5 \cos \theta \sin \theta = 0 \Rightarrow \cos \theta (1 - \sin \theta) = 0$$

$$\therefore \cos \theta = 0 \text{ or } 1 - \sin \theta = 0, \sin \theta = 1$$

For $\cos \theta = 0$, $\cos^{-1}(0) = 90^\circ$

$$\therefore \theta = \pm 90^\circ$$

For $\sin \theta = 1$, $\sin^{-1}(1) = 90^\circ$

$$\therefore \theta = 90^\circ, 270^\circ$$

d) $\sec \theta = 2 \cos \theta \Rightarrow \frac{1}{\cos \theta} = 2 \cos \theta \text{ or } 1 = 2 \cos^2 \theta$

$$\Rightarrow 2 \cos^2 \theta - 1 = 0, \text{ solving yields } \cos \theta = \pm 1/\sqrt{2}$$

For $\cos \theta = +1/\sqrt{2}$, $\cos^{-1}(1/\sqrt{2}) = 45^\circ$

$$\therefore \theta = \pm 45^\circ$$

For $\cos \theta = -1/\sqrt{2}$, $\cos^{-1}(-1/\sqrt{2}) = 135^\circ$

$$\therefore \theta = \pm 135^\circ$$

$$\therefore \theta = \pm 45^\circ, \pm 135^\circ$$

e) $\tan \theta + 2 \cot \theta = 3 \Rightarrow \tan \theta + \frac{2}{\tan \theta} = 0 \Rightarrow \tan^2 \theta - 3 \tan \theta + 2 = 0$

Solving the quadratic equation yields $\tan \theta = 2, 1$

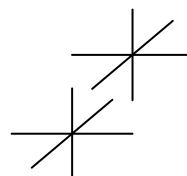
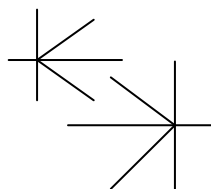
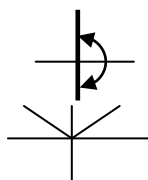
For $\tan \theta = 2$, $\tan^{-1}(2) = 63.4^\circ$

$$\therefore \theta = 63.4^\circ, -116.5^\circ$$

For $\tan \theta = 1$, $\tan^{-1}(1) = 45^\circ$

$$\therefore \theta = 45^\circ, -135^\circ$$

$$\therefore \theta = 63.4^\circ, -116.5^\circ, 45^\circ, -135^\circ$$



TRIGONOMETRIC IDENTITIES:

The fundamental identity:

This is derived basing on the Pythagoras theorem. Consider the figure below.

By Pythagoras theorem

$$x^2 + y^2 = r^2 \text{ (dividing by } r^2)$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \text{ but } \cos \theta = \frac{x}{r}, \sin \theta = \frac{y}{r}$$

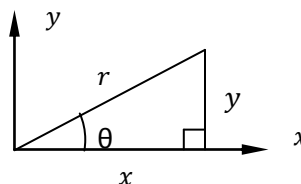
$$\Rightarrow (\cos \theta)^2 + (\sin \theta)^2 = 1 \quad \therefore \cos^2 \theta + \sin^2 \theta = 1$$

In gener $\cos^2(m\theta) + \sin^2(m\theta) = 1$ this is the fundamental identity.

Dividing the fundamental identity by $\cos^2 \theta$ or $\sin^2 \theta$,

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \Rightarrow 1 + \tan^2 \theta = \sec^2 \theta$$

Now dividing by $\sin^2 \theta$, $\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$



$$\Rightarrow 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

Also in general,

$$\begin{aligned} 1 + \tan^2(m\theta) &= \sec^2(m\theta) \\ 1 + \cot^2(m\theta) &= \operatorname{cosec}^2(m\theta) \end{aligned}$$

Application of the identities:

- Solving trigonometric equations
- Proof of identities
- Simplifying trigonometric expressions
- Elimination of parameters.

Solution to Trigonometrical Equations

Solve for $0^\circ < \theta < 360^\circ$ the equations.

i) $4\cot^2 \theta + 39 = 24\operatorname{cosec} \theta$	iv) $4\sec^2 \theta = 3\tan \theta + 5$
ii) $5\sec \theta - 2\sec^2 \theta = \tan^2 \theta + 1$	v) $(2\tan x - 1)^2 = 3(\sec^2 x - 2)$
iii) $5\cos \theta = 2(1 + 2\sin^2 \theta)$	vi) $4\cos x + 5 = 6\sin^2 x$

Solution

i) $4\cot^2 \theta + 39 = 24\operatorname{cosec} \theta$ (1)

Identity relating $\operatorname{cosec} \theta$ and $\cot^2 \theta$ is $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

$\Rightarrow \cot^2 \theta = \operatorname{cosec}^2 \theta - 1$ (2)

Put (2) in to (1) gives

$$4(\operatorname{cosec}^2 \theta - 1) + 39 = 24\operatorname{cosec} \theta$$

$$4\operatorname{cosec}^2 \theta - 4 + 39 = 24\operatorname{cosec} \theta \text{ or } 4\operatorname{cosec}^2 \theta - 24\operatorname{cosec} \theta + 35 = 0$$

Let $u = \operatorname{cosec} \theta \Rightarrow 4u^2 - 24u + 35 = 0 \therefore u = 3.5, 2.5$ (cal)

$\Rightarrow \operatorname{cosec} \theta = 3.5$ or $\frac{1}{\sin \theta} = 3.5 \therefore \sin \theta = \frac{2}{7}$ and $\operatorname{cosec} \theta = 2.5$ or

$\frac{1}{\sin \theta} = 2.5 \therefore \sin \theta = \frac{2}{5}$

Dealing with $\sin \theta = \frac{2}{7}$

$$\sin^{-1}(2/7) = 16.6^\circ$$

$$\Rightarrow \theta = 16.6^\circ, 163.4^\circ$$

Dealing with $\sin \theta = \frac{2}{5}$

$$\sin^{-1}(2/5) = 23.6^\circ$$

$$\Rightarrow \theta = 23.6^\circ, 156.4^\circ$$

$\therefore \theta = 16.6^\circ, 163.4^\circ, 23.6^\circ, 156.4^\circ$

ii) $5\sec \theta - 2\sec^2 \theta = \tan^2 \theta + 1$ (1)

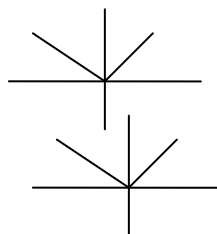
Identity relating $\sec \theta$ and $\tan \theta$ is $1 + \tan^2 \theta = \sec^2 \theta$ (2)

Put (2) in to (1) gives

$$5\sec \theta - 2\sec^2 \theta = \sec^2 \theta \Rightarrow 5\sec \theta - 3\sec^2 \theta = 0$$

$$\sec \theta(5 - 3\sec \theta) = 0 \Rightarrow \sec \theta = 0 \text{ or } (5 - 3\sec \theta) = 0 \Rightarrow \sec \theta = \frac{5}{3}$$

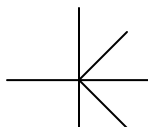
Dealing with $\sec \theta = \frac{1}{\cos \theta} = 0$, hence no values of the equation



Dealing with $\sec \theta = \frac{1}{\cos \theta} = \frac{5}{3} \Rightarrow \cos \theta = \frac{3}{5}$

$$\cos^{-1}(3/5) = 53.1^\circ$$

$$\therefore \theta = 53.1^\circ, 306.9^\circ$$



iii) $5\cos\theta = 2(1 + 2\sin^2\theta)$ (1)

Identity relating $\cos\theta$ and $\sin\theta$ is $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$$
..... (2)

Put (2) in to (1) gives

$$5\cos\theta = 2(1 + 2(1 - \cos^2\theta)) \text{ or } 5\cos\theta = 2 + 4 - 4\cos^2\theta$$

$$\Rightarrow 4\cos^2\theta + 5\cos\theta - 6 = 0$$

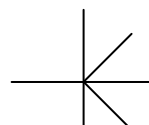
$$\text{Let } u = \cos\theta \text{ then } 4u^2 + 5u - 6 = 0 \Rightarrow u = 0.75, -2$$

From $\cos\theta = 0.75$

$$\cos^{-1}(0.75) = 41.4^\circ$$

$$\therefore \theta = 41.4^\circ, 318.6^\circ$$

From $\cos\theta = -2$, no values of θ since $-1 \leq \cos\theta \leq +1$



iv) Left as an exercise.

v) $(2 \tan x - 1)^2 = 3(\sec^2 x - 2)$

$$\text{Expanding gives } 4\tan^2 x - 4 \tan x + 1 = 3\sec^2 x - 6$$
..... (1)

Identity relating $\tan x$ and $\sec x$ is $1 + \tan^2 x = \sec^2 x$

Put (2) in to (1) gives

$$4\tan^2 x - 4 \tan x + 1 = 3(1 + \tan^2 x) - 6 \text{ or}$$

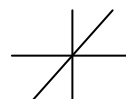
$$4\tan^2 x - 4 \tan x + 1 = 3 + 3\tan^2 x - 6$$

$$\Rightarrow \tan^2 x - 4 \tan x + 4 = 0 \text{ or } (\tan x - 2)^2 = 0$$

Dealing with $\tan x = 2$

$$\tan^{-1}(2) = 63.4^\circ$$

$$\therefore x = 63.4^\circ, 243.4^\circ$$



vi) Left as an exercise. **Ans: 78.8°, 281.2°, 149.4°, 210.6°**

Solve $\cot^2\theta = 5(\operatorname{cosec}\theta + 1)$ for $0^\circ \leq \theta < 360^\circ$ (**Uneb 2000**)

Solution

$$\cot^2\theta = 5(\operatorname{cosec}\theta + 1), \text{ using } 1 + \cot^2\theta = \operatorname{cosec}^2\theta \Rightarrow \cot^2\theta = \operatorname{cosec}^2\theta - 1$$

$$\Rightarrow \operatorname{cosec}^2\theta - 1 = 5\operatorname{cosec}\theta + 5 \text{ or } \operatorname{cosec}^2\theta - 5\operatorname{cosec}\theta - 6 = 0$$

$$\text{Let } u = \operatorname{cosec}\theta \Rightarrow u^2 - 5u - 6 = 0 \text{ solving } u = 6, -1$$

$$\therefore \operatorname{cosec}\theta = 6 \text{ or } \frac{1}{\sin\theta} = 6 \Rightarrow \sin\theta = \frac{1}{6}$$

$$\text{And } \operatorname{cosec}\theta = -1 \text{ or } \sin\theta = -1$$

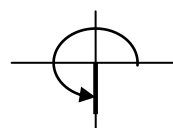
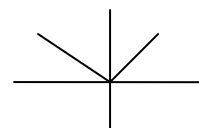
$$\text{Dealing with } \sin\theta = \frac{1}{6}, \sin^{-1}(1/6) = 9.6^\circ$$

$$\Rightarrow \theta = 9.6^\circ, 170.6^\circ$$

$$\text{Dealing with } \sin\theta = -1, \sin^{-1}(1) = 90^\circ$$

$$\Rightarrow \theta = 270^\circ$$

$$\therefore \theta = 9.6^\circ, 170.6^\circ, 270^\circ$$



Trigonometric equations and identities involving algebraic surds

Example

1. a) If $\sec \theta + \tan \theta = 4$, find $\sin \theta$, $\cos \theta$, $\tan \theta$ and $\sec \theta$
 b) $\tan \theta = 2 - \sqrt{3}$, If find $\sin \theta$, $\cos \theta$ and $\sec \theta$

Solution

a) $\sec \theta + \tan \theta = 4 \Rightarrow \tan \theta = 4 - \sec \theta$

Squaring both sides

$$\tan^2 \theta = 16 - 8\sec \theta + \sec^2 \theta$$

But $1 + \tan^2 \theta = \sec^2 \theta$ then $\sec^2 \theta - 1 = 16 - 8\sec \theta + \sec^2 \theta$

$$\Rightarrow -1 = 16 - 8\sec \theta \text{ or } \sec \theta = \frac{17}{8}$$

But $\sec \theta = \frac{1}{\cos \theta} \Rightarrow \cos \theta = \frac{8}{17}$

Now from $\cos^2 \theta + \sin^2 \theta = 1$ then $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - (8/17)^2}$

$$\therefore \sin \theta = \pm \frac{15}{17}$$

Now $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\pm 15/17}{8/17} = \pm \frac{15}{8}$

Alternatively

$$\sec \theta + \tan \theta = 4 \dots \dots \dots (1)$$

Creating $\sec \theta - \tan \theta$

$$\begin{aligned} \sec \theta - \tan \theta &= (\sec \theta - \tan \theta) \frac{(\sec \theta - \tan \theta)}{(\sec \theta - \tan \theta)} = \frac{\sec^2 \theta - \tan^2 \theta}{\sec \theta + \tan \theta} \\ &= \frac{1}{\sec \theta + \tan \theta} = \frac{1}{4} \quad \left[\begin{array}{l} \sec^2 \theta - \tan^2 \theta = 1 \\ \therefore \sec^2 \theta = 1 + \tan^2 \theta \end{array} \right] \\ \therefore \sec \theta - \tan \theta &= \frac{1}{4} \dots \dots \dots (2) \end{aligned}$$

Eliminating $\tan \theta$, (1) + (2) $\Rightarrow 2\sec \theta = 4 + \frac{1}{4} = \frac{17}{4} \therefore \sec \theta = \frac{17}{4}$,

But $\cos \theta = \frac{1}{\sec \theta} \Rightarrow \cos \theta = \frac{4}{17}$

Eliminating $\sec \theta$, (1) - (2) $\Rightarrow 2\tan \theta = 4 - \frac{1}{4} = \frac{15}{4} \therefore \tan \theta = \frac{15}{8}$

Using $\cos^2 \theta + \sin^2 \theta = 1$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - (4/17)^2} = \pm \frac{15}{17} \text{ as before}$$

b) $\tan \theta = 2 - \sqrt{3}$ and using $1 + \tan^2 \theta = \sec^2 \theta$

$$\sec^2 \theta = 1 + (2 - \sqrt{3})^2 = 1 + 4 - 4\sqrt{3} + 3 = 8 - 4\sqrt{3}$$

$$\therefore \frac{1}{\cos^2 \theta} = 8 - 4\sqrt{3} \Rightarrow \cos^2 \theta = \frac{1}{8 - 4\sqrt{3}} \therefore \cos \theta = \pm \frac{1}{\sqrt{4(2 - \sqrt{3})}} = \pm \frac{1}{2(2 - \sqrt{3})}$$

Also using $\cos^2 \theta + \sin^2 \theta = 1$

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{1}{8 - 4\sqrt{3}} = \frac{8 - 4\sqrt{3} - 1}{8 - 4\sqrt{3}} = \frac{7 - 4\sqrt{3}}{8 - 4\sqrt{3}}$$

Rationalizing denominator

$$= \frac{(7 - 4\sqrt{3})(8 + 4\sqrt{3})}{(8 - 4\sqrt{3})(8 + 4\sqrt{3})} = \frac{56 + 28\sqrt{3} - 32\sqrt{3} - 48}{64 - 16 \times 3} = \frac{8 - 4\sqrt{3}}{16} = \frac{2 - \sqrt{3}}{4}$$

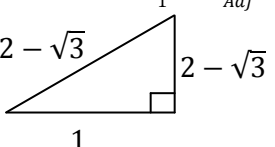
$$\therefore \sin \theta = \pm \frac{\sqrt{2 - \sqrt{3}}}{2}$$

Alternatively

$$\tan \theta = 2 - \sqrt{3} = \frac{2 - \sqrt{3}}{1} = \frac{Opp}{Adj}$$

$$h^2 = 1^2 + (2 - \sqrt{3})^2$$

$$= 1 + 4 - 4\sqrt{3} + 3 = 8 - 4\sqrt{3}$$

$$\therefore h = \sqrt{4(2 - \sqrt{3})}$$


Note θ lies in 1^{st} and 3^{rd} quadrant

$$\cos \theta = \pm \frac{1}{2(2 - \sqrt{3})}, \sin \theta = \pm \frac{2 - \sqrt{3}}{2\sqrt{2 - \sqrt{3}}} = \pm \frac{\sqrt{2 - \sqrt{3}}}{2} \text{ and}$$

$$\sec \theta = \frac{1}{\cos \theta} = \pm 2\sqrt{2 - \sqrt{3}}$$

2. If $\operatorname{cosec} \theta + \cot \theta = 2 - \sqrt{3}$, find $\sin \theta$, $\tan \theta$ and $\cos \theta$

Solution

$$\operatorname{cosec} \theta + \cot \theta = 2 - \sqrt{3} \dots \dots \dots (1)$$

Creating $\operatorname{cosec} \theta - \cot \theta$

$$\begin{aligned} \operatorname{cosec} \theta - \cot \theta &= (\operatorname{cosec} \theta - \cot \theta) \frac{(\operatorname{cosec} \theta + \cot \theta)}{(\operatorname{cosec} \theta + \cot \theta)} \\ &= \frac{\operatorname{cosec}^2 \theta - \cot^2 \theta}{\operatorname{cosec} \theta + \cot \theta} \quad \left[\because 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta \right] \\ &= \frac{1}{2 - \sqrt{3}} = \frac{2 + \sqrt{3}}{(2 - \sqrt{3})(2 + \sqrt{3})} \quad \text{Rationalising denominator} \\ \therefore \operatorname{cosec} \theta - \cot \theta &= 2 + \sqrt{3} \dots \dots \dots (2) \end{aligned}$$

Adding (1) and (2)

$$2 \operatorname{cosec} \theta = 4 \text{ or } \sin \theta = \frac{1}{2}$$

Subtracting (1) and (2)

$$2 \cot \theta = -2\sqrt{3} \text{ or } \tan \theta = -\frac{1}{\sqrt{3}}$$

Since $\sin \theta = +ve$ and $\tan \theta = -ve$, then $90^\circ < \theta < 180^\circ$

$$\cos \theta = \sqrt{\cos^2 \theta} = \sqrt{1 - \sin^2 \theta}, \sin \theta = \frac{1}{2} = \pm \sqrt{1 - \frac{1}{4}} = \pm \frac{\sqrt{3}}{2}$$

$$\therefore \cos \theta = -\frac{\sqrt{3}}{2}$$

ELIMINATION OF PARAMETERS

Usually two parametric equations in terms of trig ratios of an angle, say θ are given. The parameter is the angle θ , to be eliminated.

Example

1. Show that the parametric equations defined by the pair of equations represent the given curves

i) $x = a \cos \theta, y = b \sin \theta$, curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

ii) $x = a \tan \theta, y = b \sec \theta$, curve is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

iii) $x = 1 + \tan \theta, y = \cos \theta$, curve is $y^2(x - 1)^2 + y^2 = 1$

iv) $x = a \operatorname{cosec} \theta, y = b \sec \theta$, curve is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

v) $x = a \sec \theta, y = b + c \cos \theta$, curve is $x(y - b) = ac$

Solution

i) $x = a \cos \theta \Rightarrow \cos \theta = \frac{x}{a}$ (1)

$y = b \sin \theta \Rightarrow \sin \theta = \frac{y}{b}$ (2)

Relating $\cos \theta$ and $\sin \theta$ using identity $\cos^2 \theta + \sin^2 \theta = 1$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as required

ii) $x = a \tan \theta \Rightarrow \tan \theta = \frac{x}{a}$ (1)

$y = b \sec \theta \Rightarrow \sec \theta = \frac{y}{b}$ (2)

Relating $\tan \theta$ and $\sec \theta$ using the identity $1 + \tan^2 \theta = \sec^2 \theta$

$\Rightarrow 1 + \frac{x^2}{a^2} = \frac{y^2}{b^2} \therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ as required

iii) $x = 1 + \tan \theta \Rightarrow \tan \theta = x - 1$ (1)

$y = \cos \theta$ or $\frac{1}{y} = \frac{1}{\cos \theta} = \sec \theta \Rightarrow \sec \theta = \frac{1}{y}$ (2)

Using $1 + \tan^2 \theta = \sec^2 \theta$

$\Rightarrow 1 + (x - 1)^2 = \frac{1}{y^2} \therefore y^2(x - 1)^2 + y^2 = 1$

iv) $x = a \operatorname{cosec} \theta \Rightarrow \operatorname{cosec} \theta = \frac{x}{a} = \frac{1}{\sin \theta} \Rightarrow \sin \theta = \frac{a}{x}$ (1)

$y = a \sec \theta \Rightarrow \sec \theta = \frac{y}{a} = \frac{1}{\cos \theta} \Rightarrow \cos \theta = \frac{a}{y}$ (2)

Using $\cos^2 \theta + \sin^2 \theta = 1 \Rightarrow \frac{a^2}{x^2} + \frac{a^2}{y^2} = 1$ as required

v) $x = a \sec \theta \Rightarrow \sec \theta = \frac{x}{a}$ or $\cos \theta = \frac{a}{x}$ (1)

$y = b + c \cos \theta \Rightarrow \frac{y-b}{c} = \cos \theta$ (2)

Equating (1) and (2)

$\Rightarrow \frac{y-b}{c} = \frac{a}{x} \therefore x(y - b) = ac$

2. a) If $\tan \theta + \sin \theta = x$ and $\tan \theta - \sin \theta = b$ prove that $(x^2 - y^2)^2 = 16xy$

b) If $p \sin \theta + q \cos \theta = a$ and $p \cos \theta - q \sin \theta = b$ show that $\frac{p+a}{q+b} + \frac{q-b}{p-a} = 0$

c) If $\sin \theta - \cos \theta = a$ and $\operatorname{cosec} \theta + \sec \theta = b$ prove that

$b^2(1 - a^2)^2 = 4(2 - a^2)$

b) If $x \cos \theta + y \sin \theta = a$ and $x \sin \theta - y \cos \theta = b$ prove that

$\tan \theta = \frac{bx+ay}{ax-by}$ and $x^2 + y^2 = a^2 + b^2$

Solution

a) $\tan \theta + \sin \theta = x$ (1)

$\tan \theta - \sin \theta = b$ (2)

Eliminating $\sin \theta$, (1) + (2)

$2 \tan \theta = x + b \Rightarrow \tan \theta = \frac{x+b}{2}$ (3)

Eliminating $\tan \theta$, (1) - (2)

$2 \sin \theta = x - b \Rightarrow \sin \theta = \frac{x-b}{2}$

But $\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{2}{x-y}$ and from (3) $\cot \theta = \frac{1}{\tan \theta} = \frac{2}{x+y}$

Now using $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

$$\begin{aligned} \Rightarrow 1 + \frac{4}{(x+y)^2} &= \frac{4}{(x-y)^2} \quad \therefore 1 = 4 \left(\frac{1}{(x-y)^2} - \frac{1}{(x+y)^2} \right) = \frac{4[(x+y)^2 - (x-y)^2]}{(x-y)^2(x+y)^2} \\ &= \frac{4[(x+y+x-y)(x+y-x+y)]}{(x^2-y^2)^2}, \text{ Difference of squares} \\ &= \frac{4(2x)(2y)}{(x^2-y^2)^2} \\ \therefore (x^2 - y^2)^2 &= 16xy \end{aligned}$$

b) $p \sin \theta + q \cos \theta = a$ (1)

$p \cos \theta - q \sin \theta = b$ (2)

Squaring equations and adding

$$\begin{aligned} p^2 \sin^2 \theta + q^2 \cos^2 \theta + 2pq \sin \theta \cos \theta &= a^2 \\ p^2 \cos^2 \theta + q^2 \sin^2 \theta - 2pq \sin \theta \cos \theta &= b^2 \\ + \quad p^2(\sin^2 \theta + \cos^2 \theta) + (q^2 \cos^2 \theta + \sin^2 \theta) &= a^2 + b^2 \\ p^2 + q^2 &= a^2 + b^2 \Rightarrow p^2 - a^2 + q^2 - b^2 = 0 \\ (p-a)(p+a) + (q-b)(q+b) &= 0, \text{ dividing through by } (p-a)(q+b) \\ \Rightarrow \frac{(p-a)(p+a)}{(p-a)(q+b)} + \frac{(q-b)(q+b)}{(p-a)(q+b)} &= 0 \quad \therefore \frac{p+a}{q+b} + \frac{q-b}{p-a} = 0 \end{aligned}$$

c) $\sin \theta - \cos \theta = a$, $\operatorname{cosec} \theta + \sec \theta = b$ to prove $b^2(1-a^2)^2 = 4(2-a^2)$

Beginning from a complicated side L.H.S

$$\begin{aligned} b^2(1-a^2)^2 &= (\operatorname{cosec} \theta + \sec \theta)^2(1-(\sin \theta - \cos \theta)^2)^2 \\ &= (\operatorname{cosec}^2 \theta + 2\operatorname{cosec} \theta \sec \theta + \sec^2 \theta)(1-(\cos^2 \theta + \sin^2 \theta - 2 \sin \theta \cos \theta)) \\ &= \left(\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} + \frac{2}{\sin \theta \cos \theta} \right) (1 - (1 - 2 \sin \theta \cos \theta))^2 \\ &= \left(\frac{\cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta}{\sin^2 \theta \cos^2 \theta} \right) (2 \sin \theta \cos \theta)^2 = 4(1 + \sin \theta \cos \theta) \end{aligned}$$

Now expressing $2 \sin \theta \cos \theta$ in terms of a

From $(\sin \theta - \cos \theta) = a$, squaring gives

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta &= a^2 \Rightarrow 1 - 2 \sin \theta \cos \theta = a^2 \\ \therefore 2 \sin \theta \cos \theta &= 1 - a^2 \Rightarrow b^2(1-a^2)^2 = 4(1 + 1 - a^2) = 4(2-a^2) \end{aligned}$$

d) $x \cos \theta + y \sin \theta = a$ (1)

$x \sin \theta - y \cos \theta = b$ (2)

Eliminating constants a and b , $(1) \times b - (2) \times a$

$$\begin{aligned} \Rightarrow bx \cos \theta + by \sin \theta &= ab \\ ax \sin \theta - ay \cos \theta &= ab \end{aligned}$$

$$bxcos\theta - ax \sin \theta + by \sin \theta + aycos\theta = 0$$

$$\cos \theta (bx + ay) + (by - ax) \sin \theta = 0$$

Creating $\tan \theta$ by division by $\cos \theta$

$$\frac{\cos \theta}{\cos \theta} (bx + ay) + (by - ax) \frac{\sin \theta}{\sin \theta} = 0 \Rightarrow (bx + ay) + (by - ax) \tan \theta = 0$$

$$\therefore \tan \theta (ax - by) = bx + ay \Rightarrow \tan \theta = \frac{bx+ay}{ax-by} \text{ as required}$$

To prove $x^2 + y^2 = a^2 + b^2$, see previous example in © above

Task: If $\sec \theta - \cos \theta = a$ and $\operatorname{cosec} \theta - \sin \theta = b$, show that $a^2 b^2 (a^2 + b^2 + 3) = 1$

PROOF OF IDENTITIES

A trigonometric equation which is true for all permissible values of the angles involved is called a *trigonometric identity*.

Suggested steps in verifying Identities

1. Start with a more complicated side of the identity and transform it in to a simpler side
2. Try algebraic operations such as multiplying, factoring and combining fractions and splitting fractions
3. If the other steps fail, express each function in terms of sine and cosine function and then perform appropriate algebraic operations
4. At each step keep the other side of the identity in mind. This reveals what you should do to get there.
5. Try to manipulate either side of the equation until it is identical to the other side.

Examples

1. Prove the following identities

$$\text{i)} \sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x \quad \text{vi)} \frac{\cot x}{\cos x} + \frac{\sec x}{\cot x} = \sec^2 x \cdot \operatorname{cosec} x$$

$$\text{ii)} \sin x = \tan x \cos x \quad \text{vii)} \frac{1 + \sin x - \sin^2 x}{\cos x} = \cos x + \tan x$$

$$\text{iii)} \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = \frac{\sec x}{\tan x - \sec x}$$

$$\text{iv)} \operatorname{cosec}^4 x - \cot^4 x = \operatorname{cosec}^2 x + \cot^2 x \quad \text{viii)} \frac{\cos x + 1}{\sin^3 x} = \frac{\operatorname{cosec} x}{1 - \cos x}$$

$$\text{v)} \frac{\cos^2 x}{\sin x} + \sin x = \operatorname{cosec} x \quad \text{ix)} \frac{\tan x}{\sec x} + \frac{\cot x}{\operatorname{cosec} x} = \sin x + \cos x$$

Solution

$$\text{i)} \sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$$

$$\begin{aligned} \text{L.H.S: } \sin^4 x - \cos^4 x &= (\sin^2 x)^2 - (\cos^2 x)^2 \quad [\text{Difference of squares}] \\ &= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) \\ &= 1 \cdot (\sin^2 x - \cos^2 x) = (\sin^2 x - \cos^2 x) = \text{R.H.S} \end{aligned}$$

$$\text{ii)} \sin x = \tan x \cos x$$

$$\text{R.H.S: } \tan x \cdot \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x = \text{L.H.S}$$

$$\text{iii)} \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = \frac{\sec x}{\tan x - \sec x}, \text{ The tactic is to express both sides to the same expression. i.e. same trig ratios}$$

$$\text{L.H.S: } \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = \frac{1/\sin x}{1 - 1/\sin x} = \frac{1}{\sin x} \times \frac{\sin x}{\sin x - 1} = \frac{1}{\sin x - 1}$$

$$\text{R.H.S: } \frac{\sec x}{\tan x - \sec x} = \frac{1/\cos x}{\frac{\sin x}{\cos x} - \frac{1}{\cos x}} = \frac{1}{\cos x} \times \frac{\cos x}{\sin x - 1} = \frac{1}{\sin x - 1}$$

$$\text{Consequently } \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = \frac{\sec x}{\tan x - \sec x}$$

$$\text{iv)} \text{ Left as an exercise. Hint: see (i) above}$$

$$\text{v)} \frac{\cos^2 x}{\sin x} + \sin x = \operatorname{cosec} x$$

$$\text{L.H.S: } \frac{\cos^2 x}{\sin x} + \frac{\sin x}{1} = \frac{\cos^2 x + \sin^2 x}{\sin x} = \frac{1}{\sin x} = \operatorname{cosec} x = \text{R.H.S}$$

$$\text{vi)} \frac{\cot x}{\cos x} + \frac{\sec x}{\cot x} = \sec^2 x \cdot \operatorname{cosec} x$$

Expressing L.H.S in terms of sines and cosines

$$\begin{aligned}\frac{\cot x}{\cos x} + \frac{\sec x}{\cot x} &= \frac{\cos x}{\sin x \cdot \cos x} + \frac{1}{\cos x \cdot \frac{\cos x}{\sin x}} = \frac{1}{\sin x} + \frac{\sin x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\sin x \cdot \cos^2 x} = \frac{1}{\sin x} \cdot \frac{1}{\cos^2 x} = \operatorname{cosec} x \cdot \sec^2 x = \mathbf{R.H.S}\end{aligned}$$

vii) $\frac{1 + \sin x - \sin^2 x}{\cos x} = \cos x + \tan x$

L.H.S: $\frac{1 + \sin x - \sin^2 x}{\cos x} = \frac{1 + \sin x - (1 - \cos^2 x)}{\cos x} \quad [\because \cos^2 x + \sin^2 x = 1]$
 $= \frac{\sin x + \cos^2 x}{\cos x} = \frac{\sin x}{\cos x} + \frac{\cos^2 x}{\cos x} = \tan x + \cos x = \mathbf{R.H.S}$

viii) $\frac{\cos x + 1}{\sin^3 x} = \frac{\operatorname{cosec} x}{1 - \cos x}$

The numerator of LHS gives rise to denominator of the RHS through rationalization

L.H.S: $\frac{\cos x + 1}{\sin^3 x} = \frac{(\cos x + 1)(\cos x - 1)}{\sin^3 x \cdot (\cos x - 1)} = \frac{\cos^2 x - 1}{\sin^3 x \cdot (\cos x - 1)} \dots \dots \dots \left[\times \left(\frac{-1}{-1} \right) \right]$
 $= \frac{1 - \cos^2 x}{\sin^3 x \cdot (\cos x - 1)} = \frac{\sin^2 x}{\sin^3 x \cdot (\cos x - 1)} = \frac{1}{\sin x (1 - \cos x)} = \frac{1}{\sin x} \cdot \frac{1}{1 - \cos x} = \frac{\operatorname{cosec} x}{1 - \cos x}$

ix) $\frac{\tan x}{\sec x} + \frac{\cot x}{\operatorname{cosec} x} = \sin x + \cos x$

Since the RHS is in terms of $\sin x$ and $\cos x$, expressing LHS in terms of $\sin x$ and $\cos x$,

L.H.S: $\frac{\tan x}{\sec x} + \frac{\cot x}{\operatorname{cosec} x} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} + \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \frac{\sin x}{\cos x} \cdot \cos x + \frac{\cos x}{\sin x} \cdot \sin x$
 $= \sin x + \cos x = \mathbf{R.H.S}$

2. Prove the following identities

a) $\frac{1 - \cos \theta}{\sin \theta} = \frac{1}{\operatorname{cosec} \theta + \cot \theta}$

e) $\sec \theta - \sin \theta = \frac{\tan^2 \theta + \cos^2 \theta}{\sec \theta + \sin \theta}$

b) $\frac{\tan \theta + \cot \theta}{\sec \theta + \operatorname{cosec} \theta} = \frac{1}{\sin \theta + \cos \theta}$

f) $\frac{1 - \sin \theta + \cos \theta}{1 - \sin \theta} = \frac{1 + \sin \theta + \cos \theta}{\cos \theta}$

c) $\sec^2 \theta = \frac{\operatorname{cosec} \theta}{\operatorname{cosec} \theta - \sin \theta}$

g) $\frac{\sin^3 \theta + \cos^3 \theta}{1 - 2 \cos^2 \theta} = \frac{\sec \theta - \sin \theta}{\tan \theta - 1}$

d) $\frac{1 + \sin \theta}{1 - \sin \theta} = (\sec \theta + \tan \theta)^2$

h) $\frac{1 - 2 \sin \theta - 3 \sin^2 \theta}{\cos^2 \theta} = \frac{1 - 3 \sin \theta}{1 - \sin \theta}$

Solution

a) $\frac{1 - \cos \theta}{\sin \theta} = \frac{1}{\operatorname{cosec} \theta + \cot \theta}$

L.H.S: $\frac{1 - \cos \theta}{\sin \theta} = \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} = \operatorname{cosec} \theta - \cot \theta$

Now **L.H.S** = $\operatorname{cosec} \theta - \cot \theta$

The denominator of **R.H.S** comes from $\operatorname{cosec} \theta - \cot \theta$ through Rationalization

L.H.S = $\operatorname{cosec} \theta - \cot \theta = (\operatorname{cosec} \theta - \cot \theta) \cdot \frac{(\operatorname{cosec} \theta + \cot \theta)}{(\operatorname{cosec} \theta + \cot \theta)}$
 $= \frac{\operatorname{cosec}^2 \theta - \cot^2 \theta}{\operatorname{cosec} \theta + \cot \theta} = \frac{1}{\operatorname{cosec} \theta + \cot \theta} \quad [\because \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta]$

b) $\frac{\tan \theta + \cot \theta}{\sec \theta + \operatorname{cosec} \theta} = \frac{1}{\sin \theta + \cos \theta}$

L.H.S: Expressing in terms of $\cos \theta$ and $\sin \theta$

L.H.S = $\frac{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}}{\frac{1}{\cos \theta} + \frac{1}{\sin \theta}} = \left(\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \right) \div \left(\frac{\sin \theta + \cos \theta}{\cos \theta \sin \theta} \right)$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta + \cos \theta} = \frac{1}{\sin \theta + \cos \theta} = \mathbf{R. H. S}$$

c) $\sec^2 \theta = \frac{\operatorname{cosec} \theta}{\operatorname{cosec} \theta - \sin \theta}$

R, H, S: $\frac{\operatorname{cosec} \theta}{\operatorname{cosec} \theta - \sin \theta} = \frac{\frac{1}{\sin \theta}}{\frac{1}{\sin \theta} - \sin \theta} = \frac{1}{\sin \theta} \cdot \frac{\sin \theta}{1 - \sin^2 \theta} = \frac{1}{1 - \sin^2 \theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta$

d) $\frac{1 + \sin \theta}{1 - \sin \theta} = (\sec \theta + \tan \theta)^2$

R. H. S.: $(\sec \theta + \tan \theta)^2 = \left(\frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} \right)^2 = \frac{(1 + \sin \theta)^2}{\cos^2 \theta}$
 $= \frac{(1 + \sin \theta)^2}{1 - \sin^2 \theta} \quad [\because \cos^2 \theta + \sin^2 \theta = 1]$

$$= \frac{(1 + \sin \theta)(1 + \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} \quad [\text{Denominator, difference of squares}]$$

$$= \frac{1 + \sin \theta}{1 - \sin \theta} = \mathbf{L. H. S}$$

e) $\sec \theta - \sin \theta = \frac{\tan^2 \theta + \cos^2 \theta}{\sec \theta + \sin \theta}$

Denominator of **R.H.S** originates from $\sec \theta - \sin \theta$ on the **L. H.S** and vis versa

L. H. S: $\sec \theta - \sin \theta = (\sec \theta - \sin \theta) \frac{(\sec \theta + \sin \theta)}{(\sec \theta + \sin \theta)}$

$$= \frac{\sec^2 \theta - \sin^2 \theta}{\sec \theta - \sin \theta} = \frac{(1 + \tan^2 \theta) - (1 - \cos^2 \theta)}{\sec \theta + \sin \theta}$$

$$= \frac{\tan^2 \theta + \cos^2 \theta}{\sec \theta + \sin \theta} = \mathbf{R. H. S}$$

f) $\frac{1 - \sin \theta + \cos \theta}{1 - \sin \theta} = \frac{1 + \sin \theta + \cos \theta}{\cos \theta}$, Starting from any side

L. H. S: $\frac{1 - \sin \theta + \cos \theta}{1 - \sin \theta} = \frac{1 - \sin \theta}{1 - \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} = 1 + \frac{\cos \theta}{1 - \sin \theta}$

Now $\frac{\cos \theta}{1 - \sin \theta}$ can be rationalized since denominator yields

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$\therefore \frac{\cos \theta}{1 - \sin \theta} = \cos \theta \cdot \frac{(1 + \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} = \cos \theta \cdot \frac{1 + \sin \theta}{1 - \sin^2 \theta}$$

$$= \cos \theta \cdot \frac{1 + \sin \theta}{\cos^2 \theta} = \frac{1 + \sin \theta}{\cos \theta}$$

$$\Rightarrow \mathbf{L. H. S:} \frac{1 - \sin \theta + \cos \theta}{1 - \sin \theta} = 1 + \frac{1 + \sin \theta}{\cos \theta} = \frac{1 + \sin \theta + \cos \theta}{\cos \theta} = \mathbf{R. H. S}$$

g) $\frac{\sin^3 \theta + \cos^3 \theta}{1 - 2\cos^2 \theta} = \frac{\sec \theta - \sin \theta}{\tan \theta - 1}$

L. H. S: $\frac{\sin^3 \theta + \cos^3 \theta}{1 - 2\cos^2 \theta} = \frac{(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)}{1 - 2\cos^2 \theta} \quad \left[\because a^3 + b^3 = a^2 - ab + b^2 \right]$
 $\quad \quad \quad \text{an identity}$
 $= \frac{(\sin \theta + \cos \theta)(\sin^2 \theta + \cos^2 \theta - \sin \theta \cos \theta)}{1 - 2\cos^2 \theta} = \frac{(\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta)}{1 - 2\cos^2 \theta}$

Dividing the Numerator and Denominator by $\cos^2 \theta$

$$= \frac{(\sin \theta + \cos \theta) \frac{(1 - \sin \theta \cos \theta)}{\cos^2 \theta}}{\frac{1}{\cos^2 \theta} - 2} = \frac{\left(\frac{\sin \theta + \cos \theta}{\cos \theta} \right) \cdot \left(\frac{1 - \sin \theta \cos \theta}{\cos \theta} \right)}{\sec^2 \theta - 2}$$

$$= \frac{(\tan \theta + 1)(\sec \theta - \sin \theta)}{1 + \tan^2 \theta - 2} = \frac{(\tan \theta + 1)(\sec \theta - \sin \theta)}{\tan^2 \theta - 1} = \frac{(\tan \theta + 1)}{(\tan \theta + 1)} \cdot \frac{(\sec \theta - \sin \theta)}{(\tan \theta - 1)}$$

$$= \frac{\sec \theta - \sin \theta}{\tan \theta - 1} = \mathbf{R. H. S}$$

$$\begin{aligned} \text{h) } \frac{1-2\sin\theta-3\sin^2\theta}{\cos^2\theta} &= \frac{1-3\sin\theta}{1-\sin\theta} \\ \text{L.H.S: } \frac{1-2\sin\theta-3\sin^2\theta}{\cos^2\theta} &= \frac{1-2\sin\theta-3\sin^2\theta}{1-\sin^2\theta} \\ &= \frac{(1-3\sin\theta)(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = \frac{1-3\sin\theta}{1-\sin\theta} \end{aligned}$$

Note: $1 - 2\sin\theta - 3\sin^2\theta \equiv 1 - 2x - 3x^2$, factoring gives $(1+x)(1-3x)$

Task

Prove the following identities

$$\begin{aligned} \text{a) } \frac{\sec\theta - \csc\theta}{\tan\theta - \cot\theta} &= \frac{\tan\theta + \cot\theta}{\sec\theta + \csc\theta} & \text{d) } \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} &= 1 + \frac{2\tan\theta}{1-\tan\theta} \\ \text{b) } \frac{\cot\alpha + \tan\beta}{\cot\beta + \tan\alpha} &= \cot\alpha \tan\beta & \text{e) } \frac{\cos\theta - 1}{\sec\theta + \tan\theta} + \frac{\cos\theta + 1}{\sec\theta - \tan\theta} &= 2(1 + \tan\theta) \\ \text{c) } \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} &= 2\csc^2\theta \end{aligned}$$

3. Prove the following identities

$$\begin{aligned} \text{a) } \frac{\sin\theta}{\cot\theta + \csc\theta} &= 2 + \frac{\sin\theta}{\cot\theta - \csc\theta} \\ \text{b) } \frac{\tan\theta - \sec\theta + 1}{\tan\theta + \sec\theta - 1} &= \frac{\cos\theta}{1+\sin\theta} \\ \text{c) } \cot\theta \cos\theta + \tan\theta \sin\theta &= (\csc\theta + \sec\theta)(1 - \sin\theta \cos\theta) \\ \text{d) } \csc^2\theta - \csc\theta &= \frac{\cot^2\theta}{1+\sin\theta} \end{aligned}$$

Solution

$$\text{a) } \frac{\sin\theta}{\cot\theta + \csc\theta} = 2 + \frac{\sin\theta}{\cot\theta - \csc\theta}$$

Since the **R.H.S** is containing an isolated constant 2, then

Re arranging identity for simplicity,

$$\frac{\sin\theta}{\cot\theta + \csc\theta} - \frac{\sin\theta}{\cot\theta - \csc\theta} = 2, \text{ now proving from the } \text{L.H.S}$$

$$\begin{aligned} \text{L.H.S: } \frac{\sin\theta}{\cot\theta + \csc\theta} - \frac{\sin\theta}{\cot\theta - \csc\theta} &= \sin\theta \cdot \left[\frac{1}{\cot\theta + \csc\theta} - \frac{1}{\cot\theta - \csc\theta} \right] \\ &= \sin\theta \cdot \left[\frac{\cot\theta - \csc\theta - \cot\theta - \csc\theta}{\cot^2\theta - \csc^2\theta} \right] \\ &= \sin\theta \cdot \left[\frac{-2\csc\theta}{-1} \right] = 2 = \text{R.H.S} \end{aligned}$$

Alternatively

With out re-arranging the identity,

$$\begin{aligned} \text{L.H.S: } \frac{\sin\theta}{\cot\theta + \csc\theta} &= \frac{\sin\theta}{\cot\theta + \csc\theta} - 2 + 2 \text{ (creating the 2 in R.H.S)} \\ &= 2 + \left(\frac{\sin\theta}{\cot\theta + \csc\theta} - 2 \right) \\ &= 2 + \left(\frac{\sin\theta - 2(\cot\theta + \csc\theta)}{\cot\theta + \csc\theta} \right) = 2 + \frac{\left(\sin\theta - \left(\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} \right) \right)}{\cot\theta + \csc\theta} \\ &= 2 + \left(\frac{\sin^2\theta - 2\cos\theta - 2}{\cot\theta + \csc\theta} \right) \cdot \frac{1}{\sin\theta} \\ &= 2 + \left(\frac{1 - \cos^2\theta - 2\cos\theta - 2}{\cot\theta + \csc\theta} \right) \cdot \frac{1}{\sin\theta} \\ &= 2 + \left(\frac{\cos^2\theta + 2\cos\theta + 1}{\cot\theta + \csc\theta} \right) \cdot \frac{1}{\sin\theta} \end{aligned}$$

$$\begin{aligned}
 &= 2 + \frac{(\cos \theta + 1)^2}{(\cot \theta + \operatorname{cosec} \theta) \cdot \sin \theta} \\
 &= 2 + \frac{-(\cos \theta + 1)^2 (\cot \theta - \operatorname{cosec} \theta)}{(\cot \theta + \operatorname{cosec} \theta)(\cot \theta - \operatorname{cosec} \theta) \cdot \sin \theta} \\
 &= 2 + \frac{-(\cos \theta + 1)^2 \left(\frac{\cos \theta}{\sin \theta} - \frac{1}{\sin \theta} \right)}{\left(\frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta} \right)(\cot \theta - \operatorname{cosec} \theta) \cdot \sin \theta} = 2 + \frac{-(\cos \theta + 1)^2 (\cos \theta - 1)}{\sin^2 \theta \left(\frac{\cos \theta + 1}{\sin \theta} \right)(\cot \theta - \operatorname{cosec} \theta)} \\
 &= 2 + \frac{-(\cos \theta + 1)(\cos \theta - 1)}{(\cot \theta - \operatorname{cosec} \theta) \cdot \sin \theta} = 2 + \frac{-(\cos^2 \theta - 1)}{(\cot \theta - \operatorname{cosec} \theta) \cdot \sin \theta} \\
 &= \frac{\sin^2 \theta}{(\cot \theta - \operatorname{cosec} \theta) \cdot \sin \theta} \\
 &= 2 + \frac{\sin \theta}{(\cot \theta - \operatorname{cosec} \theta)} = \mathbf{R.H.S}
 \end{aligned}$$

b) $\frac{\tan \theta - \sec \theta + 1}{\tan \theta + \sec \theta - 1} = \frac{\cos \theta}{1 + \sin \theta}$

L.H.S: $\frac{\tan \theta - \sec \theta + 1}{\tan \theta + \sec \theta - 1} = \frac{\tan \theta - \sec \theta + (\sec^2 \theta - \tan^2 \theta)}{\tan \theta + \sec \theta - 1} \quad [\because \sec^2 \theta = 1 + \tan^2 \theta]$

$$\begin{aligned}
 &= \frac{(\tan \theta - \sec \theta) - (\tan^2 \theta - \sec^2 \theta)}{\tan \theta + \sec \theta - 1} \\
 &= \frac{(\tan \theta - \sec \theta) - (\tan \theta - \sec \theta)(\tan \theta + \sec \theta)}{\tan \theta + \sec \theta - 1} = \frac{(\tan \theta - \sec \theta)[1 - (\tan \theta + \sec \theta)]}{\tan \theta + \sec \theta - 1} \\
 &= \frac{-(\tan \theta - \sec \theta)(\tan \theta + \sec \theta - 1)}{(\tan \theta + \sec \theta - 1)} = \sec \theta - \tan \theta = \frac{1}{\cos \theta} - \frac{1}{\sin \theta} \\
 &= \frac{(1 - \sin \theta)(1 + \sin \theta)}{\cos \theta(1 + \sin \theta)} = \frac{1 - \sin^2 \theta}{\cos \theta(1 + \sin \theta)} = \frac{\cos^2 \theta}{\cos \theta(1 + \sin \theta)} = \frac{\cos \theta}{1 + \sin \theta}
 \end{aligned}$$

NOTE:

The same proof can be done using half angle identities. See proof under this topic.

c) $\cot \theta \cos \theta + \tan \theta \sin \theta = (\operatorname{cosec} \theta + \sec \theta)(1 - \sin \theta \cos \theta)$

L.H.S: $\cot \theta \cos \theta + \tan \theta \sin \theta = \frac{\cos \theta}{\sin \theta} \cdot \cos \theta + \frac{\sin \theta}{\cos \theta} \cdot \sin \theta$

$$\begin{aligned}
 &= \frac{\cos^2 \theta}{\sin \theta} + \frac{\sin^2 \theta}{\cos \theta} = \frac{\cos^3 \theta + \sin^3 \theta}{\sin \theta \cos \theta} = \frac{(\cos \theta + \sin \theta)(\cos^2 \theta - \sin \theta \cos \theta + \sin^2 \theta)}{\sin \theta \cos \theta} \\
 &= \frac{(\cos \theta + \sin \theta)(\cos^2 \theta + \sin^2 \theta - \sin \theta \cos \theta)}{\sin \theta \cos \theta} = \left(\frac{\cos \theta + \sin \theta}{\sin \theta \cos \theta} \right) \cdot (1 - \sin \theta \cos \theta) \\
 &= \left(\frac{\cos \theta}{\sin \theta \cos \theta} + \frac{\sin \theta}{\sin \theta \cos \theta} \right) \cdot (1 - \sin \theta \cos \theta) \\
 &= (\operatorname{cosec} \theta + \sec \theta) \cdot (1 - \sin \theta \cos \theta) = \mathbf{R.H.S}
 \end{aligned}$$

d) $\operatorname{cosec}^2 \theta - \operatorname{cosec} \theta = \frac{\cot^2 \theta}{1 + \sin \theta}$

$$\begin{aligned}
 \operatorname{cosec}^2 \theta - \operatorname{cosec} \theta &= \frac{1}{\sin^2 \theta} - \frac{1}{\sin \theta} = \frac{1 - \sin \theta}{\sin^2 \theta} \\
 &= \frac{(1 - \sin \theta)(1 + \sin \theta)}{\sin^2 \theta(1 + \sin \theta)} = \frac{1 - \sin^2 \theta}{\sin^2 \theta(1 + \sin \theta)} = \frac{\cos^2 \theta}{\sin^2 \theta(1 + \sin \theta)} = \frac{\cot^2 \theta}{1 + \sin \theta}
 \end{aligned}$$

Task

Verify that

- $\frac{\tan^2 A}{\sec A + 1} = \sec A - 1$
- $\frac{\cos^2 \theta \tan \theta}{\sin \theta} + \frac{\sin^3 \theta \cot \theta}{\cos^2 \theta} = \sec \theta$
- $\frac{2 \tan A (\tan A - \cot A)}{\tan^2 A - 1} = 2$
- $\frac{\cot x \sin^3 x + \cos x (1 + \tan^2 x)}{\tan x \cos^2 x \sin x} - 1 = \sec^2 x \operatorname{cosec}^2 x$

$$e) \frac{(\operatorname{cosec}^2 x - 1) \sin^2 x + \sin x \cos^2 x}{\cos^2 x} = 1 + \sin x$$

$$f) 2 \operatorname{cosec} x = \frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x}$$

Simplifying Trigonometric expressions

Example

1. Simplify if $s = \sin \theta$ and $c = \cos \theta$

$$a) \frac{1-s^2}{1-c^2} \quad c) \frac{s}{c^2-1} \quad d) \frac{c}{s} + \frac{s}{c} \quad e) \frac{c^4-s^4}{c^2-s^2}$$

$$b) \frac{sc}{\sqrt{1-s^2}}$$

Solution

$$a) \frac{1-s^2}{1-c^2} = \frac{1-\sin^2 \theta}{1-\cos^2 \theta} = \frac{\cos^2 \theta + \sin^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta - \cos^2 \theta} = \frac{\cos^2 \theta}{\sin^2 \theta} = \cot^2 \theta$$

$$b) \frac{sc}{\sqrt{1-s^2}} = \frac{\sin \theta \cdot \cos \theta}{\sqrt{1-\sin^2 \theta}} = \frac{\sin \theta \cdot \cos \theta}{\sqrt{\cos^2 \theta}} = \frac{\sin \theta \cdot \cos \theta}{\cos \theta} = \sin \theta \quad [\because \cos^2 \theta + \sin^2 \theta = 1]$$

$$c) \frac{s}{c^2-1} = \frac{\sin \theta}{\cos^2 \theta - 1} = \frac{\sin \theta}{-\sin^2 \theta} = \frac{1}{-\sin \theta} = -\operatorname{cosec} \theta$$

$$d) \text{ left as an exercise}$$

$$e) \frac{c^4-s^4}{c^2-s^2} = \frac{(c^2)^2 - (s^2)^2}{c^2-s^2} = \frac{(c^2+s^2)(c^2-s^2)}{c^2-s^2} = c^2 + s^2 = \cos^2 \theta + \sin^2 \theta = 1$$

2. Simplify using identities

$$a) \sin x(\sin x - 1) + \cos^2 x - 1 \quad f) \frac{\sin x (\sin x - \cos x \tan x \cot^2 x)}{\cos x (\tan x + 1)} + \cos x$$

$$b) (1 - \sin A)(1 + \sin A) - \cos^2 A \quad g) \frac{\sin x}{1 - \sin x} + \frac{\cos^2 x}{\sin x + 1} - \frac{1 + \sin x}{\cos^2 x}$$

$$c) 3 \sin^2 B + 3 \cos^2 B - 2 \quad h) \frac{\operatorname{cosec}^2 x - \cot^2 x \cdot \operatorname{cosec}^2 x}{\cot^2 x} + \operatorname{cosec}^2 x$$

$$d) \sin x + \cos^2 x - \sin x(1 - \sin x)$$

$$e) \left(\frac{\sec \varphi}{\tan \varphi} - \sin \varphi \right) \cdot \sec \varphi$$

Solution

$$a) \sin x(\sin x - 1) + \cos^2 x - 1 = \sin^2 x - \sin x + \cos^2 x - 1$$

$$= \sin^2 x + \cos^2 x - \sin x - 1 = 1 - \sin x - 1 = -\sin x$$

$$b) (1 - \sin A)(1 + \sin A) - \cos^2 A = 1 - \sin^2 A - \cos^2 A$$

$$= 1 - (\sin^2 A + \cos^2 A) = 1 - 1 = 0$$

$$c) 3 \sin^2 B + 3 \cos^2 B - 2 = 3 \sin^2 B + 3 \cos^2 B - 2(1)$$

$$= 3 \sin^2 B + 3 \cos^2 B - 2(\cos^2 B + \sin^2 B)$$

$$= 3 \sin^2 B + 3 \cos^2 B - 2 \cos^2 B - 2 \sin^2 B$$

$$= \sin^2 B + \cos^2 B = 1$$

$$d) \sin x + \cos^2 x - \sin x(1 - \sin x) = \sin x + \cos^2 x - \sin x + \sin^2 x$$

$$= \cos^2 x + \sin^2 x = 1$$

$$e) \left(\frac{\sec \varphi}{\tan \varphi} - \sin \varphi \right) \cdot \sec \varphi = \left(\frac{\frac{1}{\cos \varphi}}{\frac{\sin \varphi}{\cos \varphi}} - \sin \varphi \right) \cdot \frac{1}{\cos \varphi}$$

$$= \left(\frac{1}{\cos \varphi} \times \frac{\cos \varphi}{\sin \varphi} - \sin \varphi \right) \cdot \frac{1}{\cos \varphi}$$

$$= \left(\frac{1}{\sin \varphi} - \sin \varphi \right) \cdot \frac{1}{\cos \varphi} = \frac{1 - \sin^2 \varphi}{\sin \varphi} \cdot \frac{1}{\cos \varphi} = \frac{\cos^2 \varphi}{\sin \varphi} \cdot \frac{1}{\cos \varphi} = \frac{\cos \varphi}{\sin \varphi} = \cot \varphi$$

$$\begin{aligned}
 \text{f) } \frac{\sin x (\sin x - \cos x \tan x \cot^2 x)}{\cos x (\tan x + 1)} + \cos x &= \frac{\sin^2 x - \sin x \cos x \cdot \frac{\sin x \cos^2 x}{\cos x \sin^2 x}}{\cos x \cdot \frac{\sin x}{\cos x} + \cos x} + \cos x \\
 &= \frac{\sin^2 x - \cos^2 x}{\sin x + \cos x} = \frac{(\sin x + \cos x)(\sin x - \cos x)}{\sin x + \cos x} = \sin x - \cos x \\
 \text{g) } \frac{\sin x}{1 - \sin x} + \frac{\cos^2 x}{\sin x + 1} - \frac{1 + \sin x}{\cos^2 x} &= \left(\frac{\sin x}{1 - \sin x} + \frac{\cos^2 x}{\sin x + 1} \right) - \frac{(1 + \sin x)}{\cos^2 x} \\
 &= \frac{(\sin^2 x + \cos^2 x) + \sin x - \sin x \cos^2 x}{\cos^2 x} - \frac{(1 + \sin x)}{\cos^2 x} \\
 &= \frac{1 + \sin x - \sin x \cos^2 x - 1 - \sin x}{\cos^2 x} = \frac{-\sin x \cos^2 x}{\cos^2 x} = -\sin x \\
 \text{h) } \frac{\operatorname{cosec}^2 x - \cot^2 x \cdot \operatorname{cosec}^2 x}{\cot^2 x} + \operatorname{cosec}^2 x &= \frac{\operatorname{cosec}^2 x - \cot^2 x \cdot \operatorname{cosec}^2 x + \cot^2 x \cdot \operatorname{cosec}^2 x}{\cot^2 x} \\
 &= \frac{\operatorname{cosec}^2 x}{\cot^2 x} = \operatorname{cosec}^2 x \cdot \tan^2 x = \frac{1}{\sin^2 x} \cdot \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
 \end{aligned}$$

Task

Using suitable identities simplify

- a) $\frac{\sin A + \cos A}{\sin A} - \frac{\cos A - \sin A}{\cos A}$
- b) $\frac{1 - 2\cos^2 x}{\sin x \cos x} + \cot x$
- c) $\left(\frac{\sin x - \sin^3 x}{\sin x \cos x + \cos^2 x} \right) \cdot \left(\frac{1 + 2 \sin x \cos x}{\cos x \sin x} \right)$
- d) $\left(\frac{\sin x + \sin x \cos x}{1 - \sin x} \right) \cdot \left(\frac{\cos^2 x}{1 + \cos x} \cdot \operatorname{cosec} x \right)$

NOTE:

This application of simplifying trig functions is very applicable in proof of identities. It is most suitable when dealing with trig ratios expressed in terms of cosines and sines

SKETCHING TRIGONOMETRIC CURVES

The major trigonometric curves to be considered are

$$y = a \cos Ax, y = b \sin Bx \quad \text{and} \quad y = c \tan Cx$$

Example

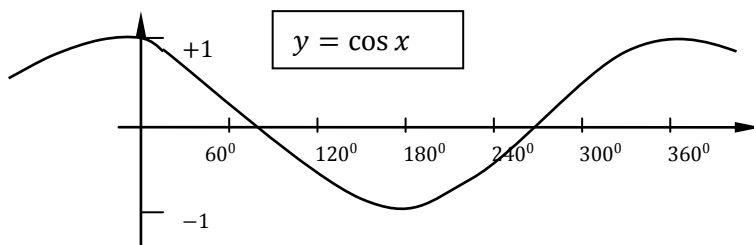
1. Sketch the following curves for $0^\circ \leq \theta \leq 360^\circ$

- a) $y = \cos x$
- b) $y = \sin x$
- c) $y = \tan x$

Mathematics for an A-Level Student Volume II

Solution

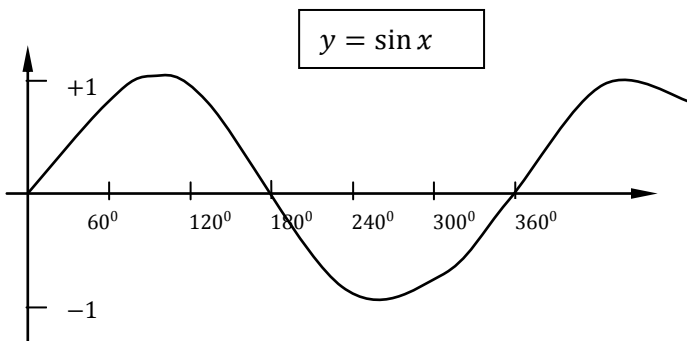
x^0	0	30	60	90	120	150	180	210	240	270	300	330	360
$y = \cos x$	1	$\frac{\sqrt{3}}{2} = 0.9$	$\frac{1}{2} = 0.5$	0	$\frac{1}{2} = 0.5$	$\frac{-\sqrt{3}}{2} = -0.9$	-1	$\frac{-\sqrt{3}}{2} = -0.9$	$\frac{-1}{2} = -0.5$	0	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} = 0.9$	1
$y = \sin x$	0	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} = 0.9$	1	$\frac{\sqrt{3}}{2} = 0.9$	$\frac{1}{2} = 0.5$	0	$\frac{-1}{2} = -0.5$	$\frac{-\sqrt{3}}{2} = -0.9$	-1	$\frac{-\sqrt{3}}{2} = -0.9$	$\frac{-1}{2} = -0.5$	0
$y = \tan x$	0	$\frac{1}{\sqrt{3}} = 0.6$	$\sqrt{3} = 1.7$	$+\infty$	$-\sqrt{3} = -1.7$	$\frac{-1}{\sqrt{3}} = -0.6$	0	$\frac{1}{\sqrt{3}} = 0.6$	$\sqrt{3} = 1.7$	$-\infty$	$-\sqrt{3} = -1.7$	$\frac{-1}{\sqrt{3}} = -0.6$	0



Deduction

Maximum value of $\cos x = 1$ and occurs at $x = 0^0$ or 360^0 and minimum value is -1 and occurs at $x = 180^0$

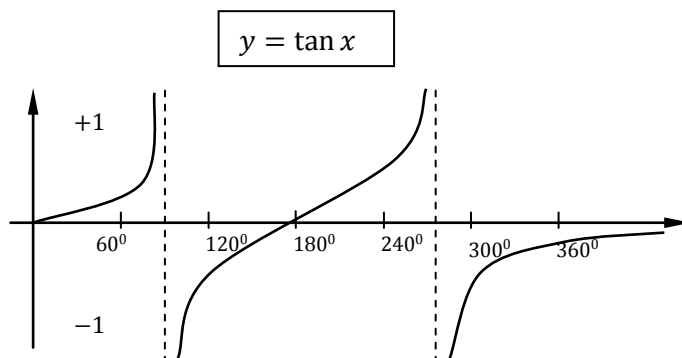
In general $-1 \leq \cos(\text{Angle}) \leq +1$



Deduction

Maximum value of $\sin x = 1$ and occurs at $x = 90^0$ or 360^0 and minimum value is -1 and occurs at $x = 270^0$

In general $-1 \leq \sin(\text{Angle}) \leq +1$



Deduction

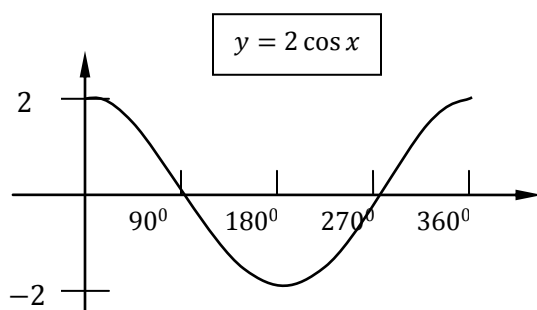
1. The curve $y = \tan x$ is un defined at $x = 90^\circ, 270^\circ$ for $0^\circ \leq \theta \leq 360^\circ$ i.e for $\tan x = \frac{\sin x}{\cos x}$, the curve is undefined at $\cos x = 0^\circ$ i.e.at $90^\circ, 270^\circ, 450^\circ$ etc
2. The curve $y = \tan x$ has no maximum and minimum values

2. Sketch the curves $0^\circ \leq \theta \leq 360^\circ$
- a) $y = 2\cos x$ b) $y = 3\sin x$

Solution

Using intervals of 45°

x°	0	45	90	135	180	225	270	315	360
$y = 2 \cos x$	2.0	1.4	0.0	-1.4	-2.0	-1.4	0.0	1.4	2.0
$y = 3 \sin x$	0.0	2.1	3.0	2.12	0.0	-2.1	-3.0	-2.1	0.0

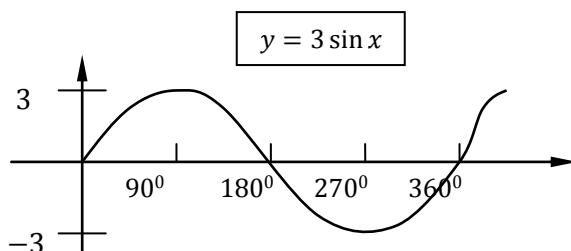


Deduction

Maximum value of $2\cos x$ is 2 and occurs at $x = 0^\circ$ or 360° and minimum value is -2 and occurs at $x = 180^\circ$.

In general for

$y = a \cos(\text{Angle})$, the maximum value is $+a$ and minimum value is $-a$ for $a > 0$



Deduction

Maximum value of $3\sin x$ is 3 and occurs at $x = 90^\circ$ and minimum value is -3 and occurs at $x = 270^\circ$.

In general for

$y = a\sin(\text{Angle})$, the maximum value is $+a$ and minimum value is $-a$ for $a > 0$

NOTE

The above deductions are used in deducing maximum and minimum values of trigonometric functions

STATING MAXIMUM AND MINIMUM VALUES OF

$y = a \pm b\cos(m\theta + \alpha)$ AND $y = a \pm b\sin(m\theta + \alpha)$ AND THEIR RECIPRICALS AND VALUES OF θ FOR WHICH THEY OCCUR

The following rules may be used

$\max(x + y) = \max(x) + \max(y)$	$\max\left(\frac{x}{y}\right) = \frac{\max(x)}{\min(y)}$ $\min\left(\frac{x}{y}\right) = \frac{\min(x)}{\max(y)}$
$\min(x + y) = \min(x) + \min(y)$	
$\max(x - y) = \max(x) - \min(y)$	
$\min(x - y) = \min(x) - \max(y)$	
$\max(x \cdot y) = \max(x) \cdot \max(y)$ [for $x, y > 0$]	
$\min(x \cdot y) = \min(x) \cdot \min(y)$ [for $x, y > 0$]	

Example

Write down the maximum and minimum value of the following expressions giving the smallest positive or zero value of θ for which they occur.

- i) $\frac{1}{3}\cos\frac{1}{2}\theta$ v) $\frac{1}{4-3\cos 2\theta}$ ix) $-\frac{1}{2}\sin 2\theta$
 ii) $\frac{1}{4}\sin 2\theta$ vi) $\frac{1}{2+3\sin \theta}$ x) $-\frac{1}{4}\cos 2\theta$
 iii) $1 - 2\sin \theta$ vii) $\sec^3 \frac{3}{2}\theta$
 iv) $3 + 2\cos 2\theta$ viii) $\operatorname{cosec} 3\theta$

Solution

i) $\max\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3} \max\left(\cos\frac{1}{2}\theta\right)$ but $-1 \leq \cos\frac{1}{2}\theta \leq +1$

$$\therefore \max\left(\cos\frac{1}{2}\theta\right) = 1 \Rightarrow \max\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3} \times 1 = \frac{1}{3}$$

Finding value of $\theta_{\max}$

$$\frac{1}{3}\cos\frac{1}{2}\theta = \frac{1}{3}(\text{max value}) \Rightarrow \cos\frac{1}{2}\theta = 1, \text{ solving the equation}$$



$$\cos^{-1}(1) = 0^{\circ}, 360^{\circ}$$

$$\therefore \frac{1}{2}\theta = 0^{\circ}, 360^{\circ} \Rightarrow \theta_{\max} = 0^{\circ}$$

$$\min\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3}\min\left(\cos\frac{1}{2}\theta\right) \text{ but } \min\left(\cos\frac{1}{2}\theta\right) = -1$$

$$\therefore \min\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3} \times -1 = -\frac{1}{3}$$

Finding value of $\theta_{\min}$

$$\frac{1}{3}\cos\frac{1}{2}\theta = -\frac{1}{3}(\text{min value}) \Rightarrow \cos\frac{1}{2}\theta = -1, \text{ solving the equation}$$

$$\frac{1}{2}\theta = 180^{\circ} \Rightarrow \theta_{\min} = 360^{\circ}$$

ii) $\max\left(\frac{1}{4}\sin 2\theta\right) = \frac{1}{4}\max(\sin 2\theta)$ but $-1 \leq \sin 2\theta \leq +1$

$$\therefore \max(\sin 2\theta) = 1, \min(\sin 2\theta) = -1$$

$$\Rightarrow \max\left(\frac{1}{4}\sin 2\theta\right) = \frac{1}{4} \times 1 = \frac{1}{4}$$

Finding value of $\theta_{\max}$

$$\frac{1}{4}\sin 2\theta = \frac{1}{4}(\text{max value}) \Rightarrow \sin 2\theta = 1, \text{ solving the equation}$$

$$\sin^{-1}(1) = 90^{\circ}$$

$$\therefore 2\theta = 90^{\circ} \Rightarrow \theta_{\max} = 45^{\circ}$$

$$\min\left(\frac{1}{4}\sin 2\theta\right) = \frac{1}{4}\min(\sin 2\theta) = \frac{1}{4} \times -1 = -\frac{1}{4}$$

Finding value of $\theta_{\min}$

$$\frac{1}{4}\sin 2\theta = -\frac{1}{4}(\text{min value}) \Rightarrow \sin 2\theta = -1$$

$$\therefore 2\theta = 270^{\circ}, \theta_{\min} = 135^{\circ}$$

iii) $\max(1 - 2\sin\theta) = \max(1) - \min(2\sin\theta)$

$$= 1 - 2\min(\sin\theta) = 1 - 2(-1) = 3$$

Finding value of $\theta_{\max}$

$$1 - 2\sin\theta = 3(\text{max value}) \Rightarrow \sin\theta = -1, \text{ solving the equation}$$

$$\therefore \theta_{\max} = 270^{\circ}$$

$$\min(1 - 2\sin\theta) = \min(1) - \max(2\sin\theta)$$

$$= 1 - 2\max(\sin\theta) = 1 - 2(1) = -1$$

Finding value of $\theta_{\min}$

$$1 - 2\sin\theta = -1(\text{min value}) \Rightarrow \sin\theta = 1, \text{ solving the equation}$$

$$\therefore \theta_{\min} = 90^{\circ}$$

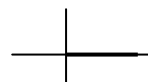
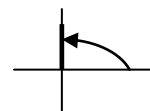
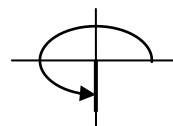
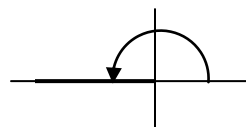
iv) $\max(3 + 2\cos 2\theta) = \max(3) + \max(2\cos 2\theta)$

$$= 3 + 2\max(\cos 2\theta) = 3 + 2(1) = 5$$

Finding value of $\theta_{\max}$

$$3 + 2\cos 2\theta = 5 \Rightarrow \cos 2\theta = 1, \text{ solving the equation}$$

$$\Rightarrow 2\theta = 0^{\circ}, 360^{\circ}$$

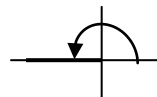


$$\therefore \theta_{\max} = 0^\circ, 180^\circ$$

$$\begin{aligned} \min(3 + 2\cos 2\theta) &= \min(3) + \min(2\cos 2\theta) \\ &= 3 + 2\min(\cos 2\theta) = 3 + 2(-1) = 1 \end{aligned}$$

Now finding value of $\theta_{\min}$

$$\begin{aligned} 3 + 2\cos 2\theta &= 1 \Rightarrow \cos 2\theta = -1, \text{ solving the equation} \\ \Rightarrow 2\theta &= 180^\circ \therefore \theta_{\min} = 90^\circ \end{aligned}$$



$$\begin{aligned} \text{v) } \max\left(\frac{1}{4-3\cos 2\theta}\right) &= \frac{\max(1)}{\min(4-3\cos 2\theta)} = \frac{1}{\min(4) - \max(3\cos 2\theta)} \\ &= \frac{1}{4-3\max(\cos 2\theta)} = \frac{1}{4-3(+1)} = 1 \end{aligned}$$

Finding value of $\theta_{\max}$

$$\frac{1}{4-3\cos 2\theta} = 1(\text{max value}) \text{ or } 4-3\cos 2\theta = 1 \Rightarrow \cos 2\theta = 1$$

Solving the equations as before $2\theta = 90^\circ \Rightarrow \theta_{\max} = 45^\circ$

$$\begin{aligned} \min\left(\frac{1}{4-3\cos 2\theta}\right) &= \frac{\min(1)}{\max(4-3\cos 2\theta)} = \frac{1}{\max(4) - \min(3\cos 2\theta)} \\ &= \frac{1}{4-3\min(\cos 2\theta)} = \frac{1}{4-3(-1)} = \frac{1}{7} \end{aligned}$$

Now finding value of $\theta_{\min}$

$$\frac{1}{4-3\cos 2\theta} = \frac{1}{7}(\text{min value}) \Rightarrow 4-3\cos 2\theta = 7 \Rightarrow \cos 2\theta = -1$$

Solving the equations as before $2\theta = 180^\circ \Rightarrow \theta_{\min} = 90^\circ$

$$\begin{aligned} \text{vi) } \max\left(\frac{1}{2+3\sin \theta}\right) &= \frac{\max(1)}{\min(2+3\sin \theta)} = \frac{1}{2+3\min(\sin \theta)} \\ &= \frac{1}{2+3(-1)} = -1 \end{aligned}$$

Finding value of $\theta_{\max}$

$$\frac{1}{2+3\sin \theta} = -1 \text{ or } -1 = 2+3\sin \theta \Rightarrow \sin \theta = -1$$

Solving the equations as before $\theta_{\max} = 270^\circ$

$$\begin{aligned} \min\left(\frac{1}{2+3\sin \theta}\right) &= \frac{\min(1)}{\max(2+3\sin \theta)} = \frac{1}{2+3\max(\sin \theta)} \\ &= \frac{1}{2+3(+1)} = \frac{1}{5} \end{aligned}$$

Now finding value of $\theta_{\min}$

$$\frac{1}{2+3\sin \theta} = \frac{1}{5} \Rightarrow 5 = 2+3\sin \theta \text{ or } \sin \theta = 1$$

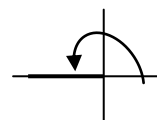
Solving the equations as before $\theta_{\min} = 90^\circ$

$$\text{vii) } \max\left(\sec \frac{3}{2}\theta\right) = \max\left(\frac{1}{\cos \frac{3}{2}\theta}\right) = \frac{\max(1)}{\min(\cos \frac{3}{2}\theta)} = \frac{1}{-1} = -1$$

Finding value of $\theta_{\max}$

$$-1 = \frac{1}{\cos \frac{3}{2}\theta} \Rightarrow \cos \frac{3}{2}\theta = -1, \text{ Solving the equations as before}$$

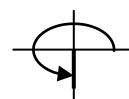
$$\therefore \frac{3}{2}\theta = 180^\circ, \theta_{\max} = 120^\circ, \text{ min values is left as an exercise}$$



$$\text{viii) } \max(\operatorname{cosec} 3\theta) = \max\left(\frac{1}{\sin 3\theta}\right) = \frac{1}{\min(\sin 3\theta)} = \frac{1}{-1} = -1$$

Finding the value of $\theta_{\max}$

$$\begin{aligned} \frac{1}{\sin 3\theta} &= -1 \Rightarrow \sin 3\theta = -1 \\ \Rightarrow 3\theta &= 270^\circ \therefore \theta_{\max} = 90^\circ \end{aligned}$$



$$\min(\operatorname{cosec} 3\theta) = \min\left(\frac{1}{\sin 3\theta}\right) = \frac{1}{\max(\sin 3\theta)} = \frac{1}{+1} = 1$$

Finding the value of $\theta_{\min}$

$$\frac{1}{\sin 3\theta} = 1 \Rightarrow \sin 3\theta = 1$$

$$\therefore 3\theta_{\min} = 90^\circ \Rightarrow \theta_{\min} = 30^\circ$$

ix) $-\frac{1}{2}\sin 2\theta = 0 - \frac{1}{2}\sin 2\theta$

[Re-writing the expression, since the multiplying constant is -ve]

$$\begin{aligned}\max\left(-\frac{1}{2}\sin 2\theta\right) &= \max\left(0 - \frac{1}{2}\sin 2\theta\right) = \max(0) - \frac{1}{2}\min(\sin 2\theta) \\ &= 0 - \frac{1}{2}(-1) = \frac{1}{2}\end{aligned}$$

Finding the value of $\theta_{\max}$

$$\frac{1}{2} = -\frac{1}{2}\sin 2\theta \text{ or } \sin 2\theta = -1$$

$$\Rightarrow 2\theta = 270^\circ \therefore \theta_{\max} = 135^\circ$$

$$\begin{aligned}\min\left(-\frac{1}{2}\sin 2\theta\right) &= \min\left(0 - \frac{1}{2}\sin 2\theta\right) = \min(0) - \frac{1}{2}\max(\sin 2\theta) \\ &= 0 - \frac{1}{2}(+1) = -\frac{1}{2}\end{aligned}$$

Finding the value of $\theta_{\min}$

$$-\frac{1}{2} = -\frac{1}{2}\sin 2\theta \text{ or } \sin 2\theta = 1$$

$$\Rightarrow 2\theta = 90^\circ \therefore \theta_{\min} = 45^\circ$$

x) Left as an exercise

Answer $\frac{1}{4}, 135^\circ; -\frac{1}{4}, 45^\circ$

Task

Find the maximum and minimum value and values of θ for which they occur between 0° and 360°

i) $\frac{1}{1+\operatorname{cosec} \theta}$ [**HINT:** $\max\left(\frac{1}{1+\operatorname{cosec} \theta}\right) = \frac{1}{\min(1+\operatorname{cosec} \theta)} \Rightarrow$ **Ans:** $\frac{1}{2}, 90^\circ$
 $\min\left(\frac{1}{1+\operatorname{cosec} \theta}\right) = \frac{1}{\max(1+\operatorname{cosec} \theta)} \Rightarrow$ **Ans: no min**]

ii) $\frac{\cos \theta}{\cos \theta + \sin \theta}$ [**HINT:** making numerator 1, dividing both numerator and denominator by $\cos \theta$ gives $\frac{1}{1+\tan \theta}$, since $\tan \theta$ has no max and min value then **Ans: None**]

iii) $\frac{2}{3-2\cot 2\theta}$ [**Ans: None**]

TRIGONOMETRIC IDENTITIES INVOLVING FUNCTIONS OF THE SUM OR DIFFERENCE OF TWO ANGLES

The formulae $\sin(\varphi \pm \theta)$, $\cos(\varphi \pm \theta)$, $\tan(\varphi \pm \theta)$ and $\cot(\varphi \pm \theta)$

The following figures shall be made of use

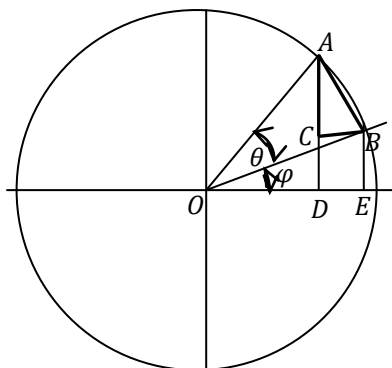


Fig.1

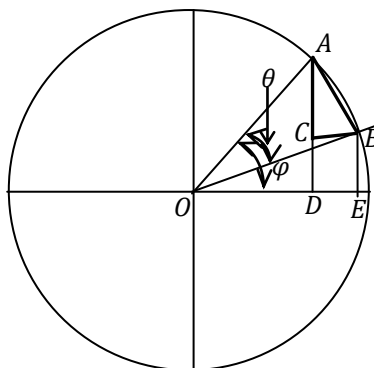


Fig.2

Functions of sum of two angles

a) $\sin(\varphi + \theta)$

Referring to fig.1, where $AB \perp OB$ and $AD \perp OD$

From $\triangle OAB$, $AB = OA \sin \theta$ & $BAC = \varphi$

From $\triangle ABC$, $AC = AB \cos \varphi$

$AC = OA \sin \theta \cos \varphi$

From $\triangle OAB$, $OB = OA \cos \theta$

From $\triangle OBE$, $BE = OB \sin \varphi \Rightarrow BE = OA \cos \theta \sin \varphi$

But $BE = CD \therefore CD = OA \cos \theta \sin \varphi$

From $\triangle OAD$, $\sin(\varphi + \theta) = \frac{AC + CD}{OA} = \frac{OA \sin \theta \cos \varphi + OA \cos \theta \sin \varphi}{OA}$

$$\therefore \sin(\varphi + \theta) = \sin \varphi \cos \theta + \cos \varphi \sin \theta$$

NOTE: $\sin(\varphi \pm \theta) \neq \sin \varphi + \sin \theta$

b) $\cos(\varphi + \theta)$

Again referring to fig.1

From $\triangle OAB$, $OB = OA \cos \theta$

From $\triangle OBE$, $OE = OB \cos \varphi \Rightarrow OE = OA \cos \theta \cos \varphi$

From $\triangle OAB$, $AB = OA \sin \theta$

From $\triangle ABC$, $CB = AB \sin \varphi$

$DE = CB = OA \sin \theta \sin \varphi$

From $\triangle OAD$, $\cos(\varphi + \theta) = \frac{OE - DE}{OA} = \frac{OA \cos \theta \cos \varphi - OA \sin \theta \sin \varphi}{OA}$

$$\therefore \cos(\varphi + \theta) = \cos \theta \cos \varphi - \sin \theta \sin \varphi$$

NOTE: $\cos(\varphi + \theta) \neq \cos \varphi + \cos \theta$

c) $\tan(\varphi + \theta)$

$$\tan(\varphi + \theta) = \frac{\sin(\varphi + \theta)}{\cos(\varphi + \theta)}$$

From the above two deduced results,

$$\tan(\varphi + \theta) = \frac{\sin(\varphi + \theta)}{\cos(\varphi + \theta)} = \frac{\sin\varphi\cos\theta + \cos\varphi\sin\theta}{\cos\varphi\cos\theta - \sin\varphi\sin\theta}$$

Dividing Numerator and Denominator by $\cos\varphi\cos\theta$

$$\begin{aligned}\tan(\varphi + \theta) &= \frac{\frac{\sin\varphi}{\cos\varphi} + \frac{\sin\theta}{\cos\theta}}{1 - (\sin\varphi/\cos\varphi)(\sin\theta/\cos\theta)} \\ &= \frac{\tan\varphi + \tan\theta}{1 - \tan\varphi\tan\theta}\end{aligned}$$

$$\therefore \tan(\varphi + \theta) = \frac{\tan\varphi + \tan\theta}{1 - \tan\varphi\tan\theta}$$

NOTE: $\tan(\varphi + \theta) \neq \tan\varphi + \tan\theta$

d) $\cot(\varphi + \theta)$

$$\cot(\varphi + \theta) = \frac{1}{\tan(\varphi + \theta)} = \frac{1 - \tan\varphi\tan\theta}{\tan\varphi + \tan\theta}$$

$$\text{From } \cot\alpha = \frac{1}{\tan\alpha} \Rightarrow \tan\alpha = \frac{1}{\cot\alpha}$$

$$\Rightarrow \cot(\varphi + \theta) = \frac{1 - \frac{1}{\cot\varphi} \cdot \frac{1}{\cot\theta}}{\frac{1}{\cot\varphi} + \frac{1}{\cot\theta}} = \frac{\cot\varphi\cot\theta - 1}{\cot\varphi + \cot\theta}$$

$$\therefore \cot(\varphi + \theta) = \frac{\cot\varphi\cot\theta - 1}{\cot\varphi + \cot\theta}$$

NOTE: $\cot(\varphi + \theta) \neq \cot\varphi + \cot\theta$

Functions of difference between two angles

Referring to fig.2, where $OA \perp AB$ and $AD \perp OD$

a) $\sin(\varphi - \theta)$

From $\triangle OAB$, $AB = OB \sin\theta$

From $\triangle ABC$, $AC = AB \cos\varphi \Rightarrow AC = OB \sin\theta \cos\varphi$

From $\triangle OAD$, $AD = OA \sin\varphi$

From $\triangle OAB$, $OA = OB \cos\theta$

$\Rightarrow AD = OB \sin\varphi \cos\theta$

$$\text{Form } \triangle OBE, \sin(\varphi - \theta) = \frac{AD - AC}{OB} = \frac{OB \sin\varphi \cos\theta - OB \cos\varphi \sin\theta}{OB}$$

$$\therefore \sin(\varphi - \theta) = \sin\varphi \cos\theta - \cos\varphi \sin\theta$$

NOTE: $\sin(\varphi - \theta) \neq \sin\varphi - \sin\theta$

b) $\cos(\varphi - \theta)$

Again referring to fig.2

From $\triangle OAB$, $OA = OB \cos\theta$

From $\triangle OAD$, $OD = OA \cos\varphi \Rightarrow OD = OB \cos\varphi \cos\theta$

From $\triangle OAB$, $AB = OB \sin \theta$

From $\triangle ABC$, $CB = AB \sin \varphi \Rightarrow CB = OB \sin \varphi \sin \theta$

Form $\triangle OBE$, $\cos(\varphi - \theta) = \frac{OD+CB}{OB} = \frac{OB \cos \varphi \cos \theta + OB \sin \varphi \sin \theta}{OB}$

$$\therefore \cos(\varphi - \theta) = \cos \varphi \cos \theta + \sin \varphi \sin \theta$$

NOTE: $\cos(\varphi - \theta) \neq \cos \varphi - \cos \theta$

c) $\tan(\varphi - \theta)$

$$\tan(\varphi - \theta) = \frac{\sin(\varphi - \theta)}{\cos(\varphi - \theta)}$$

From the above two deduced results,

$$\tan(\varphi - \theta) = \frac{\sin(\varphi - \theta)}{\cos(\varphi - \theta)} = \frac{\sin \varphi \cos \theta - \cos \varphi \sin \theta}{\cos \varphi \cos \theta + \sin \varphi \sin \theta}$$

Dividing Numerator and Denominator by $\cos \varphi \cos \theta$

$$\tan(\varphi - \theta) = \frac{\frac{\sin \varphi}{\cos \varphi} - \frac{\sin \theta}{\cos \theta}}{1 + (\sin \varphi / \cos \varphi)(\sin \theta / \cos \theta)} = \frac{\tan \varphi - \tan \theta}{1 + \tan \varphi \tan \theta}$$

$$\therefore \tan(\varphi - \theta) = \frac{\tan \varphi - \tan \theta}{1 + \tan \varphi \tan \theta}$$

NOTE: $\tan(\varphi - \theta) \neq \tan \varphi - \tan \theta$

a) $\cot(\varphi - \theta)$

$$\cot(\varphi - \theta) = \frac{1}{\tan(\varphi - \theta)} = \frac{1 + \tan \varphi \tan \theta}{\tan \varphi - \tan \theta}$$

$$\text{From } \cot \alpha = \frac{1}{\tan \alpha} \Rightarrow \tan \alpha = \frac{1}{\cot \alpha}$$

$$\Rightarrow \cot(\varphi - \theta) = \frac{1 + \frac{1}{\cot \varphi} \cdot \frac{1}{\cot \theta}}{\frac{1}{\cot \varphi} - \frac{1}{\cot \theta}} = \frac{\cot \varphi \cot \theta + 1}{\cot \varphi - \cot \theta}$$

$$\therefore \cot(\varphi - \theta) = \frac{\cot \varphi \cot \theta + 1}{\cot \varphi - \cot \theta}$$

NOTE: $\cot(\varphi - \theta) \neq \cot \varphi - \cot \theta$

Application of the above identities

- Finding exact values of trig ratios involving the use of special angles
- Solving trig equations
- Proof of identities and more further identities
- integration

Finding exact values of trig ratios involving the use of special angles

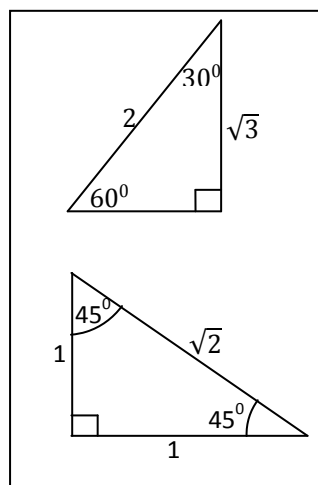
Examples

Show that

- 1) $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$
- 2) $\cos 75^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$
- 3) $\sin 195^\circ = \frac{\sqrt{2}-\sqrt{6}}{4}$
- 4) $\tan 225^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}$
- 5) $\tan 195^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Solution

- 1) $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$
 $15^\circ = 60^\circ - 45^\circ$ [Both special angles]
 or $= 45^\circ - 30^\circ$ [both special angles]
 Using any of the above expressions
 $\cos 15^\circ = \cos(60^\circ - 45^\circ)$
 $= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$
 $= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ As required
- 2) $\cos 75^\circ = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$
 $\cos 75^\circ = \cos(30^\circ + 45^\circ)$
 $= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$
 $= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \dots \dots \dots \left(\times \frac{\sqrt{2}}{\sqrt{2}} \right)$
 $= \frac{\sqrt{2}}{4}(\sqrt{3}-1)$ As required
- 3) $\sin 195^\circ = \frac{\sqrt{2}-\sqrt{6}}{4}$
 $\sin 195^\circ = \sin(180^\circ + 15^\circ)$
 $= \sin 180^\circ \cos 15^\circ + \cos 180^\circ \sin 15^\circ$
 $= \cos 180^\circ \sin 15^\circ$ [$\because \sin 180^\circ = 0$]
 $= -\sin 15^\circ$ [$\because \cos 180^\circ = -1$]
 Now $\sin 15^\circ = \sin(45^\circ - 30^\circ)$
 $= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$
 $= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$
 $\therefore \sin 195^\circ = -\left(\frac{\sqrt{3}-1}{2\sqrt{2}} \right) = \frac{1-\sqrt{3}}{2\sqrt{2}} \dots \dots \dots \times \left(\frac{\sqrt{2}}{\sqrt{2}} \right)$
 $= \frac{\sqrt{2}-\sqrt{6}}{4}$ As required
- 4) $\tan 255^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}$
 $\tan 255^\circ = \tan(180^\circ + 75^\circ)$



$$= \frac{\tan 180^\circ + \tan 75^\circ}{1 - \tan 180^\circ \tan 75^\circ} = \frac{0 + \tan 75^\circ}{1 - 0} = \tan 75^\circ$$

But $\tan 75^\circ = \tan(30^\circ + 45^\circ)$

$$= \frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ} = \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} = \frac{1 + \sqrt{3}}{\sqrt{3} - 1}$$

$\therefore \tan 255^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ As required

5) Left as an exercise. Proceed as above

Task

Show that

i) $\tan^2 225^\circ = 2 + \sqrt{3}$

ii) $\sin^2 195^\circ = \frac{1}{4}(2 - \sqrt{3})$

iii) $\cos^2 435^\circ = \frac{1}{4}(\sqrt{3} - 2)$

[**HINT:** First get $\tan 225^\circ$, $\sin 195^\circ$ and $\cos 435^\circ$ then use algebra to find the required answer]

Solving trigonometric equations

Solve for values of x between 0° and 360° inclusive

a. $2 \sin x = \cos(x + 60^\circ)$

e. $\sin(x + 30^\circ) = \cos(x + 30^\circ)$

b. $\cos(x + 45^\circ) = \cos x$

f. $\tan(x - 45^\circ) = \frac{1}{3}$

c. $\sin(x - 30^\circ) = \frac{1}{2} \cos x$

g. $\tan(45^\circ + x) = 2$

d. $3 \sin(x + 10^\circ) = 4 \cos(x - 10^\circ)$

Solution

NOTE: The equations (a) to (e) take the form

$$a \sin(px + q) = b \cos(px + c)$$

The equations are solved by creation of **tanpx**.

a. $2 \sin x = \cos(x + 60^\circ)$

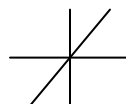
$$2 \sin x = \cos x \cos 60^\circ - \sin x \sin 60^\circ = \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x$$

$$\Rightarrow 4 \sin x = \cos x - \sqrt{3} \sin x \dots \dots \dots (\text{dividing through by } \cos x)$$

$$\Rightarrow 4 \tan x = 1 - \sqrt{3} \tan x \Rightarrow (4 + \sqrt{3}) \tan x = 1$$

$$\therefore \tan x = \frac{1}{4 + \sqrt{3}} = 0.1745 \quad (\text{Note: } \sqrt{3} \text{ is taken +ve})$$

Solving the equation $x = 9.9^\circ, 189.9^\circ$



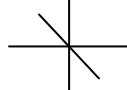
$\tan^{-1}(0.1745) = 9.9$

b. $\cos(x + 45^\circ) = \cos x$

$$\cos x \cos 45^\circ - \sin x \sin 45^\circ = \cos x$$

$$\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x = \cos x \text{ or } \cos x - \sin x = \sqrt{2} \cos x \dots \dots \dots (\div \cos x)$$

$$1 - \tan x = \sqrt{2} \text{ or } \tan x = 1 - \sqrt{2} = -0.4142 \text{ (for only +ve } \sqrt{2})$$



$\tan^{-1}(0.4142) = 22.5^\circ$
 $\therefore x = 157.5^\circ, 337.5^\circ$

c. (c) and (e) are left as an exercise

Ans: (c) $49.1^0, 229.1^0$ (e) $15^0, 195^0$

d. $3 \sin(x + 10^0) = 4 \cos(x - 10^0)$

$$3(\sin x \cos 10^0 + \cos x \sin 10^0) = 4(\cos x \cos 10^0 + \sin x \sin 10^0)$$

$$3 \sin x \cos 10^0 + 3 \cos x \sin 10^0 = 4 \cos x \cos 10^0 + 4 \sin x \sin 10^0$$

Dividing through by $\cos x$

$$3 \cos 10^0 \cdot \tan x + 3 \sin 10^0 = 4 \cos 10^0 + 4 \sin 10^0 \cdot \tan x$$

$$\tan x \cdot (3 \cos 10^0 - 4 \sin 10^0) = 4 \cos 10^0 - 3 \sin 10^0$$

$$\therefore \tan x = \frac{4 \cos 10^0 - 3 \sin 10^0}{3 \cos 10^0 - 4 \sin 10^0} = 1.5126, \text{ solving } x = 56.5^0, 236.5^0$$

e. $\tan(x - 45^0) = \frac{1}{3}$

$$\frac{\tan x - \tan 45}{1 + \tan x \tan 45} = \frac{1}{3} \text{ or } 3 \tan x - 3 = 1 + \tan x$$

$$\Rightarrow \tan x = 2, \text{ solving } x = 63.4^0, 243.4^0$$

Alternatively

$$\text{Let } x - 45^0 = B \Rightarrow \tan B = \frac{1}{3}, \text{ solving the equation } B = 18.4^0, 198.4^0$$

$$\Rightarrow x - 45^0 = 18.4^0, 198.4^0$$

$$\therefore x = (18.4^0, 198.4^0) + 45^0 \text{ or } x = 63.4^0, 243.4^0$$

g. Left an exercise **Answer:** $18.4^0, 198.4^0$

Proof of identities

1. Prove the following identities

a) $\frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} = 0$

b) $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$

c) $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

d) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ where A, B and C are angles of a triangle

e) $\cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2 \alpha - \cos^2 \beta$

Solution

a) $\frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} = 0$

L. H. S: $\frac{\sin(A-B)}{\cos A \cos B} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} = \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}$
 $= \tan A - \tan B$

Similarly $\frac{\sin(B-C)}{\cos B \cos C} = \tan B - \tan C$ Also $\frac{\sin(C-A)}{\cos C \cos A} = \tan C - \tan A$

$\therefore \frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} = \tan A - \tan B + \tan B - \tan C - \tan C - \tan A = 0$

b) $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$

R. H. S: $\frac{\sin(A+B)}{\cos A \cos B} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B} = \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}$
 $= \tan A + \tan B = \text{L. H. S}$

c) $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

L. H. S: $\sin(A+B) + \sin(A-B) = \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$$= 2\sin A \cos B = \mathbf{R.H.S}$$

d) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, $A + B + C = 180^\circ$

$$A + B = 180 - C \Rightarrow \tan(A + B) = \tan(180 - C)$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan 180 - \tan C}{1 + \tan 180 \tan C} = -\tan C \quad [\because \tan 180 = 0]$$

$$\tan A + \tan B = -\tan C + \tan A \tan B \tan C \Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

e) $\cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2 \alpha - \cos^2 \beta$

$$\begin{aligned} \mathbf{L.H.S:} \cos(\alpha + \beta) \cos(\alpha - \beta) &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta)(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= (\cos \alpha \cos \beta)^2 - (\sin \alpha \sin \beta)^2 = \cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta \end{aligned}$$

In the required proof **R.H.S** does not have $\cos \beta$ and $\sin \alpha$

$$\begin{aligned} \therefore \mathbf{L.H.S} &= \cos^2 \alpha [1 - \sin^2 \beta] - [1 - \cos^2 \alpha] \sin^2 \beta \\ &= \cos^2 \alpha - \cos^2 \alpha \sin^2 \beta - \sin^2 \beta + \cos^2 \alpha \sin^2 \beta \\ &= \cos^2 \alpha - \sin^2 \beta = \mathbf{R.H.S} \end{aligned}$$

2. Verify the following identities

a) $\sin(90^\circ + x) = \cos x$

g) $\frac{\cot \theta}{\cot \theta - \cot 3\theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} = 1$

b) $\tan(270^\circ + \theta) = -\cot \theta$

c) $\tan(90^\circ + \theta) = -\cot \theta$

d) $\tan(90^\circ - \theta) = \cot \theta$

e) $\sec(90^\circ + \theta) = -\operatorname{cosec} \theta$

f) $\operatorname{cosec}(180^\circ - \theta) = \operatorname{cosec} \theta$

Solution

a) $\sin(90^\circ + x) = \sin 90 \cos x + \cos 90 \sin x = \cos x$ [$\because \sin 90 = 1, \cos 90 = 0$]

b) $\tan(270^\circ + \theta) = \frac{\sin(270 + \theta)}{\cos(270 + \theta)} = \frac{\sin 270 \cos \theta + \cos 270 \sin \theta}{\cos 270 \cos \theta - \sin 270 \sin \theta} = -\frac{\cos \theta}{\sin \theta} = -\cot \theta$
 $[\because \sin 270^\circ = -1, \cos 270^\circ = 0]$

c) (c), (d) and (g) are left as an exercise

e) $\sec(90^\circ + \theta) = \frac{1}{\cos(90 + \theta)} = \frac{1}{\cos 90 \cos \theta - \sin 90 \sin \theta} = \frac{1}{-\sin \theta} = -\operatorname{cosec} \theta$

f) Left as an exercise

Task

Prove the following identities

a) $\sin(\alpha + \beta) \cdot \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$

b) $\sin(\alpha + \beta) \cdot \sin(\alpha - \beta) = \cos^2 \alpha - \cos^2 \beta$

c) If $\sin(x + \alpha) = \cos(x - \beta)$, then $\tan x = \frac{\cos \beta - \sin \alpha}{\cos \alpha - \sin \beta}$

d) If $\sin(x + \alpha) = 2 \cos(x - \beta)$, then $\tan x = \frac{2 - \tan \alpha}{1 - 2 \tan \alpha}$

e) $\frac{\sin(A-B)}{\sin A \sin B} + \frac{\sin(B-C)}{\sin B \sin C} + \frac{\sin(C-A)}{\sin C \sin A} = 0$

Finding exact value of $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$, $\tan(\alpha \pm \beta)$ or $\cot(\alpha \pm \beta)$ given $\sin(\alpha \text{ or } \beta)$ or $\cos(\alpha \text{ or } \beta)$ Or $\tan(\alpha \text{ or } \beta)$

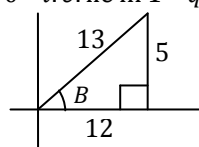
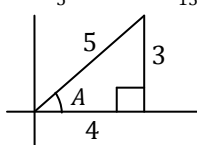
Example

1. If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$ where A and B are acute angles, find the value of

- a) $\sin(A + B)$ b) $\cos(A - B)$ c) $\cot(A - B)$

Solution

Given: $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$, $A, B < 90^\circ$ i.e. lie in 1^{st} quadrant



a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13} = \frac{56}{65}$$

b) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13} = \frac{63}{65}$$

c) $\cot(A - B) = \frac{1}{\tan(A - B)} = \frac{1 + \tan A \tan B}{\tan A - \tan B} = \frac{1 + \frac{3}{4} \times \frac{5}{12}}{\frac{3}{4} - \frac{5}{12}} = \frac{63}{16}$

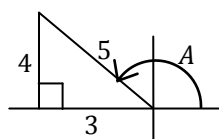
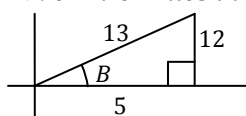
2. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ where B is acute and A is obtuse, find the values of

- a) $\tan(A + B)$ b) $\tan(A - B)$ c) $A + B$

Solution

Given: $\sin A = \frac{4}{5}$, $90^\circ < A < 180^\circ$ i.e. A lies in 2^{nd} quadrant $\cos B = \frac{12}{13}$,

$0 < B < 90^\circ$ i.e. B lies in 1^{st} quadrant



a) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{4}{3} + \frac{12}{5}}{1 - \frac{4}{3} \times \frac{12}{5}} = \frac{16}{63}$

$$\left[\because \tan A = \frac{4}{3} \left(\begin{array}{l} -ve \text{ since } A \text{ is in } 2^{nd} \\ \text{quadrant where tangents are} \end{array} \right) \right] = \frac{16}{63}$$

b) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{4}{3} - \frac{12}{5}}{1 + \left(\frac{4}{3}\right) \times \frac{12}{5}} = \frac{56}{33}$

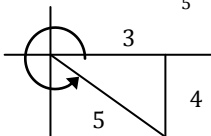
c) $A + B = ?$ From (a) above, $A + B = \tan^{-1}\left(\frac{16}{63}\right)$

3. If $\cos A = \frac{3}{5}$ and $\tan B = \frac{12}{5}$ where both A and B are reflex angles, find the values of

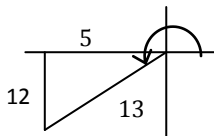
- a) $\sin(A - B)$ b) $\tan(A - B)$ c) $\cos(A + B)$

Solution

Given: $\cos A = \frac{3}{5}$, $\tan B = \frac{12}{5}$, $180^\circ < (A \text{ or } B) < 360^\circ$



In 4th quadrant $\therefore \cos A = +ve$



In 3rd quadrant $\therefore \tan B = +ve$

$$a) \sin(A - B) = \sin A \cos B - \cos A \sin B = \left(-\frac{4}{5}\right) \times \left(-\frac{5}{13}\right) - \frac{3}{5} \times \left(-\frac{12}{13}\right) = \frac{56}{65}$$

$$b) \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{-\frac{4}{5} - \frac{12}{5}}{1 + \left(-\frac{4}{5}\right)\left(\frac{12}{5}\right)} = \frac{56}{33}$$

$$c) \cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{3}{5} \times \left(-\frac{5}{13}\right) - \left(-\frac{4}{5}\right) \times \left(-\frac{12}{13}\right) = -\frac{63}{65}$$

Task

1. If $\tan A = -\frac{1}{7}$ and $\tan B = \frac{3}{4}$ where A is obtuse and B is acute, find $\tan(A - B)$ hence find $A - B$.

Ans: $\tan(A - B) = -1$; $A - B = (135^\circ, 315^\circ) + 180^\circ n$

2. Prove that $\frac{\tan(a-b) + \tan b}{1 - \tan(a-b)\tan b} = \tan a$

MULTIPLE ANGLE IDENTITIES

a) Double Angle Identities

i) $\sin 2\varphi$

Using $\sin(\varphi + \theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$

Let $\varphi = \theta \Rightarrow \sin(\varphi + \varphi) = \sin\varphi\cos\varphi + \cos\varphi\sin\varphi$

$$\therefore \sin 2\varphi = 2 \sin \varphi \cos \varphi$$

In general

$$\therefore \sin 2m\varphi = 2 \sin m \varphi \cos m \varphi$$

ii) $\cos 2\varphi$

Using $\cos(\varphi + \theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$

Let $\theta = \varphi \Rightarrow \cos(\varphi + \varphi) = \cos\varphi\cos\varphi - \sin\varphi\sin\varphi$

$$\therefore \cos 2\varphi = \cos^2 \varphi - \sin^2 \varphi$$

Further manipulation using $\cos^2 \varphi + \sin^2 \varphi = 1$

$$\cos^2 \varphi = 1 - \sin^2 \varphi \Rightarrow \cos 2\varphi = 1 - \sin^2 \varphi - \sin^2 \varphi$$

$$\therefore \cos 2\varphi = 1 - 2\sin^2 \varphi$$

$$\text{or } \sin^2 \varphi = \frac{1}{2}(1 - \cos 2\varphi)$$

$$\sin^2 \varphi = 1 - \cos^2 \varphi \Rightarrow \cos 2\varphi = \cos^2 \varphi - (1 - \cos^2 \varphi)$$

$$\therefore \cos 2\varphi = 2\cos^2 \varphi - 1$$

$$\text{or } \cos^2 \varphi = \frac{1}{2}(1 + \cos 2\varphi)$$

In general

$$\begin{aligned}\cos(2m)\varphi &= \cos^2 m\varphi - \sin^2 m\varphi \\ &= 1 - 2\sin^2 m\varphi \\ &= 2\cos^2 m\varphi - 1\end{aligned}$$

$$\begin{aligned}\sin^2 m\varphi &= \frac{1}{2}(1 - \cos 2m\varphi) \\ \cos^2 m\varphi &= \frac{1}{2}(1 + \cos 2m\varphi)\end{aligned}$$

iii) $\tan 2\varphi$

$$\text{Using } \tan(\varphi + \theta) = \frac{\tan\varphi + \tan\theta}{1 - \tan\varphi\tan\theta}$$

$$\text{Let } \theta = \varphi \Rightarrow \tan(\varphi + \varphi) = \frac{\tan\varphi + \tan\varphi}{1 - \tan\varphi\tan\varphi} = \frac{2\tan\varphi}{1 - \tan^2\varphi}$$

$$\therefore \tan 2\varphi = \frac{2\tan\varphi}{1 - \tan^2\varphi}$$

In general

$$\therefore \tan 2m\varphi = \frac{2\tan m\varphi}{1 - \tan^2 m\varphi}$$

iv) $\cot 2\varphi$

$$\text{Using } \cot(\varphi + \theta) = \frac{\cot\varphi \cot\theta - 1}{\cot\varphi + \cot\theta}$$

$$\text{Let } \theta = \varphi \Rightarrow \cot(\varphi + \varphi) = \frac{\cot\varphi \cot\varphi - 1}{\cot\varphi + \cot\varphi} = \frac{\cot^2\varphi - 1}{2\cot\varphi}$$

$$\therefore \cot 2\varphi = \frac{\cot^2\varphi - 1}{2\cot\varphi}$$

$$\text{In general } \cot 2m\varphi = \frac{\cot^2 m\varphi - 1}{2\cot m\varphi}$$

Application of double angle identities

- Solution to trig equations
- Proof of identities and derivation of other multiple angle identities
- Elimination of parameters
- Simplifying trig expressions
- integration

Solution to trig equations

1. solve for $0^\circ < \theta < 360^\circ$ the equations

- $3\cos 2\theta + \sin\theta = 1$
- $3\cot 2\theta + \cot\theta = 1$
- $4\tan\theta \cdot \tan 2\theta = 1$
- $\sin 2\theta \cdot \cos\theta + \sin^2\theta = 1$
- $3\tan\theta = \tan 2\theta$
- $2\sin\theta(5\cos 2\theta + 1) = 3\sin 2\theta$

Solution

a) $3\cos 2\theta + \sin\theta = 1$

Expressing the equation in terms of only $\sin\theta$

$$3(1 - 2\sin^2\theta) + \sin\theta = 1$$

$$-6\sin^2\theta + \sin\theta + 2 = 0 \quad [\text{Let } u = \sin\theta, -6u^2 + u + 2 = 0]$$

Solving algebraically, $\sin \theta = -\frac{1}{2}, \frac{2}{3}$

Dealing with $\sin \theta = -\frac{1}{2}$

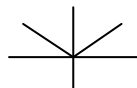
$$\Rightarrow \theta = 210^\circ, 330^\circ$$



$$\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$$

Dealing with $\sin \theta = \frac{2}{3}$

$$\Rightarrow \theta = 41.8^\circ, 138.2^\circ$$



$$\sin^{-1}\left(\frac{2}{3}\right) = 41.8^\circ$$

$$\therefore \theta = 41.8^\circ, 138.2^\circ, 210^\circ, 330^\circ$$

b) $3\cot 2\theta + \cot \theta = 1$

Expressing the equation in terms of only $\cot \theta$

$$3\left[\frac{\cot^2 \theta - 1}{2\cot \theta}\right] + \cot \theta = 1 \dots \dots \dots (\times \cot \theta)$$

$$3\cot^2 \theta - 3 + 2\cot^2 \theta = 2\cot \theta \Rightarrow 5\cot^2 \theta - 2\cot \theta - 3 = 0$$

$$\Rightarrow 5\cot^2 \theta - 5\cot \theta + 3\cot \theta - 3 = 0 \Rightarrow 5\cot \theta (\cot \theta - 1) + 3(\cot \theta - 1) = 0$$

$$\therefore (5\cot \theta + 3)(\cot \theta - 1) = 0 \text{ or } \cot \theta = -\frac{3}{5}, 1$$

But $\cot \theta = \frac{1}{\tan \theta} = -\frac{3}{5}, 1 \Rightarrow \tan \theta = -\frac{5}{3}, 1$

Dealing with $\tan \theta = 1$

$$\Rightarrow \theta = 45^\circ, 225^\circ$$



$$\tan^{-1}(1) = 45^\circ$$

Dealing with $\tan \theta = -\frac{5}{3}$

$$\Rightarrow \theta = 121^\circ, 301^\circ$$



$$\tan^{-1}\left(\frac{5}{3}\right) = 59^\circ$$

$$\therefore \theta = 45^\circ, 121^\circ, 225^\circ, 301^\circ$$

c) $4 \tan \theta \cdot \tan 2\theta = 1$

Expressing in terms of $\tan \theta$

$$4\tan \theta \cdot \frac{2\tan \theta}{1-\tan^2 \theta} = 1 \Rightarrow 8\tan \theta = 1 - \tan^2 \theta \Rightarrow 9\tan^2 \theta = 1 \therefore \tan \theta = \pm \frac{1}{3}$$

Dealing with $\tan \theta = \frac{1}{3}$

$$\Rightarrow \theta = 18.4^\circ, 198.4^\circ$$



$$\tan^{-1}\left(\frac{1}{3}\right) = 18.4^\circ$$

Dealing with $\tan \theta = -\frac{1}{3}$

$$\Rightarrow \theta = 161.6^\circ, 341.6^\circ$$



$$\tan^{-1}\left(\frac{1}{3}\right) = 18.4^\circ$$

$$\therefore \theta = 18.4^\circ, 161.6^\circ, 198.4^\circ, 341.6^\circ$$

d) $\sin 2\theta \cdot \cos \theta + \sin^2 \theta = 1$

$$2 \sin \theta \cos \theta \cdot \cos \theta = 1 - \sin^2 \theta = \cos^2 \theta$$

$$2\sin \theta (\cos^2 \theta) - \cos^2 \theta = 0 \Rightarrow \cos^2 \theta (2\sin \theta - 1) = 0$$

[Do not divide through by $\cos^2 \theta$, you limit the required values of θ]

$$\Rightarrow \cos^2 \theta = 0 \text{ or } \cos \theta = 0 \text{ and } 2\sin \theta - 1 = 0 \text{ or } \sin \theta = \frac{1}{2}$$

Dealing with $\cos \theta = 0$

Dealing with $\sin \theta = \frac{1}{2}$

$$\Rightarrow \theta = 30^\circ, 150^\circ$$

$$\therefore \theta = 30^\circ, 90^\circ, 150^\circ, 270^\circ$$



$$\therefore \theta = 90^\circ, 270^\circ$$



$$\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$$

e) Left as an exercise **Answer: $0^\circ, 30^\circ, 150^\circ, 180^\circ, 210^\circ, 330^\circ$** . 360° rejected

$\therefore$ it is outside the interval $0^\circ < \theta < 360^\circ$

f) $2 \sin \theta (5 \cos 2\theta + 1) = 3 \sin 2\theta$

$$2 \sin \theta (5 \cos 2\theta + 1) - 3 \sin 2\theta = 0$$

$$\Rightarrow 2 \sin \theta (5 \cos 2\theta + 1) - 3 (2 \sin \theta \cos \theta) = 0$$

$$2 \sin \theta \{ (5 \cos 2\theta + 1) - 3 \cos \theta \} = 0$$

$$\Rightarrow 2 \sin \theta = 0 \text{ or } \sin \theta = 0 \quad \text{and} \quad 5 \cos 2\theta + 1 - 3 \cos \theta = 0$$

Dealing with $\sin \theta = 0$

$$\therefore \theta = 0^\circ, 360^\circ$$

Dealing with $5 \cos 2\theta + 1 - 3 \cos \theta = 0$

Expressing in terms of $\cos \theta$

$$5(2 \cos^2 \theta - 1) + 1 - 3 \cos \theta = 0 \Rightarrow 10 \cos^2 \theta - 3 \cos \theta - 4 = 0$$

Solving the equation gives $\cos \theta = 0.8, -\frac{1}{2}$

Dealing with $\cos \theta = 0.8$

$$\Rightarrow \theta = 36.9^\circ, 323.1^\circ$$

Dealing with $\cos \theta = -\frac{1}{2}$

$$\Rightarrow \theta = 120^\circ, 300^\circ$$

$$\therefore \theta = 0^\circ, 36.9^\circ, 120^\circ, 300^\circ, 323.1^\circ \quad 360^\circ \text{ rejected since } 0^\circ < \theta < 360^\circ$$



$$\cos^{-1}(0.8) = 36.9^\circ$$



$$\cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$$

Task

1. Solve for $-180^\circ \leq \theta \leq +180^\circ$ the equations

a) $7 \tan 2\theta + \cot 2\theta = 5 \sec 2\theta$

b) $4 \cos \theta - 2 \cos 2\theta - 3 = 0$

c) $\tan 4\theta \cdot \tan 2\theta = 1$ [HINT: $\tan 4\theta = \tan 2(2\theta) = \frac{2 \tan 2\theta}{1 - \tan^2 2\theta}$]

2. Solve $\sin 2\theta = \cos \theta$ for $0^\circ \leq \theta \leq 90^\circ$ (Uneb 2012)

Answers: (a) $15^\circ, 75^\circ, 9.8^\circ, 80.3^\circ, -99.8^\circ, -105^\circ$ (b) $\pm 60^\circ$

(c) $15^\circ, 105^\circ, -75^\circ, -165^\circ$, (2) (Uneb 2012) $\theta = 30^\circ, 150^\circ$

Proof of identities

- 1) Proof the following identities

a) $\frac{\tan\left(\frac{\pi}{4} + \theta\right) - \tan\left(\frac{\pi}{4} - \theta\right)}{\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)} = \sin 2\theta$

b) $\frac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} = \tan 2\alpha + \sec 2\alpha$

c) $\cot 2\theta = \frac{1 + \cos 4\theta}{\sin 4\theta}$

d) $\sec 2\theta = \frac{\csc^2 \theta}{\csc^2 \theta - 2} = \frac{\sec^2 \theta}{2 - \sec^2 \theta}$

e) $\cos 4\theta = \frac{\tan^4 \theta - 6 \tan^2 \theta + 1}{\tan^4 \theta + 2 \tan^2 \theta + 1}$

f) $\sin 3\theta = 3 \sin^3 \theta - 4 \sin \theta$

g) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

h) $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$

i) $\tan\left(\frac{\pi}{4} + \theta\right) - \tan\left(\frac{\pi}{4} - \theta\right) = 2 \tan 2\theta$ (Uneb 2006)

j) $\frac{\sin A}{\sin B} + \frac{\cos A}{\cos B} = \frac{2 \sin(A+B)}{\sin 2B}$

k) $\frac{1}{\cos A + \sin A} + \frac{1}{\cos A - \sin A} = \tan 2A \csc A$

$$1) (\sin 2\alpha - \sin 2\beta) \tan(\alpha + \beta) = 2(\sin^2 \alpha - \sin^2 \beta)$$

Solution

$$a) \frac{\tan(\frac{\pi}{4} + \theta) - \tan(\frac{\pi}{4} - \theta)}{\tan(\frac{\pi}{4} + \theta) + \tan(\frac{\pi}{4} - \theta)} = \sin 2\theta \quad \text{Note: } 1 \text{ radian} = 180^\circ, \Rightarrow \frac{\pi}{4} = \frac{180^\circ}{4} = 45^\circ$$

$$\begin{aligned} \text{L. H. S: } \frac{\tan(\frac{\pi}{4} + \theta) - \tan(\frac{\pi}{4} - \theta)}{\tan(\frac{\pi}{4} + \theta) + \tan(\frac{\pi}{4} - \theta)} &= \frac{\frac{\tan(\frac{\pi}{4}) + \tan \theta}{1 - \tan(\frac{\pi}{4})\tan \theta} - \frac{\tan(\frac{\pi}{4}) - \tan \theta}{1 + \tan(\frac{\pi}{4})\tan \theta}}{\frac{\tan(\frac{\pi}{4}) + \tan \theta}{1 - \tan(\frac{\pi}{4})\tan \theta} + \frac{\tan(\frac{\pi}{4}) - \tan \theta}{1 + \tan(\frac{\pi}{4})\tan \theta}} = \frac{\frac{1 + \tan \theta}{1 - \tan \theta} - \frac{1 - \tan \theta}{1 + \tan \theta}}{\frac{1 + \tan \theta}{1 - \tan \theta} + \frac{1 - \tan \theta}{1 + \tan \theta}} \\ &= \frac{\frac{(1 + \tan \theta)^2 - (1 - \tan \theta)^2}{1 - \tan^2 \theta}}{\frac{(1 + \tan \theta)^2 + (1 - \tan \theta)^2}{1 - \tan^2 \theta}} = \frac{(1 + \tan \theta + 1 - \tan \theta)(1 + \tan \theta - 1 + \tan \theta)}{(1 + \tan \theta)^2 + (1 - \tan \theta)^2} \\ &= \frac{2(2 \tan \theta)}{1 + 2 \tan \theta + \tan^2 \theta + 1 - 2 \tan \theta + \tan^2 \theta} = \frac{4 \tan \theta}{2 + 2 \tan^2 \theta} = \frac{4 \tan \theta}{2(1 + \tan^2 \theta)} \\ &= \frac{4 \sin \theta / \cos \theta}{2 \sec^2 \theta} = \frac{2 \sin \theta / \cos \theta}{1 / \cos^2 \theta} = \frac{2 \sin \theta}{\cos \theta} \cdot \cos^2 \theta = 2 \sin \theta \cos \theta = \sin 2\theta \end{aligned}$$

Alternatively

$$\begin{aligned} \frac{\tan(\frac{\pi}{4} + \theta) - \tan(\frac{\pi}{4} - \theta)}{\tan(\frac{\pi}{4} + \theta) + \tan(\frac{\pi}{4} - \theta)} &= \frac{\frac{\sin(\frac{\pi}{4} + \theta)}{\cos(\frac{\pi}{4} + \theta)} - \frac{\sin(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} - \theta)}}{\frac{\sin(\frac{\pi}{4} + \theta)}{\cos(\frac{\pi}{4} + \theta)} + \frac{\sin(\frac{\pi}{4} - \theta)}{\cos(\frac{\pi}{4} - \theta)}} = \frac{\frac{\sin(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta) - \sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} + \theta)}{\cos(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta)}}{\frac{\sin(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta) + \sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} + \theta)}{\cos(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta)}} \\ &= \frac{\sin(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta) - \sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} + \theta)}{\sin(\frac{\pi}{4} + \theta)\cos(\frac{\pi}{4} - \theta) + \sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} + \theta)} = \frac{\sin[(\frac{\pi}{4} + \theta) - (\frac{\pi}{4} - \theta)]}{\sin[(\frac{\pi}{4} + \theta) + (\frac{\pi}{4} - \theta)]} \\ &= \frac{\sin 2\theta}{\sin(\frac{\pi}{2})} \text{ But } \sin(\frac{\pi}{2}) = \sin 90^\circ = 1 \therefore \text{L. H. S} = \text{R. H. S} = \sin 2\theta \end{aligned}$$

$$b) \frac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} = \tan 2\alpha + \sec 2\alpha$$

$$\begin{aligned} \text{R. H. S: } \tan 2\alpha + \sec 2\alpha &= \frac{\sin 2\alpha}{\cos 2\alpha} + \frac{1}{\cos 2\alpha} = \frac{\sin 2\alpha + 1}{\cos 2\alpha} \\ &= \frac{2 \sin 2\alpha \cos \alpha + \cos^2 \alpha + \sin^2 \alpha}{\cos^2 \alpha - \sin^2 \alpha} = \frac{(\cos \alpha + \sin \alpha)^2}{(\cos \alpha - \sin \alpha)(\cos \alpha + \sin \alpha)} = \frac{(\cos \alpha + \sin \alpha)}{(\cos \alpha - \sin \alpha)} = \text{L. H. S} \end{aligned}$$

$$c) \cot 2\theta = \frac{1 + \cos 4\theta}{\sin 4\theta}, \text{ let } 2\theta = A \text{ Now to prove } \cot A = \frac{1 + \cos 2A}{\sin 2A}$$

$$\text{R. H. S: } \frac{1 + \cos 2A}{\sin 2A} = \frac{1 + (2 \cos^2 A - 1)}{2 \sin A \cos A} = \frac{2 \cos^2 A}{2 \sin A \cos A} = \frac{\cos A}{\sin A} = \cot A \therefore \cot 2\theta = \frac{1 + \cos 4\theta}{\sin 4\theta}$$

$$d) \sec 2\theta = \frac{\operatorname{cosec}^2 \theta}{\operatorname{cosec}^2 \theta - 2} = \frac{\sec^2 \theta}{2 - \sec^2 \theta}$$

$$\text{R. H. S: } \frac{\operatorname{cosec}^2 \theta}{\operatorname{cosec}^2 \theta - 2} = \frac{1/\sin^2 \theta}{\frac{1}{\sin^2 \theta} - 1} = \frac{1}{1 - 2 \sin^2 \theta} = \frac{1}{\cos 2\theta} = \sec 2\theta = \text{L. H. S}$$

$$\text{Also } \sec 2\theta = \frac{\sec^2 \theta}{2 - \sec^2 \theta} = \frac{1/\cos^2 \theta}{2 - \frac{1}{\cos^2 \theta}} = \frac{1}{2 \cos^2 \theta - 1} = \frac{1}{\cos 2\theta} = \sec 2\theta = \text{L. H. S}$$

$$e) \cos 4\theta = \frac{\tan^4 \theta - 6 \tan^2 \theta + 1}{\tan^4 \theta + 2 \tan^2 \theta + 1}$$

$$\begin{aligned} \text{L. H. S: } \cos 4\theta &= \cos 2(2\theta) = \cos^2 2\theta - \sin^2 2\theta = \frac{\cos^2 2\theta - \sin^2 2\theta}{1^2} \\ &= \frac{\cos^2 2\theta - \sin^2 2\theta}{(\cos^2 2\theta + \sin^2 2\theta)^2} = \frac{(\cos^2 2\theta - \sin^2 2\theta)^2 - (2 \sin \theta \cos \theta)^2}{\cos^4 \theta + \cos^2 \theta \sin^2 \theta + \sin^4 \theta} \\ &= \frac{\cos^4 \theta - 2 \sin^2 \theta \cos^2 \theta - 4 \sin^2 \theta \cos^2 \theta + \sin^4 \theta}{\cos^4 \theta + \cos^2 \theta \sin^2 \theta + \sin^4 \theta} \end{aligned}$$

Dividing the numerator and denominator by $\cos^4 \theta$

$$\therefore \cos 4\theta = \frac{\cos^4 \theta / \cos^4 \theta - 6 \sin^2 \theta \cos^2 \theta / \cos^4 \theta + \sin^4 \theta / \cos^4 \theta}{\cos^4 \theta / \cos^4 \theta + \cos^2 \theta \sin^2 \theta / \cos^4 \theta + \sin^4 \theta / \cos^4 \theta} = \frac{1 - 6 \tan^2 \theta + \tan^4 \theta}{1 + 2 \tan^2 \theta + \tan^4 \theta} = \mathbf{R.H.S}$$

Task

Show that $\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$

f) $\sin 3\theta = 3 \sin^3 \theta - 4 \sin \theta$

L.H.S: $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$
 $= 2 \sin \theta \cos \theta \cdot \cos \theta + \cos 2\theta \sin \theta = 2 \sin \theta \cos^2 \theta + (1 - 2 \sin^2 \theta) \cdot \sin \theta$
 $= 2 \sin \theta (1 - \sin^2 \theta) + \sin \theta - 2 \sin^3 \theta$
 $= 2 \sin \theta - 2 \sin^3 \theta + \sin \theta - 2 \sin^3 \theta = 3 \sin \theta - 4 \sin^3 \theta = \mathbf{R.H.S}$

g) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

L.H.S: $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$
 $= (2 \cos^2 \theta - 1) \cdot \cos \theta - (2 \sin \theta \cos \theta) \cdot \sin \theta$
 $= 2 \cos^3 \theta - \cos \theta - 2 \sin^2 \theta \cos \theta = 2 \cos^3 \theta - \cos \theta - 2(1 - \cos^2 \theta) \cos \theta$
 $= 2 \cos^3 \theta - \cos \theta - 2 \cos \theta + 2 \cos^3 \theta = 4 \cos^3 \theta - 3 \cos \theta = \mathbf{R.H.S}$

h) $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$

$$\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} = \frac{\frac{2 \tan \theta}{1 - \tan^2 \theta} + \tan \theta}{1 - \frac{2 \tan \theta \cdot \tan \theta}{1 - \tan^2 \theta}}$$

$$= \frac{(2 \tan \theta + \tan \theta - \tan^3 \theta) / (1 - \tan^2 \theta)}{(1 - \tan^2 \theta - 2 \tan^2 \theta) / (1 - \tan^2 \theta)} = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} = \mathbf{R.H.S}$$

NOTE:

The multiple identities of thrice the angles are very useful.

Note them carefully

$$\begin{aligned} \cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta \\ \sin 3\theta &= 3 \sin^3 \theta - 4 \sin \theta \\ \tan 3\theta &= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \end{aligned}$$

i) left as an exercise. See (a) above

j) $\frac{\sin A}{\sin B} + \frac{\cos A}{\cos B} = \frac{2 \sin(A+B)}{\sin 2B}$

R.H.S: $\frac{2 \sin(A+B)}{\sin 2B} = \frac{2 \sin A \cos B + 2 \cos A \sin B}{\sin 2B}$
 $= \frac{2 \sin A \cos B}{2 \sin B \cos B} + \frac{2 \cos A \sin B}{2 \sin B \cos B} = \frac{\sin A}{\sin B} + \frac{\cos A}{\cos B} = \mathbf{L.H.S}$

k) $\frac{1}{\cos A + \sin A} + \frac{1}{\cos A - \sin A} = \tan 2A \operatorname{cosec} A$

L.H.S: $\frac{1}{\cos A + \sin A} + \frac{1}{\cos A - \sin A} = \frac{\cos A + \sin A + \cos A - \sin A}{\cos^2 A - \sin^2 A} = \frac{2 \cos A}{\cos 2A}$

Now creating $\tan 2A$ by suitable manipulation

$$\frac{1}{\cos A + \sin A} + \frac{1}{\cos A - \sin A} = \frac{2 \cos A}{\cos 2A} \cdot \frac{\sin 2A}{\sin 2A} = \left(\frac{\sin 2A}{\cos 2A} \right) \cdot \frac{2 \cos A}{\sin 2A} = \tan 2A \cdot \frac{2 \cos A}{2 \sin A \cos A}$$

$$= \tan 2A \operatorname{cosec} A = \mathbf{R.H.S}$$

l) $(\sin 2\alpha - \sin 2\beta) \tan(\alpha + \beta) = 2(\sin^2 \alpha - \sin^2 \beta)$

L.H.S: $(\sin 2\alpha - \sin 2\beta) \tan(\alpha + \beta) = (\sin 2\alpha - \sin 2\beta) \cdot \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$

$$\begin{aligned}
 &= \frac{(2\sin\alpha\cos\alpha - 2\sin\beta\cos\beta)(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
 &= \frac{2\sin^2\alpha\cos\alpha\cos\beta + 2\sin\alpha\cos^2\alpha\sin\beta - 2\sin\alpha\sin\beta\cos^2\beta - 2\cos\alpha\cos\beta\sin^2\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
 &= \frac{2\cos\alpha\cos\beta[\sin^2\alpha - \sin^2\beta] + 2\sin\alpha\sin\beta[\cos^2\alpha - \cos^2\beta]}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
 &= \frac{2\cos\alpha\cos\beta[\sin^2\alpha - \sin^2\beta] + 2\sin\alpha\sin\beta[(1 - \sin^2\alpha) - (1 - \sin^2\beta)]}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
 &= 2(\sin^2\alpha - \sin^2\beta) \cdot \frac{[\cos\alpha\cos\beta - \sin\alpha\sin\beta]}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = 2(\sin^2\alpha - \sin^2\beta) = \text{R.H.S}
 \end{aligned}$$

2. Conditional proofs involving Double Angle Identities

- If $\tan^2\alpha = 2\tan^2\beta + 1$, prove that $\cos 2\beta = 1 + 2\cos 2\alpha$
- If $\tan^2\theta = 2\tan^2\varphi + 1$, then show that $\cos 2\theta + \sin^2\varphi = 0$
- If $\tan\theta = \frac{b}{a}$, prove that $a\cos 2\theta + b\sin 2\theta = a$
- If $\tan\theta = \frac{p}{q}$, prove that $p\cos 2\theta + q\sin 2\theta = \frac{p(3q^2 - p^2)}{p^2 + q^2}$
- If $\tan\theta = \lambda\tan(A - \theta)$, show that $(\lambda - 1)\sin A = (\lambda + 1)\sin(2\theta - A)$
- If $\tan^2\alpha - 2\tan^2\beta = 1$, prove that $\cos 2\alpha + \sin^2\beta = 0$

Solution

- a) $\tan^2\alpha = 2\tan^2\beta + 1$, to prove $\cos 2\beta = 1 + 2\cos 2\alpha$

$$\text{L.H.S: } \cos 2\beta = \frac{\cos^2\beta - \sin^2\beta}{1} = \frac{\cos^2\beta - \sin^2\beta}{\cos^2\beta + \sin^2\beta}$$

Dividing the numerator and denominator by $\cos^2\beta$ to create $\tan\beta$

$$\Rightarrow \cos 2\beta = \frac{\cos^2\beta / \cos^2\beta - \sin^2\beta / \cos^2\beta}{\cos^2\beta / \cos^2\beta + \sin^2\beta / \cos^2\beta} = \frac{1 - \tan^2\beta}{1 + \tan^2\beta}$$

$$\text{But from } \tan^2\alpha = 2\tan^2\beta + 1 \Rightarrow \tan^2\beta = \frac{\tan^2\alpha - 1}{2}$$

$$\therefore \cos 2\beta = \frac{1 - \frac{\tan^2\alpha - 1}{2}}{1 + \frac{\tan^2\alpha - 1}{2}} = \frac{2 - \tan^2\alpha + 1}{2 + \tan^2\alpha - 1} = \frac{3 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{3 - \tan^2\alpha}{\sec^2\alpha}$$

$$= \frac{3 - \frac{\sin^2\alpha}{\cos^2\alpha}}{\frac{1}{\cos^2\alpha}} = 3\cos^2\alpha - \sin^2\alpha = 3\cos^2\alpha - (1 - \cos^2\alpha)$$

$$= 4\cos^2\alpha - 1, \text{ from double angle identities, } \cos 2\alpha = 2\cos^2\alpha - 1$$

$$\Rightarrow \cos 2\alpha = 4\left[\frac{\cos 2\alpha + 1}{2}\right] - 1 = 2\cos 2\alpha + 2 - 1 = 2\cos 2\alpha + 1 = \text{R.H.S}$$

- b) $\tan^2\theta = 2\tan^2\varphi + 1$, to prove $\cos 2\theta + \sin^2\varphi = 0$

$$\text{L.H.S: } \cos 2\theta + \sin^2\varphi = \frac{\cos^2\theta - \sin^2\theta}{1} + \sin^2\varphi = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta} + \sin^2\varphi$$

Creating $\tan\theta$ as in previous example

$$\cos 2\theta + \sin^2\varphi = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} + \sin^2\varphi$$

$$\text{From } \tan^2\theta = 2\tan^2\varphi + 1 \Rightarrow \cos 2\theta + \sin^2\varphi = \frac{1 - (2\tan^2\varphi + 1)}{1 + (2\tan^2\varphi + 1)} + \sin^2\varphi$$

$$\begin{aligned}
 \cos 2\theta + \sin^2\varphi &= \frac{-2\tan^2\varphi}{2(1 + \tan^2\varphi)} + \sin^2\varphi = -\frac{\left(\frac{\sin^2\varphi}{\cos^2\varphi}\right)}{\sec^2\varphi} + \sin^2\varphi \\
 &= -\frac{\sin^2\varphi}{\cos^2\varphi} \cdot \cos^2\varphi + \sin^2\varphi = 0 = \text{R.H.S}
 \end{aligned}$$

c) $\tan \theta = \frac{b}{a}$, to prove $a \cos 2\theta + b \sin 2\theta = a$

$$\begin{aligned} \text{L.H.S: } a \cos 2\theta + b \sin 2\theta &= a \cdot \frac{(\cos^2 \theta - \sin^2 \theta)}{1} + b \cdot \frac{2 \sin \theta \cos \theta}{1} \\ &= a \cdot \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} + b \cdot \frac{2 \sin \theta \cos \theta}{\cos^2 \theta + \sin^2 \theta} \end{aligned}$$

Dividing the numerator and denominator by $\cos^2 \theta$

$$\begin{aligned} a \cos 2\theta + b \sin 2\theta &= a \cdot \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} + b \cdot \frac{2 \tan \theta}{1 + \tan^2 \theta} \\ &= \frac{a \left(1 - \frac{b^2}{a^2}\right) + 2b \cdot \frac{b}{a}}{1 + \frac{b^2}{a^2}} = \frac{\left(a - \frac{b^2}{a}\right) \cdot a^2}{a^2 + b^2} = \frac{a^2}{a^2 + b^2} \cdot \left(\frac{a^2 - b^2 + 2b^2}{a}\right) = \frac{a^2}{a^2 + b^2} \cdot \frac{a^2 + b^2}{a} = a \end{aligned}$$

d) (d), (e) and (f) are left as an exercise

Task

Proceeding as above,

a) If $\sin 3\theta = p$, $\sin^2 \theta = \frac{3}{4} - q$, prove that $p^2 + 16q^2 = 12q^2$

b) If $2 \cos \theta = x + \frac{1}{x}$, show that $2 \cos 3\theta = x^3 + \frac{1}{x^3}$

Half Angle Identities

a) $\sin \frac{\varphi}{2}$

Using $\sin(\varphi + \theta) = \sin \varphi \cos \theta + \cos \varphi \sin \theta$

$$\sin \varphi = \sin \left(\frac{\varphi}{2} + \frac{\varphi}{2}\right) = \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} + \cos \frac{\varphi}{2} \sin \frac{\varphi}{2}$$

$$\sin \varphi = 2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2}$$

Recall: $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta$ substituting $\theta = \frac{\varphi}{2}$

$$\therefore \cos \varphi = \cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2} = 1 - 2 \sin^2 \frac{\varphi}{2}$$

$$\therefore \sin \frac{\varphi}{2} = \pm \sqrt{\frac{1 - \cos \varphi}{2}},$$

The “+” is used when $\frac{\varphi}{2}$ is in 1st or 2nd quadrant and otherwise the “-” is used

b) $\cos \frac{\varphi}{2}$

$$\left. \begin{aligned} \cos \varphi &= \cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2} \\ &= 2 \cos^2 \frac{\varphi}{2} - 1 \\ &= 1 - 2 \sin^2 \frac{\varphi}{2} \end{aligned} \right\} \left[\because \cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2} = 1 \right]$$

From $\cos \varphi = 2 \cos^2 \frac{\varphi}{2} - 1$

$$\therefore \cos \frac{\varphi}{2} = \pm \sqrt{\frac{1 + \cos \varphi}{2}},$$

The “+” is used when $\frac{\varphi}{2}$ is in 1st or 4th quadrant and otherwise the “-” is used

c) $\tan \frac{\varphi}{2}$

$$\tan \varphi = \tan \left(\frac{\varphi}{2} + \frac{\varphi}{2} \right) = \frac{\tan \frac{\varphi}{2} + \tan \frac{\varphi}{2}}{1 - \tan^2 \frac{\varphi}{2}}$$

$$\text{Also } \tan \frac{\varphi}{2} = \frac{\sin \frac{\varphi}{2}}{\cos \frac{\varphi}{2}} = \frac{\pm \sqrt{\frac{1 - \cos \varphi}{2}}}{\pm \sqrt{\frac{1 + \cos \varphi}{2}}}$$

$$\therefore \tan \frac{\varphi}{2} = \pm \sqrt{\frac{1 - \cos \varphi}{1 + \cos \varphi}}$$

Where $\cos \varphi \neq -1$ and the "+" is used when $\frac{\varphi}{2}$ is in 1st or 3rd quadrant and otherwise the "-" is used

d) $\cot \frac{\varphi}{2}$

$$\cot \frac{\varphi}{2} = \frac{1}{\tan \frac{\varphi}{2}}$$

$$\therefore \cot \frac{\varphi}{2} = \pm \sqrt{\frac{1 + \cos \varphi}{1 - \cos \varphi}}$$

Application

- Solution to trigonometric equations
- Proof of identities
- Elimination of parameters
- Simplifying trigonometric equations
- Integration
- Derivation of the t – formulae

Derivation of the t -formulae

$$\begin{aligned} \text{i) From } \cos \varphi &= \cos \left(\frac{\varphi}{2} + \frac{\varphi}{2} \right) = \cos \frac{\varphi}{2} \cos \frac{\varphi}{2} - \sin \frac{\varphi}{2} \sin \frac{\varphi}{2} \\ &= \frac{\cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2}}{1} = \frac{\cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2}}{\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}} \end{aligned}$$

Dividing the Numerator and Denominator by $\cos^2 \frac{\varphi}{2}$

$$\cos \varphi = \frac{\cos^2 \frac{\varphi}{2} / \cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2} / \cos^2 \frac{\varphi}{2}}{\cos^2 \frac{\varphi}{2} / \cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2} / \cos^2 \frac{\varphi}{2}} = \frac{1 - \tan^2 \frac{\varphi}{2}}{1 + \tan^2 \frac{\varphi}{2}} \quad \text{Let } t = \tan \frac{\varphi}{2}$$

$$\therefore \cos \varphi = \frac{1 - t^2}{1 + t^2}, t = \tan \frac{\varphi}{2}$$

$$\text{In general } \cos(\text{Angle}) = \frac{1 - t^2}{1 + t^2}, t = \tan \left(\frac{\text{Angle}}{2} \right)$$

$$\begin{aligned} \text{ii) From } \sin \varphi &= \sin \left(\frac{\varphi}{2} + \frac{\varphi}{2} \right) = \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} + \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \\ &= 2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \sin \varphi = \frac{2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2}}{1} = \frac{2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2}}{\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2}} \end{aligned}$$

Dividing the Numerator and Denominator by $\cos^2 \frac{\varphi}{2}$

$$\Rightarrow \sin \varphi = \frac{2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} / \cos^2 \frac{\varphi}{2}}{\cos^2 \frac{\varphi}{2} / \cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2} / \cos^2 \frac{\varphi}{2}} = \frac{2 \tan \frac{\varphi}{2}}{1 + \tan^2 \frac{\varphi}{2}}, \text{ let } t = \tan \frac{\varphi}{2}$$

$$\therefore \sin \varphi = \frac{2t}{1 + t^2}, t = \tan \frac{\varphi}{2}$$

$$\text{In general } \sin(\text{Angle}) = \frac{2t}{1 + t^2}, t = \tan \left(\frac{\text{Angle}}{2} \right)$$

iii) From $\tan \varphi = \tan \left(\frac{\varphi}{2} + \frac{\varphi}{2} \right) = \frac{\tan \frac{\varphi}{2} + \tan \frac{\varphi}{2}}{1 - \tan \frac{\varphi}{2} \tan \frac{\varphi}{2}} = \frac{2 \tan \frac{\varphi}{2}}{1 - \tan^2 \frac{\varphi}{2}}$, let $t = \tan \frac{\varphi}{2}$

$$\therefore \tan \varphi = \frac{2t}{1-t^2}, t = \tan \frac{\varphi}{2}$$

$$\text{In general } \tan(\text{Angle}) = \frac{2t}{1-t^2}, t = \tan \left(\frac{\text{Angle}}{2} \right)$$

$$\cot \varphi = \frac{1}{\tan \varphi} = \frac{1-t^2}{2t}, t = \tan \frac{\varphi}{2}$$

NOTE:

It is very important to note carefully the generalized t – formulae of the trig ratios above and their definitions.

Application

- Solution to trigonometric equations
- Proof of identities
- Simplifying trigonometric equations
- Integration

These will be tackled later.

Proof of identities

Prove the following identities

a) $\operatorname{cosec} \frac{1}{2} \theta = \pm \frac{\sqrt{2-2\cos\theta}}{1-\cos\theta} = \pm \frac{\sqrt{2+2\cos\theta}}{\sin\theta}$

b) $\sec \frac{1}{2} \theta = \pm \frac{\sqrt{2+2\cos\theta}}{1+\cos\theta}$

c) $2 \cot^2 \frac{1}{2} A + \tan A = \tan A \cot^2 \frac{1}{2} A$

d) $\tan^2 \frac{1}{2} x = -2 \pm \sqrt{5}$ if $\tan \frac{1}{2} x = \operatorname{cosec} x - \sin x$

e) $2 \cot 2A \cot \frac{1}{2} A = \frac{\cos A}{1-\cos A} - \frac{1+\cos A}{\cos A}$

f) $\tan^2 \frac{\theta}{2} = \frac{2\sin\theta - \sin 2\theta}{2\sin\theta + \sin 2\theta}$

g) $\cot \frac{1}{2} \varphi = \frac{\sin \varphi}{1-\cos \varphi}$

h) $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \tan \left(\frac{\pi}{4} + \frac{A}{2} \right)$

i) $\frac{1+\sin\theta}{1-\sin\theta} = \tan^2 \left(45^\circ + \frac{\theta}{2} \right)$

j) $\cos \varphi = \frac{\cos \theta - e}{1 - e \cos \theta}$ if $\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \cdot \tan \frac{\varphi}{2}$

k) **Uneb2010.** Show that $\tan \left(\frac{\pi}{4} + \frac{A}{2} \right) = \frac{1+\sin A}{\cos A}$

l) $\frac{1+\sin x + \cos x}{1+\sin x - \cos x} = \cot \frac{x}{2}$, hence solve the equation $\frac{1+\sin x + \cos x}{1+\sin x - \cos x} = \sqrt{2}$

m) $2 \cot \frac{A}{2} - \sin A = \cot^2 \frac{A}{2} \sin A$

Solution

a) $\operatorname{cosec} \frac{1}{2}\theta = \pm \frac{\sqrt{2-2\cos\theta}}{1-\cos\theta} = \pm \frac{\sqrt{2+2\cos\theta}}{\sin\theta}$

$$\begin{aligned} \text{R.H.S: } \pm \frac{\sqrt{2-2\cos\theta}}{1-\cos\theta} &= \pm \frac{\sqrt{2-2\left(\frac{1-\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}\right)}}{1-\left(\frac{1-\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}\right)} = \pm \frac{\sqrt{\frac{2+2\tan^2 \frac{\theta}{2}-2+2\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}}}{\frac{1+\tan^2 \frac{\theta}{2}-1+\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}} \\ &= \pm \frac{\sqrt{\frac{4\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}}}{\frac{2\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}} = \pm \frac{\sqrt{\frac{4\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}}}{\frac{2\tan^2 \frac{\theta}{2}}{1+\tan^2 \frac{\theta}{2}}} \cdot \frac{1+\tan^2 \frac{\theta}{2}}{2\tan^2 \frac{\theta}{2}} \\ &= \pm \frac{1}{\sqrt{\sec^2 \frac{\theta}{2}}} \cdot \frac{\sec^2 \frac{\theta}{2}}{\tan^2 \frac{\theta}{2}} = \frac{1}{\sec \frac{\theta}{2}} \cdot \sec^2 \frac{\theta}{2} \cdot \frac{1}{\tan^2 \frac{\theta}{2}} = \sec \frac{\theta}{2} \cdot \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} \\ &= \cos \frac{\theta}{2} \times \frac{\cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} = \frac{1}{\sin \frac{\theta}{2}} = \operatorname{cosec} \frac{\theta}{2} \end{aligned}$$

Also $\operatorname{cosec} \frac{\theta}{2} = \pm \frac{\sqrt{2+2\cos\theta}}{\sin\theta}$

R.H.S: $\pm \frac{\sqrt{2+2\cos\theta}}{\sin\theta} = \frac{\sqrt{2}\sqrt{1+\cos\theta}}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}$, but $\cos^2 \frac{\theta}{2} = \frac{1+\cos\theta}{2} \Rightarrow 1+\cos\theta = 2\cos^2 \frac{\theta}{2}$

R.H.S $= \frac{\sqrt{2} \cdot \sqrt{2\cos^2 \frac{\theta}{2}}}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{2\cos \frac{\theta}{2}}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{1}{\sin \frac{\theta}{2}} = \operatorname{cosec} \frac{\theta}{2}$

b) $\sec \frac{1}{2}\theta = \pm \frac{\sqrt{2+2\cos\theta}}{1+\cos\theta}$

R.H.S $= \frac{\sqrt{2}\sqrt{1+\cos\theta}}{1+\cos\theta} = \frac{\sqrt{2}}{\sqrt{1+\cos\theta}}$, but $\cos^2 \frac{\theta}{2} = \frac{1+\cos\theta}{2} \Rightarrow 1+\cos\theta = 2\cos^2 \frac{\theta}{2}$

$\Rightarrow$ **R.H.S** $= \frac{\sqrt{2}}{\sqrt{1+\cos\theta}} = \frac{\sqrt{2}}{\sqrt{2\cos^2 \frac{\theta}{2}}} = \frac{1}{\cos \frac{\theta}{2}} = \sec \frac{1}{2}\theta$

c) $2 \cot \frac{1}{2}A + \tan A = \tan A \cot^2 \frac{1}{2}A$

$$\begin{aligned} \text{L.H.S: } 2\cot \frac{1}{2}A + \tan A &= \frac{2}{\tan \frac{A}{2}} + \frac{2 \tan \frac{A}{2}}{1-\tan^2 \frac{A}{2}} = \frac{2-2\tan^2 \frac{A}{2}+2\tan^2 \frac{A}{2}}{\tan \frac{A}{2} \cdot (1-\tan^2 \frac{A}{2})} \\ &= \frac{2}{\tan \frac{A}{2} (1-\tan^2 \frac{A}{2})} \dots \dots \dots \left(\times \frac{\tan \frac{A}{2}}{\tan \frac{A}{2}} \right) \\ &= \frac{2 \tan \frac{A}{2}}{1-\tan^2 \frac{A}{2}} \cdot \frac{1}{\tan^2 \frac{A}{2}} = \tan A \cdot \cot^2 \frac{A}{2} = \text{R.H.S} \end{aligned}$$

d) To prove $\tan^2 \frac{1}{2}x = -2 \pm \sqrt{5}$ given $\tan \frac{1}{2}x = \operatorname{cosec} x - \sin x$

From $\tan \frac{1}{2}x = \operatorname{cosec} x - \sin x = \frac{1}{\sin x} - \sin x = \frac{1-\sin^2 x}{\sin x} = \frac{\cos^2 x}{\sin x}$

But $\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$, $\sin x = \frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$

$$\begin{aligned}\therefore \tan \frac{1}{2}x &= \frac{\left(\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)^2}{\frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}} = \frac{(1-\tan^2 \frac{x}{2})^2}{2 \tan \frac{x}{2}(1+\tan^2 \frac{x}{2})} \\ \Rightarrow 2 \tan^2 \frac{x}{2} &= \frac{(1-\tan^2 \frac{x}{2})^2}{(1+\tan^2 \frac{x}{2})} \quad \therefore 2 \tan^2 \frac{x}{2} + 2 \tan^4 \frac{x}{2} = 1 - 2 \tan^2 \frac{x}{2} + \tan^4 \frac{x}{2} \\ \Rightarrow \tan^4 \frac{x}{2} + 4 \tan^2 \frac{x}{2} - 1 &= 0 \quad \text{Solving the equation let } u = \tan^2 \frac{x}{2} \\ \Rightarrow u^2 + 4u - 1 &= 0 \quad \therefore u = \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm \sqrt{20}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5} \\ \therefore \tan^2 \frac{x}{2} &= -2 \pm \sqrt{5}\end{aligned}$$

e) $2 \cot 2A \cot \frac{1}{2}A = \frac{\cos A}{1-\cos A} - \frac{1+\cos A}{\cos A}$

R. H. S: $\frac{\cos A}{1-\cos A} - \frac{1+\cos A}{\cos A} = \frac{\cos^2 A - (1-\cos^2 A)}{\cos A (1-\cos A)} = \frac{2\cos^2 A - 1}{\cos A (1-\cos A)} = \frac{\cos 2A}{\cos A (1-\cos A)}$

But $\sin^2 \frac{A}{2} = \frac{1-\cos A}{2}$ or $2\sin^2 \frac{A}{2} = 1 - \cos A$

$$\begin{aligned}\therefore \frac{\cos A}{1-\cos A} &= \frac{\cos 2A}{\cos A \cdot 2\sin^2 \frac{A}{2}} \dots\dots\dots \left(\times \frac{\sin 2A}{\sin 2A} \right) \\ &= \frac{\cos 2A}{\sin 2A} \cdot \frac{\sin 2A}{\cos A \cdot 2\sin^2 \frac{A}{2}} = \cot 2A \cdot \frac{2 \sin A \cos A}{\cos A \cdot 2\sin^2 \frac{A}{2}} = \cot 2A \cdot \frac{\sin A}{\sin^2 \frac{A}{2}} \\ &= \cot 2A \cdot \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{\sin^2 \frac{A}{2}} = \cot 2A \cdot \frac{2 \cos \frac{A}{2}}{\sin \frac{A}{2}} = \cot 2A \cdot \cot \frac{A}{2} = \text{L. H. S}\end{aligned}$$

f) (f), (g) Left as an exercise

h) $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \tan\left(\frac{\pi}{4} + \frac{A}{2}\right)$

L. H. S: $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{\frac{\sin A}{\cos A} + \frac{1}{\cos A} - 1}{\frac{\sin A}{\cos A} - \frac{1}{\cos A} + 1} = \frac{\sin A + 1 - \cos A}{\sin A - 1 + \cos A}$

$$\begin{aligned}&= \frac{\sin A + (1 - \cos A)}{\sin A - (1 - \cos A)}, \text{ but } 1 - \cos A = 2\sin^2 \frac{A}{2} \\ \therefore \frac{\sin A + (1 - \cos A)}{\sin A - (1 - \cos A)} &= \frac{\sin A + 2\sin^2 \frac{A}{2}}{\sin A - 2\sin^2 \frac{A}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2} + 2\sin^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2} - 2\sin^2 \frac{A}{2}} = \frac{2 \sin \frac{A}{2} (\cos \frac{A}{2} + \sin \frac{A}{2})}{2 \sin \frac{A}{2} (\cos \frac{A}{2} - \sin \frac{A}{2})} \\ &= \frac{\cos \frac{A}{2} + \sin \frac{A}{2}}{\cos \frac{A}{2} - \sin \frac{A}{2}} \quad \text{Dividing the Numerator and Denominator by } \cos \frac{A}{2} \\ &= \frac{\cos \frac{A}{2} + \sin \frac{A}{2}}{\cos \frac{A}{2} - \sin \frac{A}{2}} = \frac{1 + \tan \frac{A}{2}}{1 - \tan \frac{A}{2}} = \frac{1 + \tan \frac{A}{2}}{1 - \tan \frac{A}{2}} = \frac{\tan \frac{\pi}{4} + \tan \frac{A}{2}}{1 - \tan \frac{\pi}{4} \tan \frac{A}{2}} = \tan\left(\frac{\pi}{4} + \frac{A}{2}\right) = \text{R. H. S}\end{aligned}$$

i) $\frac{1+\sin\theta}{1-\sin\theta} = \tan^2\left(45^\circ + \frac{\theta}{2}\right)$

L. H. S: $\frac{1+\sin\theta}{1-\sin\theta} = \frac{\left(1 + \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}\right)}{\left(1 - \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}\right)} = \frac{1 + \tan^2 \frac{\theta}{2} + 2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2} - 2 \tan \frac{\theta}{2}} = \frac{(1 + \tan \frac{\theta}{2})^2}{(1 - \tan \frac{\theta}{2})^2}$

$$= \left(\frac{\tan 45^\circ + \tan \frac{\theta}{2}}{\tan 45^\circ - \tan \frac{\theta}{2}}\right)^2 = \tan^2\left(45^\circ + \frac{\theta}{2}\right)$$

j) $\cos \varphi = \frac{\cos \theta - e}{1 - e \cos \theta}$ if $\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \cdot \tan \frac{\varphi}{2}$

L. H. S: $\cos \varphi = \frac{1 - \tan^2 \frac{\varphi}{2}}{1 + \tan^2 \frac{\varphi}{2}}$ Now from $\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \cdot \tan \frac{\varphi}{2}$, squaring both sides

$$\tan^2 \frac{\theta}{2} = \left(\frac{1-e}{1+e} \right) \tan^2 \frac{\varphi}{2} \text{ or } \tan^2 \frac{\varphi}{2} = \tan^2 \frac{\theta}{2} \cdot \left(\frac{1+e}{1-e} \right)$$

$$\therefore \cos \varphi = \frac{1 - \tan^2 \frac{\theta}{2} \cdot \left(\frac{1+e}{1-e} \right)}{1 + \tan^2 \frac{\theta}{2} \cdot \left(\frac{1+e}{1-e} \right)} = \frac{1-e - (1+e) \cdot \tan^2 \frac{\theta}{2}}{1-e + (1+e) \cdot \tan^2 \frac{\theta}{2}}$$

$$= \frac{(1-e) \cdot \cos^2 \frac{\theta}{2} - (1+e) \sin^2 \frac{\theta}{2}}{(1-e) \cdot \cos^2 \frac{\theta}{2} + (1+e) \sin^2 \frac{\theta}{2}} = \frac{\cos^2 \frac{\theta}{2} - e \cdot \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} - e \cdot \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - e \cdot \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} + e \cdot \sin^2 \frac{\theta}{2}}$$

$$= \frac{(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}) + e \cdot (-\sin^2 \frac{\theta}{2} - \cos^2 \frac{\theta}{2})}{(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}) + e \cdot (\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2})} = \frac{\cos \theta - e \cdot (\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2})}{1 - e \cdot \cos \theta}$$

$$= \frac{\cos \theta - e}{1 - e \cdot \cos \theta} = \text{L. H. S}$$

Task

Prove that $\frac{\sin \alpha \sin \beta}{\cos \alpha + \cos \beta} = \frac{2 \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{1 - \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2}}$

Finding Exact value of Trigonometric Ratios of Special Angles

Show that

- a) $\sin^2 \frac{\pi}{4} = \frac{2-\sqrt{2}}{2}$
- b) $\tan^2 \frac{\pi}{8} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$
- c) $\cos^2 \frac{\pi}{8} = \frac{2+\sqrt{2}}{2}$
- d) $\sin^2 \frac{5\pi}{8} = \frac{2-\sqrt{2-\sqrt{3}}}{4}$
- e) $\tan \frac{3\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$
- f) $\cos^2 \frac{11\pi}{12} = \frac{2+\sqrt{3}}{4}$

Solution

a) $\sin^2 \frac{\pi}{4} = \frac{2-\sqrt{2}}{2}, \frac{\pi}{8} \text{ radians} = \frac{180^\circ}{8} = \frac{45^\circ}{2}$

From $\sin^2 \frac{\theta}{2} = \frac{1-\cos \theta}{2}$, let $\theta = 45^\circ = \frac{\pi}{4}$

$$\Rightarrow \sin^2 \left(\frac{45^\circ}{2} \right) = \sin^2 \frac{\pi}{8} = \frac{1-\cos 45^\circ}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}} \dots \dots \dots \left(\times \frac{\sqrt{2}}{\sqrt{2}} \right)$$

$$= \frac{2-\sqrt{2}}{4}$$

b) $\tan^2 \frac{\pi}{8} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$

$\tan^2 \frac{\pi}{8} = \tan^2 \left(\frac{45^\circ}{2} \right)$ From $\frac{\sin^2 \frac{\pi}{8}}{\cos^2 \frac{\pi}{8}} = \tan^2 \frac{\pi}{8}$

Using $\sin^2 \frac{\theta}{2} = \frac{1-\cos \theta}{2}$, $\cos^2 \frac{\theta}{2} = \frac{1+\cos \theta}{2}$, let $\theta = 45^\circ = \frac{\pi}{4}$

$$\Rightarrow \tan^2 \frac{\pi}{8} = \frac{(1-\cos 45^\circ)/2}{(1+\cos 45^\circ)/2} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} \dots \dots \dots \left(\times \frac{\sqrt{2}}{\sqrt{2}} \right)$$

$$\therefore \tan^2 \frac{\pi}{8} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$$

c) Left as an exercise

$$\text{d) } \sin^2 \frac{5\pi}{8} = \frac{2-\sqrt{2-\sqrt{3}}}{4}$$

$$\frac{5\pi}{24} \text{ radians} = \frac{5 \times 180^\circ}{4} = \frac{75^\circ}{2}$$

Using $\sin^2 \frac{\theta}{2} = \frac{1-\cos\theta}{2}$, let $\theta = 75^\circ$

$$\sin^2 \frac{5\pi}{8} = \sin^2 \left(\frac{75^\circ}{2} \right) = \frac{1-\cos 75^\circ}{2}$$

$$\text{But } \cos 75^\circ = \cos(30^\circ + 45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\begin{aligned} \Rightarrow \sin^2 \frac{5\pi}{8} &= \sin^2 \left(\frac{75^\circ}{2} \right) = \frac{1-\frac{\sqrt{3}-1}{2\sqrt{2}}}{2} = \frac{2\sqrt{2}-\sqrt{3}+1}{4\sqrt{2}} \dots\dots\dots \left(\times \frac{\sqrt{2}}{\sqrt{2}} \right) \\ &= \frac{4-\sqrt{6}+\sqrt{2}}{8} = \frac{4-(\sqrt{6}-\sqrt{2})}{8} \end{aligned}$$

From algebra $\sqrt{6}-\sqrt{2} = \sqrt{(\sqrt{6}-\sqrt{2})^2}$ to create the root in the required proof

$$\begin{aligned} \therefore \sqrt{6}-\sqrt{2} &= \sqrt{(6-2\sqrt{12}+2)} = \sqrt{8-2.2\sqrt{3}} \\ &= 2.\sqrt{2-\sqrt{3}} \end{aligned}$$

$$\therefore \sin^2 \frac{5\pi}{8} = \frac{4-(\sqrt{6}-\sqrt{2})}{8} = \frac{4-2.\sqrt{2-\sqrt{3}}}{8} = \frac{2-\sqrt{2-\sqrt{3}}}{4}$$

e) left as an exercise

$$\text{f) } \cos^2 \frac{11\pi}{12} = \frac{2+\sqrt{3}}{4}$$

$$\frac{11\pi}{12} \text{ radians} = \frac{11 \times 180^\circ}{12} = 165^\circ$$

$$\Rightarrow \cos^2 \frac{11\pi}{12} = \cos^2 165^\circ = \frac{2+\sqrt{3}}{4}$$

But $165^\circ = \frac{330^\circ}{2}$ i.e. making use of half angles

$$\therefore \cos^2 \frac{11\pi}{12} = \cos^2 165^\circ = \cos^2 \left(\frac{330^\circ}{2} \right) = \frac{2+\sqrt{3}}{4}$$

Using $\cos^2 \frac{\theta}{2} = \frac{1+\cos\theta}{2}$ let $\theta = 330^\circ$

$$\Rightarrow \cos^2 \frac{11\pi}{12} = \frac{1+\frac{\sqrt{3}}{2}}{2} = \frac{2+\sqrt{3}}{4}$$

Task

Show that

$$\text{i) } \tan^2 \frac{\pi}{16} = \frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}$$

$$\text{ii) } \cos \frac{7\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Solution to Trigonometric Equations

The use of half angle identities in solution to trigonometric equations is done using the t – formulae for convenience.

The equations to be solved by t – formulae usually take the form

$$a \cos(m\theta) + b \sin(m\theta) + c = 0$$

Note: The angle is the same for cosine and sine trig ratios, and a , b , and c are constants.

Example

1. Solve the equation for $0^\circ \leq \theta \leq 360^\circ$

- a) $3\cos\theta + 4\sin\theta = 2$
- b) $2\cos 2\theta + 3\sin 2\theta - 2 = 0$
- c) $2\sin\theta + 7\cos\theta = 4$
- d) $3\tan\theta - 2\sec\theta = 4$
- e) $5\sin 3\theta - 12\cos 3\theta = 6$
- f) $\cos \frac{1}{2}\theta - 7\sin \frac{1}{2}\theta = 2$

Solution

a) $3\cos\theta + 4\sin\theta = 2$

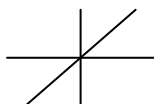
$$\text{From } \cos\theta = \frac{1-t^2}{1+t^2}, \sin\theta = \frac{2t}{1+t^2}, t = \tan \frac{\theta}{2}$$

$$\Rightarrow 3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 2 \dots\dots\dots (\times (1+t^2))$$

$$\Rightarrow 3 - 3t^2 + 8t = 2 + 2t^2 \text{ or } 5t^2 - 8t - 1 = 0$$

$$\therefore t = \frac{8 \pm \sqrt{64+20}}{10} = \frac{8 \pm 9.165}{10} \Rightarrow t = 1.7165, -0.1165$$

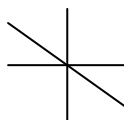
$$\text{For } \tan \frac{\theta}{2} = 1.7165$$



$$\tan^{-1}(1.7165) = 59.8^\circ$$

$$\frac{\theta}{2} = 59.8^\circ, 239.8^\circ \Rightarrow \theta = 119.6^\circ, 479.6^\circ$$

$$\text{For } \tan \frac{\theta}{2} = -0.1165$$



$$\tan^{-1}(0.1165) = 6.6^\circ$$

$$\frac{\theta}{2} = 173.4^\circ, 353.4^\circ \Rightarrow \theta = 346.8^\circ, 706.8^\circ$$

$$\therefore \theta = 119.6^\circ, 346.8^\circ$$

b) $2\cos 2\theta + 3\sin 2\theta - 2 = 0$

$$\cos 2\theta = \frac{1-t^2}{1+t^2}, \sin 2\theta = \frac{2t}{1+t^2}, t = \tan \theta \left[\because t = \tan \frac{\text{Angle}}{2} \right]$$

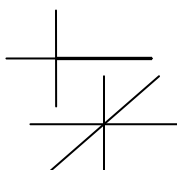
$$\Rightarrow 2\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) - 2 = 0 \dots\dots\dots (\times (1+t^2))$$

$$\Rightarrow 2 - 2t^2 + 6t - 2 - 2t^2 = 0 \text{ or } -4t^2 + 6t = 0 \text{ or } -2t^2 - 3t = 0$$

$$\text{By factorization } t = 0, \frac{3}{2}$$

$$\text{For } \tan \theta = 0$$

$$\text{For } \tan \theta = \frac{3}{2}$$



$$\tan^{-1}(0) = 0^\circ, 180^\circ, 360^\circ$$

$$\tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ$$

$$\Rightarrow \theta = 56.3^0, 236.3^0$$

$$\therefore \theta = 0^0, 56.3^0, 180^0, 236.3^0, 360^0$$

c) Left as an exercise

$$\text{Answer: } \theta = 72.6^0, 319.3^0$$

d) Left as an exercise

$$\text{Answer: } 76.7^0, 209.6^0$$

$$\left[\begin{array}{l} \text{HINT: } 3\tan\theta - 2\sec\theta = 4 \\ \frac{3\sin\theta}{\cos\theta} - \frac{2}{\cos\theta} = 4 \text{ or } \\ 3\sin\theta - 4\cos\theta - 2 = 0 \end{array} \right]$$

e) $5 \sin 3\theta - 12 \cos 3\theta = 6$

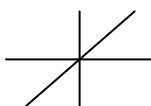
$$\cos 3\theta = \frac{1-t^2}{1+t^2}, \sin 3\theta = \frac{2t}{1+t^2}, t = \tan \frac{3}{2}\theta \left[\because t = \tan \frac{\text{Angle}}{2} \right]$$

$$\Rightarrow 5 \left(\frac{2t}{1+t^2} \right) - 12 \left(\frac{1-t^2}{1+t^2} \right) = 6 \dots \dots \dots (\times (1+t^2))$$

$$\Rightarrow 10 - 12 + 12t^2 = 6 + 6t^2 \text{ or } 6t^2 + 10t - 18 = 0$$

$$\text{or } 3t^2 + 5t - 9 = 0 \therefore t = \frac{-5 \pm \sqrt{25+108}}{6} \Rightarrow t = 1.0888, -2.7554$$

For $\tan \frac{3\theta}{2} = 1.0888$

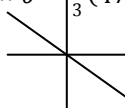


$$\tan^{-1}(1.0888) = 47.4^0$$

$$\Rightarrow \frac{3\theta}{2} = 47.4^0, 227.4^0, 407.4^0, 587.5^0 \therefore \theta = \frac{2}{3}(47.4^0, 227.4^0, 407.4^0, 587.5^0)$$

$$\therefore \theta = 31.6^0, 151.6^0, 271.6^0$$

For $\tan \frac{3\theta}{2} = -2.7554$



$$\tan^{-1}(2.7554) = 70.1^0$$

$$\Rightarrow \frac{3\theta}{2} = 109.9^0, 289.9^0, 469.9^0 \therefore \theta = \frac{2}{3}(109.9^0, 289.9^0, 469.9^0)$$

$$\therefore \theta = 73.3^0, 193.3^0, 313.3^0$$

$$\therefore \theta = 31.6^0, 73.3^0, 151.6^0, 193.3^0, 271.6^0, 313.3^0$$

f) left as an exercise. [HINT: use $t = \tan \frac{\theta}{4}$ **Ans:** No angle in this interval]

Solving the equation $a \cos(m\theta) + b \sin(m\theta) + c = 0$ using R – formulae

To solve this equation, first express

$$\begin{aligned} a \cos(m\theta) + b \sin(m\theta) &\equiv R \sin(m\theta \pm \alpha) \\ &\equiv R \cos(m\theta \pm \alpha), \alpha \text{ is Acute} \end{aligned}$$

Then solve the equation $R \cos(m\theta \pm \alpha) + c = 0$ or $R \sin(m\theta \pm \alpha)$.

Example

2. Solve the above equations in the previous example using R – formulae.

Solution

a) $3 \cos \theta + 4 \sin \theta = 2$

Let $3 \cos \theta + 4 \sin \theta \equiv R \sin(\theta + \alpha)$

$$\equiv R[\sin \theta \cos \alpha + \cos \theta \sin \alpha]$$

$$\equiv (R \cos \alpha) \sin \theta + (R \sin \alpha) \cos \theta$$

Equating corresponding coefficients of trig ratios

For $\cos \theta$: $3 = R \sin \alpha$ (1)

For $\sin \theta$: $4 = R \cos \alpha$ (2)

Solving (1) and (2) simultaneously for R and α

$$(1)^2 + (2)^2$$

$$3^2 + 4^2 = R^2 \sin^2 \alpha + R^2 \cos^2 \alpha = R^2 (\sin^2 \alpha + \cos^2 \alpha)$$

$$25 = R^2$$

$$\therefore R = 5$$

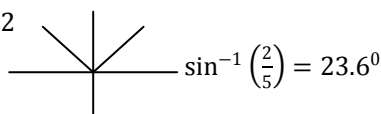
(1) $\div$ (2) to create $\tan \alpha$

$$\frac{3}{4} = \frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha$$

$$\therefore \alpha = \tan^{-1} \left(\frac{3}{4} \right) = 36.9^\circ$$

$$\Rightarrow 3 \cos \theta + 4 \sin \theta = 5 \sin(\theta + 36.9^\circ) = 2$$

$$\Rightarrow \sin(\theta + 36.9^\circ) = \frac{2}{5}$$



$$\theta + 36.9^\circ = 23.6^\circ, 156.4^\circ, 383.6^\circ \Rightarrow \theta = (23.6^\circ, 156.4^\circ, 383.6^\circ) - 36.9^\circ$$

$$\therefore \theta = 119.5^\circ, 346.7^\circ$$

b) $2 \cos 2\theta + 3 \sin 2\theta - 2 = 0$

Let $2 \cos 2\theta + 3 \sin 2\theta \equiv R \cos(2\theta - \alpha)$

$$\equiv R[\cos 2\theta \cos \alpha + \sin 2\theta \sin \alpha]$$

$$\equiv (R \cos \alpha) \cos 2\theta + (R \sin \alpha) \sin 2\theta$$

Equating corresponding coefficients of trig ratios

For $\cos 2\theta$: $2 = R \cos \alpha$ (1)

For $\sin 2\theta$: $3 = R \sin \alpha$ (2)

$$(1)^2 + (2)^2$$

$$2^2 + 3^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = R^2 (\cos^2 \alpha + \sin^2 \alpha)$$

$$13 = R^2$$

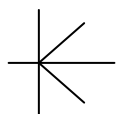
$$\therefore R = \sqrt{13}$$

(2) $\div$ (1) to create $\tan \alpha$

$$\frac{3}{2} = \frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha$$

$$\text{Now } 2\cos 2\theta + 3\sin 2\theta - 2 = 0 \Rightarrow \sqrt{13} \cos(2\theta - 56.3^\circ) - 2 = 0$$

$$\cos(2\theta - 56.3^\circ) = \frac{2}{\sqrt{13}} \quad \cos^{-1}\left(\frac{2}{\sqrt{13}}\right) = 56.3^\circ$$



$$\therefore 2\theta - 56.3^\circ = \pm 56.3^\circ, 303.7^\circ, 416.3^\circ, 776.3^\circ, 663.7^\circ$$

$$\therefore 2\theta = (\pm 56.3^\circ, 303.7^\circ, 416.3^\circ, 776.3^\circ, 663.7^\circ) + 56.3^\circ$$

$$\therefore \theta = 0^\circ, 56.3^\circ, 180^\circ, 236.3^\circ, 360^\circ$$

e) $5\sin 3\theta - 12\cos 3\theta = 6$

$$\text{Let } 5\sin 3\theta - 12\cos 3\theta \equiv R\sin(3\theta + \alpha)$$

$$\equiv R[\sin 3\theta \cos \alpha + \cos 3\theta \sin \alpha]$$

$$\equiv (R \cos \alpha) \sin 3\theta + (R \sin \alpha) \cos 3\theta$$

Equating corresponding coefficients of trig ratios

$$\text{For } \sin 3\theta: 5 = R \cos \alpha \dots\dots\dots(1)$$

$$\text{For } \cos 3\theta: -12 = R \sin \alpha \dots\dots\dots(2)$$

Solving (1) and (2) simultaneously for R and α

$$(1)^2 + (2)^2$$

$$5^2 + 12^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = R^2 (\cos^2 \alpha + \sin^2 \alpha)$$

$$169 = R^2$$

$$\therefore R = 13$$

$$(2) \div (1) \text{ to create } \tan \alpha$$

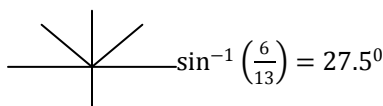
$$\frac{-12}{5} = \frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha$$

$$\therefore \alpha = -\tan^{-1}\left(\frac{12}{5}\right) = -67.4^\circ$$

$$\therefore 5 \sin 3\theta - 12 \cos 3\theta = 13 \sin(3\theta + -67.4^\circ) = 6$$

$$= 13 \sin(3\theta - 67.4^\circ) = 6$$

$$\sin(3\theta - 67.4^\circ) = \frac{6}{13}$$



$$3\theta - 67.4^\circ = 27.5^\circ, 152.5^\circ, 387.5^\circ, 512.5^\circ, 747.5^\circ, 872.5^\circ, 1107.5^\circ, 1232.5^\circ$$

$$\therefore 3\theta = (27.5^\circ, 152.5^\circ, 387.5^\circ, 512.5^\circ, 747.5^\circ, 872.5^\circ, 1107.5^\circ, 1232.5^\circ) + 67.4^\circ$$

$$\therefore \theta = 31.6^\circ, 73.3^\circ, 151.6^\circ, 193.3^\circ, 271.6^\circ, 313.3^\circ$$

Task

1. Solve $3\sin x + 4\cos x = 2$ for $-180^\circ \leq x \leq +180^\circ$ (**Uneb 2011**) **Ans:** $-29.6^\circ, 103.2^\circ$
2. Solve the equation $\sqrt{3} \cos 2\theta - \sin 2\theta = \sqrt{2}$ for any values of θ between 0° and 360° (**Jjeb2011**) **Ans:** $7.5^\circ, 142.5^\circ, 187.5^\circ, 330^\circ$
3. Given that $t = \tan \frac{\theta}{2}$, solve the equation $5\sin \theta - 2\cos 2\theta = 1$ for $0^\circ \leq \theta \leq 360^\circ$ **Ans:** $32.5^\circ, 191.1^\circ$
4. Solve for $0^\circ < \theta < 360^\circ$ in the equation $10\sin^2 \theta - 5\sin 2\theta = 4$

Ans: $63.4^0, 243.4^0, 161.6^0, 341.56^0$ [HINT: use $\sin^2 \theta = \frac{1-\cos 2\theta}{2}$]

Proof of identities using t – formulae

Prove the following identities

a) $2 \cot \frac{A}{2} - \sin A = \cot^2 \frac{A}{2} \sin A$

b) $2 \cot \frac{A}{2} + \tan A = \cot^2 \frac{A}{2} \tan A$

c) $\cot \frac{1}{2} \varphi = \sin \frac{\varphi}{1-\cos \varphi}$

Solution

a) $2 \cot \frac{A}{2} - \sin A = \cot^2 \frac{A}{2} \sin A$

$$\begin{aligned} \text{L. H. S: } 2 \cot \frac{A}{2} - \sin A &= \frac{2}{\tan \frac{A}{2}} - \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \frac{2}{t} - \frac{2t}{1+t^2} = \frac{2+2t^2-2t^2}{t(1+t^2)} = \frac{2}{t(1+t^2)} \\ &= \frac{2}{\frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} \cdot \sec^2 \frac{A}{2}} \left[\because 1 + \tan^2 \frac{A}{2} = \sec^2 \frac{A}{2} \right] \\ &= \frac{2 \cos \frac{A}{2}}{\sin \frac{A}{2} \cdot \frac{1}{\cos^2 \frac{A}{2}}} = 2 \cot \frac{A}{2} \cdot \cos^2 \frac{A}{2} \dots \dots \dots \left(\times \frac{\sin \frac{A}{2}}{\sin \frac{A}{2}} \right) \\ &= 2 \cot \frac{A}{2} \cdot \frac{\sin \frac{A}{2} \cdot \cos \frac{A}{2} \cdot \cos \frac{A}{2}}{\sin \frac{A}{2}} = \cot \frac{A}{2} \cdot \left(2 \sin \frac{A}{2} \cdot \cos \frac{A}{2} \right) \cdot \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} \\ &= \cot \frac{A}{2} \cdot \sin A \cdot \cot \frac{A}{2} = \cot^2 \frac{A}{2} \sin A = \text{R. H. S} \end{aligned}$$

b) $2 \cot \frac{A}{2} + \tan A = \cot^2 \frac{A}{2} \tan A$

$$\begin{aligned} \text{L. H. S: } 2 \cot \frac{A}{2} + \tan A &= \frac{2}{\tan \frac{A}{2}} + \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} = \frac{2}{t} + \frac{2t}{1-t^2} = \frac{2-2t^2+2t^2}{t(1-t^2)} = \frac{2}{t(1-t^2)} \\ &= \frac{2}{\tan \frac{A}{2} (1 - \tan^2 \frac{A}{2})} = \frac{2 \cot \frac{A}{2}}{(\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}) / \cos^2 \frac{A}{2}} \\ &= \frac{2 \cot \frac{A}{2} \cdot \cos^2 \frac{A}{2}}{\cos A} = \cot \frac{A}{2} \cdot \frac{2 \cos \frac{A}{2} \cdot \sin \frac{A}{2} \cdot \cos \frac{A}{2}}{\cos A \cdot \sin \frac{A}{2}} \end{aligned}$$

$$\cot \frac{A}{2} \cdot \frac{\sin A}{\cos A} \cdot \cot \frac{A}{2} = \cot^2 \frac{A}{2} \tan A = \text{R. H. S}$$

c) left as an exercise

Stating the Maximum and Minimum values of

$a \cos(m\theta) + b \sin(m\theta) + c$ and values of θ for which they occur

First express $a \cos(m\theta) + b \sin(m\theta) + c$ in the *R – formulae* and state the maximum and minimum values of the expression as it was done in $y = a \pm b \cos(m\theta + \alpha)$ and $y = a \pm b \sin(m\theta + \alpha)$

Examples

State the maximum and minimum values of the following trigonometric expression, and state the values of θ for which they occur between 0° and 360° inclusive

- a. $2\sin\theta - 5\cos\theta$
- b. $3\cos\theta + 4\sin\theta - 3$
- c. $2\cos 2\theta + 3\sin 2\theta - 2$
- d. $5\sin 3\theta - 12\cos 3\theta + 6$
- e. $3\sqrt{2}\cos(\theta + 45^\circ) + 7\sin\theta - 1$
- f. $\cos(\theta + 60^\circ) - \cos\theta - 0.5$

Solution

- a. Let $2\sin\theta - 5\cos\theta \equiv R\cos(\theta + \alpha) = (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$

Comparing corresponding coefficients

For $\sin\theta$: $2 = -R\sin\alpha$ (1)

For $\cos\theta$: $-5 = R\cos\alpha$ (2)

Solving (1) and (2) simultaneously for R and α

$(1)^2 + (2)^2$

$2^2 + 5^2 = R^2\sin^2\alpha + R^2\cos^2\alpha = R^2(\sin^2\alpha + \cos^2\alpha)$

$29 = R^2$

$\therefore R = \sqrt{29}$

$(1) \div (2)$ to create $\tan\alpha$

$\frac{2}{-5} = \frac{-R\sin\alpha}{R\cos\alpha} = -\tan\alpha$

$\therefore \alpha = \tan^{-1}\left(\frac{2}{5}\right) = 21.8^\circ$

$\therefore 2\sin\theta - 5\cos\theta = \sqrt{29}\cos(\theta + 21.8^\circ)$

$\max(2\sin\theta - 5\cos\theta) = \max[\sqrt{29}\cos(\theta + 21.8^\circ)] = \sqrt{29} \cdot \max[\cos(\theta + 21.8^\circ)]$
 $= \sqrt{29}$

Finding the value of $\theta_{\max}$

$\sqrt{29}\cos(\theta + 21.8^\circ) = \sqrt{29} \Rightarrow \cos(\theta + 21.8^\circ) = 1$

$\Rightarrow \theta + 21.8^\circ = 0^\circ, 360^\circ \therefore \theta_{\max} = 338.2^\circ$

$\min(2\sin\theta - 5\cos\theta) = \min[\sqrt{29}\cos(\theta + 21.8^\circ)] = \sqrt{29} \cdot \min[\cos(\theta + 21.8^\circ)]$
 $= -\sqrt{29}$

Finding the value of $\theta_{\min}$

$\sqrt{29}\cos(\theta + 21.8^\circ) = -\sqrt{29} \Rightarrow \cos(\theta + 21.8^\circ) = -1$

$\Rightarrow \theta + 21.8^\circ = 180^\circ \therefore \theta_{\min} = 158.2^\circ$

- b. $3\cos\theta + 4\sin\theta - 3$

$3\cos\theta + 4\sin\theta \equiv R\sin(\theta + \alpha)$ proceeding as above,

$3\cos\theta + 4\sin\theta = 5\sin(\theta + 36.9^\circ)$ (Left for students to derive)

$\max(3\cos\theta + 4\sin\theta - 3) = \max(5\sin(\theta + 36.9^\circ) - 3)$
 $= \max[5\sin(\theta + 36.9^\circ)] - \min(3)$
 $= 5(1) - 3 = 2$

Finding the value of $\theta_{\max}$

$5\sin(\theta + 36.9^\circ) - 3 = 2 \Rightarrow \sin(\theta + 36.9^\circ) = 1$

$$\begin{aligned}\Rightarrow \theta + 36.9^0 &= 90^0 \therefore \theta_{\max} = 53.1^0 \\ \min(3\cos\theta + 4\sin\theta - 3) &= \min(5\sin(\theta + 36.9^0) - 3) \\ &= \min[5\sin(\theta + 36.9^0)] - \max(3) \\ &= 5(-1) - 3 = -8\end{aligned}$$

Finding the value of $\theta_{\min}$

$$\begin{aligned}5\sin(\theta + 36.9^0) - 3 &= -8 \Rightarrow \sin(\theta + 36.9^0) = -1 \\ \Rightarrow \theta + 36.9^0 &= 270^0 \therefore \theta_{\min} = 233.1^0\end{aligned}$$

c. $2\cos 2\theta + 3\sin 2\theta - 2$

Let $2\cos 2\theta + 3\sin 2\theta \equiv R\cos(2\theta - \alpha)$ proceeding as usual,

$$2\cos 2\theta + 3\sin 2\theta = \sqrt{13}\cos(2\theta - 56.3^0)$$

$$\begin{aligned}\max(2\cos 2\theta + 3\sin 2\theta - 2) &= \max[\sqrt{13}\cos(2\theta - 56.3^0) - 2] \\ &= \sqrt{13}\max[\cos(2\theta - 56.3^0) - 2] = \sqrt{13} - 2\end{aligned}$$

Finding the value of $\theta_{\max}$

$$\sqrt{13}\cos(2\theta - 56.3^0) - 2 = \sqrt{13} - 2 \Rightarrow \cos(2\theta - 56.3^0) = 1$$

$$\Rightarrow 2\theta - 56.3^0 = 0^0, 360^0 \therefore 2\theta = 56.3^0, 416.3^0$$

$$\therefore \theta_{\max} = 28.2^0, 208.2^0$$

$$\begin{aligned}\min(2\cos 2\theta + 3\sin 2\theta - 2) &= \min[\sqrt{13}\cos(2\theta - 56.3^0) - 2] \\ &= \sqrt{13}\min[\cos(2\theta - 56.3^0) - 2] = -\sqrt{13} - 2\end{aligned}$$

Finding the value of $\theta_{\min}$

$$\sqrt{13}\cos(2\theta - 56.3^0) - 2 = -\sqrt{13} - 2 \Rightarrow \cos(2\theta - 56.3^0) = -1$$

$$\Rightarrow 2\theta - 56.3^0 = 180^0, 540^0 \text{ solving for } \theta$$

$$\therefore \theta_{\min} = 118.2^0, 298.2^0$$

d. Left as an exercise

e. (e) and (f) use special angles and expand using identities to express in the form being dealt with, then proceed as in the above examples.

Answer: (e) $\max = 4, \min = -6, \theta_{\max} = 53.1^0, \theta_{\min} = 233.1^0$

(f) $\max = 0.5, \min = -1.5, \theta_{\max} = 240^0, \theta_{\min} = 60^0$

NOTE:

Other applications mentioned and not tackled in this edition shall be handled in the 2nd edition.

Task

Express $3\cos^2\theta + 4\sin\theta\cos\theta - 5\sin^2\theta$ in the form $a + b\cos(2\theta - \beta)$ where a, b are constants and β is an acute angle. Hence prove that the maximum and minimum values of the expression

$$9 + 3\cos^2\theta + 4\sin\theta\cos\theta - 5\sin^2\theta \text{ are } 2(4 + \sqrt{5}) \text{ and } 2(4 - \sqrt{5}) \text{ respectively.}$$

$$\left[\text{HINT: } \cos^2\theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2} \text{ and } 4\sin\theta\cos\theta = 2\sin 2\theta \right]$$

THE FACTOR FORMULAE

(Sum and Product identities)

The factor formulae provides a means to convert products of certain trigonometric functions in to sums and conversely.

Factor formulae are derived using the sum and difference of identities for the sines and cosine function.

Recall

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Adding gives

$$\cos(A + B) + \cos(A - B) = 2\cos A \cos B$$

Subtracting gives

$$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$$

$$\text{Also } \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Adding gives

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B$$

Subtracting gives

$$\sin(A + B) - \sin(A - B) = 2\cos A \sin B$$

Thus if A and B are
real(measures of angles), then

- i) $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$
- ii) $\cos(A + B) - \cos(A - B) = -2\sin A \sin B$
- iii) $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$
- iv) $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$

The factor formulae can be converted in to an equivalent form by making the substitution of variables.

$$\text{let } P = A + B \quad \dots\dots\dots (1)$$

$$Q = A - B \quad \dots\dots\dots (2)$$

Solving for A and B

$$\text{Adding: } \frac{P+Q}{2} = A \quad \dots\dots\dots (3)$$

$$\text{Subtracting: } \frac{P-Q}{2} = B \quad \dots\dots\dots (4)$$

Substituting eqn. (1) to (4) in to the above factor formulae, then

$$\cos P + \cos Q = 2\cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\cos P - \cos Q = -2\sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

$$\sin P + \sin Q = 2\sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\sin P - \sin Q = 2\cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

Application

- Solution to trigonometric equations
- Proof of identities
- Factorization and simplifying trigonometric equations

Solution to trigonometric equations

1. Solve for $0^\circ \leq x \leq 360^\circ$ the equations

- i) $\sin 3x + \sin x = 0$
- ii) $\cos x + \cos 5x = 0$
- iii) $\cos 4x - \cos x = 0$
- iv) $\sin(x + 10^\circ) + \sin x = 0$
- v) $\cos(x + 20^\circ) - \cos(x - 70^\circ) = 0$
- vi) $\cos(2x + 10^\circ) + \cos(2x - 10^\circ) = 0$

Solution

i) $\sin 3x + \sin x = 0$

$$2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 0 \Rightarrow 2\sin 2x \cos x = 0$$

$$\Rightarrow \sin 2x = 0, \cos x = 0 \quad [\because a \cdot b = 0 \text{ iff } a = 0, b = 0]$$

Dealing with $\sin 2x = 0$

$$2x = 0^\circ, 180^\circ, 360^\circ, 540^\circ, 900^\circ$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, 270^\circ, 450^\circ$$

Dealing with $\cos x = 0$

$$\therefore x = 90^\circ, 270^\circ$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, 270^\circ$$

ii) $\cos x + \cos 5x = 0$

$$2\cos\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right) \Rightarrow 2\cos 3x \cos(-2x) = 0$$

$$\therefore \cos 3x \cos 2x = 0 \quad [\because \cos(-2x) = \cos 2x] \Rightarrow \cos 3x = 0, \cos 2x = 0$$

Dealing with $\cos 3x = 0$

$$\Rightarrow 3x = 90^\circ, 270^\circ, 450^\circ, 540^\circ, 630^\circ, 810^\circ, 900^\circ, 990^\circ$$

$$\therefore x = 30^\circ, 90^\circ, 150^\circ, 180^\circ, 210^\circ, 270^\circ, 300^\circ, 330^\circ$$

Dealing with $\cos 2x = 0$

$$\Rightarrow 2x = 90^\circ, 270^\circ, 450^\circ, 540^\circ$$

$$\therefore x = 45^\circ, 135^\circ, 225^\circ, 270^\circ$$

$$\therefore x = 30^\circ, 45^\circ, 90^\circ, 135^\circ, 150^\circ, 180^\circ, 210^\circ, 225^\circ, 270^\circ, 300^\circ, 315^\circ, 330^\circ$$

iii) Left as an exercise **Answer:** $0^\circ, 72^\circ, 120^\circ, 144^\circ, 216^\circ, 240^\circ, 288^\circ, 360^\circ$

iv) $\sin(x + 10^\circ) + \sin x = 0$

$$2\sin\left(\frac{x+10+x}{2}\right)\cos\left(\frac{x+10-x}{2}\right) = 0 \dots\dots\dots (\div 2)$$

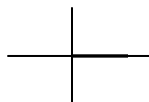
$$\Rightarrow \sin(x + 5^\circ) \cos 5^\circ = 0 \dots\dots\dots (\div \cos 5^\circ)$$

$$\Rightarrow \sin(x + 5^\circ) = 0 \therefore (x + 5^\circ) = 0^\circ, 180^\circ, 360^\circ$$

$$\therefore x = 175^\circ, 355^\circ \text{ [reject } -5^\circ \because \text{it is out of range]}$$

v) Left as an exercise **Answer:** $25^\circ, 205^\circ$

vi) Left as an exercise **Answer:** $45^\circ, 135^\circ, 225^\circ, 315^\circ$



2. Solve the equations for values of θ from 0° to 180° inclusive.

- a. $\cos\theta + \cos3\theta + \cos5\theta = 0$
- b. $\sin\theta - 2\sin2\theta + \sin3\theta = 0$
- c. $\sin\theta + \cos2\theta - \sin3\theta = 0$
- d. $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

Solution

This set of numbers is solved by factorization. Group any two trig ratios, apply factor formulae. If the third/ remaining trig ratio is a factor of the two ten factorize.

- a. $\cos\theta + \cos3\theta + \cos5\theta = 0$

Here we can group either group $(\cos\theta + \cos5\theta)$ or $(\cos3\theta + \cos5\theta)$

$$\Rightarrow \cos\theta + (\cos3\theta + \cos5\theta) = 0$$

$$\Rightarrow \cos\theta + 2\cos4\theta \cos(-\theta) = 0 \text{ or } \cos\theta [1 + 2\cos4\theta] = 0 \quad [\because \cos(-\theta) = \cos\theta]$$

$$\cos\theta = 0 \text{ or } 1 + 2\cos4\theta = 0$$

Dealing with $\cos\theta = 0$

$$\Rightarrow \theta = 90^\circ, 270^\circ$$

Dealing with $\cos4\theta = -\frac{1}{2}$

$$\Rightarrow 4\theta = 120^\circ, 240^\circ, 480^\circ, 600^\circ, 840^\circ, 960^\circ, 1200^\circ, 1320^\circ$$

$$\Rightarrow \theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ$$

$$\therefore \theta = 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ$$

- b. $\sin\theta - 2\sin2\theta + \sin3\theta = 0$

$$\sin\theta + \sin3\theta - 2\sin2\theta = 0 \Rightarrow 2\sin2\theta \cos(-\theta) - 2\sin2\theta = 0$$

$$2\sin2\theta[\cos\theta - 1] = 0 \text{ . From which } \sin2\theta = 0 \text{ or } \cos\theta - 1 = 0 \Rightarrow \cos\theta = 1$$

Dealing with $\cos\theta = 1$

$$\Rightarrow \theta = 0^\circ, 360^\circ$$

Dealing with $\sin2\theta = 0$

$$\Rightarrow 2\theta = 0^\circ, 180^\circ, 360^\circ \text{ or } \theta = 0^\circ, 90^\circ, 180^\circ$$

$$\therefore \theta = 0^\circ, 90^\circ, 180^\circ$$

- c. left as an exercise **Answer: $30^\circ, 45^\circ, 135^\circ, 150^\circ$**

- d. $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

$$\left(\cos\frac{1}{2}\theta + \cos\frac{5}{2}\theta\right) + 2\cos\frac{3}{2}\theta = 0 \Rightarrow 2\cos\frac{3}{2}\theta \cos\left(-\frac{1}{2}\theta\right) + 2\cos\frac{3}{2}\theta = 0$$

$$2\cos\frac{3}{2}\theta \left[\cos\frac{1}{2}\theta + 1\right] = 0 \text{ From which } \cos\frac{3}{2}\theta = 0 \text{ or } \cos\frac{1}{2}\theta + 1 = 0 \Rightarrow$$

$$\cos\frac{1}{2}\theta = -1$$

Dealing with $\cos\frac{3}{2}\theta = 0$

$$\Rightarrow \frac{3}{2}\theta = 90^\circ, 270^\circ \text{ or } \theta = \frac{2}{3} \times (90^\circ, 270^\circ)$$

$$\Rightarrow \theta = 60^\circ, 180^\circ$$

Dealing with $\cos\frac{1}{2}\theta = 1$

$$\Rightarrow \frac{1}{2}\theta = 180^\circ \text{ or } \theta = 360^\circ$$

$\therefore \theta = 60^\circ, 180^\circ$ [reject 360° $\because$ it is out of range]

NOTE: Factor formula is applied to two trig ratios with the same coefficient.

3. Solve for $0^\circ < \theta < 360^\circ$ the equation

a. $\sin 3\theta + 2\sin 5\theta \sin^2 \theta + \sin 7\theta = 0$

b. $\sin \theta + \sin 7\theta = \sin 4\theta$

c. $\cos 3\theta = \cos(2\theta + 45^\circ)$

d. $2\sin^2(\theta - 30^\circ) = \cos 2\theta + \frac{1}{2}$

Solution

a. (a) and (b) left as an exercise **Answer:** (a) $0^\circ, 36^\circ, 72^\circ, 90^\circ, 108^\circ, 144^\circ, 180^\circ, 216^\circ, 252^\circ, 270^\circ, 288^\circ, 324^\circ$

c. $\cos 3\theta = \cos(2\theta + 45^\circ)$

$$\cos 3\theta - \cos(2\theta + 45^\circ) = 0 \Rightarrow -2\sin\left(\frac{5\theta + 45^\circ}{2}\right)\sin\left(\frac{\theta - 45^\circ}{2}\right) = 0$$
 From which

$$\sin\left(\frac{5\theta + 45^\circ}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta - 45^\circ}{2}\right) = 0$$

$$\text{Dealing with } \sin\left(\frac{\theta - 45^\circ}{2}\right) = 0 \Rightarrow \left(\frac{\theta - 45^\circ}{2}\right) = 0^\circ, 180^\circ, 360^\circ \therefore \theta = 45^\circ, 315^\circ$$

$$\text{Dealing with } \sin\left(\frac{5\theta + 45^\circ}{2}\right) = 0 \Rightarrow \left(\frac{5\theta + 45^\circ}{2}\right) = 0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ, 900^\circ$$

$$\Rightarrow \theta = \frac{[2 \times (0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ, 900^\circ) - 45^\circ]}{5} \therefore \theta = 63^\circ, 135^\circ, 207^\circ, 279^\circ, 351^\circ$$

$$\therefore \theta = 45^\circ, 63^\circ, 135^\circ, 207^\circ, 279^\circ, 315^\circ, 351^\circ$$

d. $2\sin^2(\theta - 30^\circ) = \cos 2\theta + \frac{1}{2}$

$$\sin^2(\theta - 30^\circ) = \frac{1 - \cos 2(\theta - 30^\circ)}{2} = \frac{1 - \cos(2\theta - 60^\circ)}{2}$$

$$\Rightarrow \frac{2[1 - \cos(2\theta - 60^\circ)]}{2} - \cos 2\theta = \frac{1}{2} \text{ or } 1 - \cos(2\theta - 60^\circ) - \cos 2\theta = \frac{1}{2}$$

$$\cos(2\theta - 60^\circ) + \cos 2\theta = \frac{1}{2} \Rightarrow 2\cos(2\theta - 30^\circ)\cos(-30^\circ) = \frac{1}{2}$$

$$2 \cdot \frac{\sqrt{3}}{2} \cdot \cos(2\theta - 30^\circ) = \frac{1}{2} \Rightarrow \cos(2\theta - 30^\circ) = \frac{1}{2\sqrt{3}} \text{ taking } + \text{ve root}$$

$$2\theta - 60^\circ = 73.2^\circ, 286.8^\circ, 433.2^\circ, 646.8^\circ \Rightarrow 2\theta = (73.2^\circ, 286.8^\circ, 433.2^\circ, 646.8^\circ) + 60^\circ$$

$$\therefore \theta = \frac{(73.2^\circ, 286.8^\circ, 433.2^\circ, 646.8^\circ) + 60^\circ}{2} = 51.6^\circ, 158.4^\circ, 231.6^\circ, 338.4^\circ$$

Proof of identities

1. Prove the following identities

a. $\cos^2 A - \cos^2 B = \sin(A + B)\sin(B - A)$

b. $\frac{\cos B + \cos C}{\sin B - \sin C} = \cot \frac{B - C}{2}$

c. $\frac{\sin B - \sin C}{\sin B + \sin C} = \cot \frac{B + C}{2} \tan \frac{B - C}{2}$

d. $\frac{\cos 7t + \cos 5t}{\sin 7t - \sin 5t} = \frac{\operatorname{cosec} t}{\sec t}$

e. $\frac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x} = \frac{1}{\cot 3x}$

$$\begin{aligned} \text{f. } \frac{\sin 3x - \sin x}{\cos 3x + \cos x} &= \tan x \\ \text{g. } \frac{\sin 3t - \sin t}{\sin 3t + \sin t} &= \frac{2}{1 - \tan^2 t} \\ \text{h. } \frac{\cos 9x + \cos 5x}{\sin 9x - \sin 5x} &= \frac{1 - \tan^2 x}{2 \tan x} \end{aligned}$$

Solution

$$\begin{aligned} \text{a. } \cos^2 A - \cos^2 B &= \sin(A+B)\sin(B-A) \\ \text{L.H.S: } \cos^2 A - \cos^2 B &= (\cos A + \cos B)(\cos A - \cos B) \\ &= \left[2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \right] \cdot \left[-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \right] \\ &= -\left[2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A+B}{2}\right) \right] \cdot \left[2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A-B}{2}\right) \right] \\ &= -\sin\left[2\left(\frac{A+B}{2}\right) \right] \cdot \sin\left[2\left(\frac{A-B}{2}\right) \right] \\ &= -\sin(A+B)\sin(A-B) = \sin(A+B)\sin(B-A) \quad [\because \sin(-\theta) = -\sin\theta] \\ \text{b. } \frac{\cos B + \cos C}{\sin B - \sin C} &= \cot \frac{B-C}{2} \\ \text{L.H.S: } \frac{\cos B + \cos C}{\sin B - \sin C} &= \frac{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}} = \frac{\cos\frac{B-C}{2}}{\sin\frac{B-C}{2}} = \cot \frac{B-C}{2} = \text{R.H.S} \\ \text{c. } \frac{\sin B - \sin C}{\sin B + \sin C} &= \cot \frac{B+C}{2} \tan \frac{B-C}{2} \\ \text{L.H.S: } \frac{\sin B - \sin C}{\sin B + \sin C} &= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} = \frac{\cos\frac{B+C}{2}}{\sin\frac{B+C}{2}} \cdot \frac{\sin\frac{B-C}{2}}{\cos\frac{B-C}{2}} = \cot \frac{B+C}{2} \tan \frac{B-C}{2} = \text{R.H.S} \\ \text{d. } \frac{\cos 7t + \cos 5t}{\sin 7t - \sin 5t} &= \frac{\operatorname{cosec}}{\sec} \\ \text{L.H.S: } \frac{\cos 7t + \cos 5t}{\sin 7t - \sin 5t} &= \frac{2\cos\left(\frac{7t+5t}{2}\right)\cos\left(\frac{7t-5t}{2}\right)}{\cos\left(\frac{7t+5t}{2}\right)\sin\left(\frac{7t-5t}{2}\right)} = \frac{\cos 6t \cos t}{\cos 6t \sin t} = \frac{\operatorname{cosec}}{\sec} = \text{R.H.S} \\ \text{e. (e), (f) left as an exercise} \\ \text{g. } \frac{\sin 3t + \sin t}{\sin 3t - \sin t} &= \frac{2}{1 - \tan^2 t} \\ \text{L.H.S: } \frac{\sin 3t + \sin t}{\sin 3t - \sin t} &= \frac{2\sin\left(\frac{3t+t}{2}\right)\cos\left(\frac{3t-t}{2}\right)}{\cos\left(\frac{3t+t}{2}\right)\sin\left(\frac{3t-t}{2}\right)} = \frac{\sin 2t \cos t}{\cos 2t \sin t} = \frac{2\sin t \cos t \cdot \cos t}{\frac{1 - \tan^2 t}{1 + \tan^2 t} \cdot \sin t} \left[\because \sin 2t = 2\cos t \sin t \right] \\ &= \frac{2(1 + \tan^2 t) \cdot \cos^2 t}{1 - \tan^2 t} = \frac{2\sec^2 t \cdot \cos^2 t}{1 - \tan^2 t} = \frac{2\left(\frac{1}{\cos^2 t} \cdot \cos^2 t\right)}{1 - \tan^2 t} = \frac{2}{1 - \tan^2 t} = \text{R.H.S} \\ \text{h. } \frac{\cos 9x + \cos 5x}{\sin 9x - \sin 5x} &= \frac{1 - \tan^2 x}{2 \tan x} \\ \text{L.H.S: } \frac{\cos 9x + \cos 5x}{\sin 9x - \sin 5x} &= \frac{2\cos\frac{9x+5x}{2}\cos\frac{9x-5x}{2}}{2\cos\frac{9x+5x}{2}\sin\frac{9x-5x}{2}} = \frac{\cos 7x \cos 2x}{\cos 7x \sin 2x} = \frac{\cos 2x}{\sin 2x} \\ &= \frac{\left(\frac{1 - \tan^2 x}{1 + \tan^2 x}\right)}{2 \tan x} = \frac{1 - \tan^2 x}{2 \tan x} = \text{R.H.S} \end{aligned}$$

2. Prove the following identities

$$\begin{aligned} \text{a. } \frac{\sin x + \sin 2x + \sin 3x}{\sin 2x} &= 2\cos x + 1 \\ \text{b. } \cos x + \sin 2x - \cos 3x &= \sin 2x(2\sin x + 1) \\ \text{c. } \frac{\cos 3\theta + \cos 5\theta + \cos 7\theta}{\cos 5\theta} &= 2\cos \theta + 1 \\ \text{d. } 1 + 2\cos 2\theta + \cos 4\theta &= 4\cos^2 \theta \cos 2\theta \\ \text{e. } \sin x - \sin(x + 60^\circ) + \sin(x + 120^\circ) &= 0 \\ \text{f. } \cos x + \cos(x + 120^\circ) + \cos(x + 240^\circ) &= 0 \end{aligned}$$

g. $\sin \alpha + \sin \left(\alpha + \frac{2\pi}{3} \right) + \sin \left(\alpha + \frac{4\pi}{3} \right) = 0$

h. $\cos \theta + \cos \left(\theta + \frac{2\pi}{3} \right) + \cos \left(\theta + \frac{4\pi}{3} \right) = 0$

Solution

a. $\frac{\sin x + \sin 2x + \sin 3x}{\sin 2x} = 2\cos x + 1$

Since $\sin 2x$ is a factor of $(\sin x + \sin 3x)$, then

L.H.S: $\frac{\sin x + (\sin 2x + \sin 3)x}{\sin 2x} = \frac{\sin 2x + (2\sin 2x \cos x)}{\sin 2x} = \frac{\sin 2x [1 + 2\cos 2x]}{\sin 2x} = 2\cos x + 1 = \text{R.H.S}$

b. $\cos x + \sin 2x - \cos 3x = \sin 2x(2\sin x + 1)$

L.H.S: $(\cos x - \cos 3x) + \sin 2x = -2\sin 2x \sin(-x) + \sin 2x$
 $= 2\sin 2x \sin x + \sin 2x = \sin 2x(2\sin x + 1) = \text{R.H.S}$

c. Left as an exercise

d. $1 + 2\cos 2\theta + \cos 4\theta = 4\cos^2 \theta \cos 2\theta$

L.H.S: $1 + 2\cos 2\theta + \cos 4\theta = (1 + \cos 4\theta) + 2\cos 2\theta$

But $1 = \cos 0^\circ = \cos 360^\circ$, Replacing 1 with $\cos 0^\circ$ to allow usage of factor formulae

$\Rightarrow 1 + 2\cos 2\theta + \cos 4\theta = (\cos 0 + \cos 4\theta) + 2\cos 2\theta = 2\cos 2\theta \cos(-2\theta) + 2\cos 2\theta$
 $= 2\cos 2\theta(\cos 2\theta + 1) = 2\cos 2\theta(2\cos^2 \theta - 1 + 1) = 4\cos^2 \theta \cos 2\theta = \text{R.H.S}$

e. $\sin x - \sin(x + 60^\circ) + \sin(x + 120^\circ) = 0$

Combining $\sin x + \sin(x + 120^\circ)$ to give $\sin(x + 60^\circ)$ as a factor

L.H.S: $\sin x - \sin(x + 60^\circ) + \sin(x + 120^\circ)$
 $= [\sin x + \sin(x + 120^\circ)] - \sin(x + 60^\circ)$
 $= 2\sin(x + 60^\circ) \cos(-60^\circ) - \sin(x + 60^\circ)$
 $= \sin(x + 60^\circ) = [2\cos 60^\circ - 1] = 0$

f. Left as an exercise

g. $\sin \alpha + \sin \left(\alpha + \frac{2\pi}{3} \right) + \sin \left(\alpha + \frac{4\pi}{3} \right) = 0$

Re-writing in terms of degrees

$\sin \alpha + \sin(\alpha + 120^\circ) + \sin(\alpha + 240^\circ) = 0$
 $\Rightarrow \sin \alpha + \sin(\alpha + 120^\circ) + \sin(\alpha + 240^\circ) = [\sin \alpha + \sin(\alpha + 240^\circ)] + \sin(\alpha + 120^\circ)$
 $= 2\sin(\alpha + 120^\circ) \cos(-120^\circ) + \sin(\alpha + 120^\circ) =$
 $\sin(\alpha + 120^\circ)[2\cos(-120^\circ) + 1] = 0$

3. Prove the following identities

a. $\frac{\sin 3A \sin 6A + \sin A \sin 2A}{\sin 3A \cos 6A + \sin A \cos 2A} = \tan 5A$

b. $\frac{\sin 3\theta \cos 4\theta - \sin \theta \cos 2\theta}{\sin \theta \sin 4\theta + \cos \theta \cos 6\theta} = \tan 2\theta$

c. $\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$

d. $\frac{\sin \theta \sin 2\theta + \sin 3\theta \sin 6\theta + \sin 4\theta \sin 13\theta}{\sin \theta \cos 2\theta + \sin 3\theta \cos 6\theta + \sin 4\theta \cos 13\theta} = \tan 9\theta$

Solution

a. $\frac{\sin 3A \sin 6A + \sin A \sin 2A}{\sin 3A \cos 6A + \sin A \cos 2A} = \tan 5A$

Here express the product of trig ratios as a sum or difference of cosines/sines using the factor formulae.

The factor formulae to be used are;

If A and B are real(measures of angles), then

- i) $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$
- ii) $\cos(A + B) - \cos(A - B) = -2\sin A \sin B$
- iii) $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$
- iv) $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$

Using $\cos(A + B) - \cos(A - B) = -2\sin A \sin B$ or $\sin A \sin B$

$$\begin{aligned} &= -\frac{1}{2}[\cos(A + B) - \cos(A - B)] \\ \Rightarrow \sin 3A \sin 6A &= -\frac{1}{2}[\cos(3A + 6A) - \cos(3A - 6A)] = -\frac{1}{2}[\cos 9A - \cos(-3A)] \\ &= -\frac{1}{2}[\cos 9A - \cos 3A] \\ \Rightarrow \sin A \sin 2A &= -\frac{1}{2}[\cos(A + 2A) - \cos(A - 2A)] = -\frac{1}{2}[\cos 3A - \cos(-A)] \\ &= -\frac{1}{2}[\cos 3A - \cos A] \end{aligned}$$

Using $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$ or $\sin A \cos B$

$$\begin{aligned} &= \frac{1}{2}[\sin(A + B) + \sin(A - B)] \\ \Rightarrow \sin 3A \cos 6A &= \frac{1}{2}[\sin(3A + 6A) + \sin(3A - 6A)] = \frac{1}{2}[\sin 9A + \sin(-3A)] \\ &= \frac{1}{2}[\sin 9A - \sin 3A] \quad [\because \sin(-3A) = -\sin 3A] \\ \Rightarrow \sin A \cos 2A &= \frac{1}{2}[\sin(A + 2A) + \sin(A - 2A)] = \frac{1}{2}[\sin 3A + \sin(-A)] \\ &= \frac{1}{2}[\sin 3A - \sin A] \end{aligned}$$

$$\begin{aligned} \therefore \frac{\sin 3A \sin 6A + \sin A \sin 2A}{\sin 3A \cos 6A + \sin A \cos 2A} &= \frac{-\frac{1}{2}[\cos 9A - \cos 3A] + -\frac{1}{2}[\cos 3A - \cos A]}{\frac{1}{2}[\sin 9A - \sin 3A] + \frac{1}{2}[\sin 3A - \sin A]} = \frac{-\cos 9A + \cos 3A - \cos 3A + \cos A}{\sin 9A - \sin 3A + \sin 3A - \sin A} \\ &= \frac{\cos A - \cos 9A}{\sin 9A - \sin A} = \frac{-2\sin 5A \sin(-4A)}{2\cos 5A \sin 4A} = \frac{2\sin 5A \sin 4A}{2\cos 5A \sin 4A} = \tan 5A \end{aligned}$$

b. $\frac{\sin 3\theta \cos 4\theta - \sin \theta \cos 2\theta}{\sin \theta \sin 4\theta + \cos \theta \cos 6\theta} = \tan 2\theta$

$$\begin{aligned} \text{L.H.S: } \frac{\sin 3\theta \cos 4\theta - \sin \theta \cos 2\theta}{\sin \theta \sin 4\theta + \cos \theta \cos 6\theta} &= \frac{\frac{1}{2}[\sin 7\theta + \sin(-\theta)] - \frac{1}{2}[\sin 3\theta + \sin(-\theta)]}{-\frac{1}{2}[\cos 5\theta - \cos(-3\theta)] + \frac{1}{2}[\cos 7\theta + \cos(-5\theta)]} \\ &= \frac{\sin 7\theta - \sin \theta - \sin 3\theta + \sin \theta}{\cos 3\theta - \cos 5\theta + \cos 7\theta + \cos 5\theta} = \frac{\sin 7\theta - \sin 3\theta}{\cos 7\theta + \cos 3\theta} = \frac{2\cos 5\theta \sin 2\theta}{2\cos 5\theta \cos 2\theta} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta \end{aligned}$$

c. left as an exercise

d. $\frac{\sin \theta \sin 2\theta + \sin 3\theta \sin 6\theta + \sin 4\theta \sin 13\theta}{\sin \theta \cos 2\theta + \sin 3\theta \cos 6\theta + \sin 4\theta \cos 13\theta} = \tan 9\theta$

$$\begin{aligned} \text{L.H.S: } &= \frac{-\frac{1}{2}[\cos 3\theta - \cos(-\theta)] + \frac{1}{2}[\cos 9\theta - \cos(-3\theta)] + \frac{1}{2}[\cos 17\theta - \cos(-9\theta)]}{\frac{1}{2}[\sin 3\theta + \sin(-\theta)] + \frac{1}{2}[\sin 9\theta + \sin(-3\theta)] + \frac{1}{2}[\sin 17\theta + \sin(-9\theta)]} \\ &= \frac{-\cos 3\theta + \cos \theta - \cos 9\theta + \cos 3\theta - \cos 17\theta + \cos 9\theta}{\sin 3\theta - \sin \theta + \sin 9\theta - \sin 3\theta + \sin 17\theta - \sin 9\theta} = \frac{\cos \theta - \cos 17\theta}{\sin 17\theta - \sin \theta} \\ &= \frac{-2\sin 9\theta \sin(-8\theta)}{2\cos 9\theta \sin 8\theta} = \frac{\sin 9\theta}{\cos 9\theta} \cdot \frac{\sin 8\theta}{\sin 8\theta} = \tan 9\theta \end{aligned}$$

PROOF OF IDENTITIES INVOLVING ANGLES OF A TRIANGLE

Example

1. If A, B , and C are angles of a triangle, prove that

- $\cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
- $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$
- $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$
- $\sin(B + C - A) + \sin(C + A - B) - \sin(A + B - C) = 4 \cos A \cos B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- $\cos 4A - \cos 4B - \cos 4C = 4 \sin 2B \sin 2C \cos 2A - 1$ (Uneb 2012)
- $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$
- $\frac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1} = 8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$
- $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$
- $\frac{\cot A + \cot B}{\tan A + \tan B} + \frac{\cot B + \cot C}{\tan B + \tan C} + \frac{\cot C + \cot A}{\tan C + \tan A} = 1$

Solution

a. $\cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

L.H.S: $\cos A + \cos B + \cos C - 1 = (\cos A + \cos B) + (\cos C - 1)$
 $= 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + (\cos C - \cos 0^\circ) = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + -2 \sin \frac{C}{2} \sin \frac{C}{2}$
 $= 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) - 2 \sin \frac{C}{2} \sin \frac{C}{2}$, but $A + B = 180^\circ - C \Rightarrow \frac{A+B}{2} = 90^\circ - \frac{C}{2}$

L.H.S $= 2 \cos \left(90^\circ - \frac{C}{2} \right) \cos \left(\frac{A-B}{2} \right) - 2 \sin \frac{C}{2} \sin \frac{C}{2}$,

But $\cos \left(90^\circ - \frac{C}{2} \right) = \cos 90^\circ \cos \frac{C}{2} + \sin 90^\circ \sin \frac{C}{2} = \sin \frac{C}{2}$

$\Rightarrow$ **L.H.S** $= 2 \sin \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \sin \frac{C}{2} \right]$, but $A + B + C = 180^\circ$, $\frac{C}{2} = 90^\circ - \left(\frac{A+B}{2} \right)$

$\Rightarrow$ **L.H.S** $= 2 \sin \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \sin \left[90^\circ - \left(\frac{A+B}{2} \right) \right] \right]$,

But $\sin \left[90^\circ - \left(\frac{A+B}{2} \right) \right] = \sin 90^\circ \cos \left(\frac{A+B}{2} \right) - \cos 90^\circ \sin \left(\frac{A+B}{2} \right) = \cos \left(\frac{A+B}{2} \right)$

$\Rightarrow$ **L.H.S** $= 2 \sin \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \cos \left(\frac{A+B}{2} \right) \right] 2 \sin \frac{C}{2} \left[-2 \sin \left(\frac{\frac{A-B}{2} + \frac{A+B}{2}}{2} \right) \sin \left(\frac{\frac{A-B}{2} - \frac{A+B}{2}}{2} \right) \right]$
 $= 4 \sin \frac{C}{2} \left[-\sin \frac{A}{2} \sin \left(-\frac{B}{2} \right) \right] = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

b. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

L.H.S: $(\sin A + \sin B) + \sin C = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + 2 \sin \frac{C}{2} \sin \frac{C}{2}$

$\left[\because \sin C = \sin C + \sin 0 = 2 \sin \frac{C}{2} \sin \frac{C}{2} \right]$, but $A+B+C = 180^\circ$, $\frac{C}{2} = 90^\circ - \left(\frac{A+B}{2} \right)$

$$\begin{aligned}\therefore \sin\left(\frac{A+B}{2}\right) &= \sin\left(90^\circ - \frac{C}{2}\right) = \sin 90^\circ \cos \frac{C}{2} - \cos 90^\circ \sin \frac{C}{2} = \cos \frac{C}{2} \\ \Rightarrow \text{L.H.S} &= 2 \cos \frac{C}{2} \left[\cos\left(\frac{A-B}{2}\right) + \sin \frac{C}{2} \right], \text{ also } \sin \frac{C}{2} = \sin\left[90^\circ - \left(\frac{A+B}{2}\right)\right] = \cos\left(\frac{A+B}{2}\right) \\ \Rightarrow \text{L.H.S} &= 2 \cos \frac{C}{2} \left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right) \right] \\ &= 2 \cos \frac{C}{2} \left[2 \cos\left(\frac{\frac{A-B}{2} + \frac{A+B}{2}}{2}\right) \cos\left(\frac{\frac{A-B}{2} - \frac{A+B}{2}}{2}\right) \right] \\ &= 4 \cos \frac{C}{2} \left[\cos \frac{A}{2} \cos\left(-\frac{B}{2}\right) \right] = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}\end{aligned}$$

c. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

L.H.S: $(\sin 2A + \sin 2B) + \sin 2C = 2 \sin(A+B) \cos(A-B) + 2 \cos C \sin C$

But $\sin(A+B) = \sin(180^\circ - C) = \sin 180^\circ \cos C - \cos 180^\circ \sin C = \sin C$

$$\begin{aligned}\therefore \sin 2A + \sin 2B + \sin 2C &= 2 \sin C \cos(A-B) + 2 \cos C \sin C \\ &= 2 \sin C [\cos(A-B) + \cos C]\end{aligned}$$

But $C = 180^\circ - (A+B) \Rightarrow \cos C = \cos(180^\circ - (A+B)) = \cos 180^\circ \cos(A+B) + \sin 180^\circ \sin(A+B)$

$$\begin{aligned}\text{L.H.S} &= 2 \sin C [\cos(A-B) - \cos(A+B)] \\ &= 2 \sin C \left[-2 \sin\left(\frac{A+B+A-B}{2}\right) \sin\left(\frac{A+B-A+B}{2}\right) \right] \\ &= 2 \sin C [-2 \sin A \sin(-B)] = 4 \sin A \sin B \sin C\end{aligned}$$

d. *left as an exercise*

e. $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$

L.H.S: $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = \frac{\cos A \sin A + \cos B \sin B + \cos C \sin C}{\sin A \sin B \sin C}$

Multiplying Numerator and Denominator by 2 to create double angles of sine ratios

L.H.S $= \frac{2 \cos A \sin A + 2 \cos B \sin B + 2 \cos C \sin C}{2 \sin A \sin B \sin C} = \frac{\sin 2A + \sin 2B + \sin 2C}{2 \sin A \sin B \sin C}$

Simplifying Numerator just like in (c) above, then $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

$\therefore \text{L.H.S} = \frac{4 \sin A \sin B \sin C}{2 \sin A \sin B \sin C} = 2$

f. $\sin(B+C-A) + \sin(C+A-B) - \sin(A+B-C) = 4 \cos A \cos B \sin C$

From $A+B+C = 180^\circ \Rightarrow A+B = 180^\circ - C, C+A = 180^\circ - B, B+C = 180^\circ - A$

$\sin(B+C-A) = \sin(180^\circ - 2A) = \sin 180^\circ \cos 2A - \cos 180^\circ \sin 2A = \sin 2A$

$\sin(C+A-B) = \sin(180^\circ - 2B) = \sin 180^\circ \cos 2B - \cos 180^\circ \sin 2B = \sin 2B$

$\sin(A+B-C) = \sin(180^\circ - 2C) = \sin 180^\circ \cos 2C - \cos 180^\circ \sin 2C = \sin 2C$

$\therefore \text{L.H.S} = \sin 2A + \sin 2B - \sin 2C = 2 \sin(A+B) \cos(A-B) - 2 \cos C \sin C$

But $\sin(A+B) = \sin(180^\circ - C) = \sin 180^\circ \cos C - \cos 180^\circ \sin C$
 $= \sin C$

Also $C = 180^\circ - (A+B)$

$\Rightarrow \cos C = \cos(180^\circ - (A+B)) = \cos 180^\circ \cos(A+B) + \sin 180^\circ \sin(A+B)$
 $= -\cos(A+B)$

$$\begin{aligned}\therefore \sin 2A + \sin 2B - \sin 2C &= 2 \sin C [\cos(A-B) + \cos(A+B)] \\ &= 2 \sin C [2 \cos A \cos(-B)] \\ &= 4 \cos A \cos B \sin C\end{aligned}$$

g. *left as an exercise*

h. $\cos 4A - \cos 4B - \cos 4C = 4\sin 2B \sin 2C \cos 2A - 1$

$$\mathbf{L.H.S} = -2\sin(2A + 2B)\sin(2A - 2B) + (\cos 0 - \cos 4A) - 1 [\because \cos 0 - 1 = 1 - 1]$$

$$\text{But } A + B + C = 180^\circ \Rightarrow 2A + 2B = 360^\circ - 2C$$

$$\sin(2A + 2B) = \sin(360^\circ - 2C) = \sin 360^\circ \cos 2C - \cos 360^\circ \sin 2C = -\sin 2C$$

$$\mathbf{L.H.S} = -[-2\sin 2C \sin(2A - 2B)] + -2\sin 2C \sin(-2C) - 1$$

$$= 2\sin 2C [\sin(2A - 2B) + \sin 2C] - 1 = 2\sin 2C [(\sin(2A - 2B) + \sin 2C)] - 1$$

$$= 2\sin 2C [2\sin(A - B + C)\cos(A - B - C)] - 1 = 2\sin 2C [2\sin(180^\circ - 2B)\cos(2A - 180^\circ)]$$

$$\text{But } \sin(180^\circ - 2B) = \sin 180^\circ \cos 2B + \cos 180^\circ \sin 2B = -\sin 2B,$$

$$\cos(2A - 180^\circ) = \cos 2A \cos 180^\circ - \sin 180^\circ \sin 2A = -\cos 2A$$

$$\therefore \mathbf{L.H.S} = 2\sin 2C [-2\sin 2B \cdot -\cos 2A] - 1 = 4\sin 2B \sin 2C \cos 2A - 1$$

i. $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$

$$\mathbf{L.H.S} = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos C + (\cos 0 - 1)$$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + (\cos 0 - \cos C) - 1$$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \left[-2\sin\frac{C}{2}\sin\left(-\frac{C}{2}\right)\right] - 1$$

$$= 2\cos\left(90^\circ - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\sin\frac{C}{2} - 1$$

$$= 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\sin\frac{C}{2} - 1$$

$$= 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}\right] - 1, \text{ but } \sin\frac{C}{2} = \sin\left(90^\circ - \frac{A+B}{2}\right)$$

$$= \cos\left(\frac{A+B}{2}\right)$$

$$\mathbf{L.H.S} = 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right] - 1$$

$$= 2\sin\frac{C}{2}\left[2\cos\left(\frac{\frac{A-B}{2} + \frac{A+B}{2}}{2}\right)\cos\left(\frac{\frac{A-B}{2} - \frac{A+B}{2}}{2}\right)\right]$$

$$= 2\sin\frac{C}{2}\left[2\cos\frac{A}{2}\cos\left(-\frac{B}{2}\right)\right] = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

j. $\frac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1} = 8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

$$\mathbf{L.H.S} = \frac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1} = \frac{4\sin A \sin B \sin C}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \left[\begin{array}{l} \text{proof of } \sin 2A + \sin 2B + \sin 2C \\ \text{and } \cos A + \cos B + \cos C - 1 \text{ are} \\ \text{left for the students or refer to} \\ \text{(c) and (a) above} \end{array} \right]$$

$$\mathbf{L.H.S} = \frac{4\left[2\sin\frac{A}{2}\cos\frac{A}{2}\right]\left[2\sin\frac{B}{2}\cos\frac{B}{2}\right]\left[2\sin\frac{C}{2}\cos\frac{C}{2}\right]}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = 8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

k. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$A + B + C = 180^\circ \text{ or } A + B = 180^\circ - C, \text{ Now introducing tangent ratio}$$

$$\tan(A + B) = \tan(180^\circ - C) \Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan 180^\circ - \tan C}{1 + \tan 180^\circ \tan C} = -\tan C$$

$$[\because \tan 180^\circ = 0]$$

$$\tan A + \tan B = -\tan C + \tan A \tan B \tan C \therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

1. $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

Again

$$A + B + C = 180^\circ \text{ or } \frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2}, \text{ Now introducing cotangent ratio}$$

$$\cot \left(\frac{A}{2} + \frac{B}{2} \right) = \cot \left(90^\circ - \frac{C}{2} \right) \Rightarrow \frac{1}{\tan \left(\frac{A}{2} + \frac{B}{2} \right)} = \frac{\cos \left(90^\circ - \frac{C}{2} \right)}{\sin \left(90^\circ - \frac{C}{2} \right)}$$

$$\Rightarrow \frac{1 - \tan \frac{A}{2} \tan \frac{B}{2}}{\tan \frac{A}{2} + \tan \frac{B}{2}} = \frac{\cos 90^\circ \cos \frac{C}{2} + \sin 90^\circ \sin \frac{C}{2}}{\sin 90^\circ \cos \frac{C}{2} - \cos 90^\circ \sin \frac{C}{2}} = \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} = \tan \frac{C}{2}$$

$$\Rightarrow 1 - \tan \frac{A}{2} \tan \frac{B}{2} = \tan \frac{A}{2} \tan \frac{C}{2} + \tan \frac{B}{2} \tan \frac{C}{2} \text{ or } \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{C}{2} \tan \frac{B}{2} + \tan \frac{A}{2} \tan \frac{C}{2} = 1$$

Dividing through by $\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$

$$\Rightarrow \frac{\tan \frac{A}{2} \tan \frac{B}{2}}{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} + \frac{\tan \frac{C}{2} \tan \frac{B}{2}}{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} + \frac{\tan \frac{A}{2} \tan \frac{C}{2}}{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} = \frac{1}{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}$$

$$\Rightarrow \frac{1}{\tan \frac{C}{2}} + \frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{B}{2}} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \therefore$$

$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

m. proceed as in (l) above. Left as an exercise. HINT: Introduce $\tan$ /

$$\cot \text{ on } \frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2}$$

n. $\frac{\cot A + \cot B}{\tan A + \tan B} + \frac{\cot B + \cot C}{\tan B + \tan C} + \frac{\cot C + \cot A}{\tan C + \tan A} = 1$

$$\begin{aligned} \text{L. H. S} &= \frac{\frac{1}{\tan A} + \frac{1}{\tan B}}{\tan A + \tan B} + \frac{\frac{1}{\tan B} + \frac{1}{\tan C}}{\tan B + \tan C} + \frac{\frac{1}{\tan C} + \frac{1}{\tan A}}{\tan C + \tan A} \\ &= \frac{\frac{\tan A + \tan B}{\tan A \tan B}}{\tan A + \tan B} + \frac{\frac{\tan B + \tan C}{\tan B \tan C}}{\tan B + \tan C} + \frac{\frac{\tan A + \tan C}{\tan A \tan C}}{\tan A + \tan C} \\ &= \frac{\tan A \tan B \tan C}{\tan A \tan B} + \frac{\tan A \tan B \tan C}{\tan B \tan C} + \frac{\tan A \tan B \tan C}{\tan A \tan C} \\ &= \frac{1}{\tan A \tan B} + \frac{1}{\tan B \tan C} + \frac{1}{\tan A \tan C} \dots \dots \dots (1) \end{aligned}$$

From $A + B + C = 180^\circ$ or $A + B = 180^\circ - C$, Now introducing tangent ratio

$$\tan(A + B) = \tan(180^\circ - C)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan 180^\circ - \tan C}{1 + \tan 180^\circ \tan C} = -\tan C \quad [\because \tan 180^\circ = 0]$$

$$\tan A + \tan B = -\tan C + \tan A \tan B \tan C \therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Dividing through by $\tan A \tan B \tan C$

$$\Rightarrow \frac{1}{\tan B \tan C} + \frac{1}{\tan A \tan C} + \frac{1}{\tan A \tan B} = 1 \dots \dots \dots (2)$$

$\therefore$ (2) in to (1), **L. H. S = R. H. S**

2. If A, B , and C are angles of a triangle, prove that

a. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \sin C$

b. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$

c. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$

d. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \sin C$

$$\begin{aligned}
 e. \quad & \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2\cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} \\
 f. \quad & \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\
 g. \quad & \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = 2\cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} \\
 h. \quad & \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 \left(1 + \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)
 \end{aligned}$$

Solution

When proving these identities, it is convenient to use half or double angle identities

$$i. e \sin^2 A = \frac{1 - \cos 2A}{2} \text{ and } \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$a. \sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \sin C$$

$$\text{L. H. S} = \frac{1 - \cos 2A}{2} + \frac{1 - \cos 2B}{2} - \frac{1 - \cos 2C}{2} = \frac{1}{2} [1 - (\cos 2A + \cos 2B - \cos 2C)]$$

$$= \frac{1}{2} [1 - (2\cos(A+B)\cos(A-B) - \cos 2C)]$$

$$= \frac{1}{2} [1 - (2\cos(180^\circ - C)\cos(A-B) - \cos 2C)]$$

$$= \frac{1}{2} [1 - (-2\cos C \cdot \cos(A-B) - (2\cos^2 C - 1))] = \frac{1}{2} [1 + 2\cos C \cdot \cos(A-B) + 2\cos^2 C - 1]$$

$$= \frac{1}{2} [1 + 2\cos C \cdot \cos(A-B) + 2\cos^2 C - 1]$$

$$= \cos C \cdot [\cos(A-B) + \cos C], \text{ but } \cos C = \cos[180^\circ - (A+B)]$$

$$\therefore \sin^2 A + \sin^2 B - \sin^2 C = \cos C \cdot [\cos(A-B) - \cos(A+B)]$$

$$= \cos C \cdot [-2\sin A \sin(-B)] = 2\sin A \sin B \sin C = \text{R. H. S}$$

$$b. \sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$$

$$\text{L. H. S} = \frac{1 - \cos 2A}{2} + \frac{1 - \cos 2B}{2} + \frac{1 - \cos 2C}{2} = \frac{1}{2} [3 - (\cos 2A + \cos 2B + \cos 2C)]$$

$$= \frac{1}{2} [3 - (2\cos(A+B)\cos(A-B) + \cos 2C)]$$

$$= \frac{1}{2} [3 - (2\cos(180^\circ - C)\cos(A-B) + \cos 2C)]$$

$$= \frac{1}{2} [3 - (-2\cos C \cdot \cos(A-B) + (2\cos^2 C - 1))] = \frac{1}{2} [3 + 2\cos C \cdot \cos(A-B) - 2\cos^2 C + 1]$$

$$= \frac{1}{2} [3 + 2\cos C \cdot \cos(A-B) - 2\cos^2 C + 1]$$

$$= 2 + \cos C \cdot [\cos(A-B) - \cos C], \text{ but } \cos C = \cos[180^\circ - (A+B)]$$

$$\therefore \sin^2 A + \sin^2 B + \sin^2 C = \cos C \cdot [\cos(A-B) + \cos(A+B)]$$

$$= 2 + \cos C [2\cos A \cos B] = 2 + 2\cos A \cos B \cos C = \text{R. H. S}$$

$$c. \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$$

$$\text{L. H. S} = \frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2} = \frac{1}{2} [3 + \cos 2A + \cos 2B + \cos 2C]$$

$$= \frac{1}{2} [3 + (2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1)]$$

$$= \frac{1}{2} [3 + (2\cos(180^\circ - C)\cos(A-B) + 2\cos^2 C - 1)]$$

$$= \frac{1}{2} [2 + (2\cos(180^\circ - C)\cos(A-B) + 2\cos^2 C)]$$

$$= 1 + -\cos C \cos(A-B) + \cos^2 C = 1 + \cos C [\cos C - \cos(A-B)]$$

$$= 1 + \cos C [\cos(180^\circ - (A+B)) - \cos(A-B)]$$

$$= 1 - \cos C [\cos(A+B) + \cos(A-B)]$$

$$= 1 - \cos C [2\cos A \cos(-B)] = 1 - 2\cos A \cos B \cos C = \text{R. H. S}$$

d. Left as an exercise

e. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2\cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$

$$\begin{aligned} \text{L.H.S} &= \frac{1-\cos A}{2} + \frac{1-\cos B}{2} - \frac{1-\cos C}{2} = \frac{1}{2}[1 - (\cos A + \cos B - \cos C)] \\ &= \frac{1}{2}\left[1 - \left(2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos C\right)\right] \\ &= \frac{1}{2}\left[1 - \left(2\cos\left(90^\circ - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos C\right)\right] \\ &= \frac{1}{2}\left[1 - \left(2\sin \frac{C}{2}\cos\left(\frac{A-B}{2}\right)\right) + \left(1 - 2\sin^2 \frac{C}{2}\right)\right] \\ &= 1 - \sin \frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin \frac{C}{2}\right] \\ &= 1 - \sin \frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\left[90^\circ - \frac{(A+B)}{2}\right]\right] \\ &= 1 - \sin \frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right] = 1 - \sin \frac{C}{2}\left[2\cos \frac{A}{2}\cos\left(-\frac{B}{2}\right)\right] \\ &= 1 - 2\cos \frac{A}{2}\cos \frac{B}{2}\sin \frac{C}{2} = \text{R.H.S} \end{aligned}$$

f. (f), (g) and (h) left as an exercise

Task

- 1) If $A + B + C = 180^\circ$, prove that $\frac{\sin(B+C) + \sin(A+C)\sin(A+B)}{\cos\left(\frac{\pi}{2}-B\right)\cos\left(\frac{3\pi}{2}-C\right)\left(\frac{2\pi}{2}-A\right)} = 1$
- 2) If in a $\triangle ABC$, $\sin 2A = \sin 2B + \sin 2C$, prove that either B or C is a right angle

Conditional proofs in factor formulae

1. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, prove that $\tan\left(\frac{A-B}{2}\right) = \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$
2. Given that $\sin x + \sin y = \lambda_1$, and $\cos x + \cos y = \lambda_2$, show that

$$\begin{aligned} \text{I. } \tan\left(\frac{x+y}{2}\right) &= \frac{\lambda_1}{\lambda_2} \\ \text{II. } \cos(x+y) &= \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2} \quad (\text{uneb1994}) \end{aligned}$$

3. If $\sin \theta + \sin 2\theta = a$ and $\cos \theta + \cos 2\theta = b$, prove that $(a^2 + b^2)(a^2 + b^2 - 3) = 2b$

Solution

1. $\sin A + \sin B = a$, $\cos A + \cos B = b$, To prove that $\tan\left(\frac{A-B}{2}\right) = \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

$$\text{from R.H.S } \sqrt{\frac{4-a^2-b^2}{a^2+b^2}} = \sqrt{\frac{4-(a^2+b^2)}{a^2+b^2}}$$

$$\begin{aligned} a^2 + b^2 &= (\sin A + \sin B)^2 + (\cos A + \cos B)^2 \\ &= (\sin^2 A + 2\sin A \sin B + \sin^2 B) + (\cos^2 A + 2\cos A \cos B + \cos^2 B) \\ &= (\sin^2 A + \cos^2 A) + (\sin^2 B + \cos^2 B) + 2(\sin A \sin B + \cos A \cos B) \\ &= 1 + 1 + 2\cos(A-B) = 2 + 2\cos(A-B) \end{aligned}$$

$$\therefore \sqrt{\frac{4-a^2-b^2}{a^2+b^2}} = \sqrt{\frac{4-[2+2\cos(A-B)]}{2+2\cos(A-B)}} = \sqrt{\frac{2-2\cos(A-B)}{2+2\cos(A-B)}} = \sqrt{\frac{1-\cos(A-B)}{1+\cos(A-B)}}, \text{ now let } A-B = \theta$$

$$\Rightarrow \sqrt{\frac{1-\cos(A-B)}{1+\cos(A-B)}} = \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} = \sqrt{\frac{2\sin^2 \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2}}} = \tan \frac{\theta}{2}$$

$$\therefore \sqrt{\frac{4-a^2-b^2}{a^2+b^2}} = \tan\left(\frac{A-B}{2}\right)$$

2. $\sin x + \sin y = \lambda_1$, and $\cos x + \cos y = \lambda_2$, To show that $\tan\left(\frac{x+y}{2}\right) = \frac{\lambda_1}{\lambda_2}$

$$\text{R.H.S: } \frac{\lambda_1}{\lambda_2} = \frac{\sin x + \sin y}{\cos x + \cos y} = \frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = \frac{\sin\left(\frac{x+y}{2}\right)}{\cos\left(\frac{x+y}{2}\right)} = \tan\left(\frac{x+y}{2}\right)$$

$$\text{To show that } \cos(x+y) = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2}$$

$$\begin{aligned} \text{R.H.S: } \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2 + \lambda_1^2} &= \frac{(\cos x + \cos y)^2 - (\sin x + \sin y)^2}{(\cos x + \cos y)^2 + (\sin x + \sin y)^2} \\ &= \frac{(\cos^2 x + 2\cos x \cos y + \cos^2 y) - (\sin^2 x + 2\sin x \sin y + \sin^2 y)}{(\cos^2 x + 2\cos x \cos y + \cos^2 y) + (\sin^2 x + 2\sin x \sin y + \sin^2 y)} \\ &= \frac{(\cos^2 x - \sin^2 x) + (\cos^2 y - \sin^2 y) + 2(\cos x \cos y - \sin x \sin y)}{1 + 1 + 2\cos(x-y)} = \frac{\cos 2x + \cos 2y + 2\cos(x+y)}{2 + 2\cos(x-y)} \\ &= \frac{2\cos(x+y)\cos(x-y) + 2\cos(x+y)}{2 + 2\cos(x-y)} = \frac{\cos(x+y)[\cos(x-y) + 1]}{[\cos(x-y) + 1]} = \cos(x+y) \end{aligned}$$

3. Left as an exercise.

FINDING ANGLES

(The inverse notation)

A given trig equation may have *infinite number* of solutions if no restriction for the angles is done. e.g. $-\pi$ to $+\pi$ or 0 to 2π e.t.c

When the solution is infinitely stated, the solution is called *general solution*.

Example

1. Solve

$$\cos x = \frac{\sqrt{2}}{2}$$

Taking $\frac{\sqrt{2}}{2} = +ve$ value, then

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ = \frac{\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{4} \pm 2\pi n, -\frac{\pi}{4} \pm 2\pi n$$

$$\therefore \left\{x: x = \frac{\pi}{4} \pm 2\pi n \text{ or } -\frac{\pi}{4} \pm 2\pi n\right\}, \text{ where } n \in \mathbb{Z}. \text{ i.e, } n \text{ is an integer.}$$

Thus this infinite set is denoted as

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \text{ or } \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \pm 2\pi n \text{ or } -\frac{\pi}{4} \pm 2\pi n$$

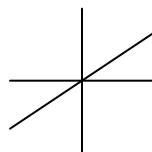
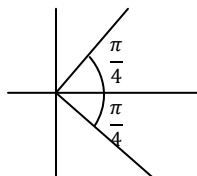
$$\text{NOTE: } \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \neq \left[\cos\left(\frac{\sqrt{2}}{2}\right)\right]^{-1}$$

2. Solve

$$\tan x = \frac{\sqrt{3}}{2}$$

Taking $\frac{\sqrt{3}}{2} = +ve$ value, then

$$\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ = \frac{\pi}{6}$$



$$\Rightarrow x = \frac{\pi}{6} \pm \pi n$$

$$\therefore \left\{ x: x = \frac{\pi}{6} \pm \pi n \right\}$$

Thus this infinite set is denoted as

$$\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \text{ or } \arctan\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \pm \pi n$$

3. Solve

$$\sin x = \frac{1}{\sqrt{2}}$$

Taking $\frac{1}{\sqrt{2}} = +ve \text{ value}$, then

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ = \frac{\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{4} + 2\pi n \text{ or } \left(\pi - \frac{\pi}{4}\right) + 2\pi n = (2n + 1)\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{\pi}{4} + \pi n, \text{ for } n \text{ is even or } -\frac{\pi}{4} + \pi n \text{ for } n \text{ is odd}$$

$$\therefore \left\{ x: \pi n + (-1)^n \frac{\pi}{4} \text{ for } n \in \mathbb{Z} \right\}$$

The solution set is denoted as

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \text{ or } \arcsin\left(\frac{1}{\sqrt{2}}\right) = \pi n + (-1)^n \frac{\pi}{4}$$

In general, the key to identifying all members of

a set like $\sin^{-1}(x) = y \text{ or } \cot^{-1}(x) = y$ is to find the real members of y

such that $0 \leq y \leq 2\pi \text{ or } 0 \leq y \leq 360^\circ$ and then using $2\pi n \text{ or } \pi n$, period for the function to determine the remaining members

Example

1. Evaluate

i) $\sin\left(\arccos\frac{\sqrt{2}}{2}\right)$

ii) $\sec(\arctan(1))$

iii) $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$

iv) $\sin\left(\arcsin\left(\frac{4}{5}\right)\right)$

v) $\sec\left(\arcsin\left(-\frac{7}{25}\right)\right)$

vi) $\cot\left(\operatorname{arccosec}\left(\frac{25}{24}\right)\right)$

vii) $\cos(\operatorname{arccot}(-4))$

Solution

i) $\sin\left(\arccos\frac{\sqrt{2}}{2}\right) = \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Dealing with the bracket

Let $x = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \text{ or } \cos x = \frac{\sqrt{2}}{2}$

Now solving $\cos x = \frac{\sqrt{2}}{2}$ in general solution

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ = \frac{\pi}{4}$$

$$\therefore x = \frac{\pi}{4} \pm 2\pi n, -\frac{\pi}{4} \pm 2\pi n$$

$$\begin{aligned} \therefore \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) &= \sin\left(\frac{\pi}{4} \pm 2\pi n\right) \text{ or } \sin\left(-\frac{\pi}{4} \pm 2\pi n\right) \\ &= \sin\left(\frac{\pi}{4}\right) \text{ or } \sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \text{ or } -\frac{\sqrt{2}}{2} \end{aligned}$$

Alternatively

$$\sin\left(\arccos\frac{\sqrt{2}}{2}\right) = \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$$

Dealing with the bracket, and taking $\frac{\sqrt{2}}{2} = +ve$

$$\text{Let } x = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \text{ or } \cos x = \frac{\sqrt{2}}{2}$$

$$\therefore \text{In } 1^{\text{st}} \text{ quadrant, } \sin x = +\frac{\sqrt{2}}{2}$$

$$\therefore \text{In } 4^{\text{th}} \text{ quadrant, } \sin x = -\frac{\sqrt{2}}{2}$$

ii) $\sec(\arctan(1))$

$$\sec(\arctan(1)) = \sec(\tan^{-1}(1))$$

$$\text{Let } \tan^{-1}(1) = \theta \text{ or } \tan \theta = 1$$

$$\Rightarrow \sec(\tan^{-1}(1)) = \sec \theta = \frac{1}{\cos \theta}$$

$$\text{In } 1^{\text{st}} \text{ quadrant } \cos \theta = \frac{1}{\sqrt{2}} \therefore \sec \theta = \sqrt{2}$$

$$\text{In } 3^{\text{rd}} \text{ quadrant } \cos \theta = -\frac{1}{\sqrt{2}} \therefore \sec \theta = -\sqrt{2}$$

$$\therefore \sec(\arctan(1)) = \pm \sqrt{2}$$

iii) $\sin\left(\arccos\left(\frac{5}{13}\right)\right) = \sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

$$\text{Let } \cos^{-1}\left(\frac{5}{13}\right) = \theta \text{ or } \cos \theta = \frac{5}{13}$$

$$\therefore \sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin \theta$$

$$\text{In } 1^{\text{st}} \text{ quadrant, } \sin \theta = \frac{12}{13}$$

$$\text{In } 4^{\text{th}} \text{ quadrant, } \sin \theta = -\frac{12}{13}$$

$$\therefore \sin\left(\arccos\left(\frac{5}{13}\right)\right) = \pm \frac{12}{13}$$

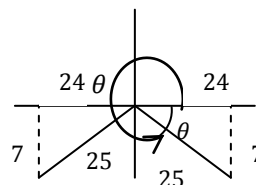
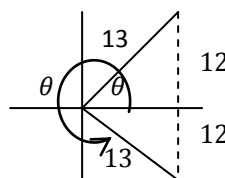
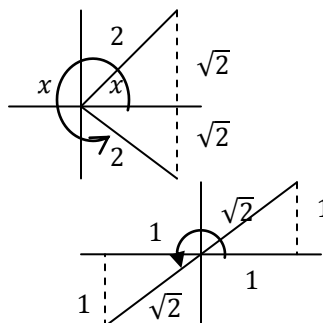
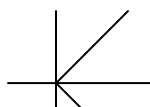
iv) Left as an exercise **Answer:** $\frac{4}{5}$

v) $\sec\left(\arcsin\left(-\frac{7}{25}\right)\right) = \sec\left(\sin^{-1}\left(-\frac{7}{25}\right)\right)$

$$\text{Let } \sin^{-1}\left(-\frac{7}{25}\right) = \theta \text{ or } \sin \theta = -\frac{7}{25} \Rightarrow \sec \theta = \frac{1}{\cos \theta}$$

$$\text{In } 3^{\text{rd}} \text{ quadrant, } \cos \theta = -\frac{24}{25} \therefore \sec \theta = -\frac{25}{24}$$

$$\text{In } 4^{\text{th}} \text{ quadrant, } \cos \theta = +\frac{24}{25} \therefore \sec \theta = +\frac{25}{24}$$



$$\text{vi) } \cot \left(\operatorname{arc cosec} \left(\frac{25}{24} \right) \right) = \cot \left(\operatorname{cosec}^{-1} \left(\frac{25}{24} \right) \right)$$

$$\text{Let } \operatorname{cosec}^{-1} \left(\frac{25}{24} \right) = \theta \text{ or } \operatorname{cosec} \theta = \frac{25}{24}$$

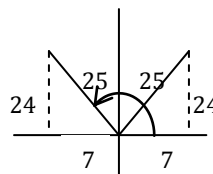
$$\text{But } \operatorname{cosec} \theta = \frac{1}{\sin \theta} \Rightarrow \sin \theta = \frac{24}{25}$$

$$\therefore \cot \left(\operatorname{arc cosec} \left(\frac{25}{24} \right) \right) = \cot \theta = \frac{1}{\tan \theta}$$

$$\text{In } 1^{\text{st}} \text{ quadrant, } \tan \theta = \frac{24}{7} \therefore \cot \theta = \frac{7}{24}$$

$$\text{In } 2^{\text{nd}} \text{ quadrant, } \tan \theta = -\frac{24}{7} \therefore \cot \theta = -\frac{7}{24}$$

$$\therefore \cot \left(\operatorname{arc cosec} \left(\frac{25}{24} \right) \right) = \pm \frac{7}{24}$$



vii) Left as an exercise

2. Evaluate

$$\text{i) } \cos \left(\operatorname{arc sin} \left(\frac{3}{5} \right) + \operatorname{arc tan} \left(\frac{12}{13} \right) \right)$$

$$\text{ii) } \sin \left(\operatorname{arc cos} \left(\frac{3}{5} \right) + \operatorname{arc sec}(3) \right)$$

$$\text{iii) } \sin \left(2 \operatorname{arc sin} \left(\frac{4}{5} \right) \right)$$

$$\text{iv) } \cos \left(2 \operatorname{arc cot} \left(\frac{5}{13} \right) \right)$$

$$\text{v) } \tan \left(\frac{1}{2} \operatorname{arc cos} \left(\frac{1}{2} \right) \right)$$

$$\text{vi) } -2 \sin(\operatorname{arc tan}(1)) \cdot \sin(\operatorname{arc tan}(\sqrt{3}))$$

$$\text{vii) } 2 \cos(\operatorname{arc sec}(2)).$$

Solution

$$\text{i) } \cos \left(\operatorname{arc sin} \left(\frac{3}{5} \right) + \operatorname{arc tan} \left(\frac{12}{13} \right) \right) = \cos \left(\sin^{-1} \left(\frac{3}{5} \right) + \tan^{-1} \left(\frac{12}{13} \right) \right)$$

$$\text{Let } \sin^{-1} \left(\frac{3}{5} \right) = \alpha \Rightarrow \sin \alpha = \frac{3}{5} \text{ and } \tan^{-1} \left(\frac{12}{13} \right) = \beta \Rightarrow \tan \beta = \frac{12}{13}$$

$$\therefore \cos \left(\sin^{-1} \left(\frac{3}{5} \right) + \tan^{-1} \left(\frac{12}{13} \right) \right) = \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\text{For } \sin \alpha = \frac{3}{5}$$

$$\text{In } 1^{\text{st}} \text{ quadrant } \cos \alpha = +\frac{4}{5}$$

$$\text{In } 2^{\text{nd}} \text{ quadrant } \cos \alpha = -\frac{4}{5}$$

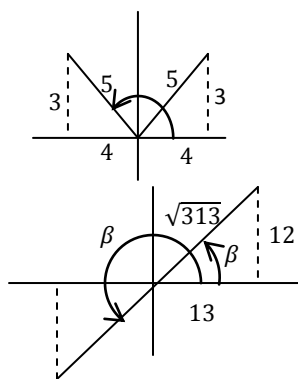
$$\text{For } \tan \beta = \frac{12}{13}$$

$$\text{In } 1^{\text{st}} \text{ quadrant } \cos \beta = +\frac{13}{\sqrt{313}}$$

$$\text{In } 3^{\text{rd}} \text{ quadrant } \cos \beta = -\frac{13}{\sqrt{313}}$$

$$\begin{aligned} &\Rightarrow \cos \left(\sin^{-1} \left(\frac{3}{5} \right) + \tan^{-1} \left(\frac{12}{13} \right) \right) \\ &= \pm \frac{4}{5} \cdot \pm \frac{13}{\sqrt{313}} - \frac{3}{5} \cdot \pm \frac{12}{\sqrt{313}} = \pm \frac{52}{5\sqrt{313}} \pm \frac{36}{5\sqrt{313}} \end{aligned}$$

$$\text{ii) } \sin \left(\operatorname{arc cos} \left(\frac{3}{5} \right) + \operatorname{arc sec}(3) \right) = \sin \left(\cos^{-1} \left(\frac{3}{5} \right) + \sec^{-1}(3) \right)$$



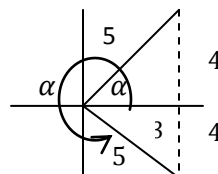
Let $\cos^{-1}\left(\frac{3}{5}\right) = \alpha \Rightarrow \cos \alpha = \frac{3}{5}$ and $\sec^{-1}(3) = \beta \Rightarrow \sec \beta = 3, \sec \beta = \frac{1}{\cos \beta} \therefore$

$$\cos \beta = \frac{1}{3}$$

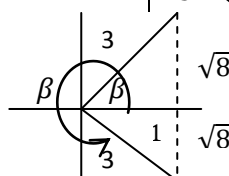
$$\Rightarrow \sin\left(\arccos\left(\frac{3}{5}\right) + \operatorname{arcsec}(3)\right) = \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\text{For } \cos \alpha = \frac{3}{5} \Rightarrow \sin \alpha = \pm \frac{4}{5}$$

(In 1st and 4th quadrant)



$$\text{For } \cos \beta = \frac{1}{3} \Rightarrow \sin \beta = \pm \frac{2}{3}\sqrt{2}$$



$$\therefore \sin\left(\arccos\left(\frac{3}{5}\right) + \operatorname{arcsec}(3)\right) = \pm \frac{4}{5} \cdot \frac{1}{3} + \frac{1}{3} \cdot \pm \frac{2}{3}\sqrt{2} = \pm \frac{4}{15} \pm \frac{2}{3}\sqrt{2}$$

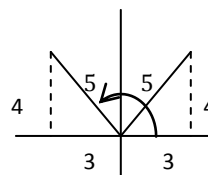
$$\text{iii) } \sin\left(2 \arcsin\left(\frac{4}{5}\right)\right) = \sin\left(2 \sin^{-1}\left(\frac{4}{5}\right)\right)$$

$$\text{Let } \theta = \sin^{-1}\left(\frac{4}{5}\right) \Rightarrow \sin \theta = \frac{4}{5}$$

$$\cos \theta = \pm \frac{3}{5}$$

$$\Rightarrow \sin\left(2 \arcsin\left(\frac{4}{5}\right)\right) = \sin 2\theta = 2 \sin \theta \cos \theta$$

$$= 2 \cdot \pm \frac{3}{5} \cdot \frac{4}{5} = \pm \frac{24}{25}$$



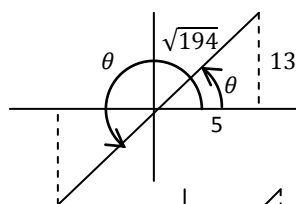
$$\text{iv) } \cos\left(2 \operatorname{arccot}\left(\frac{5}{13}\right)\right) = \cos\left(2 \cot^{-1}\left(\frac{5}{13}\right)\right)$$

$$\text{Let } \theta = \cot^{-1}\left(\frac{5}{13}\right) \Rightarrow \cot \theta = \frac{5}{13} \text{ or } \tan \theta = \frac{13}{5}$$

$$\cos \theta = \pm \frac{5}{\sqrt{194}}$$

$$\therefore \cos\left(2 \operatorname{arccot}\left(\frac{5}{13}\right)\right) = \cos 2\theta = 2\cos^2 \theta - 1$$

$$= 2\left(\pm \frac{5}{\sqrt{194}}\right)^2 - 1 = -\frac{72}{97}$$



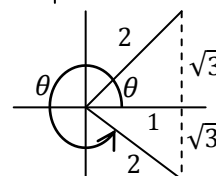
$$\text{v) } \tan\left(\frac{1}{2} \arccos\left(\frac{1}{2}\right)\right) = \tan\left(\frac{1}{2} \cos^{-1}\left(\frac{1}{2}\right)\right)$$

$$\text{Let } \theta = \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \tan\left(\frac{1}{2} \arccos\left(\frac{1}{2}\right)\right) = \tan \frac{1}{2} \theta$$

$$= \pm \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \quad [\text{see half angle identities}]$$

$$= \pm \sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}} = \pm \frac{1}{\sqrt{3}}$$



$$\text{vi) } -2 \sin(\arctan(1)) \cdot \sin(\arctan(\sqrt{3})) = -2 \sin(\tan^{-1}(1)) \cdot \sin(\tan^{-1}(\sqrt{3}))$$

Let $\tan^{-1}(1) = \alpha \Rightarrow \tan \alpha = 1$ and $\tan^{-1}(\sqrt{3}) = \beta \Rightarrow \tan \beta = \sqrt{3}$

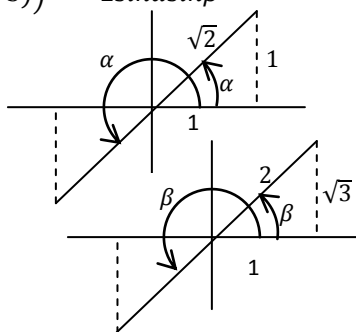
$$\Rightarrow -2\sin(\arctan(1)) \cdot \sin(\arctan(\sqrt{3})) = -2\sin \alpha \sin \beta$$

For $\tan \alpha = 1$

$$\Rightarrow \sin \alpha = \pm \frac{1}{\sqrt{2}}$$

For $\tan \beta = \sqrt{3}$

$$\Rightarrow \sin \beta = \pm \frac{\sqrt{3}}{2}$$



$$\therefore -2\sin(\arctan(1)) \cdot \sin(\arctan(\sqrt{3})) = -2 \cdot \pm \frac{1}{2} \cdot \pm \frac{\sqrt{3}}{2} = \pm \frac{\sqrt{6}}{2}$$

vii) Left as an exercise

NOTE:

In the above examples, the angles in **TWO** quadrants have been used in the evaluation of trigonometric functions. Now let us look at inverse of trigonometric functions requiring only the *principal value*, the *numerically smallest value*.

Conventional definitions

1.

The unique number y such that $y \in \sin^{-1} x$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, is called *the principal value of arc sin x* and is denoted as $y = \text{Arc sin } x$.

The capital "A" denotes the Principal value.

The Arc sin function is

$$\{(x, y): y = \text{Arc sin } x, |x| \leq 1, x \in \mathbb{R}\}$$

2.

The principal value of Arc cos x, $-1 \leq x \leq +1$ is the unique number $y \in \cos^{-1} x$, $0 \leq y \leq \frac{\pi}{2}$. the principal value is denoted *Arc cos x*.

The Arc cos function is

$$\{(x, y): y = \text{Arc cos } x, |x| \leq 1, x \in \mathbb{R}\}$$

The unique member $y, -\frac{\pi}{2} < y < \frac{\pi}{2}, y \in \tan^{-1} x, x \in \mathbb{R}$, is the *principal value of arc tanx*; it is denoted *Arc tanx*.

The *Arc tan* function is

$$3. \{(x, y): y = \text{Arc tan} x, x \in \mathbb{R}\}$$

The unique member $y, 0 < y < \pi, y \in \cot^{-1} x, x \in \mathbb{R}$, is the principal value of *arc cotx*; denoted *Arc cotx*.

The *Arc cot* function is

$$\{(x, y): y = \text{Arc cot} x, x \in \mathbb{R}\}$$

The *principal values of arc secx and arc cosecx* are denoted by

$$4. \text{Arc sec} x = \text{Arc cos} \frac{1}{x}, |x| \geq 1$$

$$\text{Arc cosec} x = \text{Arc sin} \frac{1}{x}, |x| \geq 1$$

Note:

Much emphasis must be put on the above definitions *i.e* angle ranges for each trigonometric function.

Examples

1. Evaluate

a. $\sin \left[\text{Arc cos} \left(-\frac{7}{25} \right) \right]$

b. $\cos \left[\text{Arc sin} \left(-\frac{1}{5} \right) \right]$

c. $\tan [\text{Arc tan} 4]$

d. $\cot \left[\text{Arc cos} \left(-\frac{11}{61} \right) \right]$

e. $\sec [\text{Arc cosec} 4]$

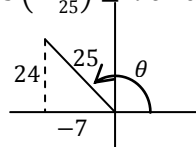
Solution

a. $\sin \left[\text{Arc cos} \left(-\frac{7}{25} \right) \right]$

Let $\text{Arc cos} \left(-\frac{7}{25} \right) = \theta, \cos \theta = -\frac{7}{25}$. Since $0 \leq \text{Arc cos} \left(-\frac{7}{25} \right) \leq \pi$ or $0 \leq \theta \leq \pi$,

Then θ lies in 2nd quadrant

$$\therefore \sin \left[\text{Arc cos} \left(-\frac{7}{25} \right) \right] = \sin \theta = +\frac{24}{25}$$

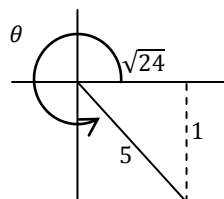


b. $\cos \left[\text{Arc sin} \left(-\frac{1}{5} \right) \right]$

Let $\text{Arc sin} \left(-\frac{1}{5} \right) = \theta, \sin \theta = -\frac{1}{5}$. But $-\frac{\pi}{2} \leq \text{Arc sin} \left(-\frac{1}{5} \right) \leq \frac{\pi}{2}$ or $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,

and since $\sin \theta = -\frac{1}{5}$, then θ lies in 4th quadrant

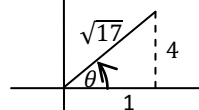
$$\therefore \cos \left[\text{Arc sin} \left(-\frac{1}{5} \right) \right] = \cos \theta = +\frac{\sqrt{24}}{5}$$



c. $\tan [\text{Arc tan} 4]$

Let $\text{Arc tan} 4 = \theta$, $\tan \theta = 4$. But $-\frac{\pi}{2} \leq \text{Arc tan } 4 \leq \frac{\pi}{2}$ or $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, hence θ is a measure in 1st quadrant

$$\therefore \tan[\text{Arc tan} 4] = \tan \theta = +4$$

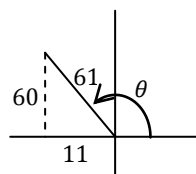


d. $\cot \left[\text{Arc cos} \left(-\frac{11}{61} \right) \right]$

Let $\text{Arc cos} \left(-\frac{11}{61} \right) = \theta$, $\cos \theta = -\frac{11}{61}$. Since $0 \leq \text{Arc cos} \left(-\frac{11}{61} \right) \leq \pi$ or $0 \leq \theta \leq \pi$,

Then θ lies in 2nd quadrant

$$\therefore \cot \left[\text{Arc cos} \left(-\frac{11}{61} \right) \right] = \cot \theta = \frac{1}{\tan \theta} = \frac{1}{\frac{60}{11}} = -\frac{11}{60}$$



e. left as an exercise **Answer:** $\frac{4}{\sqrt{11}}$

2. Evaluate

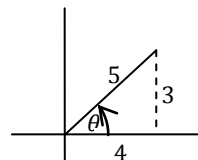
- $\sin \left[\text{Arc cos} \left(\frac{4}{5} \right) + \frac{\pi}{2} \right]$
- $\cos \left[\text{Arc cos} \left(-\frac{1}{2} \right) - \text{Arc sin} \left(\frac{3}{5} \right) \right]$
- $\sin \left[\text{Arc cos} \left(\frac{1}{2} \right) - \text{Arc sin} \left(\frac{5}{13} \right) \right]$
- $\tan \left[\text{Arc sin} \left(\frac{1}{\sqrt{10}} \right) + \text{Arc cos} \left(-\frac{4}{5} \right) \right]$
- $\cos \left[\text{Arc tan} \left(-\frac{4}{3} \right) - \text{Arc sin} \left(\frac{11}{61} \right) \right]$
- $\cot \left[\text{Arc cot} \left(\frac{2}{3} \right) + \text{Arc cot} \left(\frac{1}{3} \right) \right]$
- $\cos \left[\frac{1}{2} \text{Arc sin} \left(-\frac{15}{17} \right) \right]$
- $\tan \left[2 \text{Arc cot} \left(\frac{24}{7} \right) \right]$
- $\tan \left[\frac{1}{2} \text{Arc sec} \left(-\frac{13}{5} \right) \right]$

Solution

a. $\sin \left[\text{Arc cos} \left(\frac{4}{5} \right) + \frac{\pi}{2} \right]$, let $\text{Arc cos} \left(\frac{4}{5} \right) = \theta \Rightarrow \cos \theta = \frac{4}{5}$

Since $0 \leq \text{Arc cos} \left(\frac{4}{5} \right) \leq \pi$ or $0 \leq \theta \leq \pi$, then θ is in the 1st quadrant

$$\begin{aligned} \sin \left[\text{Arc cos} \left(\frac{4}{5} \right) + \frac{\pi}{2} \right] &= \sin \left[\theta + \frac{\pi}{2} \right] \\ &= \sin \theta \cos \frac{\pi}{2} + \cos \theta \sin \frac{\pi}{2} = \cos \theta = +\frac{4}{5} \end{aligned}$$



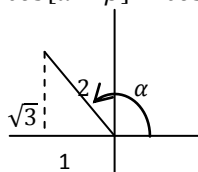
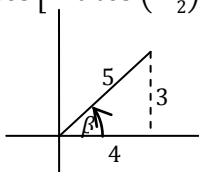
b. $\cos \left[\text{Arc cos} \left(-\frac{1}{2} \right) - \text{Arc sin} \left(\frac{3}{5} \right) \right]$

Let $\text{Arc cos} \left(-\frac{1}{2} \right) = \alpha$, $\cos \alpha = -\frac{1}{2}$ and $\text{Arc sin} \left(\frac{3}{5} \right) = \beta \Rightarrow \sin \beta = \frac{3}{5}$

Since $0 \leq \text{Arc cos} \left(-\frac{1}{2} \right) \leq \pi$, $-\frac{\pi}{2} \leq \text{Arc sin} \left(\frac{3}{5} \right) \leq \frac{\pi}{2}$, then

α lies in 2nd quadrant, β in 1st quadrant

$$\Rightarrow \cos \left[\text{Arc cos} \left(-\frac{1}{2} \right) - \text{Arc sin} \left(\frac{3}{5} \right) \right] = \cos[\alpha - \beta] = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$



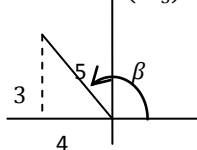
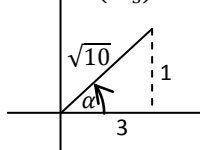
$$= -\frac{1}{2} \times \frac{4}{5} + \frac{\sqrt{3}}{2} \times \frac{3}{5} = \frac{3\sqrt{3}-4}{10}$$

c. Left as an exercise **Ans:** $\frac{12\sqrt{3}-5}{26}$

d. $\tan \left[\text{Arc sin} \left(\frac{1}{\sqrt{10}} \right) + \text{Arc cos} \left(-\frac{4}{5} \right) \right]$

Let $\text{Arc sin} \left(\frac{1}{\sqrt{10}} \right) = \alpha$, $\sin \alpha = \frac{1}{\sqrt{10}}$ for $-\frac{\pi}{2} \leq \text{Arc sin} \left(\frac{1}{\sqrt{10}} \right) \leq \frac{\pi}{2}$,

$\text{Arc cos} \left(-\frac{4}{5} \right) = \beta \Rightarrow \cos \beta = -\frac{4}{5}$ for $0 \leq \text{Arc cos} \left(-\frac{4}{5} \right) \leq \pi$



$$\Rightarrow \tan \left[\text{Arc sin} \left(\frac{1}{\sqrt{10}} \right) + \text{Arc cos} \left(-\frac{4}{5} \right) \right] = \tan[\alpha + \beta] = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{1}{3} \times \frac{3}{4}}{1 - \frac{1}{3} \times \frac{3}{4}} = -\frac{5}{12} \div \frac{15}{12} = -\frac{1}{3}$$

e. Left as an exercise.

Ans: $\frac{136}{305}$

f. Try it out

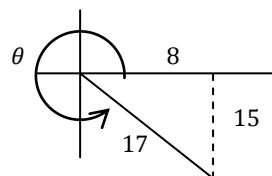
g. $\cos \left[\frac{1}{2} \text{Arc sin} \left(-\frac{15}{17} \right) \right]$, let $\text{Arc sin} \left(-\frac{15}{17} \right) = \theta \Rightarrow \sin \theta = -\frac{15}{17}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\cos \left[\frac{1}{2} \text{Arc sin} \left(-\frac{15}{17} \right) \right] = \cos \frac{1}{2} \theta = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

Since the angle is in the 4th quadrant,

$$\cos \frac{1}{2} \theta = +ve$$

$$\Rightarrow \cos \frac{1}{2} \theta = +\sqrt{\frac{1 + \frac{8}{17}}{2}} = +\frac{25}{\sqrt{34}}$$

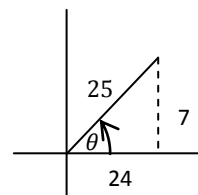


h. $\tan \left[2 \text{Arc cot} \left(\frac{24}{7} \right) \right]$

Let $\text{Arc cot} \left(\frac{24}{7} \right) = \theta \Rightarrow \cot \theta = \frac{24}{7}$ or $\tan \theta = \frac{7}{24}$, $0 < \theta < \pi$

$$\therefore \tan \left[2 \text{Arc cot} \left(\frac{24}{7} \right) \right] = \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$= \frac{2 \times \frac{7}{24}}{1 - \left(\frac{7}{24} \right)^2} = \frac{336}{527}$$



i. Left as an exercise.

3. Show that

i) $\sin[\text{Arc cos } x + \text{Arc sin } y] = \sqrt{(1-x^2)(1-y^2)} + xy$

ii) $\text{Arc cot } x = \frac{\pi}{2} - \text{Arc tan } x$

iii) $\text{Arc tan} \left(\frac{1}{2} \right) + \text{Arc tan} \left(\frac{1}{3} \right) = \frac{\pi}{4}$

iv) $\cos(\text{Arc sin } u) = \sqrt{1-u^2}$, $|u| \leq 1$

v) $\tan[\text{Arc tan}(a) + \text{Arc tan}(1)] = \frac{1+a}{1-a}$

vi) $\tan[\text{Arc tan}(b) - \text{Arc tan}(a)] = \frac{b-a}{1+b.a}$

vii) $\sin[\text{Arc sin } x + \text{Arc sin } y] = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$$\text{viii) } \text{Arc tan}\left(-\frac{3}{4}\right) + \text{Arc sin}\left(-\frac{7}{25}\right) = \text{Arc cos}\left(\frac{3}{5}\right)$$

$$\text{ix) } 2\text{Arc tan}\left(\frac{1}{3}\right) + \text{Arc tan}\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

Solution

$$\text{i) } \sin[\text{Arc cos } x + \text{Arc sin } y] = \sqrt{(1-x^2)(1-y^2)} + xy$$

$$\text{Let } \text{Arc cos } x = \alpha, \cos \alpha = x \text{ But } \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - x^2}$$

$$\text{Arc sin } y = \beta, \sin \beta = y, \cos \beta = \sqrt{1 - \sin^2 \beta} = \sqrt{1 - y^2}$$

$$\text{Now } \sin[\alpha + \beta] = \sin \alpha \cos \beta + \cos \alpha \sin \beta = \sqrt{1 - x^2} \cdot \sqrt{1 - y^2} + xy$$

$$= \sqrt{(1-x^2)(1-y^2)} + xy = \mathbf{R.H.S}$$

$$\text{ii) } \text{Arc cot } x = \frac{\pi}{2} - \text{Arc tan } x$$

$$\text{Let } \mathbf{R.H.S} = A = \frac{\pi}{2} - \text{Arc tan } x, \text{ let } \text{Arc tan } x = \theta \Rightarrow \tan \theta = x$$

$$\therefore A = \frac{\pi}{2} - \theta, \text{ now introducing cotangent on both sides}$$

$$\cot A = \cot\left(\frac{\pi}{2} - \theta\right) = \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$\Rightarrow \cot A = \tan \theta = x \text{ or } A = \cot^{-1} x = \mathbf{L.H.S}$$

$$\text{iii) } \text{Arc tan}\left(\frac{1}{2}\right) + \text{Arc tan}\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

$$\text{Let } \text{Arc tan}\left(\frac{1}{2}\right) + \text{Arc tan}\left(\frac{1}{3}\right) = D, \text{Arc tan}\left(\frac{1}{2}\right) = \alpha,$$

$$\tan \alpha = \frac{1}{2}, \text{Arc tan}\left(\frac{1}{3}\right) = \beta \text{ or } \tan \beta = \frac{1}{3}$$

$$\Rightarrow \alpha + \beta = D, \text{ now introducing tangent/sine/cosine}$$

$$\Rightarrow \tan D = \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1 \therefore D = \tan^{-1}(1) = \frac{\pi}{4} = \mathbf{R.H.S}$$

i) left as an exercise

$$\text{ii) } \tan[\text{Arc tan}(a) + \text{Arc tan}(1)] = \frac{1+a}{1-a}$$

$$\text{Let } \text{Arc tan}(a) = \alpha \text{ or } \tan \alpha = a, \text{Arc tan}(1) = \beta \text{ or } \tan \beta = 1$$

$$\text{Now } \tan[\text{Arc tan}(a) + \text{Arc tan}(1)] = \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{1+a}{1-a} = \mathbf{R.H.S}$$

iii) Left as an exercise

iv) Left as an exercise

$$\text{v) } \text{Arc tan}\left(-\frac{3}{4}\right) + \text{Arc sin}\left(-\frac{7}{25}\right) = \text{Arc cos}\left(\frac{3}{5}\right)$$

$$\text{let } \text{Arc tan}\left(-\frac{3}{4}\right) + \text{Arc sin}\left(-\frac{7}{25}\right) = D, \text{Arc tan}\left(-\frac{3}{4}\right) = \alpha \text{ or } \tan \alpha = -\frac{3}{4},$$

$$\text{Arc sin}\left(-\frac{7}{25}\right) = \beta \text{ or } \sin \beta = -\frac{7}{25},$$

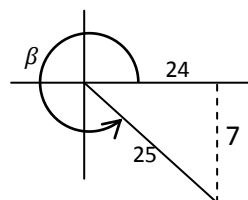
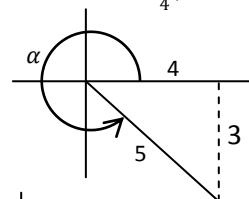
$$\text{Now } \alpha + \beta = D, \text{ introducing cosine}$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos D$$

$$\text{But for } \tan \alpha = -\frac{3}{4}, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$$

$$\text{For } \sin \beta = -\frac{7}{25}, -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$$

$$\therefore \cos D = \frac{4}{5} \times \frac{24}{25} - \left(-\frac{3}{5}\right)\left(-\frac{7}{25}\right) = \frac{96-21}{125} = \frac{3}{5}$$



$$\therefore \cos D = \frac{3}{5}, D = \cos^{-1}\left(\frac{3}{5}\right) = \mathbf{R.H.S}$$

Note

The conventional definition of inverse of trigonometric functions have been made of use in this example.

ix. $2\text{Arc tan}\left(\frac{1}{3}\right) + \text{Arc tan}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

let $\mathbf{L.H.S} = 2\text{Arc tan}\left(\frac{1}{3}\right) + \text{Arc tan}\left(\frac{1}{7}\right) = y$

Let $\text{Arc tan}\left(\frac{1}{3}\right) = \alpha, \tan \alpha = \frac{1}{3}, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and

$\text{Arc tan}\left(\frac{1}{7}\right) = \beta$ or $\tan \beta = \frac{1}{7}, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\Rightarrow 2\alpha + \beta = y$, now introducing tangent on both sides

$$\therefore \tan(2\alpha + \beta) = \tan y \Rightarrow \tan y = \frac{\tan 2\alpha + \tan \beta}{1 - \tan 2\alpha \tan \beta}$$

$$\text{Simplifying } \tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha} = \frac{2 \times \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2} = \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$$

$$\therefore \tan y = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}} = \frac{25}{28} \div \frac{25}{28} = 1 \Rightarrow y = \mathbf{L.H.S} = \tan^{-1}(1) = 45^\circ = \frac{\pi}{4} = \mathbf{R.H.S}$$

REVISION TASK

Show that

a) $\tan^{-1}(-x) = -\tan^{-1}(x)$

b) $\cos^{-1}(-x) = \pi - \cos^{-1}(x) = -\cos^{-1}(x)$

c) $\sin^{-1}(-x) = -\sin^{-1}(x)$

d) $\cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2}$

e) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

f) $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

g) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

h) If $\cot^{-1}\left(\frac{1}{a}\right) + \cot^{-1}\left(\frac{1}{b}\right) + \cot^{-1}\left(\frac{1}{c}\right) = \frac{\pi}{2}$ then $bc + ca + ab = 1$

i) If $\cos^{-1}(a) + \cos^{-1}(b) + \cos^{-1}(c) = \pi$, then $a^2 + b^2 + c^2 = 1$

j) $\sin\left[2\tan^{-1}\left(\frac{1}{4}\right) + \cot^{-1}\left(\frac{23}{7}\right)\right] = \frac{1}{\sqrt{2}}$

k) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

l) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

m) $\sin^{-1} x = \operatorname{cosec}^{-1} \frac{1}{x}$

n) $\cos^{-1} x = \sec^{-1} \frac{1}{x}$

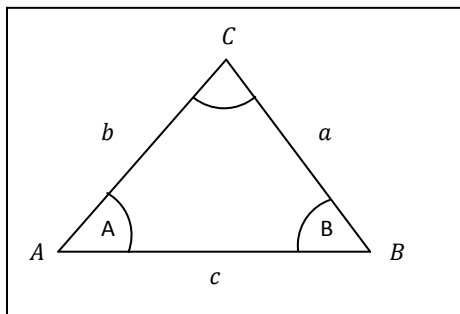
o) $\tan^{-1} x = \cot^{-1} \frac{1}{x}$

RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

Solution to triangles

Euler's notation of Triangle

Consider the figure below



If A, B and C denote angles of a triangle, then by Euler's notation, a, b and c denote the sides of the triangle opposite to these angles.

When three elements, one at least of which is a side, are known, the other three can be found. The process of calculating for the other unknown elements of a triangle when the three of the elements are given, is termed as *solution to a triangle*.

, this also is used in solving a

For any triangle

$$A + B + C = 180^\circ$$

Laws used in solving a triangle

These include:

- Sine rule
- Cosine rule
- Tangent rule

These are useful in dealing with non-right triangles.

THE SINE RULE

States that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

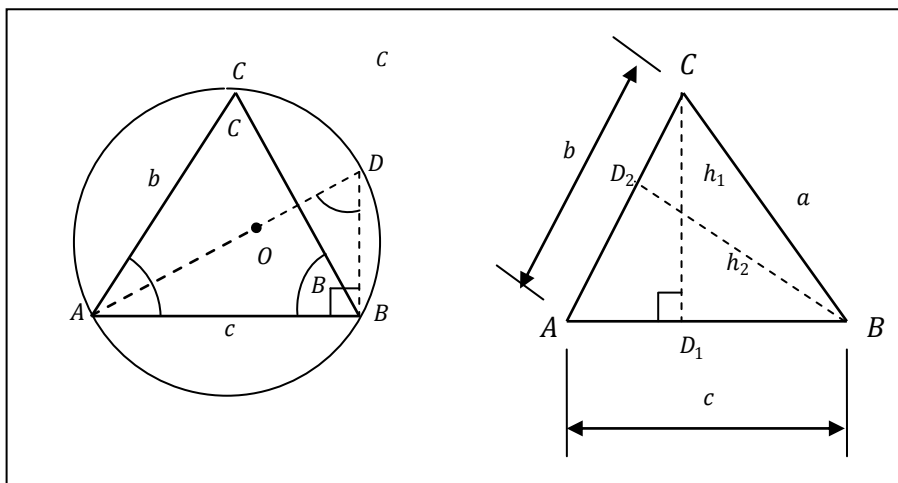
$R = \text{radius of circumscribing circle of triangle}$

Or

The sides of a triangle are proportional to the sines of angles opposite to them

Proof: consider *two* cases

Case I: Acute angle



Consider $\triangle ABC$ with acute angles. h_1 and h_2 are altitudes from C and B respectively.

From fig. a; in $\triangle ABD$

$$\sin C = \frac{AB}{AD} = \frac{c}{2R} \quad \therefore 2R = \frac{c}{\sin C} \quad \dots\dots\dots(1)$$

$$\text{From fig. b; in } \triangle ACD_1; \sin A = \frac{h_1}{b} \quad \therefore h_1 = b \sin A \quad \dots\dots\dots(2)$$

$$\text{In } \triangle AD_1B, \sin B = \frac{h_1}{a} \quad \therefore h_1 = a \sin B \quad \dots\dots\dots(3)$$

Eqns: (1) = (3)

$$b \sin A = a \sin B \Rightarrow \frac{b}{\sin B} = \frac{a}{\sin A} \quad \dots\dots\dots(4)$$

$$\text{In } \triangle AD_2B; \sin A = \frac{h_2}{c} \Rightarrow h_2 = c \sin A \quad \dots\dots\dots(5)$$

$$\text{In } \triangle CD_2B; \sin C = \frac{h_2}{a} \Rightarrow h_2 = a \sin C \quad \dots\dots\dots(6)$$

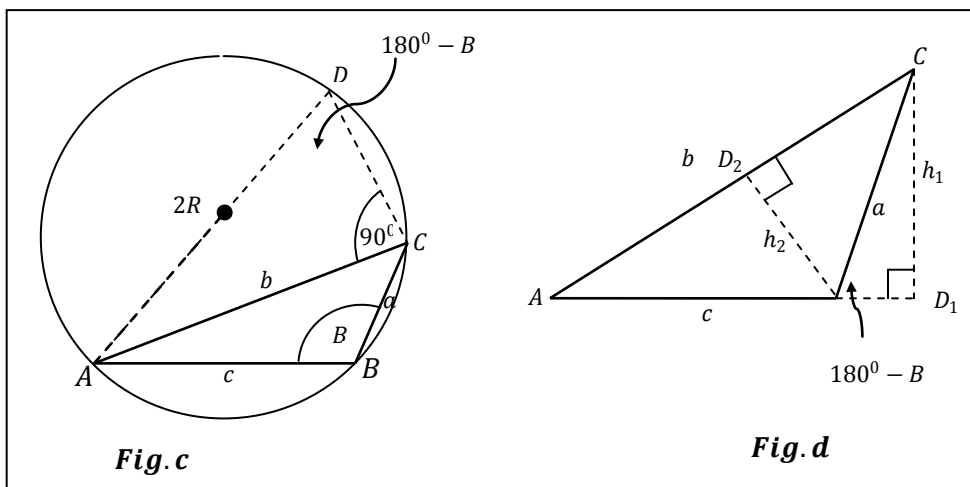
Eqns: (5) = (6)

$$c \sin A = a \sin C \Rightarrow \frac{a}{\sin A} = \frac{c}{\sin C} \quad \dots\dots\dots(7)$$

Now Eqns: (7), (4) and (1) are the same. i. e

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \text{ the sine rule}$$

Case II: Obtuse angle



Consider $\triangle ABC$ with $\angle B$ obtuse, h_1, h_2 are altitudes from C and B respectively. In *fig. c*, in $\triangle ADC$

$$\sin D = \frac{b}{2R}, 2R = \frac{b}{\sin D}. \text{ But } \sin D = \sin(180^\circ - B) = \sin B$$

$$\therefore 2R = \frac{b}{\sin B} \dots\dots\dots (1)$$

In *fig. c*, in $\triangle AD_1C$

$$\sin A = \frac{h_1}{b}, h_1 = b \sin A \dots\dots\dots (2)$$

$$\text{In } \triangle BCD, \sin(180^\circ - B) = \sin B = \frac{h_1}{a}, h_1 = a \sin B \dots\dots\dots (3)$$

$$\text{Eqns. (2) = (3)} \Rightarrow b \sin A = a \sin B, \frac{a}{\sin A} = \frac{b}{\sin B} \dots\dots\dots (4)$$

In *fig. d*, in $\triangle ABD_2$

$$\sin A = \frac{h_2}{c}, h_2 = c \sin A \dots\dots\dots (5)$$

In $\triangle BD_2C$

$$\sin C = \frac{h_2}{a}, h_2 = a \sin C \dots\dots\dots (6)$$

$$\text{Eqns. (5) = (6)} \Rightarrow c \sin A = a \sin C, \frac{a}{\sin A} = \frac{c}{\sin C} \dots\dots\dots (7)$$

Now Eqns. (7), (4) and (1) are the same

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \text{ the sine rule}$$

When to apply sine rule

- One side and two angles are given. Here get the 3rd angle by use of $A + B + C = 180^\circ$
- Two sides and non-included angle are given i.e. the non-included angle is opposite to either given side. Here sometimes an ambiguous case arises, so take care when dealing with this. See examples on sine rule.

When not to apply sine rule

- Three sides are given
- Two sides and their included angle

THE COSINE RULE

States that

In any triangle, the square of a side is equal to the sum of their squares of the other sides minus twice the product of these sides times the cosine of the measure of their included angle. *i.e.* In a $\triangle ABC$

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos A$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos A$$

Proof

Two cases are to be considered

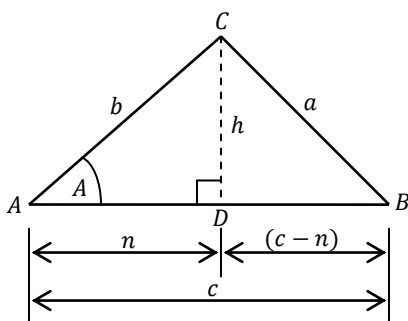


Fig. I Acute Angled Triangle

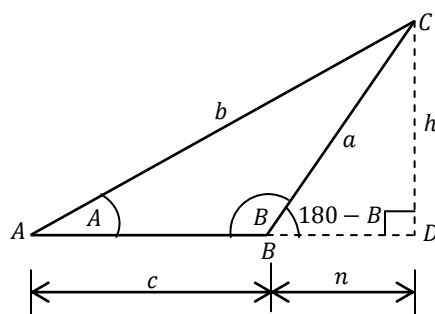


Fig. II Acute Angled Triangle

Considering **Fig. I**

In $\triangle ADC$, $h^2 = b^2 - n^2$ (1)

In $\triangle CDB$, $h^2 = a^2 - (c - n)^2 = a^2 - c^2 + 2cn - n^2$ (2)

Subtracting (1) and (2) gives

$$0 = b^2 - a^2 + c^2 - 2cn, \text{ but from } \triangle ACD, \cos A = \frac{n}{b}, n = b \cdot \cos A$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cdot \cos A, \text{ the cosine rule}$$

Considering **Fig. II**

In $\triangle BCD$, $h^2 = a^2 - n^2$ (1)

In $\triangle ADC$, $h^2 = b^2 - (c + n)^2 = b^2 - c^2 - 2cn - n^2$ (2)

Subtracting (1) and (2) gives

$$0 = a^2 - b^2 + c^2 + 2cn, \text{ but from } \triangle BDC, \cos(180 - B) = -\cos B = \frac{n}{a}, n = a \cdot \cos B$$

$$\therefore b^2 = a^2 + c^2 - 2ac \cdot \cos A, \text{ the cosine rule}$$

When to apply cosine rule

- Two sides and the included angle given
- Three sides are given

When not to apply cosine rule

- Two sides and non-included angle given
- One side and two of the angles given

Application of cosine and sine rule

- Solution to triangles
- Proof of identities related to triangles
- Finding area of a triangle

General examples on sine rule

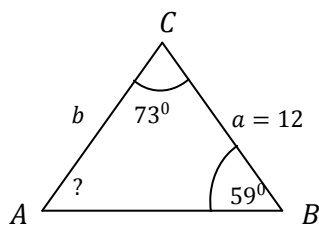
1. Solve the triangle if

- i) $a = 12, B = 59^\circ, C = 73^\circ$
- ii) $A = 75.6^\circ, b = 5.6, C = 48.3^\circ$

Solution

NOTE: The angles in this question are all acute.

i)



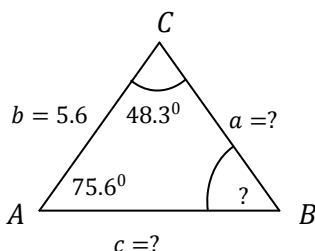
Finding $\angle A$

$$\angle A + 73 + 59 = 180$$

$$\therefore \angle A = 48^\circ$$

$$\begin{aligned} \text{From } \frac{a}{\sin A} &= \frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{12}{\sin 48} = \frac{b}{\sin 59} = \frac{c}{\sin 73} \\ \Rightarrow c &= \frac{12 \sin 73}{\sin 48} = 15.4 \\ \Rightarrow b &= \frac{12 \sin 59}{\sin 48} = 13.8 \end{aligned}$$

ii)



$$\therefore A = 48^\circ, b = 13.8, c = 15.4$$

Finding $\angle B$

$$\angle B + 48.3 + 75.6 = 180$$

$$\angle B = 56.1^\circ$$

$$\begin{aligned} \text{From } \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 75.6} = \frac{5.6}{\sin 56.1} = \frac{c}{\sin 48.3} \\ \Rightarrow a &= \frac{5.6 \sin 75.6}{\sin 56.1} = 6.5 \end{aligned}$$

$$\Rightarrow c = \frac{5.6 \sin 48.3}{\sin 56.1} = 5.0$$

$$\therefore \angle B = 56.1^\circ, a = 6.5, c = 5.0$$

2. Solve the triangle if

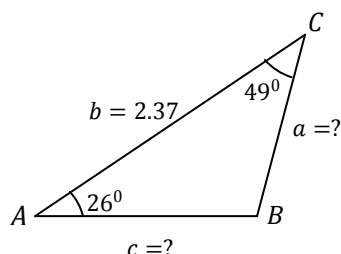
a) $A = 26^\circ, b = 2.37, C = 49^\circ$

b) $a = 136, B = 104.2^\circ, C = 43.1^\circ$

Solution

NOTE: one of the angles given is obtuse

a)



Finding $\angle B$

$$\angle B + 26 + 49 = 180$$

$$\angle B = 105^\circ$$

By sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{136}{\sin 32.7} = \frac{b}{\sin 104.2} = \frac{c}{\sin 43.1}$

$$\therefore b = \frac{136 \sin 104.2}{\sin 32.7} = 244.04$$

$$c = \frac{136 \sin 43.1}{32.7} = 172.01$$

$$\therefore \angle A = 32.7^\circ, b = 244.05, c = 172.01$$

Task

1. If in $\triangle ABC$, $A = 34^\circ, C = 120^\circ$ and $b = 200\text{m}$, how long are the sides BC and AB ? **Ans:** $AB = 395\text{m}, BC = 255\text{m}$

2. Solve the triangle if $A = 71^\circ, B = 42^\circ, c = 15$.

Ans $a = 15.41, b = 10.90, C = 67^\circ$

The ambiguous case

Two sides and non-included angle given

Let A, b and a be the given elements in $\triangle ABC$.

The different cases that might arise are given below for $A < 90^\circ$ and for

$A \geq 90^\circ$

Case I: for $A < 90^\circ$

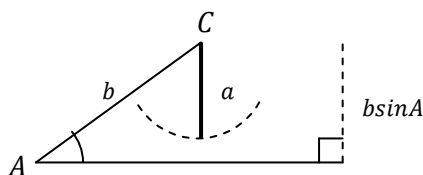


Fig. No solution: $a < b \sin A$

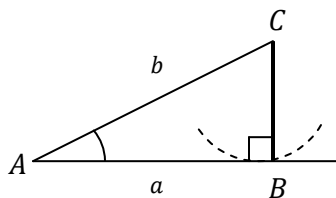


Fig. One solution: $a = b \sin A$

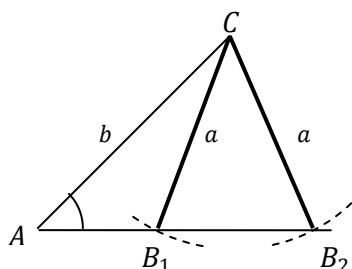


Fig. Two solutions: $b \sin A < a < b$

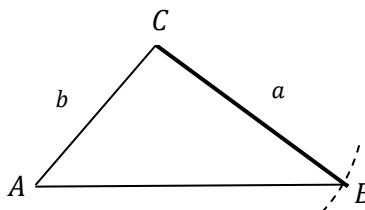


Fig. One solution: $a \geq b$

Case II: For $A \geq 90^\circ$

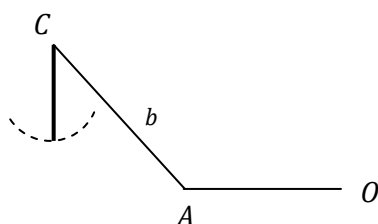


Fig. No solution: $a \geq b$

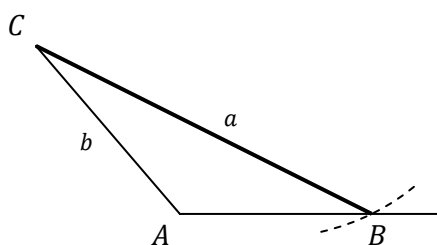


Fig. One solution: $a > b$

NOTE: sine rule can be used to solve a triangle for the data set is ambiguous, but first determine whether the solution exists.

Example

1. Given $\triangle ABC$ with $A = 30^\circ$, $a = 15$, $b = 20$, find B, C

Solution

$\sin 30^\circ = \frac{1}{2}$ and $b \sin 30^\circ = 10$, hence $b \sin 30^\circ < a < b$. i.e. $10 < 15 < 20$

There are two solutions.

Solution I

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin B_1}, \text{ for } \triangle AB_1C$$

$$\Rightarrow \frac{15}{1/2} = \frac{20}{\sin B_1}, \sin B_1 = \frac{10}{15}, B_1 = 41.8^\circ$$

$$\therefore \angle ACB_1 = 180 - (30 + 41.8) = 108.2^\circ$$

$$\text{and } c = \frac{\sin 108.2^\circ \times 15}{\sin 30^\circ} = 28.5$$

Solution II

Since $\angle AB_2C$ and $\angle CB_2B_1$ are

Supplementary and $\angle CB_2B_1 = \angle CB_1B_2 = 41.8^\circ$

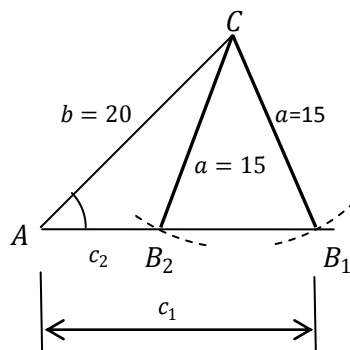
$$\therefore \angle AB_2C = 180 - 41.8 = 138.2^\circ$$

$$\text{Now } \angle ACB_2 = 180 - (30 + 138.2) = 11.8^\circ$$

$$\text{In } \triangle ACB_2, \frac{a}{\sin A} = \frac{c_2}{\sin C} \Rightarrow \frac{15}{\sin 30^\circ} = \frac{c_2}{\sin 11.8^\circ} \text{ or } c_2 = \frac{15 \times \sin 11.8^\circ}{\sin 30^\circ} = 6.2$$

2. Solve the triangle completely

$$\text{a. } b = 17.6, C = 48.3^\circ, c = 15.3$$



b. $B = 129^\circ, b = 7.89, c = 4.56$

Solution

a. By sine rule $\frac{a}{\sin A} = \frac{17.6}{\sin B} = \frac{15.3}{\sin 48.3}$

From the given data, $c < b$ and

$$17.6 \sin 48.3 = 13.14 < c$$

$\therefore b \sin C < c < b$, hence two solutions

From $\triangle AB_1C$,

$$\frac{b}{\sin B_1} = \frac{c}{\sin C} \Rightarrow \frac{17.6}{\sin B_1} = \frac{15.3}{\sin 48.3}$$

$$\therefore B_1 = 59.2^\circ = \angle AB_1C$$

$$\text{Also } \angle A = 180 - (48.3 + 59.2) = 72.5^\circ$$

$$\text{But } \angle AB_2B_1 = \angle AB_1C = 59.2^\circ$$

$$\text{Then } \angle CB_2A = 180 - 59.2 = 120.8^\circ$$

$$\text{From } \triangle ACB_2, \frac{a_2}{\sin A} = \frac{b}{\sin B_2} \Rightarrow \frac{a_2}{\sin A} = \frac{17.6}{\sin 120.8}$$

$$\angle CB_2A = 180 - (48.3 + 120.8) = 10.9^\circ$$

$$\therefore a_2 = \frac{17.6 \sin 10.9}{\sin 120.8} = 3.87$$

$$\text{From } \triangle ACB_1, \frac{c}{\sin C} = \frac{a_1}{\sin A} \Rightarrow \frac{15.3}{\sin 48.3} = \frac{a_1}{\sin 72.5} \therefore a_1 = 19.54$$

$$\therefore B = 59.2^\circ, A = 72.5^\circ, a = 19.54; B = 120.8^\circ, A = 10.9^\circ, a = 3.87$$

- b. Since $\angle B = 129^\circ > 90^\circ$, then looking at $b > c$ then there is only one solution.

$$\text{By sine rule } \frac{a}{\sin A} = \frac{7.89}{\sin 129} = \frac{4.59}{\sin C} \Rightarrow \sin C = \frac{4.59 \sin 129}{7.89}$$

$$\therefore C = 26.9^\circ$$

$$\therefore \angle A = 180 - (26.9 + 129) = 24.1^\circ$$

$$\text{Also } \frac{a}{\sin 24.1} = \frac{7.89}{\sin 129} \Rightarrow a = 4.15$$

$$\therefore C = 26.9^\circ, A = 24.1^\circ, a = 4.15$$

3. Solve the triangle completely in which

$$b = 15.6m, c = 12.3m, C = 34.3^\circ$$

Solution

$$\text{By sine rule } \frac{a}{\sin A} = \frac{15.6}{\sin B} = \frac{12.3}{\sin 34.3}$$

From the given data, $c < b$ and $b \sin C = 15.6 \sin 34.3 < c$

$\therefore b \sin C < c < b$, hence two solutions.

$$\text{From } \triangle AB_1C, \frac{a_1}{\sin A} = \frac{15.6}{\sin B_1} = \frac{12.3}{\sin 34.3} \Rightarrow \sin B_1 = \frac{15.6 \sin 34.3}{12.3}$$

$$\therefore B_1 = 45.6^\circ$$

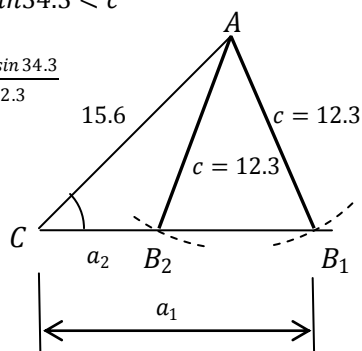
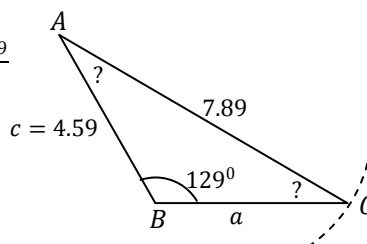
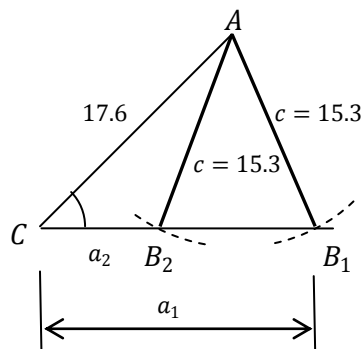
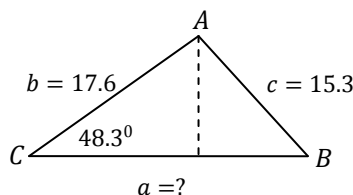
$$\angle CAB_1 = A = 180 - (34.3 + 45.6) = 100.1^\circ$$

$$\therefore a_1 = \frac{12.3 \times \sin 100.1}{\sin 34.3} = 21.49$$

$$\text{From } \triangle ACB_2, \frac{12.3}{\sin 34.3} = \frac{a_2}{\sin \angle CAB_2}$$

$$\angle CB_2A = 180 - 45.6^\circ = 134.4^\circ$$

$$\angle CAB_2 = 180 - (34.3 + 134.4) = 11.3^\circ$$



$$\therefore \frac{12.3}{\sin 34.3} = \frac{a_2}{\sin 11.3} \Rightarrow a_2 = 4.28$$

$$\therefore B = 45.6^\circ, A = 100.1^\circ, a = 21.49m; B = 134.4^\circ, A = 11.3^\circ, a = 4.28$$

Task

Solve the triangle if $b = 2.718, c = 3.142, B = 54.3^\circ$

Answer: $A = 55.9^\circ, C = 69.9^\circ, a = 2.770; A = 15.6^\circ, C = 110.2^\circ, a = 0.897$

General examples on cosine rule

Solve the triangle in which

i) $b = 10.67m, c = 21.7m$ and $A = 44.5^\circ$

ii) $a = 5m, a = 8m$ and $C = 40^\circ$

iii) $a = 9, c = 5$ and $B = 120^\circ$

iv) $a = 1, b = 2$ and $c = 2$

v) $A = 3, b = 7$ and $c = 5$

Solution

i) By cosine rule $a^2 = b^2 + c^2 - 2bc \cdot \cos A$
 $a^2 = 10.67^2 + 21.7^2 - 2 \times 10.67 \times 21.7 \cos 44.5$
 $\therefore a^2 = 254.4483085 \Rightarrow a = 15.97m$

The other angles may be got using sine rule.

$$\frac{15.97}{\sin 44.5} = \frac{10.67}{\sin B} \text{ or } \sin B = \frac{10.67 \sin 44.5}{15.97} \therefore B = 27.9^\circ$$

$$\therefore \angle C = 180 - (27.9 + 44.5) = 107.6^\circ$$

$$\therefore \angle C = 107.6^\circ, 27.9^\circ, a = 15.97$$

ii) Left as an exercise **Ans:** $c = 5.26, A = 37.7^\circ, B = 102.3^\circ$

iii) Left as an exercise **Ans:** $b = 12.29, A = 39.4^\circ, C = 20.6^\circ$

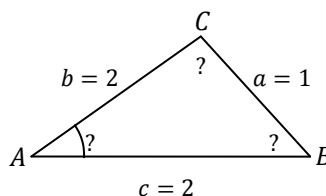
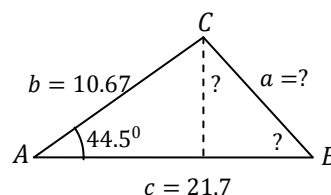
iv) from $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$$\therefore \cos A = \frac{2^2 + 2^2 - 1^2}{2 \times 2 \times 2} \Rightarrow A = 28.96^\circ$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2 \cdot a \cdot c} = \frac{1^2 + 2^2 - 2^2}{2 \times 1 \times 2} \Rightarrow B = 75.52^\circ$$

$$\therefore \angle C = 180 - (28.96 + 75.52) = 75.52^\circ$$

v) Left as an exercise



APPLICATION OF SINE RULE AND COSINE RULE

Proof of identities related to triangles

Examples on SINE RULE

In a $\triangle ABC$, prove that

- i) $\frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$
- ii) $\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$
- iii) $\frac{a+b-c}{a+b+c} = \tan\frac{A}{2}\tan\frac{B}{2}$
- iv) $\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \tan B \cot C$
- v) $\sin\frac{1}{2}(B-C) = \frac{b-c}{a} \cdot \cos\frac{A}{2}$
- vi) $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Solution

- i) $\frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$$\frac{b-c}{a} = \frac{2R\sin B - 2R\sin C}{2R\sin A} = \frac{\sin B - \sin C}{\sin A} = \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}};$$

$$\frac{B+C}{2} = 90 - \frac{A}{2} \therefore \cos\left(\frac{B+C}{2}\right) = \cos\left(90 - \frac{A}{2}\right) = \sin\frac{A}{2}$$

$$\therefore \frac{b-c}{a} = \frac{2\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$$
- ii) $\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$

$$\frac{a+b}{a-b} = \frac{2R\sin A + 2R\sin B}{2R\sin A - 2R\sin B} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} = \frac{\sin\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} \cdot \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}$$

$$\therefore \frac{a+b}{a-b} = \tan\left(\frac{A+B}{2}\right) \cdot \cot\left(\frac{A-B}{2}\right) = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$
- iii) $\frac{a+b-c}{a+b+c} = \tan\frac{A}{2}\tan\frac{B}{2}$

$$\frac{a+b-c}{a+b+c} = \frac{2R\sin A + 2R\sin B - 2R\sin C}{2R\sin A + 2R\sin B + 2R\sin C} = \frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$\text{But } \sin\left(\frac{A+B}{2}\right) = \sin\left(90 - \frac{C}{2}\right) = \cos\frac{C}{2}$$

$$\therefore \frac{a+b-c}{a+b+c} = \frac{2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2}}{2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}}{\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}}$$

$$\text{But } \sin\frac{C}{2} = \sin\left(90 - \left(\frac{A+B}{2}\right)\right) = \cos\left(\frac{A+B}{2}\right)$$

$$\Rightarrow \frac{a+b-c}{a+b+c} = \frac{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)} = \frac{-2\sin\left[\frac{\left(\frac{A-B}{2}\right) + \left(\frac{A+B}{2}\right)}{2}\right]\sin\left[\frac{\left(\frac{A-B}{2}\right) - \left(\frac{A+B}{2}\right)}{2}\right]}{2\cos\left[\frac{\left(\frac{A-B}{2}\right) + \left(\frac{A+B}{2}\right)}{2}\right]\cos\left[\frac{\left(\frac{A-B}{2}\right) - \left(\frac{A+B}{2}\right)}{2}\right]} = \frac{-2\sin\left(\frac{A}{2}\right)\sin\left(-\frac{B}{2}\right)}{2\cos\left(\frac{A}{2}\right)\cos\left(-\frac{B}{2}\right)}$$

$$\therefore \frac{a+b-c}{a+b+c} = \frac{\sin \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} = \tan \frac{A}{2} \tan \frac{B}{2}$$

$$\text{iv) } \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \tan B \cot C$$

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{(2R\sin A)^2 + (2R\sin B)^2 - (2R\sin C)^2}{(2R\sin A)^2 - (2R\sin B)^2 + (2R\sin C)^2} = \frac{\sin^2 A + \sin^2 B - \sin^2 C}{\sin^2 A - \sin^2 B + \sin^2 C}$$

From double angle identities

$$\sin^2 A = \frac{1-\cos 2A}{2}, \sin^2 B = \frac{1-\cos 2B}{2}, \sin^2 C = \frac{1-\cos 2C}{2}$$

$$\Rightarrow \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} - \frac{1-\cos 2C}{2}}{\frac{1-\cos 2A}{2} - \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}} = \frac{1-\cos 2A + 1-\cos 2B - 1 + \cos 2C}{1-\cos 2A - 1 + \cos 2B + 1 - \cos 2C}$$

$$= \frac{1 - [\cos 2A + \cos 2B - \cos 2C]}{1 - [\cos 2A - \cos 2B + \cos 2C]} = \frac{1 - [2\cos(A+B)\cos(A-B) - \cos 2C]}{1 - [-2\sin(A+B)\sin(A-B) + \cos 2C]}$$

But $\cos(A+B) = \cos(180 - C) = -\cos C$;

Also $\sin(A+B) = \sin(180 - C) = \sin C$;

$$\therefore \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{1 - [-2\cos C \cdot \cos(A-B) - \cos 2C]}{1 - [-2\sin C \cdot \sin(A-B) + (1-2\sin^2 C)]} = \frac{1 - [-2\cos C (\cos(A-B) + \cos C) + 1]}{1 - [-2\sin C (\sin(A-B) + \sin C) + 1]}$$

$$= \frac{1 + 2\cos C \cdot [\cos(A-B) + \cos C] - 1}{1 + 2\sin C \cdot [\sin(A-B) + \sin C] - 1} = \frac{\cos C}{\sin C} \cdot \left[\frac{\cos(A-B) + \cos C}{\sin(A-B) + \sin C} \right]$$

$$= \cot C \cdot \left[\frac{2\cos\left(\frac{A-B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)}{2\sin\left(\frac{A-B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)} \right]$$

$$\text{But } \frac{A-B+C}{2} = \frac{A+C}{2} - \frac{B}{2} = \frac{180-B}{2} - \frac{B}{2} = 90 - B, \therefore \cos\left(\frac{A-B+C}{2}\right) = \cos(90 - B) = \sin B$$

$$\text{Also } \sin\left(\frac{A-B+C}{2}\right) = \sin(90 - B) = \cos B$$

$$\therefore \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \cot C \cdot \frac{\sin B}{\cos B} = \tan B \cdot \cot C$$

v) Left as an exercise. Refer to (i) above.

$$\text{vi) } \frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\frac{a^2-b^2}{c^2} = \frac{(2R\sin A)^2 - (2R\sin B)^2}{(2R\sin C)^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C} = \frac{\frac{1-\cos 2A}{2} - \frac{1-\cos 2B}{2}}{\sin^2 C} = \frac{\cos 2B - \cos 2A}{2\sin^2 C}$$

$$\frac{a^2-b^2}{c^2} = \frac{-2\sin(B+A)\sin(B-A)}{2\sin C \cdot \sin C}$$

But $\sin(B+A) = \sin(180 - C) = \sin C$

$$\therefore \frac{a^2-b^2}{c^2} = \frac{-2\sin C \cdot \sin(B-C)}{2\sin C \cdot \sin C} = \frac{\sin(B-C)}{\sin C} = \frac{\sin(B-C)}{\sin(A+B)}$$

Task

For any $\triangle ABC$, prove that $\tan \frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right) \cdot \cot \frac{C}{2}$

Examples on COSINE RULE

In $\triangle ABC$, prove that

$$\text{i) } a^2 = (b-c)^2 + 4bc \cdot \sin^2 \frac{1}{2}A$$

$$\text{ii) } a^2 + b^2 + c^2 = 2(bc \cdot \cos A + ac \cdot \cos B + ab \cdot \cos C)$$

$$\text{iii) } \frac{b^2-c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$$

$$\text{iv) } \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \tan B \cot C$$

Solution

i) $a^2 = (b - c)^2 + 4bc \cdot \sin^2 \frac{1}{2}A$

R. H. S: $(b - c)^2 + 4bc \cdot \sin^2 \frac{1}{2}A = b^2 - 2bc + c^2 + 4bc \left(\frac{1 - \cos A}{2} \right)$

$= b^2 - 2bc + c^2 + 2bc(1 - \cos A) = b^2 + c^2 - 2bc \cdot \cos A$

But from cosine rule $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$\therefore \text{R. H. S} = a^2 = \text{L. H. S}$

ii) $a^2 + b^2 + c^2 = 2(bc \cdot \cos A + ac \cdot \cos B + ab \cdot \cos C)$

$a^2 = b^2 + c^2 - 2bc \cdot \cos A$

$b^2 = a^2 + c^2 - 2ac \cdot \cos B$

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

$a^2 + b^2 + c^2 = (2a^2 + 2b^2 + 2c^2) - 2(bc \cdot \cos A + ac \cdot \cos B + ab \cdot \cos C)$

$\therefore a^2 + b^2 + c^2 = 2(bc \cdot \cos A + ac \cdot \cos B + ab \cdot \cos C)$

iii) $\frac{b^2 - c^2}{a^2} = \frac{\sin(B - C)}{\sin(B + C)}$

From $b^2 = a^2 + c^2 - 2ac \cdot \cos B \Rightarrow b^2 - c^2 = a^2 - 2ac \cdot \cos B \dots\dots\dots(\div a^2)$

$\frac{b^2 - c^2}{a^2} = 1 - 2 \cdot \frac{c}{a} \cdot \cos B$

From sine rule, $c = 2R \sin C, a = 2R \sin A$

$\therefore \frac{b^2 - c^2}{a^2} = 1 - 2 \cdot \frac{2R \sin C}{2R \sin A} \cdot \cos B = 1 - \frac{2 \sin C \cos B}{\sin A} = \frac{\sin A - 2 \sin C \cos B}{\sin A}$

But $\sin A = \sin(180 - (B + C)) = \sin(B + C)$

$\therefore \frac{b^2 - c^2}{a^2} = \frac{\sin(B + C) - 2 \sin C \cos B}{\sin(B + C)} = \frac{\sin B \cos C + \cos B \sin C - 2 \sin C \cos B}{\sin(B + C)}$
 $= \frac{\sin B \cos C - \cos B \sin C}{\sin(B + C)} = \frac{\sin(B - C)}{\sin(B + C)}$

iv) $\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \tan B \cot C$

From the cosine rule

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$ or $a^2 + b^2 = c^2 + 2ab \cdot \cos C$

$b^2 = a^2 + c^2 - 2ac \cdot \cos B$ or $a^2 + c^2 = b^2 + 2ac \cdot \cos B$

$\Rightarrow \frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \frac{c^2 + 2ab \cdot \cos C - c^2}{b^2 + 2ac \cdot \cos B - b^2} = \frac{2ab \cdot \cos C}{2ac \cdot \cos B} = \frac{b}{c} \cdot \frac{\cos C}{\cos B}$

From sine rule, $c = 2R \sin C, b = 2R \sin B$

$\therefore \frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \frac{2R \sin B}{2R \sin C} \cdot \frac{\cos C}{\cos B} = \frac{\sin B}{\cos B} \cdot \frac{\sin C}{\cos C} = \tan B \cot C$

NOTE:

When the proof is involving sides and angles of a triangle, the application of the suitable rule can be determined by the sides of the triangle. *i.e.* For sides being squared, use cosine rule otherwise use sine rule.

THE TANGENT RULE

For any $\triangle ABC$ with Euler's notation

$$\begin{aligned}\tan \frac{1}{2}(B - C) &= \left(\frac{b - c}{b + c} \right) \cdot \cot \frac{A}{2} \\ \tan \frac{1}{2}(A - B) &= \left(\frac{a - b}{a + b} \right) \cdot \cot \frac{C}{2}\end{aligned}$$

Proof: this is based on sine rule.

$$\tan \frac{1}{2}(B - C) = \left(\frac{b - c}{b + c} \right) \cdot \cot \frac{A}{2}$$

$$\frac{b - c}{b + c} = \frac{2R \sin B - 2R \sin C}{2R \sin B + 2R \sin C} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2 \cos \left(\frac{B + C}{2} \right) \sin \left(\frac{B - C}{2} \right)}{2 \sin \left(\frac{B + C}{2} \right) \cos \left(\frac{B - C}{2} \right)}$$

$$\text{But } \frac{B + C}{2} = 90 - \frac{A}{2}$$

$$\therefore \cos \left(\frac{B + C}{2} \right) = \cos \left(90 - \frac{A}{2} \right) = \sin \frac{A}{2}, \text{ and } \therefore \sin \left(\frac{B + C}{2} \right) = \sin \left(90 - \frac{A}{2} \right) = \cos \frac{A}{2}$$

$$\therefore \frac{b - c}{b + c} = \frac{2 \sin \frac{A}{2} \sin \left(\frac{B - C}{2} \right)}{2 \cos \frac{A}{2} \cos \left(\frac{B - C}{2} \right)} = \tan \frac{A}{2} \cdot \tan \left(\frac{B - C}{2} \right)$$

$$\text{Now } \left(\frac{b - c}{b + c} \right) \cdot \cot \frac{A}{2} = \tan \frac{A}{2} \cdot \tan \left(\frac{B - C}{2} \right) \cdot \cot \frac{A}{2} = \tan \frac{1}{2}(B - C) = \text{L.H.S}$$

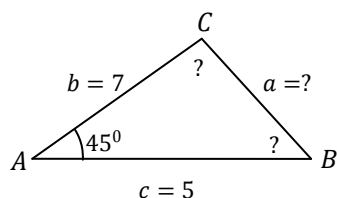
Application of Tangent rule

Used in solving a triangle when two sides and included angle are given

Example

Given that in any $\triangle ABC$, $\tan \left(\frac{B - C}{2} \right) = \frac{b - c}{b + c} \cdot \cot \frac{A}{2}$, solve the triangle with sides 5 and 7 and the included angle 45° (uneb 1999)

Solution



Let $b = 7, c = 5$ to avoid $(b - c) < 0, A = 45^\circ$

By tangent rule,

$$\tan \left(\frac{B - C}{2} \right) = \frac{7 - 5}{7 + 5} \cdot \cot \frac{45}{2} = \frac{2}{12} \times \frac{1}{\tan 22.5} = \frac{1}{6 \tan 22.5}$$

$$\therefore \frac{B - C}{2} = \tan^{-1} \left(\frac{1}{6 \tan 22.5} \right) = 21.9^\circ$$

$$\Rightarrow \frac{B - C}{2} = 21.9^\circ \dots\dots\dots (1)$$

$$\text{But } A + B + C = 180^\circ \Rightarrow \frac{B + C}{2} = 90 - \frac{A}{2} = 90 - \frac{45}{2} = 67.5^\circ$$

$$\therefore \frac{B + C}{2} = 67.5^\circ \dots\dots\dots (2)$$

Adding (1) and (2),

$$B = 89.4^\circ$$

Subtracting (1) and (2),

$$C = 45.6^\circ$$

$$\text{From sine rule } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 45} = \frac{7}{\sin 89.4}$$

$$\therefore a = \frac{7 \sin 45}{\sin 89.4} = 4.95$$

$$\therefore B = 89.4^\circ, C = 45.6^\circ, a = 4.95$$

Note

When using the tangent rule, identify included angle between the two given sides. Equations

$\frac{B+C}{2} = 90 - \frac{A}{2}$, and another containing $\frac{B-C}{2}$ from the tangent rule are usually solved simultaneously for B and C if A is given.

Task

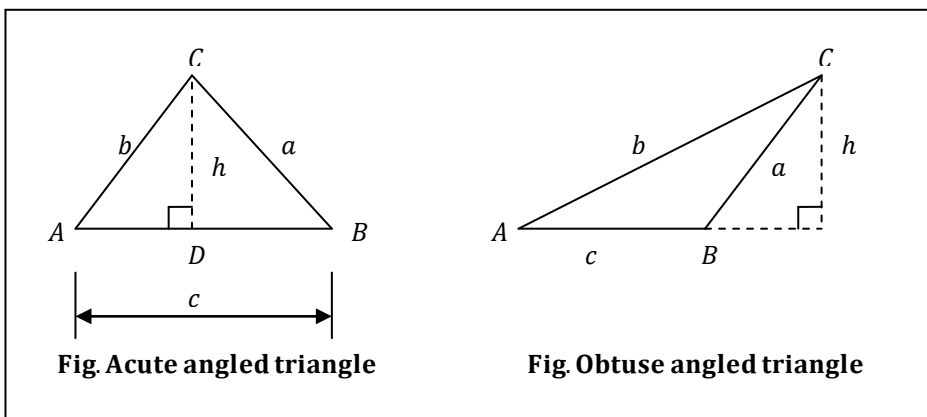
Solve the triangle in which $b = 10.67\text{m}$, $c = 21.7\text{m}$, $A = 44.8^\circ$

Answer: $a = 16.0\text{m}$, $C = 107.2^\circ$, $B = 28^\circ$

AREA OF A TRIANGLE

Recall: Area of a triangle, $\Delta = \frac{1}{2} \cdot b \cdot h$, where b = measure of one side and h = altitude to that side.

Consider the fig. below



Area, $\Delta = \frac{1}{2} b \cdot h$, for both

Three cases arise on area of a triangle

- a) **To determine the area of a triangle with two sides and an included angle given**

Let A, b , and c be given measures.

Area, $\Delta = \frac{1}{2} \cdot c \cdot h$. See figure above.

But from trigonometry, $h = b \sin A$

$$\therefore \text{Area, } \Delta = \frac{1}{2} \cdot b \cdot c \sin A$$

Similarly,

$$\text{Area, } \Delta = \frac{1}{2} \cdot a \cdot c \sin B$$

$$\text{Area, } \Delta = \frac{1}{2} \cdot a \cdot b \sin C$$

Example

Find the area of a BC , given that $B = 40^\circ$, $a = 5$ and $c = 10$

Solution

$$\text{Area}, \Delta = \frac{1}{2} \cdot a \cdot c \sin B = \frac{1}{2} \times 5 \times 10 \times \sin 40 = 16 \text{ sq. units}$$

b) To find area of a triangle given one side and two or three angles

Let A, B and C be given measures of angles

$$\therefore \text{Area}, \Delta = \frac{1}{2} \cdot b \cdot c \sin A$$

Now finding b

From sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow b = \frac{c \sin B}{\sin C}$ where $C = 180 - (A + B)$, now known,

$$\therefore b = \frac{c \sin B}{\sin C}$$

$$\therefore \text{Area}, \Delta = \frac{1}{2} c^2 \cdot \frac{\sin A \sin B}{\sin C}$$

Similarly

$$\begin{aligned} \text{Area}, \Delta &= \frac{1}{2} b^2 \cdot \frac{\sin A \sin C}{\sin B} \\ \text{Area}, \Delta &= \frac{1}{2} a^2 \cdot \frac{\sin B \sin C}{\sin A} \end{aligned}$$

Example

Find the area of ΔABC given that $a = 10$, $A = 45^\circ$, $B = 60^\circ$, $C = 35^\circ$

Solution

Using

$$\text{Area}, \Delta = \frac{1}{2} a^2 \cdot \frac{\sin B \sin C}{\sin A} \Rightarrow \text{Area}, \Delta = \frac{1}{2} \times 10^2 \times \frac{\sin 60 \sin 35}{\sin 85} = 25 \text{ sq. units}$$

c) To find area of a triangle given only the three sides

The above two cases can be made of use by transforming this question in to any of the cases (a) and (b) above.

The angles can be obtained using the cosine rule, and then using the sine rule if necessary.

Alternative approach

This requires *half angle identities*. i. e.

$$\begin{aligned} \sin \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} & \cos \frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}} \\ \sin \frac{B}{2} &= \sqrt{\frac{(s-a)(s-c)}{ac}} & \cos \frac{B}{2} &= \sqrt{\frac{s(s-b)}{ac}} \\ \sin \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} & \cos \frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}} \end{aligned}$$

Where $s = \text{semi perimeter} = \frac{1}{2}(a + b + c)$

Proof of half angle identities

Proof is based on half angle identities and cosine rule.

$$\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}, a^2 = b^2 + c^2 - 2bc \cdot \cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\begin{aligned} \sin^2 \frac{A}{2} &= \frac{1 - \left(\frac{b^2 + c^2 - a^2}{2bc} \right)}{2} = \frac{2bc - (b^2 + c^2) + a^2}{4bc} = \frac{a^2 - [b^2 - 2bc + c^2]}{4bc} \\ &= \frac{a^2 - (b - c)^2}{4bc} = \frac{(a + b - c)(a - (b - c))}{4bc} = \frac{(a + b - c)(a + c - b)}{4bc} \quad \left[\begin{array}{l} \therefore \text{By difference of two squares} \\ \text{i.e. } A^2 - B^2 = (A + B)(A - B) \end{array} \right] \end{aligned}$$

$$\text{But } s = \frac{1}{2}(a + b + c) \dots \dots \dots (*)$$

Subtracting c on both sides

$$\Rightarrow s - c = \frac{1}{2}(a + b + c) - c = \frac{1}{2}(a + b - c) \text{ or } 2(s - c) = a + b - c$$

Subtracting b on both sides of eqn. (*)

$$\Rightarrow s - b = \frac{1}{2}(a + b + c) - b = \frac{1}{2}(a + c - b) \text{ or } 2(s - b) = a + c - b$$

$$\therefore \sin^2 \frac{A}{2} = \frac{2(s - c) \cdot 2(s - b)}{4bc} = \frac{(s - b)(s - c)}{bc} \Rightarrow \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}$$

Prove the other two half angle identities of sines following the above derivation of $\sin \frac{A}{2}$.

$$\cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}, a^2 = b^2 + c^2 - 2bc \cdot \cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\begin{aligned} \cos^2 \frac{A}{2} &= \frac{1 + \left(\frac{b^2 + c^2 - a^2}{2bc} \right)}{2} = \frac{2bc + b^2 + c^2 - a^2}{4bc} = \frac{[b^2 - 2bc + c^2] - a^2}{4bc} \\ &= \frac{(b + c)^2 - a^2}{4bc} = \frac{(b + c + a)(b + c - a)}{4bc} \quad \left[\begin{array}{l} \therefore \text{By difference of two squares} \\ \text{i.e. } A^2 - B^2 = (A + B)(A - B) \end{array} \right] \end{aligned}$$

$$\text{From } s = \frac{1}{2}(a + b + c) \Rightarrow 2s = a + b + c$$

Subtracting a on both sides

$$\Rightarrow s - a = \frac{1}{2}(a + b + c) - a = \frac{1}{2}(b + c - a) \text{ or } 2(s - a) = b + c - a$$

$$\therefore \cos^2 \frac{A}{2} = \frac{2s \cdot 2(s - a)}{4bc} = \frac{s(s - a)}{bc} \Rightarrow \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$$

Prove the other two half angle identities of cosines as above.

Now area of a triangle is obtained as follows:

From case (a) above,

$$\therefore \text{Area}, \Delta = \frac{1}{2} \cdot b \cdot c \sin A = \frac{1}{2} \cdot b \cdot c \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2} = b \cdot c \cdot \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\text{But } \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} \text{ and } \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$$

$$\therefore \text{Area}, \Delta = b \cdot c \times \sqrt{\frac{(s - b)(s - c)}{bc}} \times \sqrt{\frac{s(s - a)}{bc}} = \sqrt{s(s - a)(s - b)(s - c)}$$

$$\therefore \text{Area}, \Delta = \sqrt{s(s - a)(s - b)(s - c)}$$

This formula is called **Heron's formula** or **Hero's formula of area of triangle**.

Example

1. Find the area of a ΔABC if $a = 7, b = 8$ and $c = 9$

Solution

Using $\therefore \text{Area}, \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ and $s = \frac{1}{2}(a+b+c)$

$$s = \frac{1}{2}(7+8+9) = 12$$

$$\therefore \text{Area}, \Delta = \sqrt{12 \cdot (12-7)(12-8)(12-9)} = 12\sqrt{5} \text{ sq. units}$$

2. (a) prove that in any ΔABC , $\frac{b^2-c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$

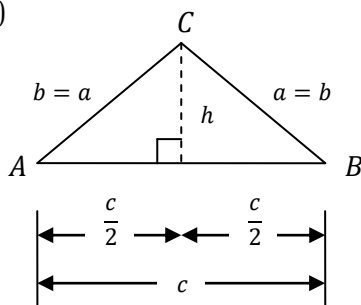
(b) Show that for any isosceles triangle ABC with $AB = c$, the base, area of the triangle is given by $\Delta = \frac{1}{2}c\sqrt{s(s-c)}$, where s is half perimeter of the triangle.

Given that $\Delta = \sqrt{3}$ and $s = 4$, determine the sides of the triangle. (Uneb 1992)

Solution

- a) See notes on cosine/sine rule application

- b)



Given: $AC = BC = a = b$

$$\text{Area}, \Delta = \frac{1}{2} \overline{AB} \cdot h = \frac{1}{2} c \cdot b \sin A$$

$$= \frac{1}{2} bc \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\therefore \Delta = bc \cdot \sin \frac{A}{2} \cos \frac{A}{2} \dots\dots\dots(1)$$

From $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, and squaring eqn. (1)

$$\Delta^2 = (bc)^2 \cdot \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}$$

$$s-b = \frac{1}{2}(a+b+c) - b = \frac{1}{2}(a+c-b)$$

But $a = b$

$$\therefore s-b = \frac{1}{2}(a+c-a) = \frac{1}{2}c$$

$$s-c = \frac{1}{2}(a+b+c) - c = \frac{1}{2}(a+b-c)$$

Also $a = b$

$$\therefore s-c = \frac{1}{2}(a+a-c) = \frac{1}{2}(2a-c)$$

$$s-a = \frac{1}{2}(a+b+c) - a = \frac{1}{2}(b+c-a)$$

But $a = b$

$$\therefore s-a = \frac{1}{2}(a+c-a) = \frac{1}{2}c$$

Also from $s = \frac{1}{2}(a+b+c)$ and $a = b \Rightarrow s = \frac{1}{2}(a+a+c) = \frac{1}{2}(2a+c)$

$$\therefore \Delta^2 = (bc)^2 \cdot \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}, \text{ becomes}$$

$$\Delta^2 = b^2 c^2 \cdot \left[\frac{\frac{1}{2}c \times \frac{1}{2}(2a-c)}{bc} \right] \cdot \left[\frac{\frac{1}{2}c \times \frac{1}{2}(2a+c)}{bc} \right] = \frac{\frac{1}{4}c^2(2a-c) \cdot \frac{1}{4}(2a+c)}{b^2 c^2} \cdot b^2 c^2$$

$$= \frac{1}{16} c^2 \cdot (2a-c)(2a+c)$$

But $s = \frac{1}{2}(2a+c) \Rightarrow 2s = 2a+c$ and also

$$s-c = \frac{1}{2}(2a-c) \Rightarrow 2(s-c) = 2a-c$$

$$\therefore \Delta^2 = \frac{1}{16} c^2 \times 2s \times 2(s-c) = \frac{1}{4} c^2 \cdot s(s-c)$$

$$\therefore \Delta = \frac{1}{2} c \sqrt{s(s-c)}$$

Given that $\Delta = \sqrt{3}$, $s = 4$, $a = b = ?$, $c = ?$

From $\Delta = \frac{1}{2} c \sqrt{s(s-c)}$, squaring both sides

$$\Delta^2 = \frac{1}{4} c^2 \cdot s(s-c) \Rightarrow 4\Delta^2 = c^2 \cdot 4(4-c) \text{ or } \Delta^2 = c^2(4-c)$$

$$\text{But } \Delta^2 = (\sqrt{3})^2 = 3, \therefore 3 = 4c^2 - c^3 \text{ or } c^3 - 4c^2 + 3 = 0$$

Solving this polynomial equation,

$$\text{Let } f(c) = c^3 - 4c^2 + 3 = 0$$

$$\text{For } f(1) = 1 - 4 + 3 = 0 \Rightarrow c - 1 \text{ is a factor of } f(c)$$

By long division

$$\begin{array}{r} c^2 - 3c - 3 \\ c-1 \overline{) c^3 - 4c^2 + 0c + 3} \\ \underline{c^3 - c^2} \\ -3c^2 + 3 \\ \underline{-3c^2 + 3c} \\ -3c + 3 \\ \underline{-3c + 3} \\ 0 \end{array}$$

$$\therefore f(c) = (c^2 - 3c - 3)(c - 1) = 0$$

$$\text{Solving } c^2 - 3c - 3 = 0$$

$$c = \frac{3 \pm \sqrt{9+12}}{2} = \frac{3 \pm 4.6}{2}$$

$$\therefore c = 1, 3.8 \quad [c = -0.8 \text{ is rejected}]$$

$$\text{Now from } s = \frac{1}{2}(a+b+c) = \frac{1}{2}(2a+c),$$

$$\text{For } s = 4, c = 1 \Rightarrow 4 = \frac{1}{2}(2a+1) \text{ or } a = 3.5 = b$$

$$\text{For } s = 4, c = 3.8 \Rightarrow 4 = \frac{1}{2}(2a+3.8) \text{ or } a = 2.1 = b$$

Half angle identities proofs of a triangle

In any $\triangle ABC$, show that

- $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$
- $(a+b+c) \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) = 2c \cot C$
- $1 - \tan \frac{A}{2} \cdot \tan \frac{B}{2} = \frac{c}{s}; s = \frac{1}{2}(a+b+c)$
- $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
- Area, $\Delta = s^2 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}; s = \frac{1}{2}(a+b+c)$
- $\frac{1}{a} \cos^2 \frac{A}{2} + \frac{1}{b} \cos^2 \frac{B}{2} + \frac{1}{c} \cos^2 \frac{C}{2} = \frac{(a+b+c)^2}{4abc}$
- Area, $\Delta = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin(B+C)}$
- $abc = 4R\Delta$, where R = radius of circumscribed circle and Δ = area of $\triangle ABC$

Solution

- $$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}, \text{ but } \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \sin A = 2 \cdot \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}} = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$
- $$(a+b+c) \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) = 2c \cot C$$

From $s = \frac{1}{2}(a+b+c) \Rightarrow 2s = a+b+c$

Also $\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$ and $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

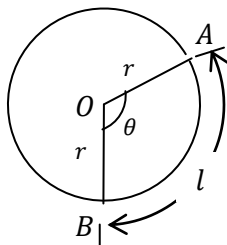
$$\begin{aligned} \therefore (a+b+c) \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) &= 2s \left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \right] \\ &= 2s \left[\frac{\sqrt{(s-b)^2(s-c)s} + \sqrt{s(s-a)^2(s-c)}}{\sqrt{s^2(s-a)(s-b)}} \right] = 2s \left[\frac{(s-b)\sqrt{s(s-c)} + (s-a)\sqrt{s(s-c)}}{s\sqrt{(s-a)(s-b)}} \right] \\ &= \left[\frac{2s\sqrt{s(s-c)}[s-b+s-a]}{s\sqrt{(s-a)(s-b)}} \right] = \frac{2\sqrt{s(s-c)}[2s-(a+b)]}{\sqrt{(s-a)(s-b)}} = \frac{2\sqrt{s(s-c)}[2s-(2s-c)]}{\sqrt{(s-a)(s-b)}} \\ &= \frac{2\sqrt{s(s-c)} \cdot c}{\sqrt{(s-a)(s-b)}} = 2c \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 2c \cdot \frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}} = 2c \cdot \frac{1}{\tan \frac{C}{2}} = 2c \cot \frac{C}{2} \end{aligned}$$
- $$1 - \tan \frac{A}{2} \cdot \tan \frac{B}{2} = \frac{c}{s}; s = \frac{1}{2}(a+b+c)$$

$$\begin{aligned} 1 - \tan \frac{A}{2} \cdot \tan \frac{B}{2} &= 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \cdot \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = 1 - \sqrt{\frac{(s-a)(s-b)(s-c)^2}{s^2(s-a)(s-b)}} \\ &= 1 - \sqrt{\frac{(s-c)^2}{s^2}} = 1 - \frac{s-c}{s} = \frac{s-s+c}{s} = \frac{c}{s} \end{aligned}$$
- (d), (e), (f) and (h), are left as an exercise.

RADIANS

The unit of angular measure is a *radian*,

Length of an arc of a circle



$$\theta \propto l \Rightarrow \theta = kl, k = \text{a constant}$$

When $\theta = 360^\circ$, then $l = \text{circumference of a circle} = 2\pi r$

$$\therefore l = k\theta \text{ becomes } 2\pi r = k \times 360^\circ \text{ or } k = \frac{\pi r}{180^\circ}$$

$$\text{Thus } l = \frac{\pi r \theta}{180^\circ}$$

$$\text{Length of an arc} = \left(\frac{\pi}{180}\right) \cdot r \theta$$

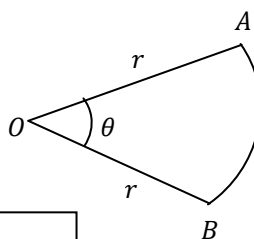
Area of a sector of circle

Area, $A \propto \theta \Rightarrow A = k\theta, k = \text{constant}$

When $\theta = 360^\circ$, area, $A = \pi r^2$

$$\therefore k = \frac{A}{\theta} = \frac{\pi r^2}{360^\circ}$$

$$\therefore \text{Area, } A = \left(\frac{\pi}{360^\circ}\right) r^2 \theta = \text{Area of a sector}$$



NOTE:

From length of an arc, $= \left(\frac{\pi}{180}\right) \cdot r \theta$, if $l = r = \text{radius of a circle}$, then $\theta = 1 \text{ radians}$

$$r = \frac{\pi}{180} \cdot r \theta \text{ or } 1 \pi \text{ rad} = 180^\circ$$

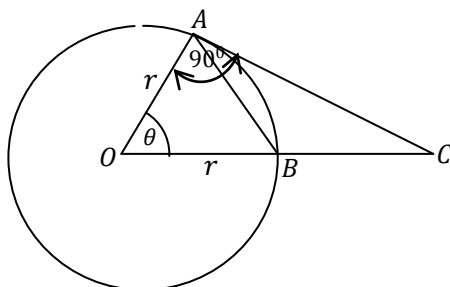
$$\therefore 1 \pi \text{ radian} = 180^\circ$$

$$\text{Length of an arc} = r \cdot \theta$$

$$\text{Area of a sector} = \frac{1}{2} r^2 \theta$$

Trigonometric ratios of small angles in Radians

Consider the figure



Let $A_1 = \text{area of } \triangle OAB$
 $A_2 = \text{area of sector } OAB$
 $A_3 = \text{area of } \triangle OAC$
 , and $\triangle OAC$ is a right triangle
 $A_1 = \frac{1}{2} \cdot \overline{OA} \cdot \overline{OB} \cdot \sin \theta = \frac{1}{2} r^2 \sin \theta$
 $A_2 = \frac{1}{2} r^2 \theta$
 $A_3 = \frac{1}{2} \cdot \overline{OA} \cdot \overline{AC} = \frac{1}{2} r \cdot r \tan \theta$
 $= \frac{1}{2} r^2 \tan \theta$

From the figure,

$$A_1 < A_2 < A_3 \Rightarrow \frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta$$

Dividing through by $\frac{1}{2} r^2 \theta$

$$\Rightarrow \sin \theta < \theta < \tan \theta \dots\dots\dots (\div \sin \theta)$$

$$\Rightarrow 1 < \frac{\theta}{\sin \theta} < \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\sin \theta}$$

$$\Rightarrow 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

Now as $\theta \rightarrow 0$, $\cos \theta \rightarrow 1$, and $\therefore \frac{\theta}{\sin \theta}$ will lie between 1 and $\frac{1}{\cos \theta}$

$$\Rightarrow 1 < \frac{\theta}{\sin \theta} < 1 \text{ or } \frac{\theta}{\sin \theta} \rightarrow 1 \text{ as } \theta \rightarrow 0$$

$$\therefore \theta \rightarrow \sin \theta \text{ as } \theta \rightarrow 0$$

$\therefore \sin \theta \approx \theta$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} \text{ [From Half angle identities]}$$

$$\text{As } \theta \rightarrow 0, \cos \theta \rightarrow 1, \sin \frac{\theta}{2} \rightarrow \frac{\theta}{2}$$

$$\therefore \cos \theta \rightarrow 1 - 2 \cdot \left(\frac{\theta}{2}\right)^2 = 1 - \frac{1}{2} \theta^2$$

$\therefore \cos \theta \approx 1 - \frac{1}{2} \theta^2$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \approx \frac{\theta}{1 - \frac{1}{2} \theta^2} = \frac{2\theta}{2 - \theta^2}$$

Now if θ is very small, then $\theta^2 \approx 0$.i.e more smaller than θ

$$\therefore \tan \theta \approx \frac{2\theta}{2} = \theta$$

$\therefore \tan \theta \approx \theta$

The above enclosed results are very vital in differentiation from first principles and in approximation of trigonometric functions.

Area of a segment

Consider the figure below

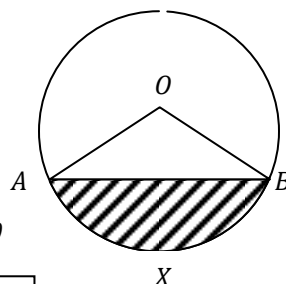
Area of segment = Area of shaded region AXB

(minor)

$$= \text{Area of sector } OAXB - \text{Area of } \triangle OAB$$

$$\Rightarrow \text{Area of segment (minor)} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$\therefore \text{Area of segment (minor)} = \frac{1}{2}r^2(\theta - \sin\theta)$$



Area of segment(AYB) = Area of circle – Area of minor segment

(major)

$$= \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$$

$$\therefore \text{Area of major segment} = \frac{1}{2}r^2(2\pi - \theta + \sin\theta)$$

Conversion of Radians to Degrees

1) Convert

i) 2 radians

ii) 4π radians to degrees.

Solution

i) Using $1\pi \text{ rad} = 180^\circ$ or $1 \text{ rad} = \frac{180^\circ}{\pi}$

$$\Rightarrow 2 \text{ radians} = \frac{180^\circ}{\pi} \times 2 = \frac{360^\circ}{\pi}$$

ii) $4\pi \text{ radians} = \frac{180^\circ}{\pi} \times 4\pi = 720^\circ$

2) Convert

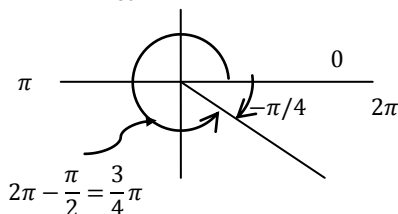
i) -45°

ii) 120° to radians

Solution

i) $1\pi \text{ rad} = 180^\circ$ or $1^\circ = \frac{\pi}{180^\circ}$

$$\Rightarrow -45^\circ = \frac{\pi}{180^\circ} \times -45^\circ = -\frac{\pi}{4} \text{ radians}$$



$$\therefore -45^\circ = \frac{3}{4}\pi \text{ radians}$$

$$\text{ii) } 1\pi \text{ rad} = 180^0, 120^0 = \frac{\pi}{180^0} \times 120^0 = \frac{2}{3}\pi \text{ radians}$$

$$\therefore 120^0 = \frac{2}{3}\pi \text{ radians}$$

Note

Angle measurement in *clockwise direction* give rise to *negative angles*, now concluding in terms of positive angles, proceed as in (i) above.

3) Convert

i) -30^0 to radians

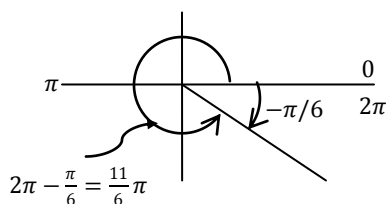
ii) $-\frac{\pi}{3}$ radians to degrees

Solution

i) $1\pi \text{ rad} = 180^0$ or $1^0 = \frac{\pi}{180^0}$

Now $-30^0 = \frac{\pi}{180^0} \times -30 = -\frac{\pi}{6}$

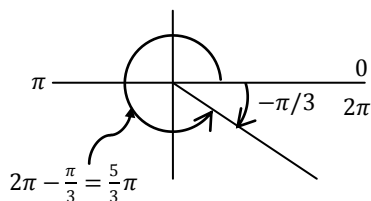
$\therefore -30^0 = 2\pi - \frac{\pi}{6} = \frac{11}{6}\pi$



ii) $1\pi \text{ rad} = 180^0$

$-\frac{\pi}{3} \text{ rad} = 180 \times -\frac{1}{3} = -60^0$

$\therefore -\frac{\pi}{3} \text{ rad} = \frac{5}{3} \times 180 = 300^0$



DIFFERENTIATION

Definition:

The process of finding the gradient function of a curve is *Differentiation*.

Note:

Gradient function is also called the derived function or derivative

Denotion:

If $y = f(x)$ is a curve, then the gradient function of the curve is given by

$$\frac{dy}{dx} = f'(x)$$

Gradient of a curve at any point

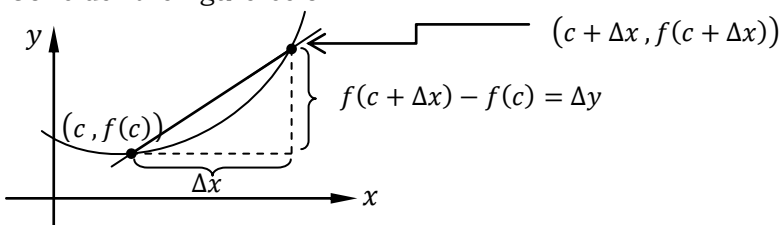
Definition:

The gradient of a curve at any point is the gradient of the tangent to the curve at that point.

When the tangent of a line is determined by means of limits leads to the definition of *derivative of a function*.

The slope of a tangent line at $P(c, f(c))$ can be approximated using a *secant line*

Consider the figure below



If $(c, f(c))$ is the point of tangency and $(c + \Delta x, f(c + \Delta x))$ is the second point on the graph of $f(x)$, the slope of the secant line through the points is given by;

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

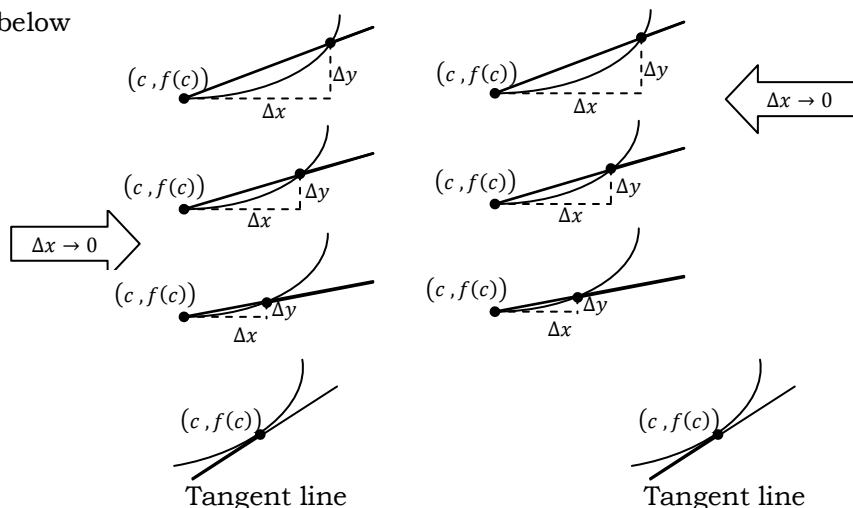
change in y
change in x

Slope of the secant line

The right hand side of this equation is a *Differential Quotient*.

The denominator Δx is the *change in x* and the numerator is the *change in Δy*. i.e. $\Delta y = f(c + \Delta x) - f(c)$.

To approximate accurately the slope of the tangent line, $\Delta x \rightarrow 0$ as in fig. below



Definition of Tangent line with slope m

If $f(x)$ is defined on an open interval containing c and if the limit

$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$, exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of $f(x)$ at the point $(c, f(c))$.

The slope of the tangent line to the graph of $f(x)$ at the point $(c, f(c))$ is also called *slope of the graph of $f(x)$ at c* .

NOTE

Since the tangent of a line is to be determined by limits the four basic limit theorems involving a combination of functions are listed below.

In each theorem we assume that

$\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exists.

Theorem 1: SUM THEOREM OF LIMITS

The limit $\lim_{x \rightarrow a} [f(x) + g(x)]$ exists, and

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

Theorem 2: CONSTANT MULTIPLE THEOREM FOR LIMITS

For any real number, the limit $\lim_{x \rightarrow a} c \cdot f(x)$ exists, and

$$\lim_{x \rightarrow a} c \cdot f(x) = c \lim_{x \rightarrow a} f(x)$$

Theorem 3: PRODUCT THEOREM FOR LIMITS

The limit $\lim_{x \rightarrow a} [f(x) \cdot g(x)]$ exists, and

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

Theorem 4: QUOTIENT THEOREM FOR LIMITS

If $\lim_{x \rightarrow a} g(x) \neq 0$, then $\lim_{x \rightarrow a} f(x) / g(x)$ exists, and $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$

Other limits to be encountered are:

- i) $\lim_{x \rightarrow a} c = c$
- ii) $\lim_{x \rightarrow a} x = a$
- iii) $\lim_{x \rightarrow a} cx = ca$
- iv) $\lim_{x \rightarrow a} x^n = a^n$
- v) $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$ for $a > 0$
- vi) $\lim_{x \rightarrow 0} \sin x = 0$
- vii) $\lim_{x \rightarrow 0} \cos x = 1$
- viii) $\lim_{x \rightarrow a} \frac{1}{x^n} = \frac{1}{a^n}$ for $a \neq 0$
- ix) $\lim_{x \rightarrow a} x^{1/n} = a^{1/n}$

Example 1

The Slope of the Graph of a Linear Function

Find the slope of the graph $f(x) = 2x - 3$ at the point $(2, 1)$

Solution

To find the slope of the graph of $f(x)$ when $c = 2$, apply the definition of the slope of the tangent as follows

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f[2(2+\Delta x) - 3] - f[2(2) - 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 2 = 2 \end{aligned}$$

Thus the slope of $f(x) = 2x - 3$ is $m = 2$

Example 2

Tangent lines to the Graph of Non linear Function

Find the slope of the tangent line to the graph of $f(x) = x^2 + 1$ at the points $(0, 1)$ and $(-1, 2)$

Solution

Let $(x, f(x))$ represent an arbitrary point on the graph of $f(x)$. The slope of the tangent at $(x, f(x))$ is given by

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^2 + 1 - (x^2 + 1)}{\Delta x} \\ \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ \lim_{\Delta x \rightarrow 0} (2x + \Delta x) &= 2x \end{aligned}$$

Thus slope at any point $(x, f(x))$ on the graph of $f(x)$ is $m = 2x$

At the point $(0, 1)$, $m = 2(0) = 0$ and at $(-1, 2)$, $m = 2(-1) = -2$

THE DERIVATIVE OF A FUNCTION

Definition of the Derivative of a Function

The *derivative* of $f(x)$ at x is given by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

, provided the limit exists

Other notations of derivative of a function $y = f(x)$

$$f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], D_x[y]$$

The notation $\frac{dy}{dx}$ using limit notation is;

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

DIFFERENTIATING FROM FIRST PRINCIPLES

(The Differential Coefficient)

The process of calculating the ratio of the incremental change in a function of y of x to the incremental change in x , .i.e. determining an expression for $\frac{\delta y}{\delta x}$ and then finding the limiting value of this ratio as δx approaches zero (i.e. $\delta x \rightarrow 0$) is known as *Differentiating From First Principles*.

NOTE

For a general function $y = f(x)$, $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Examples

1. Differentiate from first principles $f(x) = x^3 + 2x$

Solution

Using the limit process

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3 + 2x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2 + 2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 3x^2 + 3x\Delta x + (\Delta x)^2 + 2 = 3x^2 + 2 \end{aligned}$$

Alternatively

Let $f(x) = y = x^3 + 2x$

Let δy be small increment in y which gives rise to a small increment δx in x

$$y + \delta y = (x + \delta x)^3 + 2(x + \delta x)$$

$$\delta y = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 + 2x + 2\delta x - x^3 - 2x$$

$$\delta y = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 + 2\delta x$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = 3x^2 + 3x\delta x + (\delta x)^2 + 2$$

Introducing limits on both sides

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 3x^2 + 3x\delta x + (\delta x)^2 + 2$$

$$\therefore \frac{dy}{dx} = f'(x) = 3x^2 + 2$$

2. Differentiate from first principles $f(x) = \sqrt{x}$. Hence find the slope of the graph at the points $(1, 1)$ and $(4, 2)$

Solution

$$\text{Using } f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}, \text{ Rationalizing the Numerator}$$

$$= \lim_{\Delta x \rightarrow 0} \left[\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \right] \left[\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right]$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1}{(\sqrt{x+\Delta x} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

At the point $(1, 1)$, the slope is $f'(1) = \frac{1}{2}$

At the point $(4, 2)$, the slope is $f'(4) = \frac{1}{4}$

3. Find from first principles the derivative with respect to t for the function $\frac{2}{t}$

Solution

$$\text{Let } y = \frac{2}{t}$$

$$\frac{dy}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{\frac{2}{t+\Delta t} - \frac{2}{t}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{2t - 2(t+\Delta t)}{(t+\Delta t)t\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{-2\Delta t}{(t+\Delta t)t\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{-2}{(t+\Delta t)t} = -\frac{2}{t^2}$$

Basic Rules in Differentiation

Theorem 1: Constant Rule

The derivative of a constant function is 0 . i. e. If

c is a real number then $\frac{d}{dx}(c) = 0$

Proof:

Let $f(x) = c$, then by limit definition of the derivative

Theorem 2: The Power Rule

If n is a rational number, then the function

$$f(x) = x^n \text{ is differentiable and } \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\begin{aligned} \frac{d}{dx}(c) &= f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c - c}{\Delta x} = 0 \end{aligned}$$

Proof:

If n is a positive integer greater than 1, the binomial expansion produces the following

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}(\Delta x)^2 + \dots + (\Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2!}(\Delta x)^2 + \dots + (\Delta x)^n}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x \left[nx^{n-1} + \frac{n(n-1)}{2!}(\Delta x)^{2-1} + \dots + (\Delta x)^{n-1} \right]}{\Delta x} = nx^{n-1} \end{aligned}$$

NOTE:

1. For $n = 1$, $\frac{d}{dx}(x) = 1$
2. The binomial expansion is used in proving a special case of the above Power Rule. i. e.

$$(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}(\Delta x)^2 + \dots + (\Delta x)^n$$

Theorem 3: The Constant Multiple Rule

If $f(x)$ is a differentiable function and c is a real number, then $c.f(x)$ is also differentiable and

$$\frac{d}{dx}(c.f(x)) = c.f'(x)$$

Proof:

$$\begin{aligned} \frac{d}{dx}(c.f(x)) &= \lim_{\Delta x \rightarrow 0} \frac{c.f(x+\Delta x) - c.f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{c[f(x+\Delta x) - f(x)]}{\Delta x} = c \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= c.f'(x) \end{aligned}$$

NOTE:

The constant Multiple Rule states that a constant can be factored out of the differentiation process, even if it appears in the denominator.

$$\begin{aligned} \frac{d}{dx}(c.f(x)) &= c.\frac{d}{dx}(f(x)) = c.f'(x) \\ \frac{d}{dx}\left(\frac{f(x)}{c}\right) &= \frac{1}{c}.\frac{d}{dx}(f(x)) = \frac{1}{c}.f'(x) \end{aligned}$$

Theorem 4: The Sum and Difference Rules

If the sum (or difference) of two differentiable functions is differentiable and is sum(or difference) of their derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x); \text{ Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x); \text{ Difference Rule}$$

Proof:

$$\begin{aligned} \frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) + g(x+\Delta x) - [f(x) + g(x)]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] = f'(x) + g'(x) \end{aligned}$$

The sum and the difference can be extended to any number of finite functions. e.g.

If $F(x) = f(x) + g(x) - h(x) - k(x)$, then

$$F'(x) = f'(x) + g'(x) - h'(x) - k'(x)$$

Examples

Find the derivative of:

a) $f(x) = x^3$

b) $g(x) = \sqrt[3]{x}$

c) $y = \frac{1}{x^2}$

d) $y = \frac{2}{x}$

e) $f(x) = x^3 - 4x + 5$

f) $y = -\frac{x^4}{2} + 3x^3 - 2x$

e) $f(t) = \frac{4t^2}{5}$

f) $y = \frac{1}{2\sqrt[3]{x^2}}$

g) $y = -\frac{3}{2}x$

Solution

The following order can be used .i.e.

Given function	Rewrite	Differentiate	Simplify
$f(x) = x^3$		$f'(x) = 3x^2$	
$g(x) = \sqrt[3]{x}$	$g(x) = x^{\frac{1}{3}}$	$g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$	$g'(x) = \frac{1}{3\sqrt[3]{x^2}}$
$y = \frac{1}{x^2}$	$y = x^{-2}$	$\frac{dy}{dx} = (-2)x^{-3}$	$\frac{dy}{dx} = -\frac{2}{x^3}$

Given function	Rewrite	Differentiate	Simplify
$y = \frac{2}{x}$	$y = 2x^{-1}$	$\frac{dy}{dx} = (-2)x^{-2}$	$\frac{dy}{dx} = -\frac{2}{x^2}$
$f(t) = \frac{4t^2}{5}$	$f(t) = \frac{4t^2}{5}$	$f'(t) = \frac{4}{5}(2)t$	$f'(t) = \frac{8}{5}t$
$y = \frac{1}{2\sqrt[3]{x^2}}$	$y = \frac{1}{2}x^{-\frac{2}{3}}$	$\frac{dy}{dx} = \left(\frac{1}{2}\right)\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}$	$\frac{dy}{dx} = -\frac{1}{3\sqrt[3]{x^5}}$
$y = \frac{1}{x^2}$	$y = x^{-2}$	$\frac{dy}{dx} = (-2)x^{-3}$	$\frac{dy}{dx} = -\frac{2}{x^3}$
$y = -\frac{3}{2}x$	$y = -\frac{3}{2}x$	$\frac{dy}{dx} = \left(-\frac{3}{2}\right)(1)$	$\frac{dy}{dx} = -\frac{3}{2}$
$f(x) = x^3 - 4x + 5$		$f'(x) = 3x^2 - 4$	$f'(x) = 3x^2 - 4$
$y = -\frac{x^4}{2} + 3x^3 - 2x$		$\frac{dy}{dx} = -\frac{4}{2}x^3 + 3(3x^2) - 2$	$\frac{dy}{dx} = -2x^3 + 9x^2 - 2$

The Product Rule and the Quotient Rule

Theorem 5: The Product Rule

The product of two differentiable functions, $f(x)$ and $g(x)$, is itself differentiable. Moreover, the derivative of $f(x).g(x)$ is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x).f'(x)$$

Proof:

$$\frac{d}{dx}[f(x).g(x)] = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x).g(x+\Delta x) - [f(x).g(x)]}{\Delta x}$$

Now adding and subtracting $f(x+\Delta x).gx$ (Take note of this)

$$\begin{aligned} \frac{d}{dx}[f(x).g(x)] &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x).g(x+\Delta x) - f(x+\Delta x).gx + f(x+\Delta x).gx - [f(x).g(x)]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[f(x+\Delta x). \frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x). \frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[f(x+\Delta x). \frac{g(x+\Delta x) - g(x)}{\Delta x} \right] + \lim_{\Delta x \rightarrow 0} \left[g(x). \frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} f(x+\Delta x). \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} g(x). \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \end{aligned}$$

$$\therefore \frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x).f'(x)$$

Extending the Product Rule to cover more than 2 – terms .e.g if $f(x)$, $g(x)$ and $h(x)$ are differentiable functions of x , then

$$\frac{d}{dx}[f(x).g(x).h(x)] = f'(x).g(x)h(x) + g'(x).f(x).h(x) + h'(x)f(x).g(x)$$

NOTE:

The alternative statement of product rule is when $f(x) = u$ and $g(x) = v$ is:

$$\frac{d}{dx}(u \cdot v) = v \frac{du}{dx} + u \frac{dv}{dx}$$

Proving from first principles

Let $y = u \cdot v$

If $\delta y, \delta u, \delta v$ are increments in y, u and v respectively, corresponding to an increment δx in x , then

$$\begin{aligned} y + \delta y &= (u + \delta u) \cdot (v + \delta v) \\ &= uv + u\delta v + v\delta u + \delta u \cdot \delta v \end{aligned}$$

By subtraction

$$\delta y = uv + u\delta v + v\delta u + \delta u \cdot \delta v - uv$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} + \delta u \cdot \frac{\delta v}{\delta x}$$

As $\delta x \rightarrow 0, \delta u \rightarrow 0$ and $\frac{\delta u}{\delta x}, \frac{\delta v}{\delta x}, \frac{\delta y}{\delta x}$ tend respectively to $\frac{du}{dx}, \frac{dv}{dx}$ and $\frac{dy}{dx}$

$$\frac{d}{dx}(u \cdot v) = v \frac{du}{dx} + u \frac{dv}{dx}$$

Example

Find the derivative of;

a) $h(x) = (3x - 2x^2)(5 + 4x)$

b) $y = (1 + x^2)(1 - 2x^2)$

Solution

a)
$$\begin{aligned} h'(x) &= \overbrace{(3x - 2x^2)}^{\text{First}} \cdot \overbrace{\frac{d}{dx}(5 + 4x)}^{\text{Derivative of } 2^{\text{nd}}} + \overbrace{(5 + 4x)}^{\text{Second}} \cdot \overbrace{\frac{d}{dx}(3x - 2x^2)}^{\text{Derivative of } 1^{\text{st}}} \\ &= (3x - 2x^2) \cdot (4) + (5 + 4x) \cdot (3 - 4x) \\ &= 12x - 8x^2 + 15 - 8x - 16x^2 \\ &= -24x^2 + 4x + 15 \end{aligned}$$

Alternatively; Not using Product Rule

$$h(x) = (3x - 2x^2)(5 + 4x) = -8x^3 + 2x^2 + 15x$$

$$h'(x) = \frac{d}{dx}[-8x^3 + 2x^2 + 15x] = -24x^2 + 4x + 15$$

b) $y = (1 + x^2)(1 - 2x^2)$

$$\begin{aligned} \frac{dy}{dx} &= (1 + x^2) \cdot \frac{d}{dx}(1 - 2x^2) + (1 - 2x^2) \cdot \frac{d}{dx}(1 + x^2) \\ &= (1 + x^2)(-4x) + (1 - 2x^2)(2x) \\ &= -8x^3 - 2x \end{aligned}$$

The Quotient Rule

Theorem 6: the Quotient Rule

The quotient $\frac{f(x)}{g(x)}$ of two differentiable functions $f(x)$ and $g(x)$, itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of $\frac{f(x)}{g(x)}$ is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator divided by the square of the denominator.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}, g(x) \neq 0$$

Proof:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \lim_{\Delta x \rightarrow 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{g(x) \cdot f(x+\Delta x) - f(x) \cdot g(x+\Delta x)}{\Delta x \cdot g(x) \cdot g(x+\Delta x)}$$

Now adding and subtracting $f(x) \cdot g(x)$ (Take note of this)

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \lim_{\Delta x \rightarrow 0} \frac{g(x) \cdot f(x+\Delta x) - f(x) \cdot g(x) + f(x) \cdot g(x) - f(x) \cdot g(x+\Delta x)}{\Delta x \cdot g(x) \cdot g(x+\Delta x)}$$

$$\begin{aligned} &= \frac{\lim_{\Delta x \rightarrow 0} g(x) \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] - \lim_{\Delta x \rightarrow 0} f(x) \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right]}{\lim_{\Delta x \rightarrow 0} g(x) \cdot g(x+\Delta x)} \\ &= \frac{g(x) \lim_{\Delta x \rightarrow 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right] - f(x) \lim_{\Delta x \rightarrow 0} \left[\frac{g(x+\Delta x) - g(x)}{\Delta x} \right]}{g(x) \lim_{\Delta x \rightarrow 0} g(x+\Delta x)} \end{aligned}$$

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Example

Find the derivative of;

a) $y = \frac{5x-2}{x^2+1}$

b) $f(x) = \frac{1-3x^2}{2+4x^2}$

Solution

$$\begin{aligned} \text{a) } \frac{dy}{dx} &= \frac{(x^2+1) \cdot \frac{d}{dx}(5x-2) - (5x-2) \cdot \frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1)(5) - (5x-2)(2x)}{(x^2+1)^2} \\ &= \frac{5x^2+5-10x^2-4x}{(x^2+1)^2} = \frac{-5x^2+4x+5}{(x^2+1)^2} \end{aligned}$$

$$\begin{aligned} \text{b) } f'(x) &= \frac{(2+4x^2) \cdot \frac{d}{dx}(1-3x^2) - (1-3x^2) \cdot \frac{d}{dx}(2+4x^2)}{(2+4x^2)^2} = \frac{(2+4x^2) \cdot (-6x) - (1-3x^2) \cdot (8x)}{(2+4x^2)^2} \\ &= \frac{-12x-24x^3-8x+24x^3}{(2+4x^2)^2} = -\frac{20x}{[2(1+2x^2)]^2} = -\frac{5x}{(1+2x^2)^2} \end{aligned}$$

Remark

Prove that $\frac{d}{dx} [x^n] = nx^{n-1}$ for a negative integer exponent

Solution

Let $n = -k$

$$\Rightarrow \frac{d}{dx} [x^n] = \frac{d}{dx} \left[\frac{1}{x^k} \right]$$

By quotient rule,

$$\frac{d}{dx} [x^{-k}] = \frac{x^k \cdot (0) - (1) \cdot (kx^{k-1})}{x^{2k}} = -kx^{-k-1}$$

$$\therefore \frac{d}{dx} [x^{-k}] = -kx^{-k-1} = nx^{n-1}$$

The Chain Rule

Theorem 7: The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then $y = f(g(x))$ is a differentiable function of x and;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\text{Or equivalently } \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

NOTE

The chain rule requires the General Power Rule. This rule is applicable when differentiating functions of the form

$$y = [u(x)]^n$$

Theorem 8: The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1} \cdot \frac{du}{dx}$$

Or equivalently

$$\frac{d}{dx} [u^n] = nu^{n-1} \cdot u'$$

Proof:

Because $y = u^n$, apply chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du} (u^n) \cdot \frac{du}{dx}$$

Example

Find the derivative of:

a) $f(x) = (3x - 2x^2)^3$

b) $y = (1 - 3x^2)^5$

Solution

a) $f(x) = (3x - 2x^2)^3$, let $u = 3x - 2x^2$

$$\Rightarrow f(x) = u^3$$

Applying the General Power Rule

$$\begin{aligned} \Rightarrow f'(x) &= \overbrace{3}^n \cdot \overbrace{(3x - 2x^2)^2}^{u^{n-1}} \cdot \overbrace{\frac{d}{dx} (3x - 2x^2)}^{u'} \\ &= 3(3x - 2x^2)(3 - 4x) \end{aligned}$$

b) $y = (1 - 3x^2)^5$, let $u = 1 - 3x^2$

Applying the General Power Rule

$$\begin{aligned}\frac{dy}{dx} &= 5(1 - 3x^2)^4 \cdot \frac{d}{dx}(1 - 3x^2) \\ &= 5(1 - 3x^2)^4 \cdot (-6x) = -30x(1 - 3x^2)^4\end{aligned}$$

Differentiating Functions involving Radicals

Find $\frac{dy}{dx}$ if $y = \sqrt[3]{(x^2 - 1)^2}$. Hence find all points on the curve for which

$$\frac{dy}{dx} = 0$$

Solution

Rewriting the function

$$y = (x^2 - 1)^{2/3}, \text{ let } u = x^2 - 1$$

Applying the General Power Rule

$$\begin{aligned}\Rightarrow \frac{dy}{dx} &= \overbrace{\frac{2}{3} \cdot (x^2 - 1)^{-1/3}}^n \cdot \overbrace{\frac{d}{dx}(x^2 - 1)}^{u^{n-1}} \cdot \overbrace{2x}^{u'} \\ &= \frac{2}{3} (x^2 - 1)^{-1/3} (2x) = \frac{4x}{3\sqrt[3]{x^2 - 1}}\end{aligned}$$

is the point when $\frac{dy}{dx} = 0$

Differentiating Quotients with Constant Numerator

$$\text{Differentiate } g(t) = -\frac{7}{(2t-3)^2}$$

Solution

Re-writing the function

$$g(t) = -\frac{7}{(2t-3)^2} = -7 \cdot (2t-3)^{-2}$$

Applying the General Power Rule

$$\begin{aligned}g'(t) &= -7 \cdot (-2) \cdot (2t-3)^{-3} \cdot (2) \\ &= 28(2t-3)^{-3} = \frac{28}{(2t-3)^3}\end{aligned}$$

Simplifying Derivatives

The knowledge of simplifying indices is necessary

a) Simplifying by Factoring out the Least Powers

$$\text{Find } f'(x) \text{ if } f(x) = x^2\sqrt{1-x^2}$$

Solution

$$f(x) = x^2\sqrt{1-x^2} = f(x) = x^2(1-x^2)^{1/2}$$

Rewrite

$$f'(x) = x^2 \cdot \frac{d}{dx}(1-x^2)^{1/2} + (1-x^2)^{1/2} \cdot \frac{d}{dx}(x^2)$$

Product Rule

$$= x^2 \cdot \frac{1}{2} \cdot (1-x^2)^{-1/2} \cdot (-2x) + (1-x^2)^{1/2} \cdot (2x)$$

General Power Rule

$$= -x^3 \cdot (1-x^2)^{-1/2} + 2x \cdot (1-x^2)^{1/2}$$

Simplify

$$= x(1-x^2)^{-1/2}[-x^2 \cdot (1) + 2 \cdot (1-x^2)]$$

Factoring

$$= \frac{x(2-3x^2)}{\sqrt{1-x^2}}$$

Simplify

b) Simplifying Derivative of a Quotient

Given that $f(x) = \frac{x}{\sqrt[3]{x^2+4}}$, find $f'(x)$

Solution

$$f(x) = \frac{x}{\sqrt[3]{x^2+4}} = \frac{x}{(x^2+4)^{1/3}}$$

Rewrite

$$f'(x) = \frac{(x^2+4)^{1/3} \cdot (1-x \cdot \frac{1}{3}(x^2+4)^{-2/3} \cdot (2x))}{(x^2+4)^{2/3}}$$

Quotient Rule

$$= \frac{1}{3} \cdot (x^2+4)^{-2/3} \left[\frac{3(x^2+4) - 2x^2}{(x^2+4)^{2/3}} \right]$$

Factoring

$$= \frac{x^2+12}{3(x^2+4)^{4/3}}$$

Simplifying

c) Simplifying Derivative of a Power

Find $\frac{dy}{dx}$ if $y = \left(\frac{3x-1}{x^2+3}\right)^2$

Solution

$$\frac{dy}{dx} = 2 \cdot \left(\frac{3x-1}{x^2+3}\right) \cdot \frac{d}{dx} \left(\frac{3x-1}{x^2+3}\right)$$

General Power Rule

$$= \left[\frac{2(3x-1)}{x^2+3} \right] \cdot \left[\frac{(x^2+3) \cdot 3 - (3x-1) \cdot 2x}{(x^2+3)^2} \right]$$

Quotient Rule

$$= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)(x^2+3)^2}$$

Simplify

$$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$$

Simplify

Task

1. Find the first derivative of the algebraic function

a) $y = (2x - 7)^3$,

Ans: $\frac{dy}{dx} = 6(2x - 7)^2$

b) $g(x) = 3(4 - 9x)^4$,

Ans: $g'(x) = -108(4 - 9x)^3$

c) $f(x) = (9 - x^2)^{\frac{2}{3}}$,

Ans: $f'(x) = \frac{2}{9}(9 - x^2)^{-\frac{1}{3}} \cdot (-2x) = \frac{-4x}{3(9 - x^2)^{\frac{2}{3}}}$

d) $y = \sqrt[3]{(9x^2 + 4)}$,

Ans: $\frac{dy}{dx} = \frac{1}{3}(9x^2 + 4)^{-2/3} \cdot (18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$

e) $f(t) = \frac{1}{(t-3)^2}$,

Ans: $f'(t) = -2(t-3)^{-3}(1) = -\frac{2}{(t-3)^3}$

f) $y = \frac{1}{\sqrt{x+2}}$,

Ans: $\frac{dy}{dx} = -\frac{1}{2(x+2)^{3/2}}$

g) $y = \frac{x}{\sqrt{x^2+2}}$,

Ans: $\frac{dy}{dx} = \frac{1}{(x^2+2)^{3/2}}$

2. Evaluate the derivative of the function at the given point;

Function	Point	Answer
$s(t) = \sqrt{t^2 + 2t + 8}$	(2, 4)	$\frac{3}{4}$
$f(x) = \frac{3}{x^3-4}$	$(-1, \frac{5}{3})$	$-\frac{9}{25}$
$f(t) = \frac{3t+2}{t-1}$	(0, -2)	-5

3. a) Given that $y = \sqrt[3]{\frac{x-1}{(x^2-1)^2}}$, show that $\frac{dy}{dx} = \frac{1-3x}{3(x+1)^{5/3}(x-1)^{4/3}}$
- b) Given that $y = \sqrt{\frac{(x+1)^3}{x+2}}$, show that $\frac{dy}{dx} = \frac{(2x+5)\sqrt{x+1}}{2(x+2)^{3/2}}$
4. a) P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$ on the graph $y = \sqrt{x}$. Show that $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$, and hence find the gradient of the curve at the point P .
- b) A curve is defined by $y = f(x)$. Show from first principles that $\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \left\{ \frac{f(x+\delta x) - f(x)}{\delta x} \right\}$. Hence find the gradient function of the curve;
- i) $y = \frac{1}{x^2}$ **Ans:** $\frac{dy}{dx} = -\frac{2}{x^3}$
- ii) $y = \frac{1}{\sqrt[3]{8-x}}$ **Ans:** $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{(8-x)^2}}$
5. Differentiate from first principles
- a) $y = \frac{1}{x^2+1}$
- b) $y = \frac{1}{x^2+x-2}$
- c) $y = \frac{2}{\sqrt[3]{8-x^2}}$
- d) $y = \sqrt[3]{x^2+1}$

Higher-order Derivatives

When a curve $y = f(x)$ is differentiated once, $\frac{dy}{dx}$ is of first order. When it is differentiated twice, $\frac{d^2y}{dx^2}$ is of second order.

If it is differentiated thrice, $\frac{d^3y}{dx^3}$ is of third order.

In general if it is differentiated n - times, i.e. $\frac{d^ny}{dx^n}$ is of n - order.

First derivative: $f'(x) = \frac{dy}{dx} = \frac{d}{dx}[f(x)]$.

Second derivative: $f''(x) = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}[f(x)]$.

Third derivative: $f'''(x) = \frac{d^3y}{dx^3} = \frac{d^3}{dx^3}[f(x)]$

⋮

n^{th} derivative: $f^n(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}[f(x)]$.

$$\begin{aligned}
 \text{d) } \frac{dz}{dx} &= \frac{d}{dx} [x \cdot y^2] = x \cdot \frac{d}{dx} [y^2] + y^2 \cdot \frac{d}{dx} [x] && \text{Apply Product Rule} \\
 &= x \cdot \left(2y \frac{dy}{dx} \right) + y^2 \cdot (1) && \text{Apply Chain Rule} \\
 &= 2xy \frac{dy}{dx} + y^2 && \text{Simplify}
 \end{aligned}$$

Guidelines for Implicit Differentiation of an equation

1. Differentiate both sides of the equation with respect to x
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all the other terms to the right side of the equation
3. Factor $\frac{dy}{dx}$ out of the left side of the equation
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the left hand factor that does not contain $\frac{dy}{dx}$

Example

Find $\frac{dy}{dx}$ given that $y^3 + y^2 - 5y - x^2 = -4$

Solution

1. Differentiate both sides of the equation w.r.t x

$$\begin{aligned}
 \frac{d}{dx} [y^3 + y^2 - 5y - x^2] &= \frac{d}{dx} (-4) \\
 \frac{d}{dx} [y^3] + \frac{d}{dx} [y^2] - \frac{d}{dx} [5y] - \frac{d}{dx} [x^2] &= \frac{d}{dx} [-4] \\
 3y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} - 5 \frac{dy}{dx} - 2x &= 0
 \end{aligned}$$
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation
$$3y^2 \frac{dy}{dx} + 2y \frac{dy}{dx} - 5 \frac{dy}{dx} = 2x$$
3. Factor $\frac{dy}{dx}$ out of the left side of the equation
$$(3y^2 + 2y - 5) \frac{dy}{dx} = 2x$$
4. Solve for $\frac{dy}{dx}$ by dividing both sides by $(3y^2 + 2y - 5)$

$$\therefore \frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

Finding the Slope of a Graph Implicitly

1. Determine the slope of the line too the graph of $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -\frac{1}{\sqrt{2}} \right)$

Solution

Differentiating the graph w.r.t x

$$2x + 8y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{4y}$$

$$\text{Slope at } \left(\sqrt{2}, -\frac{1}{\sqrt{2}} \right) \text{ is } -\frac{\sqrt{2}}{4 \cdot (-1/\sqrt{2})} = \frac{1}{2}$$

2. Determine the slope of the graph $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$

Solution

$$\begin{aligned}\frac{d}{dx}[3(x^2 + y^2)^2] &= \frac{d}{dx}(100xy) \\ 3 \cdot (2) \cdot (x^2 + y^2) \cdot \left(2x + 2y \frac{dy}{dx}\right) &= 100 \left[x \cdot \frac{dy}{dx} + y\right] \\ 12x(x^2 + y^2) + 12y(x^2 + y^2) \cdot \frac{dy}{dx} &= 100x \frac{dy}{dx} + 100y \\ 12y(x^2 + y^2) \cdot \frac{dy}{dx} - 100x \frac{dy}{dx} &= 100y - 12x(x^2 + y^2) \\ [12y(x^2 + y^2) - 100x] \frac{dy}{dx} &= 100y - 12x(x^2 + y^2) \\ \therefore \frac{dy}{dx} &= \frac{100y - 12x(x^2 + y^2)}{12y(x^2 + y^2) - 100x} = \frac{25y - 3x(x^2 + y^2)}{3y(x^2 + y^2) - 25x}\end{aligned}$$

At (3, 1), slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)((3)^2 + (1)^2)}{3(1)((3)^2 + (1)^2) - 25(3)} = \frac{25 - 90}{30 - 75} = \frac{13}{9}$$

Task

Find $\frac{dy}{dx}$ by implicit differentiation and evaluate the derivative at the indicated point

Equation	Point	Answer
$xy = 4$	$(-4, -1)$	$-y/x, -1/4$
$y^2 = \frac{x^2 - 9}{x^2 + 9}$	$(3, 0)$	$\frac{18x}{(x^2 + y^2)y}, \text{ undefined}$
$x^{2/3} + y^{2/3} = 5$	$(8, 1)$	$-\sqrt[3]{y/x}, -\frac{1}{2}$

Finding the Second Derivative Implicitly

Given that $y^2 + x^2 = 25$, find $\frac{d^2y}{dx^2}$

Solution

$$2y \frac{dy}{dx} + 2x = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Differentiating w.r.t.x

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{y(-1) - (-x) \cdot \frac{dy}{dx}}{y^2} = \frac{-y + x(-x/y)}{y^2} \\ &= \frac{y^2 - x^2}{y^3} = -\left(\frac{x^2 + y^2}{y^3}\right) = -\frac{25}{y^3}\end{aligned}$$

Task

- If $x^2 + y^2 = y$, show that $\frac{dy}{dx} = \frac{2x}{1-2y}$. Hence show also that $\frac{d^2y}{dx^2} = \frac{2(1-8y)}{(1-2y)^3}$
- Find $\frac{d^2y}{dx^2}$ in terms of x and y if
 - $x^2 + xy = 5$ **Ans:** $10/x^3$
 - $x^2 - y^2 = 16$ **Ans:** $-16/y^3$
- Find the equations of the tangent line and normal to the circle at the indicated points. (The normal line at a point is perpendicular to the tangent line at the point.)

Circle: $x^2 + y^2 = 25$

Point: $(-3, 4), (4, 3)$

Answer:

At $(-3, 4)$

Tangent: $3x - 4y + 25 = 0$

Normal: $4x + 3y = 0$

At $(4, 3)$

Tangent: $4x + 3y - 25 = 0$

Normal: $3x - 4y = 0$

Proofs in Implicit Differentiation

- a) If $y = \sqrt{5x^2 + 3}$, show that $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$
 b) If $y = \sqrt[3]{3 - x^2}$, show that $3y \frac{d^2y}{dx^2} + 6y \left(\frac{dy}{dx}\right)^2 + 2 = 0$
 c) If $y = \frac{1}{\sqrt{x^2 - 3}}$, show that $\frac{d^2y}{dx^2} + \frac{6}{y} \frac{dy}{dx} = \frac{1}{y^5}$
 d) If $y^2 - 2y\sqrt{1 + x^2} + x^2 = 0$, show that $\frac{dy}{dx} = \frac{x}{\sqrt{1 + x^2}}$

Solution

a) $y = \sqrt{5x^2 + 3} \Rightarrow y^2 = 5x^2 + 3$
 $\Rightarrow 2y \frac{dy}{dx} = 10x \Rightarrow y \frac{dy}{dx} = 5x$

Differentiating w.r.t. x

$$y \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (y) = 5$$

$$y \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx} = 5$$

$$\therefore y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$$

b) Left as an exercise

c) Left as an exercise

d) $y^2 - 2y\sqrt{1 + x^2} + x^2 = 0$

Rewriting and differentiating w.r.t. x

$$y^2 - 2y(1 + x^2)^{1/2} + x^2 = 0$$

$$\Rightarrow 2y \frac{dy}{dx} - 2 \left[(1 + x^2)^{\frac{1}{2}} \cdot \frac{dy}{dx} + y \cdot \frac{1}{2} (1 + x^2)^{-1/2} \cdot 2x \right] + 2x = 0$$

$$\Rightarrow 2y \frac{dy}{dx} - 2\sqrt{1 + x^2} \frac{dy}{dx} - \frac{2xy}{\sqrt{1 + x^2}} + 2x = 0$$

$$\Rightarrow 2 \left[y - \sqrt{1 + x^2} \right] \frac{dy}{dx} - \frac{2xy}{\sqrt{1 + x^2}} + 2x = 0$$

$$\Rightarrow \left[y - \sqrt{1 + x^2} \right] \frac{dy}{dx} = \frac{xy}{\sqrt{1 + x^2}} - x = \frac{x \left[y - \sqrt{1 + x^2} \right]}{\sqrt{1 + x^2}} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{1 + x^2}}$$

DIFFERENTIATING PARAMETRIC EQUATIONS

If two equations $x = f(t)$ and $y = \varphi(t)$ are functions of the same variable t , t is called a *parameter*.

To differentiate .i.e. to find $\frac{dy}{dx}$, chain rule is used.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

For the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$$

Example

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if:

a) $x = t^3 + t^2$ $y = t^2 + t$

b) $x = \frac{2t}{t+2}$, $y = \frac{3t}{t+3}$

c) $x = \frac{t}{1-t}$, $y = \frac{t^2}{1-t}$

Solution

a) $x = t^3 + t^2 \Rightarrow \frac{dx}{dt} = 3t^2 + 2t$

$$\Rightarrow \frac{dy}{dt} = 2t + 1$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = (2t + 1) \times \frac{1}{3t^2 + 2t} = \frac{2t+1}{3t^2+2t}$$

$$\therefore \frac{dy}{dx} = \frac{2t+1}{3t^2+2t}$$

To find $\frac{d^2y}{dx^2}$, proceed using the formula $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$

$$\begin{aligned} \Rightarrow \frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{2t+1}{3t^2+2t} \right) \cdot \frac{dt}{dx} \\ &= \frac{(3t^2+2t) \cdot (2) - (2t+1) \cdot (6t+2)}{(3t^2+2t)^2} \cdot \frac{1}{(3t^2+2t)} \\ &= \frac{6t^2+4t - (12t^2+4t+6t+2)}{(3t^2+2t)^3} = \frac{-6t^2-6t+2}{(3t^2+2t)^3} \end{aligned}$$

b) Left as an exercise

c) Left as an exercise

2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ on the curve defined by parametric equations

$$x = \frac{t}{1+t}, y = \frac{t^3}{1+t}$$

Solution

$$x = \frac{t}{1+t} \Rightarrow \frac{dx}{dt} = \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$y = \frac{t^3}{1+t} \Rightarrow \frac{dy}{dt} = \frac{(1+t) \cdot 3t^2 - t^3}{(1+t)^2} = \frac{3t^2+2t^3}{(1+t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2+2t^3}{(1+t)^2} \times \frac{(1+t)^2}{1}$$

$$\therefore \frac{dy}{dx} = 3t^2 + 2t^3$$

Now to obtain the slope at $\left(\frac{1}{2}, \frac{1}{2}\right)$ a suitable value of t is needed.

Using $x = \frac{t}{1+t}, y = \frac{t^3}{1+t}$

The general point on the curve is $\left(\frac{t}{1+t}, \frac{t^3}{1+t}\right)$

Since $\left(\frac{1}{2}, \frac{1}{2}\right)$ is on the curve, then $\frac{t}{1+t} = \frac{1}{2} \Rightarrow 2t = 1 + t, t = 1$

$$\therefore \frac{dy}{dx} \text{ at } \left(\frac{1}{2}, \frac{1}{2}\right) = \frac{dy}{dx} \text{ at } t=1 = 3(1)^2 + 2(1)^3 = 5$$

Note

If $\frac{1}{2} = \frac{t^3}{1+t}$, the y -value being used in evaluation of t , t value would be $t =$

1. Thus you can use any equation in evaluating the t – value.

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\ &= \frac{d}{dt} (3t^2 + 2t^3) \cdot \frac{(1+t^2)}{1} \\ &= (6t + 6t^2) \cdot (1 + t^2) = 6t(1 + t)(1 + t^2) \end{aligned}$$

Since $t = 1, \frac{d^2y}{dx^2} = 6(1)(1 + 1)(1 + 1^2) = 24$

$$\therefore \frac{d^2y}{dx^2} \text{ at } t=1 = 24$$

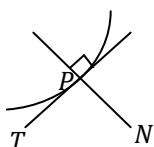
Task

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate each at the specified value of the parameter.

Parametric equations	Point	Answers
$x = 2t, y = 3t - 1$	$t = 3$	$\frac{dy}{dx} = \frac{3}{2}, \frac{d^2y}{dx^2} = 0$
$x = t + 1, y = t^2 + 3t$	$t = -1$	$\frac{dy}{dx} = 2t + 3, \frac{d^2y}{dx^2} = 2$
$x = (t^2 - 1)^2, y = t^3$	$t = \sqrt{2}$	At $t = 1, \frac{dy}{dx} = 1, \frac{d^2y}{dx^2} = 2$ $\frac{dy}{dx} = \frac{3t}{4(t^2-1)}, \frac{d^2y}{dx^2} = -\frac{3(1+t^2)}{16t(t^2-1)^3}$ At $t = \sqrt{2}, \frac{dy}{dx} = \frac{3\sqrt{2}}{4}, \frac{d^2y}{dx^2} = \frac{9\sqrt{2}}{32}$
$x = \frac{t^2}{1+t^3}, y = \frac{t^3}{1+t^3}$		$\frac{dy}{dx} = \frac{3t}{2-t^3}, \text{ for } t = 1, \frac{dy}{dx} = 2, \frac{d^2y}{dx^2} = 48$

TANGENTS AND NORMALS

Definition: A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point.



$\overline{PT}$ = Tangent at P

$\overline{PN}$ = Normal at P

From the previous discussion, if $P(x_1, y_1)$ is a point on the curve, then

$$\text{Slope of the tangent at } P(x_1, y_1) = \frac{dy}{dx} \text{ at } P(x_1, y_1)$$

$$\text{Slope of the normal at } P(x_1, y_1) = -\frac{1}{\frac{dy}{dx} \text{ at } P(x_1, y_1)}$$

Example

- Find the equation of the tangent and normal to the curve $y = 3x^2 - x + 1$ at $x = 0$

Solution

$$y = 3x^2 - x + 1, \text{ for } x = 0, y = 1 \therefore \text{point is } (0, 1)$$

Tangent

$$\frac{y-1}{x-0} = m_1, m_1 = \frac{dy}{dx} \text{ at } (0, 1)$$

$$\frac{dy}{dx} = 6x - 1 \therefore m_1 = 6(0) - 1 = -1$$

$$\frac{y-1}{x-0} = -1 \therefore y = -x + 1$$

Normal

$$\frac{y-1}{x-0} = m_2, m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$$

$$\therefore \frac{y-1}{x-0} = 1 \Rightarrow y = x + 1$$

- Find the equation of the tangents to the curve $y = x^3 - 6x^2 + 12x + 12$ which are parallel to the line $y = 3x$

Solution

Let the required tangent be $y = mx + c$

Now the tangent is parallel to the line $y = 3x \Rightarrow m = 3$. i.e. for parallel lines $m_1 = m_2$

$$\text{Now slope of the tangent at a point } P(x, y) = \frac{dy}{dx}$$

$$\Rightarrow 3 = 3x^2 - 12x + 12 \text{ or } -4x + 3 = 0$$

$$\Rightarrow (x-1)(x-3) = 0 \therefore x = 1, 3$$

Now required tangents are at points $x = 1, x = 3$

For $x = 1, y = ?$, $x = 3, y = ?$

$$\text{For } x = 1, y = 1 - 6 + 12 + 12 = 19 \therefore (1, 19)$$

$$\text{For } x = 3, y = 27 - 54 + 36 + 12 = 3 \therefore (3, 3)$$

Required tangents

$$\frac{y-19}{x-1} = 3 \Rightarrow 3x - y + 16 = 0$$

$$\frac{y-3}{x-3} = 3 \Rightarrow 3x - y - 3 = 0$$

3. Find the equation of the normal to the curve $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$ at the point

$$\left(\frac{1}{2}, \frac{1}{2}\right)$$

Solution

Point: $\left(\frac{1}{2}, \frac{1}{2}\right)$

Normal's equation

$$\frac{y-1/2}{x-1/2} = m'$$

But $m' = -\frac{1}{\text{slope of the tangent at } \left(\frac{1}{2}, \frac{1}{2}\right)}$

Slope of the tangent at $P(x, y) = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$x = \frac{t}{1+t} \Rightarrow \frac{dx}{dt} = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$y = \frac{t^3}{1+t} \Rightarrow \frac{dy}{dt} = \frac{(1+t) \cdot 3t^2 - t^3}{(1+t)^2} = \frac{3t^2 + 2t^3}{(1+t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 + 2t^3}{(1+t)^2} \times \frac{(1+t)^2}{1} = 3t^2 + 2t^3$$

To obtain the slope of the tangent at $\left(\frac{1}{2}, \frac{1}{2}\right)$, a suitable value of t is needed

Using $\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \left(\frac{t}{1+t}, \frac{t^3}{1+t}\right)$

$$\Rightarrow \frac{1}{2} = \frac{t}{1+t} \Rightarrow 2t = 1 + t \quad \therefore t = 1$$

$$\therefore \text{Slope of the tangent} = 3(1)^2 + 2(1)^3 = 5$$

$$\therefore \text{Slope of the normal, } m' = -\frac{1}{5}$$

Equation of the normal becomes

$$\frac{y-1/2}{x-1/2} = -\frac{1}{5} \Rightarrow \frac{2y-1}{2x-1} = -\frac{1}{5}$$

$$\Rightarrow 10y - 5 = -2x + 1 \quad \therefore x + 5y - 3 = 0$$

Task

1. Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point $(1, 4)$.

At what points of the curve is the tangent parallel to the line $4x + y - 3 = 0$?

Ans: $5x - y - 1 = 0$, $2, 4, 4, -8$.

2. Find the equation of the tangent of the curve $y = x^3 + \frac{x^2}{2} + 1$ at the point $\left(-1, \frac{1}{2}\right)$. Find the coordinates of another point on the curve where the tangent is parallel to that at point $\left(-1, \frac{1}{2}\right)$.

Ans: $4x - 2y + 5 = 0$; $\left(\frac{2}{3}, \frac{41}{27}\right)$.

3. Find the equation of the tangent at the point $(1, -1)$ to the curve $y = x^3 - 4x^2 + 2$. What are the coordinates of the point where the tangent meets the curve again? Find the equation of the tangent at this point.

Ans: $y = -5x + 4, (2, -6), y = -4x + 2$

4. Find the coordinates of the point of intersection of the line $x - 3y = 0$ with the curve $y = x(1 - x^2)$. If these points are in order P, O, Q , prove that the tangents to the curve at P and Q are parallel, and the tangent at O is perpendicular to them.

Ans: $\left(-\sqrt{\frac{2}{3}}, -\frac{1}{3}\sqrt{\frac{2}{3}}\right), (0, 0), \left(\sqrt{\frac{2}{3}}, \frac{1}{3}\sqrt{\frac{2}{3}}\right)$

5. Find the equation of a tangent at the point $P(3, 9)$ to the curve $y = x^3 - 6x^2 + 15x - 9$. If O is the origin, and N is the foot of the perpendicular from P to the x -axis, prove that the tangent at P passes through the midpoint of ON . Find the coordinates of another point on the curve, and the tangent at which is parallel to the tangent at $(3, 9)$. **Ans:** $6x - y - 9 = 0, (1, 1)$

6. A tangent to the parabola $x^2 = 16y$ is perpendicular to the line $x - 2y - 3 = 0$. Find the equation of this tangent and the coordinates of the point of contact.

Ans: $2x + y + 16 = 0, (-16, 16)$

7. Show that the tangent to the curve $y = 4 - 2x - x^2$ at the points $(-1, 4)$ and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$ respectively pass through the point $\left(-\frac{1}{4}, 5\frac{1}{2}\right)$

8. Find the value of constant c so that the tangent at the origin of the coordinates of the curve $y = x(c - x^2)$ makes an angle of 45° with the x -axis

APPLICATION OF DIFFERENTIATION

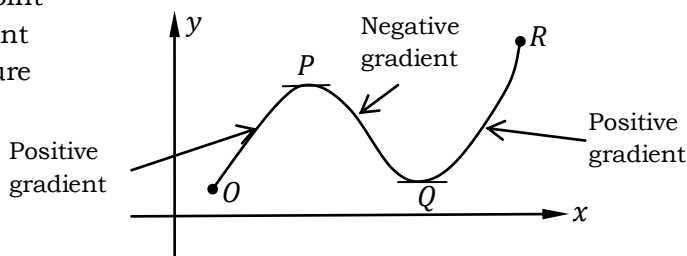
- Applied in Minimum and Maximum Problems.
- Determination of Turning Points of a curve or function.
- Small increments/changes.
- Small increment percentage.
- Finding Related Rates.
- Velocity, Acceleration and Displacement.

Turning points /stationary points of a curve

The turning points of a curve/function include:

- Maximum point
- Minimum point
- Inflexion point

Consider the figure

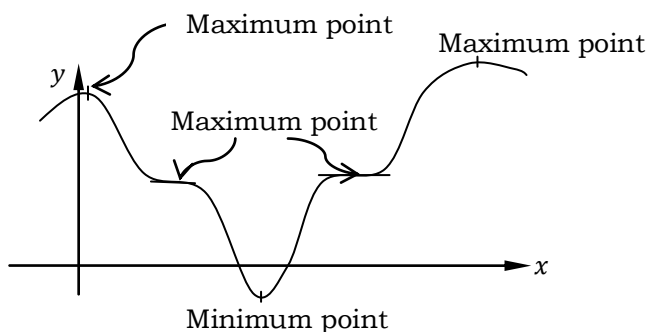


The gradient (or rate of change) of the curve changes from positive between O and P to negative between P and Q, and then positive again between Q and R. At point P, the gradient is zero, and as x increases, the gradient of the curve changes from positive just before P to negative just after. Such a point is called a maximum point. At point Q, the gradient is also zero and as x increases, the gradient of the curve changes from negative just before Q to positive just after such a point is called a minimum point. Points P and Q are given the general name of turning points.

NOTE

If the gradient on either side of that point is the same, such a point is called a point of inflexion.

Consider the figure.



Methods of determining Nature of Turning Points

- The first Derivative Test
- The second Derivative Test

Procedure for finding and distinguishing between stationary points

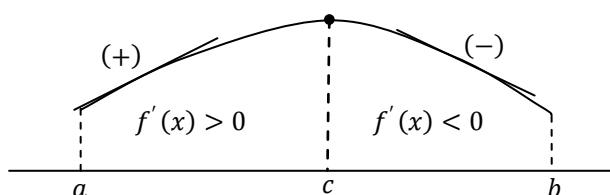
- 1) Given $y = f(x)$, determine $\frac{dy}{dx}$ (i.e. $f'(x)$).
- 2) Let $\frac{dy}{dx} = 0$, and solve for the values of x
- 3) Substitute the values of x into the original equation, $y = f(x)$, to find the corresponding y -ordinate values. This establishes the coordinates of the stationary points.
- 4) Determine the nature of the stationary points using the above derivative methods.

The First Derivative Test

Here determine the sign of the gradient of the curve just before and just after the stationary points.

If the sign change for the gradient of the curve is;

- a) Positive to negative – the point is a maximum one

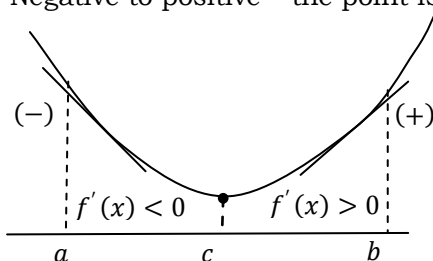


$(c, f(c))_{\text{Maximum}}$

NOTE:

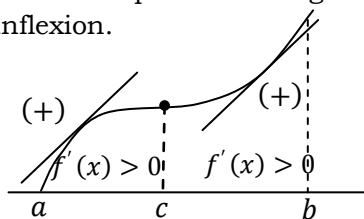
$c - a$ or $b - c$ should be a positive decimal number less than 1

- b) Negative to positive – the point is a minimum one.

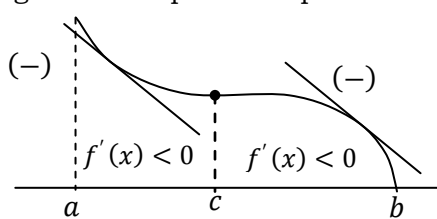


$(c, f(c))_{\text{Minimum}}$

- c) Positive to positive or negative to negative – the point is a point of inflexion.



$(c, f(c))_{\text{Inflexion}}$



$(c, f(c))_{\text{Inflexion}}$

The second Derivative Test.

Let $y = f(x)$ be the curve such that $\frac{dy}{dx} = f'(x) = 0$ at $x = c$. i.e, $f'(c) = 0$

- 1) If $f''(c) > 0$, then $f(c)$ is a minimum one
- 2) if $f''(c) < 0$, then $f(c)$ is a maximum one
- 3) if $f''(c) = 0$, then the test fails. In such cases, you can use the First Derivative Test.

Example

Determine the nature of turning points of the curve, and hence sketch the curve.

- a) $y = 3x^2 - 6x$.
- b) $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$.
- c) $f(x) = -3x^5 + 5x^3$.

Solution

- i) Since $y = 3x^2 - 6x$, $\frac{dy}{dx} = 6x - 6$.
- ii) At turning points, $\frac{dy}{dx} = 0$, hence $6x - 6 = 0 \therefore x = 1$
- iii) When $x = 1$, $y = 3(1)^2 - 6(1) = -3$
 $\therefore$ Coordinates of turning points $(1, -3)$
- iv) Using the First Derivative Test.

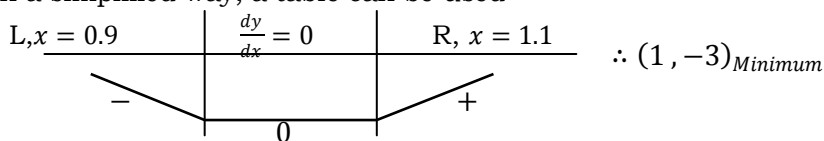


$$\left. \frac{dy}{dx} \right|_{\text{at } x=0.9} = 6(0.9) - 6 = -0.6 \text{ i.e. Negative}$$

$$\left. \frac{dy}{dx} \right|_{\text{at } x=1.1} = 6(1.1) - 6 = 0.6 \text{ i.e. Positive}$$

Since the gradient of the curve is negative just before the turning point and positive after. i.e. $(- \cup +)$, $(1, -3)$ is a *minimum*

In a simplified way, a table can be used



$$\left. \frac{dy}{dx} \right|_{\text{at } x=0.9} = 6(0.9) - 6 = -0.6 \text{ i.e. negative}$$

$$\left. \frac{dy}{dx} \right|_{\text{at } x=1.1} = 6(1.1) - 6 = 0.6 \text{ i.e. positive}$$

Alternatively: Second Derivative Test

$$y = 3x^2 - 6x$$

$$\frac{dy}{dx} = 6x - 6 \therefore \frac{d^2y}{dx^2} = 6$$

Since $\frac{d^2y}{dx^2} = 6$ is a constant of any value of x .i.e. $x = 1$, if $\frac{d^2y}{dx^2} > 0$, then $(1, -3)$ must be a *minimum*.

To sketch the curve, the following steps can be followed.

- 1) Obtain the intercepts
- 2) Obtain the turning points
- 3) Obtain the nature of turning points
- 4) Sketch the curve

Intercepts

For x -intercept, $y = 0$

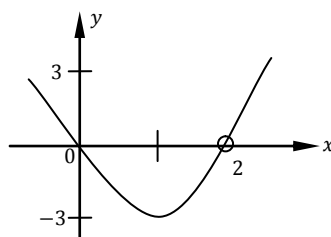
$$0 = 3x^2 - 6x \text{ or } 3x(x - 2) = 0 \therefore x = 0, 2$$

$\therefore (0,0)$ and $(2,0)$ are intercepts

For y -intercept, $x = 0 \therefore (0,0)$ is still an intercept.

Turning points and their nature are $\therefore (1, -3)_{\text{Minimum}}$

Sketch: $y = 3x^2 - 6x$



b) Since $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$, $\frac{dy}{dx} = x^2 - x - 6$

Turning points

For turning points, $\frac{dy}{dx} = 0$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x - 3)(x + 2) = 0 \therefore x = 3, -2$$

$$\text{For } x = 3, y = \frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3) + \frac{5}{3} = -11\frac{5}{6}$$

$$\text{For } x = -2, y = \frac{(-2)^3}{3} - \frac{(-2)^2}{2} - 6(-2) + \frac{5}{3} = 9$$

$\therefore$ Turning points are: $(-2, 9), (3, -11\frac{5}{6})$

Nature of turning points

Using the second derivative test

$$\text{Since } \frac{dy}{dx} = x^2 - x - 6, \text{ then } \frac{d^2y}{dx^2} = 2x - 1$$

Now when $x = -2, \frac{d^2y}{dx^2} = 2(-2) - 1 = -5$ which is negative

$\therefore (-2, 9)$ is a maximum point

When $x = 3, \frac{d^2y}{dx^2} = 2(3) - 1 = 5$ which is positive

$\therefore (3, -11\frac{5}{6})$ is a minimum point

Intercepts

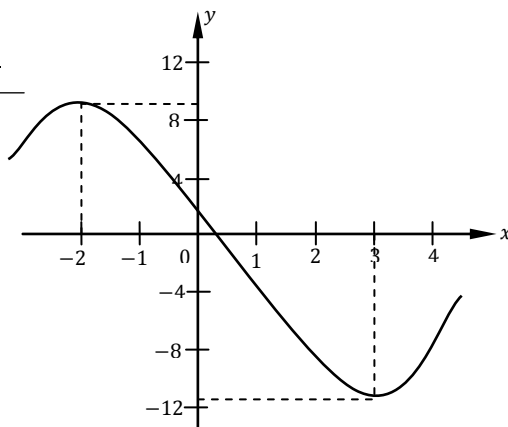
x - intercept, $y = 0$

$$\Rightarrow 0 = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3} \text{ or } 2x^3 - 3x^2 - 36x + 10 = 0$$

$$\therefore x = 4.939, -3.712, 0.273$$

y – Intercept, $x = 0, y = \frac{5}{3}$

Sketch: $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$



c) **Point**

$(-1, -2)$

$(1, 2)$

$(0, 0)$

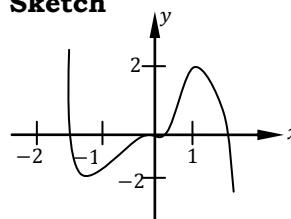
Nature

Minimum

Maximum

Inflexion

Sketch



Task

Determine the nature of turning points of the curve

$y = x^4 - 3x^3$. Hence sketch

Point

$(2, -16)$

$(0, 0)$

$(3, -27)$

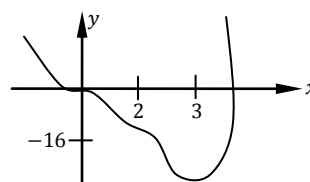
Nature

Inflexion

Inflexion

Minimum

Sketch



Applied Minimum and Maximum Problems

The common application of differentiation is determination of minimum and maximum values.

Common terms used here are:

- Greatest profit
- Least cost
- Least time
- Optimum size
- Least size
- Greatest distance

Problem Solving Strategy for Applied Minimum and Maximum Problems

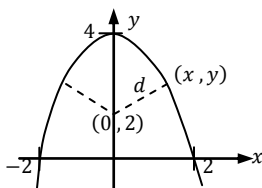
1. Assign symbols to all *given* quantities and quantities *to be determined*
2. Write a *primary equation* for the quantity that is to be maximized (or minimized). here a review of several useful formulas from geometry is needed
3. Reduce the primary equation to one having a single independent variable. This may involve the use of *secondary equations* relating the independent variables of the primary equation.
4. Determine the domain of the primary equation . *i. e.* determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques

Finding the Minimum Distance

Which points on the graph $y = 4 - x^2$ are closest to the point $(0, 2)$?

Solution

Sketching $y = 4 - x^2$



From the fig. there are two points at minimum distance from the point $(0, 2)$.
Taking the general point on the curve (x, y)

d = Distance between two points

$$d = \sqrt{(x - 0)^2 + (y - 2)^2} \quad \text{(Primary equation)}$$

Using secondary equation $y = 4 - x^2$

$$d = \sqrt{(x - 0)^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$$

For d to be small, $\frac{d}{dx}(d) = 0$

$$\Rightarrow \frac{d}{dx}(d) = \frac{1}{2}(x^4 - 3x^2 + 4)^{-1/2} \cdot (4x^3 - 6x)$$

$$\therefore 4x^3 - 6x = 0 \text{ or } 2x(2x^2 - 3) = 0$$

$$\therefore x = 0, \pm\sqrt{\frac{3}{2}}$$

Investigating the values of x

Using first derivative test,

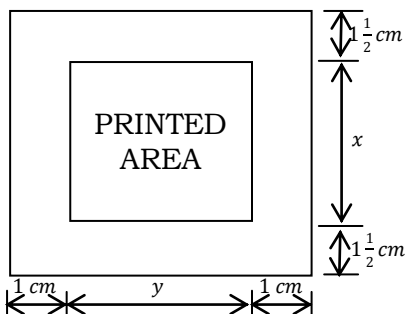
At $x = 0$ yields a maximum value, where as both $x = \pm\sqrt{\frac{3}{2}}$ yield minimum distance.

Hence, the closest points are: $(\sqrt{3/2}, 5/2)$ and $(-\sqrt{3/2}, 5/2)$

Finding the Minimum Area

1. A rectangular page is to contain 24cm^2 of print. The margins at the top and bottom of the page are each $1\frac{1}{2}\text{cm}$. The margins on each side are 1cm . What should the dimensions of the page be so that the least amount of paper is used?

Solution



Let A be the area to be minimized

$$A = (x + 3)(y + 2) \quad (\text{Primary equation})$$

The printed area inside the margins is given by:

$$24 = xy \quad (\text{Primary equation})$$

Solving the equation produces

$$y = \frac{24}{x} \quad \text{Substitution in to the primary equation}$$

$$A = (x + 3)\left(\frac{24}{x} + 2\right) = 30 + 2x + \frac{72}{x}$$

Because x must be positive, we are interested in only in values of A when $x > 0$

$$\text{For minimum area, } \frac{dA}{dx} = 0$$

$$\Rightarrow 2 - \frac{72}{x^2} = 0 \text{ or } x^2 = 36 \quad \therefore x = +6$$

Investigating $x = +6$

$$\text{By second derivative, } \frac{d^2A}{dx^2} = \frac{144}{x^3}$$

$$\text{Now when } x = +6, \frac{d^2A}{dx^2} = \frac{144}{6^3} > 0. \text{ Hence } x = 6 \text{ gives minimum area.}$$

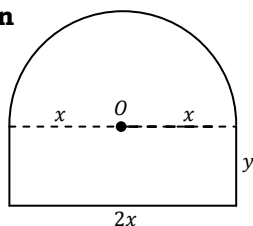
So the dimensions of the paper are:

$$x + 3 = 9 \text{ cm by } y + 2 = 6 \text{ cm}$$

Finding the Minimum Area

2. A window has the form of a rectangle surmounted by a semicircle. If the perimeter is 40 cm , find the dimensions so that the greatest amount of light may be admitted

Solution



Let the radius of semicircle be x and width of window be y

$$\begin{array}{l} \text{Area of the frame} \\ \text{for light to be admitted} \end{array} = \begin{array}{l} \text{Area of} \\ \text{semicircle} \end{array} + \begin{array}{l} \text{Area of the} \\ \text{rectangle} \end{array}$$

$$A = \frac{\pi x^2}{2} + 2xy \quad \text{(Primary equation)}$$

Perimeter of boundary = Perimeter of the rectangle + Perimeter of semicircle

$$40 = (2x + y + y) + \frac{1}{2} \times 2\pi x$$

$$40 = 2x + 2y + \pi x \quad \text{(Secondary equation)}$$

From this equation

$$y = \frac{40 - 2x - \pi x}{2}$$

∴ Primary equation becomes

$$A = \frac{\pi x^2}{2} + 2x \left(\frac{40 - 2x - \pi x}{2} \right) = \frac{\pi x^2}{2} + 40x - 2x^2 - \pi x^2$$

$$= 40x - \frac{\pi x^2}{2} - 2x^2 = 40x - \frac{x^2}{2}(\pi + 4)$$

For maximum area, $\frac{dA}{dx} = 0$

$$\Rightarrow 40 - x(\pi + 4) = 0 \quad \therefore x = \frac{40}{\pi + 4}$$

Investigating $x = \frac{40}{\pi + 4}$ to give the greatest area

$$\Rightarrow \frac{d^2A}{dx^2} = -(\pi + 4) < 0$$

∴ For maximum area, $x = \frac{40}{\pi + 4}$

Dimensions are:

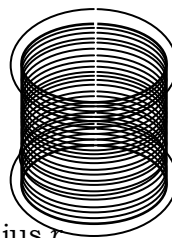
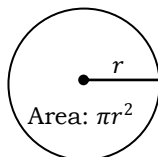
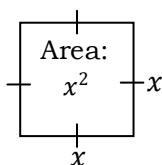
$$y = \frac{40 - 2x - \pi x}{2} = \frac{40 - \frac{80}{\pi + 4} - \frac{40\pi}{\pi + 4}}{2}$$

$$= 20 - \frac{40}{\pi + 4} - \frac{20\pi}{\pi + 4} = \frac{20\pi + 80 - 40 - 20\pi}{\pi + 4} = \frac{40}{\pi + 4}$$

∴ Radius of semicircle = Height of the rectangle = $\frac{40}{\pi + 4}$

3. Four feet of a wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Solution



Let the square be of side x , and circle of radius r

A = Area of square + Area of circle

$$A = x^2 + \pi r^2 \quad \text{(Primary equation)}$$

Because the total amount of wire = 4 ft

4 = Perimeter of square + Circumference of the circle

$$\Rightarrow 4 = 4x + 2\pi r \quad \text{(Secondary equation)}$$

$$\Rightarrow r = \frac{2(1-x)}{\pi}$$

Substituting in to the primary equation

$$A = x^2 + \pi \left(\frac{2(1-x)}{\pi} \right)^2 = x^2 + \frac{4(1-x)^2}{\pi}$$

$$\therefore A = \frac{1}{\pi} [(\pi + 4)x^2 - 8x + 4]$$

For maximum total area, $\frac{dA}{dx} = 0$

$$\therefore \frac{dA}{dx} = \frac{1}{\pi} [2x(\pi + 4) - 8] = 0$$

$$\therefore x = 4/(\pi + 4) \approx 0.56$$

Investigating whether $x = 4/(\pi + 4)$ give maximum area

$$\frac{d^2A}{dx^2} = \frac{2(\pi+4)}{\pi} > 0$$

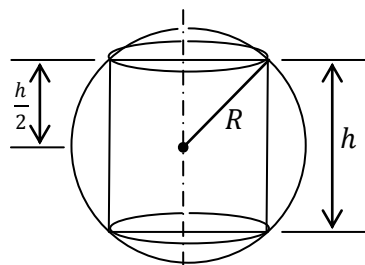
$\Rightarrow x = 4/(\pi + 4)$ gives a minimum area.

But the minimum area of a square .i.e. x^2 is zero, so taking $x = 0$, the maximum area occurs when $x = 0$.i.e. all the wire is used for a circle.

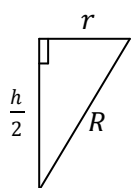
4. Inscribing a Cylinder in to a Sphere

A cylinder of volume V is to be cut from a sphere of radius R . show that the maximum volume of the cylinder is $V = \frac{4\pi R^3}{3\sqrt{3}}$

Solution



From the triangle



$$r^2 = R^2 - \frac{h^2}{4}$$

Secondary equation

$$\therefore V = \pi h \left(R^2 - \frac{h^2}{4} \right) = \pi \left(hR^2 - \frac{h^3}{4} \right), R = \text{constant}$$

Now for maximum volume, $\frac{dV}{dh} = 0$

$$\frac{dV}{dh} = \pi \left(R^2 - \frac{3h^2}{4} \right) = 0 \quad \therefore 4R^2 - 3h^2 = 0$$

$$\Rightarrow h = \sqrt{4R^2/3} = \frac{2R}{\sqrt{3}}$$

Investigating $h = \frac{2R}{\sqrt{3}}$ for maximum volume

$$\frac{d^2V}{dh^2} = \pi \left(-\frac{6h}{4} \right), \text{ since } h = +\frac{2R}{\sqrt{3}} \Rightarrow \frac{d^2V}{dh^2} = -ve$$

Thus height of the cylinder with maximum volume is $h = \frac{2R}{\sqrt{3}}$

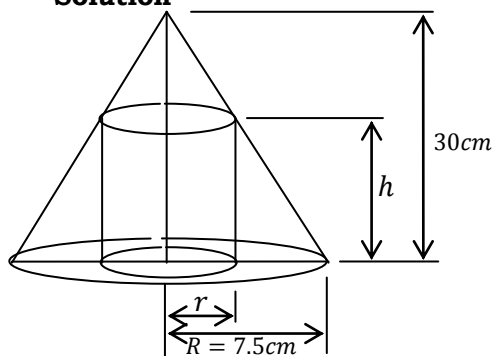
$$\text{Now maximum volume} = V = \pi \left(R^2 \cdot \frac{2R}{\sqrt{3}} - \frac{1}{4} \cdot \left(\frac{2R}{\sqrt{3}} \right)^3 \right)$$

$$\begin{aligned}
 &= \pi \left(\frac{2R^3}{\sqrt{3}} - \frac{8R^3}{12\sqrt{3}} \right) = \pi \left(\frac{2R^3}{\sqrt{3}} - \frac{2R^3}{3\sqrt{3}} \right) \\
 &= \pi \left(\frac{6R^3 - 2R^3}{3\sqrt{3}} \right) = \frac{4\pi R^3}{3\sqrt{3}}
 \end{aligned}$$

5. Inscribing a Cylinder in to a Cone

Determine the height and radius of a cylinder of maximum volume which can be cut from a cone of height 30cm and base radius 7.5cm

Solution

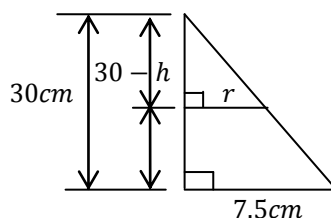


Volume of cylinder

$$V = \pi r^2 h$$

Primary equation

Obtaining the secondary equation



By similar triangles

$$\frac{30}{7.5} = \frac{30-h}{r} \Rightarrow 4r = 30 - h$$

$$\therefore h = 30 - 4r$$

$$V = \pi r^2 (30 - 4r) = \pi (30r^2 - 4r^3)$$

For maximum volume, $\frac{dV}{dr} = 0$

$$\frac{dV}{dr} = \pi (60r - 12r^2) = 0$$

$$\Rightarrow 12r(5 - r) = 0 \quad \therefore r = 0, 5$$

Investigating the values of r

$$\frac{d^2V}{dr^2} = \pi (60 - 24r)$$

$$\text{For } r = 0, \quad \frac{d^2V}{dr^2} = \pi (60 - 24(0)) = 60\pi > 0$$

$$\text{For } r = 5, \quad \frac{d^2V}{dr^2} = \pi (60 - 24(5)) = -60\pi < 0$$

Hence $r = 5$ gives the maximum value

From $h = 30 - 4r$, taking $r = 5$

$$h_{\max} = 30 - 4(5) = 10$$

$$\therefore \text{Radius} = 5\text{cm}, \quad h = 10\text{cm}$$

6. Finding the Maximum Volume

Write down the expression for volume, V and surface area, S of a cylinder of radius r and height h .

If the surface area is constant, show that the volume of the cylinder will be maximum when $h = 2r$.

Solution

$$V = \pi r^2 h, \quad S = 2\pi r^2 + 2\pi r h \quad (\text{Taking cylinder of closed ends})$$

$$\underline{\text{Given:}} \quad S = 2\pi r^2 + 2\pi r h \quad \text{Secondary equation}$$

$$\underline{\text{Required:}} \quad V = \pi r^2 h \quad \text{Primary equation}$$

Expressing the primary equation in terms of one variable

$$\text{From } S = 2\pi r^2 + 2\pi r h \Rightarrow h = \left(\frac{S - 2\pi r^2}{2\pi r} \right)$$

$$\Rightarrow V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r} \right) = \frac{r}{2} (S - 2\pi r^2)$$

$$\Rightarrow V = \frac{rS}{2} - \pi r^3$$

$$\text{For maximum volume, } \frac{dV}{dr} = 0$$

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 = 0 \quad (S = \text{constant})$$

$$r^2 = \frac{S}{6\pi} \quad \therefore S = 6\pi r^2$$

$$\text{Now from } h = \left(\frac{S - 2\pi r^2}{2\pi r} \right), \quad S = 6\pi r^2$$

$$\Rightarrow h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = 2r \quad \therefore h = 2r$$

- 7.** A right circular cylinder has a given surface S (including the ends) and a maximum volume V . show that its height is equal to its diameter of the base.

Solution

$$\underline{\text{Given:}} \quad S = 2\pi r^2 + 2\pi r h$$

Secondary equation

$$V = \pi r^2 h$$

Primary equation

From the secondary equation

$$h = \left(\frac{S - 2\pi r^2}{2\pi r} \right)$$

$$\therefore V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r} \right) = \frac{rS}{2} - \pi r^3$$

$$\text{For maximum volume, } \frac{dV}{dr} = 0$$

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 = 0 \quad (S = \text{constant})$$

$$r^2 = \frac{S}{6\pi} \quad \therefore S = 6\pi r^2 \quad \text{or} \quad r = \sqrt{\frac{S}{6\pi}}$$

$$\text{Now from } h = \left(\frac{S - 2\pi r^2}{2\pi r} \right), \quad S = 6\pi r^2$$

$$\Rightarrow h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = 2r \quad \therefore h = 2r = \text{Diameter}$$

Task

- 1.** A chemical factory wishes to make a cylindrical container of thin metal to hold 10 cm^3 , using the least possible area of metal. If the outside surface is $S\text{ cm}^2$, and the radius $r\text{ cm}$, show that $S = 2\pi r^2 + \frac{20}{r}$, and hence find the required radius and height. (leave π in your answer)

$$\underline{\text{Ans:}} \quad r = \sqrt[3]{5/\pi}, h = 2\sqrt[3]{5/\pi}$$

- 2.** A circular cylinder open at the top is to be made so as to have a volume of 1 m^3 . If $r\text{ (m)}$ is the radius of the base, prove that the total outside

surface area is $(\pi r^2 + 2/r) m^3$. Hence prove that this surface is minimum when the height equals to the radius of the base. **Ans:** $h = \sqrt[3]{1/\pi} = r$

3. A piece of wire of length of $17l$ is cut in to two parts of lengths, x and $17l - x$. The former is bent into a shape of a square, and the latter in to a rectangle of which the base is double the height.
Find an expression for the sum of the two areas of these figures. Prove that the only value of x for which this sum is a maximum or minimum is $8l$
4. A right circular cylinder is inscribed in a given sphere. Show that when the area of the curved surface is greatest, the height of the cylinder is equal to its diameter.
5. A right circular cylinder is inscribed in a sphere of given radius a . Show that the volume of the cylinder is $\pi h \left(a^2 - \frac{h^2}{4} \right)$ where h is the height of the cylinder. Find the ratio of the height to the radius of the cylinder when its volume is greatest. **Ans:** $\sqrt{2}:1$
6. Find the dimensions of a rectangle of greatest area which can be inscribed in a circle of radius r . **Ans:** $r\sqrt{2}$
7. A piece of wire, which forms the circumference of a circle of $0.12m$ radius, is cut and bent so as to form two new circles. Find the radius of each circle in order that the sum of the areas of the two circles shall be small as possible. **Ans:** each: $0.06m$
8. A prism of square section contains $64cm^3$ of clay, the side of the square being x m. Express the length of the prism in terms of x and find the total area of its faces. Show that the total area is a minimum when the prism is a cube.
Ans: Length = $\frac{64}{x^2}$; Area = $\frac{256}{x} + 2x^2$
9. A closed cylindrical can has a height h and base radius r . The volume is $0.01m^3$. show that $h = 1/100\pi r^2$. Show further that S, surface area, is given by $S = 2\pi r^2 + 1/50r$. Hence find the value of r for which S is minimum. **Ans:** $(200\pi)^{-1/3}$
10. A cylinder is cut from a sphere of radius 5 cm. If the height of the cylinder is $2h$, show that the volume of the cylinder is $2\pi h(25 - h^2)$, assuming that the curved surfaces of the cylinder reach the surface of the sphere. Find the maximum volume of the cylinder. **Ans:** $500\pi\sqrt{3}/9$
11. A cylinder is inscribed in to a cone of height h . If the volume of the cylinder is maximum show that its height is $h/3$
12. Show that the volume of the biggest right circular cone that can be inscribed in a sphere of given radius is $8/27$ times that of the sphere.
13. A given quantity of metal is to be cast in to a half cylinder with rectangular base and semi circular ends. Show that in order that the

total surface area may be minimum, the ratio of the length of the cylinder to the diameter of its semi circular ends is $\pi(\pi - 2)$

- 14.** Show that the semi vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}(\sqrt{2})$
- 15.** One corner of rectangular sheet of paper of width a is folded so as to reach the opposite edge of the sheet. Find the minimum length of the increase. **Ans:** $3\sqrt{3}a/4$
- 16.** An open box is to be made from a rectangular piece of sheet metal $12\text{cm} \times 18\text{cm}$ by cutting out equal squares from each corner and folding up the sides. Find the dimensions of the box of largest volume that can be in this manner. **Ans:** $8 + 2\sqrt{7}, 2 + 2\sqrt{7}, 5 - \sqrt{7}$
- 17.** Show that the surface area A , the volume V and radius r of a cylinder are connected by the equation $A = \frac{2V}{r} + 2\pi r^2$
- Find the minimum possible surface area of a cylinder if its volume is to be $100\pi \text{ cm}^3$. **Ans:** 255.8cm^3
 - A cylindrical block of ice is melting at a rate proportional to its surface area. Find the ratio of the length to radius if a cylindrical block of given volume is to melt as slowly as possible. **Ans:** 1:2
- 18.** A cylinder of height x fits exactly into a cone of height h and base radius r . Show that its volume is given by $v = \pi r^2 \left(1 - \frac{x}{h}\right)^2$ show that the maximum volume of the cylinder occurs when $x = \frac{1}{3}h$. Show also that the maximum curved surface area of the cylinder occurs when $x = \frac{1}{2}h$.

Finding Related Rates

This involves the use of chain rule, when a quantity is varied with respect to time or another quantity.

Guidelines for solving Related-Rate problems

- Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.
- Write an equation involving the variables whose rates of change either are given or are to be determined.
- Using the chain rule, implicitly differentiate both sides of the equation with respect to time t .
- After completing step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve the required rate of change.

Example

1. Ripples in a pond.

A pebble is dropped into a calm pond, causing ripples in form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1cm/s . when the radius is 4cm , at what rate is the total area A of the disturbed water changing?

Solution

Given the equation: $A = \pi r^2$

Given the equation: $\frac{dr}{dt} = 1$

Find: $\frac{dA}{dt}$ when $r = 4$

Differentiating $A = \pi r^2$ w.r.t. t

$$\begin{aligned}\frac{dA}{dt} &= \pi \cdot 2r \cdot \frac{dr}{dt} \\ &= 2\pi(4)(1) = 8\pi\end{aligned}$$

When $r = 4$, the area is changing at a rate $8\pi\text{cm}^2/\text{s}$

2. An inflating balloon

Air is being pumped into a spherical balloon at a rate of $4.5\text{cm}^3/\text{min}$ find the change of radius when the radius is 2cm

Solution

Let v be the volume of the balloon

r the radius

Given rate: $\frac{dv}{dt} = +\frac{9}{2}$ (constant rate)

Find: $\frac{dr}{dt}$ when $r = 2$

To find the rate of change of radius, you must find an equation that relates the radius r to the volume v

$$v = \frac{4}{3} \pi r^3$$

Implicit differentiation with respect to t

$$\frac{dv}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

Solving for $\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dv}{dt}$$

Finally when $r = 2$, the rate of change of the radius is

$$\frac{dr}{dt} = \frac{1}{16\pi} \left(\frac{9}{2} \right) \approx 0.09 \text{ cm/min}$$

- 3.** *The surface area of the sphere is decreasing at a rate of $0.9 \text{ m}^2/\text{s}$, when the radius is 0.6m . Calculate the rate of change of volume of the sphere at this instant.*

Solution

Given rate: $\frac{dA}{dt} = -0.9 \text{ m}^3/\text{s}$

Given equation: $A = 4\pi r^2$

Find: $\frac{dV}{dt}$ when $r = 0.6 \text{ m}$

Obtaining the relation between V , volume of the sphere and A , area of the sphere

$$V = \frac{4}{3}\pi r^3 = 4\pi r^2 \cdot \frac{r}{3} = \frac{Ar}{3}$$

Differentiating implicitly w.r.t time t

$$\begin{aligned}\frac{dV}{dt} &= \frac{1}{3} \left[r \cdot \frac{dA}{dt} + A \cdot \frac{dr}{dt} \right] \\ &= \frac{r}{3} \cdot \frac{dA}{dt} + \frac{A}{3} \cdot \frac{dr}{dt}\end{aligned}$$

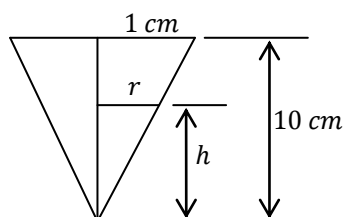
But $A = 4\pi r^2$

Differentiating implicitly w.r.t time t

$$\begin{aligned}\frac{dA}{dt} &= 4\pi \cdot 2r \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{8\pi r} \cdot \frac{dA}{dt} \\ \frac{dV}{dt} &= \frac{r}{3} \cdot \frac{dA}{dt} + \frac{A}{3} \cdot \frac{1}{8\pi r} \cdot \frac{dA}{dt} \\ &= \frac{dA}{dt} \left(\frac{r}{3} + \frac{A}{24\pi r} \right) = \frac{dA}{dt} \left(\frac{r}{3} + \frac{4\pi r^2}{24\pi r} \right) \\ &= \frac{dA}{dt} \left(\frac{r}{3} + \frac{r}{6} \right) = \frac{r}{2} \cdot \frac{dA}{dt} \\ \therefore \frac{dV}{dt} &= \frac{0.6}{2} (-0.9) = -0.27 \text{ m}^3/\text{s}\end{aligned}$$

4. A vessel in the shape of a right circular cone of height 10 cm and base radius 1 cm catching the drips from a tap leaking at the rate of $0.1 \text{ cm}^3/\text{s}$. Find the rate at which the surface area of the water is increasing when the water is half way up the cone.

Solution



Let water in the cone at any time t be at a height h and radius r

Given: $\frac{dV}{dt} = +0.1 \text{ cm}^3/\text{s}$

Given equation: $V = \frac{\pi r^2 h}{3}$

Find: $\frac{dA}{dt}$ when $h = \frac{1}{2} \times 10 = 5 \text{ cm}$

Differentiating volume w.r.t. time t

$$\begin{aligned}\frac{dV}{dt} &= \frac{\pi}{3} \left[r^2 \cdot \frac{dh}{dt} + h \cdot 2r \cdot \frac{dr}{dt} \right] \\ \frac{dV}{dt} &= \frac{\pi}{3} \left[r^2 \cdot \frac{dh}{dt} + 2hr \cdot \frac{dr}{dt} \right] \dots\dots\dots (1)\end{aligned}$$

Now $A = \pi r^2$

Differentiating area w.r.t. time t

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \dots\dots\dots (2)$$

Also by similar triangles

$$\frac{1}{r} = \frac{10}{h} \Rightarrow h = 10r$$

Differentiating area w.r.t. time t

$$\frac{dh}{dt} = 10 \cdot \frac{dr}{dt} \dots\dots\dots (3)$$

Now eqn. (3) in to eqn. (1)

$$\frac{dV}{dt} = \frac{\pi}{3} \left[r^2 \cdot 10 \cdot \frac{dr}{dt} + 2hr \cdot \frac{dr}{dt} \right] = \frac{dr}{dt} \left[\frac{10\pi r^2}{3} + \frac{2\pi hr}{3} \right]$$

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{10\pi r^2 + 2\pi hr}{3}}$$

Eqn. (4) in to eqn. (2)

$$\begin{aligned} \therefore \frac{dA}{dt} &= \frac{6\pi r \cdot \frac{dV}{dt}}{10\pi r^2 + 2\pi hr} = \frac{3 \cdot \frac{dV}{dt}}{5r + h} \\ &= \frac{3 \cdot \frac{dV}{dt}}{5 \cdot \frac{h}{10} + h} = \frac{30}{15h} \cdot \frac{dV}{dt} \\ &= \frac{2}{h} \cdot \frac{dV}{dt} = \frac{2}{5} \cdot (0.1) = 0.04 \text{ cm}^2/\text{s} \end{aligned}$$

Alternatively

Given: $\frac{dV}{dt} = +0.1 \text{ cm}^3/\text{s}$

Given equation: $V = \frac{\pi r^2 h}{3}$

Find: $\frac{dA}{dt}$ when $h = \frac{1}{2} \times 10 = 5 \text{ cm}$

Expressing volume in terms of one variable say h

By similar triangles

$$\frac{1}{r} = \frac{10}{h} \Rightarrow r = \frac{h}{10}$$

$$\therefore V = \frac{\pi r^2 h}{3} \Rightarrow V = \frac{\pi h^2 \cdot h}{300} = \frac{\pi h^3}{300}$$

Differentiating volume w.r.t. time t

$$\frac{dV}{dt} = \frac{\pi h^2}{100} \cdot \frac{dh}{dt} \Rightarrow 0.1 = \frac{\pi h^2}{100} \cdot \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{10}{\pi h^2} \dots\dots\dots (1)$$

Now $A = \pi r^2$

Expressing this area in terms of h

$$A = \frac{\pi h^2}{100}$$

Differentiating area w.r.t. time t

$$\frac{dA}{dt} = \frac{\pi h}{50} \cdot \frac{dh}{dt} \dots\dots\dots (2)$$

Eqn. (1) in to eqn. (2)

$$\frac{dA}{dt} = \frac{\pi h}{50} \cdot \frac{10}{\pi h^2} = \frac{1}{5h}$$

$$\therefore \frac{dA}{dt} = \frac{1}{5 \times 5} = 0.04 \text{ cm}^2/\text{s}$$

5. A container is in the form of an inverted circular cone, its height is 100 cm and base radius 40 cm. The container is full of water and has a small hole at its vertex. Water is flowing through the hole at a rate of $100 \text{ cm}^3/\text{s}$. Find the rate at which the water level in the container is falling when the height of water in the container is halved.

Solution

Given: $\frac{dV}{dt} = 10 \text{ cm}^3/\text{s}$

Given equation: $V = \frac{\pi r^2 h}{3}$

Find: $\frac{dh}{dt}$ when $h = \frac{1}{2} \times 100 = 50 \text{ cm}$

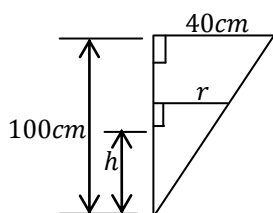
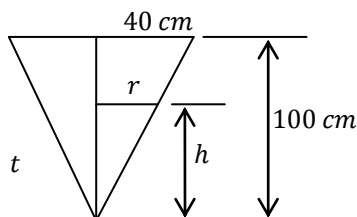
Differentiating volume implicitly w.r.t time t

$$V = \frac{\pi r^2 h}{3} \Rightarrow \frac{dV}{dt} = \frac{\pi}{3} \left[r^2 \cdot \frac{dh}{dt} + h \cdot 2r \cdot \frac{dr}{dt} \right]$$

$$\frac{dV}{dt} = \frac{\pi}{3} \left[r^2 \cdot \frac{dh}{dt} + 2hr \cdot \frac{dr}{dt} \right]$$

$$\Rightarrow r^2 \cdot \frac{dh}{dt} = \frac{3}{\pi} \cdot \frac{dV}{dt} - 2hr \cdot \frac{dr}{dt} \dots \dots \dots (1)$$

Obtaining the secondary equation



By similar triangles

$$\frac{40}{r} = \frac{100}{h} \therefore r = \frac{2}{5}h$$

Now differentiating $r = \frac{2}{5}h$ implicitly w.r.t. time t

$$\frac{dr}{dt} = \frac{2}{5} \cdot \frac{dh}{dt} \dots \dots \dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow r^2 \cdot \frac{dh}{dt} = \frac{3}{\pi} \cdot \frac{dV}{dt} - 2hr \cdot \frac{2}{5} \cdot \frac{dh}{dt}$$

But for $h = 50 \text{ cm}$, $r = \frac{2}{5} \times 50 = 20 \text{ cm}$

$$\Rightarrow (20)^2 \cdot \frac{dh}{dt} = \frac{3}{\pi} \cdot (-10) - 2(50)(20) \cdot \frac{2}{5} \cdot \frac{dh}{dt}$$

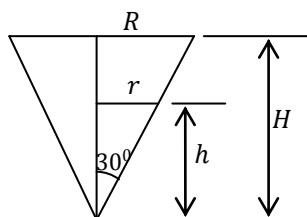
$$\Rightarrow -\frac{30}{\pi} = 400 \cdot \frac{dh}{dt} + 800 \cdot \frac{dh}{dt}$$

$$\Rightarrow -\frac{30}{\pi} = 1200 \cdot \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = -\frac{1}{40\pi} \text{ cm/s}$$

6. A vessel containing water is in the form of an inverted hollow cone with a semi vertical angle of 30° . There is a small hole at the vertex of the cone and water is running out at the rate of $3 \text{ cm}^3/\text{s}$. Find the rate at which the surface area in contact with water is changing when there are $81\pi \text{ cm}^3$ of water remaining in the cone.

Solution



Given rate: $\frac{dV}{dt} = -3$

Given equation: $V = \frac{\pi r^2 h}{3}$

Taking height of the water in the cone to be h and radius r at any time t

Find: $\frac{dS}{dt}$ when $V = 81\pi\text{cm}^3$

Differentiating volume implicitly w.r.t time t

$$V = \frac{\pi r^2 h}{3} \Rightarrow \frac{dV}{dt} = \frac{\pi}{3} \left[r^2 \cdot \frac{dh}{dt} + h \cdot 2r \cdot \frac{dr}{dt} \right]$$

$$\frac{dV}{dt} = \frac{\pi}{3} \left[r^2 \cdot \frac{dh}{dt} + 2hr \cdot \frac{dr}{dt} \right]$$

$$\Rightarrow \frac{3}{\pi} \cdot \frac{dV}{dt} = r^2 \cdot \frac{dh}{dt} + 2hr \cdot \frac{dr}{dt} \dots \dots \dots (1)$$

Obtaining the secondary equation

$$\tan 30^\circ = \frac{r}{h} \Rightarrow r = h \tan 30^\circ$$

$$\therefore r = \frac{h}{\sqrt{3}}$$

Now differentiating w.r.t. time t

$$\frac{dr}{dt} = \frac{1}{\sqrt{3}} \cdot \frac{dh}{dt} \dots \dots \dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow \frac{3}{\pi} \cdot \frac{dV}{dt} = r^2 \cdot \frac{dh}{dt} + 2hr \cdot \frac{1}{\sqrt{3}} \cdot \frac{dh}{dt} \dots \dots \dots (3)$$

$$\text{Now } S = \pi r l = \pi r \sqrt{r^2 + h^2}$$

$$\text{But } r = \frac{h}{\sqrt{3}}$$

$$\Rightarrow S = \pi \frac{h}{\sqrt{3}} \cdot \sqrt{\frac{h^2}{3} + h^2} = \frac{2\pi h^2}{\sqrt{3}}$$

Differentiating surface area implicitly w.r.t time t

$$\frac{dS}{dt} = \frac{4\pi h}{\sqrt{3}} \cdot \frac{dh}{dt} \dots \dots \dots (4)$$

From eqn. (3)

$$\Rightarrow \frac{3}{\pi} \cdot \frac{dV}{dt} = r^2 \cdot \frac{dh}{dt} + 2hr \cdot \frac{1}{\sqrt{3}} \cdot \frac{dh}{dt} = \left(r^2 + \frac{2hr}{\sqrt{3}} \right) \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{3\sqrt{3}}{\pi} \cdot \frac{dV}{dt} = \left(r^2 \sqrt{3} + 2hr \right) \cdot \frac{dh}{dt}$$

Expressing this equation in terms of only one variable say h

$$\Rightarrow \frac{3\sqrt{3}}{\pi} \cdot \frac{dV}{dt} = \left(\left(\frac{h}{\sqrt{3}} \right)^2 \sqrt{3} + 2h \left(\frac{h}{\sqrt{3}} \right) \right) \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{3\sqrt{3}}{\pi} \cdot \frac{dV}{dt} = \left(\frac{h^2}{\sqrt{3}} + \frac{2h^2}{\sqrt{3}} \right) \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{3\sqrt{3}}{\pi} \cdot \frac{dV}{dt} = \frac{3h^2}{\sqrt{3}} \cdot \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{3\sqrt{3}}{\pi} \cdot \frac{dV}{dt} \cdot \frac{\sqrt{3}}{3h^2} = \frac{3}{\pi h^2} \cdot \frac{dV}{dt}$$

Now eqn. (4) becomes

$$\frac{dS}{dt} = \frac{4\pi h}{\sqrt{3}} \cdot \frac{3}{\pi h^2} \cdot \frac{dV}{dt} = \frac{4}{h} \cdot \frac{dV}{dt}$$

$$\text{Now } V = \frac{\pi r^2 h}{3}, \quad r = \frac{h}{\sqrt{3}}$$

$$\Rightarrow V = \frac{\pi h^3}{9}$$

$$\text{But } V = 81\pi \Rightarrow 81\pi = \frac{\pi h^3}{9}$$

From which $h = 9$

$$\begin{aligned}\therefore \frac{dS}{dt} &= \frac{4}{h} \cdot \frac{dV}{dt} = \frac{4}{9} \cdot (-3) \\ \therefore \frac{dS}{dt} &= -\frac{4}{3} \text{ cm}^2/\text{s}\end{aligned}$$

Alternatively

At any time t , volume, $V = \frac{\pi r^2 h}{3}$

Now both r and h are variables, expressing $V = \frac{\pi r^2 h}{3}$ in terms of two variables

$$\tan 30^\circ = \frac{r}{h} \Rightarrow \frac{h}{\sqrt{3}} = r$$

$$\therefore V = \frac{\pi h}{3} \cdot \frac{h^2}{3} = \frac{\pi r^3 h}{9}$$

$$\text{Now } \frac{dV}{dh} = \frac{\pi h^2}{3} \dots\dots\dots (1)$$

Net surface at any time t is

$$S = \pi r l$$

$$\text{But } \cos 30^\circ = \frac{h}{l} \Rightarrow l = h \sec 30^\circ$$

$$\begin{aligned}\therefore S &= \pi \cdot \frac{h}{\sqrt{3}} \cdot h \cdot \sec 30^\circ \\ &= \frac{\pi h^2}{\sqrt{3}} \cdot \frac{1}{\cos 30^\circ} \\ &= \frac{\pi h^2}{\sqrt{3}} \cdot \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2\pi h^2}{3}\end{aligned}$$

$$\therefore \frac{dS}{dh} = \frac{4\pi h}{3}$$

$$\text{Now } \frac{dV}{dt} = -3 \text{ (volume is decreasing)}$$

And $\frac{dS}{dt}$ is required

Since there is no direct relationship between S and V , we use a 3-step link as follows

$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \frac{4\pi h}{3} \times \frac{3}{\pi h^2} \times (-3) = -\frac{12}{h}\end{aligned}$$

$$\text{But } V = 81\pi = \frac{\pi h^3}{9} \therefore h = 9$$

$$\therefore \frac{dS}{dt} = -\frac{12}{9} = -\frac{4}{3} \text{ cm}^2/\text{s}$$

$$\therefore \text{Surface area is decreasing } \frac{4}{3} \text{ cm}^2/\text{s}$$

Task

1. Sand falls on to horizontal ground at the rate of $9\text{ m}^3/\text{min}$ and forms a heap in the shape of a right circular cone vertical angle 60° . Show that 10 seconds after the sand begins to fall, the rate at which the radius of the base of the pile is increasing is $3^{\frac{1}{2}} \cdot \left(\frac{4}{\pi}\right)^{\frac{1}{3}} \text{ cm/min}$
2. A container in the shape of a right circular cone of height 20cm and radius 5cm is held vertex downward and filled with water which the drips out from the vertex at the rate of change of the height of the water in the cone when it is half empty (measured by volume) **Ans:** $-0.1011 \text{ cm}^3/\text{s}$
3. A hemisphere bowl of radius $a \text{ cm}$ is initially full of water. The water runs out through a small hole at the bottom of the bowl at a constant rate such that it empties the bowl in 24s. Given that when the depth of

the bowl is x cm and the volume of the water is $\frac{1}{3}\pi x^2(3a-x)$ cm³, show that the depth of the water at that instant is decreasing at a rate $a^3[36x(2a-x)]^{-1}$ cms⁻¹.

Find how long it will take for the depth of water to be $\frac{1}{3}a$ cm and the rate at which the depth is decreasing at that instant

Solution

Given: volume of hemispherical bowl

$$= \frac{2}{3}\pi r^3 = \frac{2}{3}\pi a^3$$

$$\text{Volume rate} = \frac{dv}{dt} = \frac{\frac{2}{3}\pi r^3}{24} = \frac{1}{36}\pi a^3 \text{ cm}^3/\text{s}$$

$$\text{Primary equation : } V = \frac{\pi x^2(3a-x)}{3}, a = \text{constant}$$

Differentiating implicitly with respect to time, t

$$\frac{dV}{dt} = \frac{\pi}{3} [6ax - 3x^2] \cdot \frac{dx}{dt}$$

Required: $\frac{dx}{dt}$ when height is x

$$\begin{aligned} \therefore \frac{dx}{dt} &= \frac{\frac{dV}{dt}}{\pi(6ax - 3x^2)} \cdot \frac{dV}{dt} \\ &= \frac{\frac{1}{36}\pi a^3}{\pi(6ax - 3x^2)} \cdot \frac{1}{36}\pi a^3 = \frac{a^3}{36(2ax - x^2)} \\ &= \frac{a^3}{36(x[2a-x])} = a^3[36x(2a-x)]^{-1} \end{aligned}$$

Find $t = ?$ when $x = \frac{1}{3}a$ cm

$$\frac{dh}{dt} \text{ when } x = \frac{1}{3}a \text{ cm.}$$

Obtaining $\frac{dh}{dt}$ when $x = \frac{1}{3}a$

$$\begin{aligned} \frac{dx}{dt} &= \frac{a^3}{36x(2a-x)} \\ \frac{dx}{dt} &= \frac{a^3}{36x(2a-\frac{x}{3})} = \frac{a^2}{12(\frac{5a}{3})} = \frac{a}{20} \text{ cm/s} \end{aligned}$$

Using volume rate = $\frac{\text{Volume}}{\text{Time}}$

Volume of water at $x = \frac{a}{3}$ cm

$$V = \frac{\pi x^2(3a-x)}{3}$$

$$\therefore \text{Volume} = \frac{\pi a^2}{27} \left(3a - \frac{a}{3}\right) = \frac{8\pi a^3}{81} \text{ cm}^3$$

Let t = Time taken to reach a depth of $x = \frac{a}{3}$

$$\text{Volume reduced in } t(s) = \frac{2\pi a^3}{3} - \frac{8\pi a^3}{81} = \frac{46\pi a^3}{81}$$

$$\therefore \text{Time} = \frac{46\pi a^3}{81} \div \frac{\pi a^3}{36} = \frac{46}{81} \times 36 = 20.444 \text{ s}$$

Task

1. A hemispherical bowl is being filled with water at a uniform rate. When the height of water is h cm, the volume is $\pi \left(rh^2 - \frac{h^3}{3} \right)$ cm³, r cm being the radius of the hemisphere. Find the rate at which the water level is rising when it is half way to the top, given that $r = 6$ and that the bowl fills in 1 minute. **Ans:** $\frac{4}{45}$ cm/s

2. An inverted right circular cone of vertical angle 120° is collecting water from a tap at a steady rate of $18\pi \text{ cm}^3/\text{min}$. find :
- The depth of water after 12 min.
 - The rate of increase of the depth at this instant. **Ans:** i) 6 cm
ii) $\frac{1}{6} \text{ cm/s}$
3. When the depth of water in a hemispherical bowl of radius r is x , the volume of water is $\frac{1}{3}\pi x^2(3r - x)$. A hemispherical bowl of radius 12 cm is initially full of water. The water runs out of the hole at rate of $96\pi \text{ cm}^3\text{s}^{-1}$. When the depth of water is $x \text{ cm}$, show that the depth is decreasing at rate of $\frac{48}{x(24-x)} \text{ cms}^{-1}$, find the rate at which the depth is decreasing when :
- The bowl is full
 - The depth is 6cm **Ans:** $\frac{39}{20} \text{ cms}^{-1}$.
4. A hemispherical bowl of radius $a \text{ cm}$ is filled with water at rate of $\frac{1}{12}\pi a^3 \text{ cm}^3\text{s}^{-1}$. When the depth of water in the hemispherical bowl of radius r is x , the volume of water is $\frac{1}{3}\pi x^2(3r - x)$. find the rate at which the level is rising when the depth of water is $\frac{1}{3}a \text{ cm}$
Ans: $\frac{39}{20} \text{ cms}^{-1}$
5. A flask in the shape of a cone is fixed with its vertex pointing downwards. The cone is filling with a liquid at a rate which at any instant is proportional to the surface area of the liquid at that instant show that the rate of increase in the depth of the liquid is constant.

Small changes or Increments

Let $p(x, y)$ be a general point on the curve $y = f(x)$, and $Q(x + \delta x, y + \delta y)$ is a point on the curve close to P

Then δx = small increase in x and δy = corresponding increase in y

From gradient of PQ . i. e.

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

So when x is so small the

$$\frac{\delta y}{\delta x} \simeq \frac{dy}{dx}$$

$$\therefore \delta y \simeq \frac{dy}{dx} \cdot \delta x$$

This approximation is used to estimate a function close to a known value i.e.

$$y + \delta y = y + \frac{dy}{dx} \cdot \delta x$$

Example

Find an approximation of

- $\sqrt{9.01}$
- $\ln(1.1)$
- $\sin 31^\circ$ and $\sin 29^\circ$ (Take $1^\circ \simeq 0.0175$)

Solution

- $\sqrt{9.01} = \sqrt{9 + 0.01} \quad \therefore x = 9, dx = 0.01$

Let $y = \sqrt{x}$

Required: $y + \delta y = y + \frac{dy}{dx} \cdot \delta x$

$$\Rightarrow y = x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \mid$$

$$\therefore \sqrt{9.01} \simeq \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot \delta x$$

$$\simeq \sqrt{9} + \frac{1}{2\sqrt{9}} \cdot 0.01$$

$$\simeq 3 + \frac{0.01}{6}$$

b) $\ln(1.1) = \ln(1 + 0.1) \quad \therefore x = 1, dx = 0.1$

Let $y = \ln x$

Required: $y + \delta y = y + \frac{dy}{dx} \cdot \delta x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \ln 1.1 = \ln 1 + \left(\frac{\delta x}{x}\right)$$

$$= \ln 1 + \frac{0.1}{1} = 0 + 0.1$$

c) $\sin 31^\circ = \sin(30 + 1)^\circ \quad \therefore x = 30^\circ, \delta x = 1^\circ = 0.0175$

Let $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

$$\therefore \sin 31^\circ = \sin(x + \delta x) = \sin x + (\cos x) \cdot \delta x$$

$$= \sin 30^\circ + \cos 30^\circ \times 0.0175$$

$$= 0.5 + 0.866 \times 0.0175 = 0.5152$$

Also $\sin 29^\circ = \sin(30 - 1)^\circ \quad \therefore x = 30^\circ, \delta x = -1^\circ = -0.0175$

Let $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

$$\therefore \sin 29^\circ = \sin(x + \delta x) = \sin x + (\cos x) \cdot \delta x$$

$$= \sin 30^\circ + \cos 30^\circ \times (-0.0175)$$

$$= 0.5 - 0.866 \times 0.0175 = -0.4848$$

NOTE:

In general, the formula used for approximation is;

$$f(x + \Delta x) \simeq f(x) + f'(x) \cdot \Delta x$$

Example

Evaluate (i) $\sqrt[4]{624}$

(ii) $\sqrt[3]{28}$

Solution

i) $\sqrt[4]{624} \approx 4.99799879888 \approx 5$

Now $5^4 = 625$

$$\Rightarrow \sqrt[4]{624} = \sqrt[4]{(625 - 1)} \Rightarrow x = 625, \Delta x = -1$$

Let $f(x) = \sqrt[4]{x} = x^{1/4}$,

$$\Rightarrow f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$$

Using $f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x$

$$f(x + \Delta x) = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}} \cdot \Delta x$$

$$\begin{aligned}\Rightarrow \sqrt[4]{624} &= \sqrt[4]{625} + \frac{1}{4\sqrt[4]{(625)^3}} \times (-1) \\ &= 5 - \frac{1}{4(5)^3} = 5 - 0.002\end{aligned}$$

$$\therefore \sqrt[4]{624} \approx 4.998$$

ii) Left as an exercise

Small Percentage Increment

When x increases by $r\%$, then the increase in x , δx is given by:

$$\% \text{ge change/increase} = \frac{\text{Increase}}{\text{Original}} \times 100$$

$$\Rightarrow r = \frac{\delta x}{x} \times 100$$

Thus

$$\delta x = \frac{r}{100} \cdot x$$

NOTE:

Suggested steps in solving numbers on small percentage increment

1. Write the formula for the percentage error with respect to the given variable
2. Make the small increment the subject in the above formula
3. Write the formula involving the required percentage error in that quantity
4. Use the formula $\delta y \approx (dy/dx) \cdot \delta x$
5. Substitute the obtained increment in to the required percentage formula.

Example

1. An error of 3% is made in measuring the radius r of a sphere. What is the resulting percentage error in calculating the volume?

Solution

$$\% \text{ge error in } r = \frac{\delta r}{r} \times 100 \Rightarrow \delta r = \frac{3}{100} r = 0.03r$$

$$\text{Now required \%ge error in volume} = \frac{\delta V}{V} \times 100$$

$$\text{But } V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} \simeq \frac{\delta V}{\delta r} = 4\pi r^2 \Rightarrow \delta V = \left(\frac{dV}{dr}\right) \cdot \delta r$$

$$\therefore \delta V = 4\pi r^2 \cdot \delta r$$

$$\% \text{ge error in volume} = 4\pi r^2 \cdot \delta r / V \times 100$$

$$= \frac{4\pi r^2 \cdot \delta r}{\frac{4}{3}\pi r^3} \times 100 = \frac{3}{r} \delta r \times 100$$

$$= \frac{3}{r} \times 0.03r \times 100 = 9\%$$

2. Find the increase in area of a circle when its radius changes from 5m to 5.1m

Solution

$$\text{Change in radius} = \delta r = 5.1 - 5 = 0.1 \text{ m}$$

$$\text{Required: } \delta A = ?$$

$$\text{But } A = \pi r^2$$

$$\delta A = \left(\frac{dA}{dr}\right) \cdot \delta r = 2\pi r \cdot \delta r$$

$$\delta A = 2\pi \times 5 \times (0.1) = \pi \text{ m}^2$$

3. The period T of a simple pendulum is calculated from the formula $T =$

$2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum and g is the acceleration due to gravity. Find the percentage change in the period caused by lengthening the pendulum by 2%.

Solution

Given: %ge change in $l = \frac{\delta l}{l} \times 100$

$$\therefore \delta l = \frac{2}{100} l = 0.02l$$

Required: %ge change in $T = \frac{\delta T}{T} \times 100$

$$T = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{\sqrt{g}} \cdot l^{1/2}$$

Differentiating and using the formula

$$\delta T = \left(\frac{dT}{dl}\right) \cdot \delta l$$

$$\Rightarrow \delta T = \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2} \cdot l^{-1/2} \cdot \delta l = \frac{\pi}{\sqrt{g}} \cdot \frac{1}{\sqrt{l}} \cdot \delta l = \frac{\pi \cdot \delta l}{\sqrt{l \cdot g}}$$

$$\therefore \% \text{ge Change in } T = \frac{\pi \cdot \delta l}{\sqrt{l \cdot g}} / 2\pi\sqrt{\frac{l}{g}} \times 100$$

$$= \frac{\delta l}{\sqrt{l \cdot g}} \times \frac{\sqrt{g}}{2\sqrt{l}} \times 100$$

$$= \frac{\delta l}{2l} \times 100 = \frac{0.02l}{2l} \times 100 = 1\%$$

4. The side of a square can be measured accurately to 0.1mm, what is the possible error in the area of the square whose side is measured to be 200mm?

Solution

Given: error in side $x = \delta x = 200\text{mm}$

Area, $A = x^2$

$$dA \simeq 2x \cdot \delta x = 2 \times 0.1 \times 200$$

$$= 40\text{mm}^2$$

5. Find the approximate percentage change in the square of a quantity when the quantity itself changes by 0.1%. Hence calculate an approximate value of $(10.01)^2$.

Solution

Let $y = x^2$, x = quantity

$$\delta y \simeq 2x \cdot \delta x$$

But $\delta x = \frac{0.1}{100} x = 0.001x$

$$\therefore \delta y \simeq 2x \cdot (0.001x) = 0.002x^2$$

Required: %ge change in $y = \frac{\delta y}{y} \times 100$

$$= \frac{0.002x^2}{x^2} \times 100 = 0.2\%$$

Approximate value of $(10.01)^2$

Using $f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x$, taking

$$x + \Delta x = 10 + 0.1 \quad \therefore x = 10, \Delta x = 0.1$$

$$\begin{aligned} f(x + \Delta x) &\approx x^2 + dy \\ &\approx 10^2 + 0.2 = 100.2 \end{aligned}$$

- 6.** If $y = x^3$, find the approximate increase in y due to the increase of 0.1% in x .

Solution

$$\text{Given: } \frac{\delta x}{x} \times 100 = 0.1 \Rightarrow \delta x = 0.001x$$

$$\text{Required: } \% \text{ge change in } y = \frac{\delta y}{y} \times 100$$

$$\text{But } \delta y \approx \left(\frac{dy}{dx} \right) \cdot \delta x$$

$$\delta y \approx 3x^2 \cdot \delta x = 3x^2 \times 0.001x = 0.003x^3$$

$$\therefore \% \text{ge Change in } y = \frac{\delta y}{y} \times 100 = \frac{0.003x^3}{x^3} \times 100 = 0.3\%$$

- 7.** The radius of a cylinder is reduced by 3% and its height increased by 4%.

Determine the approximate percentage change in

- Its volume
- In curved surface area

Solution

$$\text{a) } \% \text{ge change in volume} = \frac{\delta V}{V} \times 100$$

$$\text{Given: } \% \text{ge change in radius} = \frac{dr}{r} \times 100$$

$$\therefore \delta r = \frac{3}{100} \times r = 0.03r$$

Since there is a decrease in radius, then

$$\delta r = -0.03r$$

$$\text{Given: } \% \text{ge change in height} = \frac{dh}{h} \times 100$$

$$\therefore \delta h = \frac{4}{100} \times h = 0.04h$$

Since there is an increase in height, then

$$\delta h = +0.04h$$

$$\text{But } V = \pi r^2 h$$

Differentiating implicitly w.r.t. any variable, say r (Note this step)

$$\frac{dV}{dr} = \pi \left[r^2 \cdot \frac{dh}{dr} + h \cdot \frac{d}{dr}(r^2) \right]$$

$$\frac{dV}{dr} = \pi \left[r^2 \cdot \frac{dh}{dr} + 2hr \right], \text{ now multiplying through by } dr$$

$$\Rightarrow \frac{dV}{dr} = \pi [r^2 \cdot dh + 2hr \cdot dr]$$

By small changes

$$\begin{aligned} \therefore \delta V &\approx \pi [r^2 \cdot \delta h + 2hr \cdot \delta r] \\ &\approx \pi [r^2 \cdot (0.04h) + 2hr \cdot (-0.003r)] \\ &\approx \pi [0.04hr^2 - 0.006hr^2] \\ &\approx -0.02\pi r^2 h \end{aligned}$$

$$\therefore \%ge \text{ Change in volume} = \frac{-0.02\pi r^2 h}{\pi r^2 h} \times 100 = -2\%$$

$\therefore$ Volume reduces by 2%

b) $\%ge \text{ change in curved surface area} = \frac{\delta A}{A} \times 100$

$$A = 2\pi r h$$

Differentiating implicitly w.r.t any variable, say h

$$\frac{dA}{dh} = 2\pi \left[r \cdot \frac{dh}{dh} + h \cdot \frac{dr}{dh} \right]$$

$$= 2\pi \left[r + h \cdot \frac{dr}{dh} \right], \text{ multiplying through by } dh$$

$$dA = 2\pi [r \cdot dh + h \cdot dr]$$

By small increments,

$$\delta A = 2\pi [r \cdot \delta h + h \cdot \delta r]$$

$$\therefore \%ge \text{ Increase in curved area} = \frac{2\pi [r \cdot \delta h + h \cdot \delta r]}{2\pi r h} \times 100$$

$$= \frac{2\pi [r \cdot (0.04h) + h \cdot (-0.003r)]}{2\pi r h} \times 100$$

$$= \frac{2\pi [0.04h \cdot r - 0.003h \cdot r]}{2\pi r h} \times 100 = 0.01 \times 100 = 1\%$$

$\therefore$ Curved area increases by 1%

8. Pressure P and volume V are related by the expression $PV^3 = C$ where C is a constant. Find the approximate percentage change in C when P is increased by 2% and volume decreased by 0.8%

Solution

Given: $\delta P = \frac{2}{100P} = 0.02P$

For increase, $\delta P = +0.02P$

$$\delta V = \frac{0.8}{100V} = 0.008V$$

For a decrease, $\delta V = -0.008V$

Now $PV^3 = C$

Differentiating implicitly w.r.t. any variable, say P

$$\frac{dC}{dP} = P \cdot \frac{d}{dP} (V^3) + V^3 \cdot \frac{dP}{dP}$$

$$= \frac{dC}{dP} = 3PV^2 \cdot \frac{dV}{dP} + V^3, \text{ multiplying through by } dP$$

$$dC = 3PV^2 dV + V^3 \cdot dP$$

By small changes

$$\delta C = 3PV^2 \delta V + V^3 \cdot \delta P$$

Required: $\%ge \text{ change in } C = \frac{\delta C}{C} \times 100$

$$= \frac{3PV^2 \delta V + V^3 \cdot \delta P}{PV^3} \times 100$$

$$= \frac{3PV^2 (-0.008V) + V^3 \cdot (+0.02P)}{PV^3} \times 100$$

$$= \frac{-0.024 + 0.02}{1} \times 100 = -0.004 \times 100 = -0.4\%$$

$\therefore \%ge \text{ Change (Reduction) in } C = 0.4\%$

NOTE:

For more details on small percentage increments go to “Mathematics For an A-level Student” Volume I. Topic: Percentages and Errors in Binomial Expansion.

Task

1. The modulus of rigidity G is given by $G = \frac{R^4 \theta}{L}$ where R is the radius, θ is the angle of twist and L is the length. Find the approximate percentage in G when R is measured 1.5% too large and θ is measured 5% too small.
Ans: 1%
2. The volume of a cone is given by $V = \frac{\pi r^2 h}{3}$. If the volume increases by $\frac{2\pi}{3} \text{ cm}^3/\text{min}$ and height decreases by $0.03 \text{ cm}/\text{min}$, find the rate of change of radius when $r = 10 \text{ cm}$ and $h = 5 \text{ cm}$ **Ans:** 5
3. If the radius of a spherical balloon increases from 1 cm to 1.02 cm , find the approximate increase in volume. **Ans:** $0.08\pi \text{ cm}^3$
4. If $R = ar^n$ and an error of $x\%$ is made in measuring r , prove that an error of $nx\%$ will result in R .
5. The volume of a sphere is increased by 3%. Find the percentage increase in the radius. **Ans:** 1%
6. If L is the length of a pendulum and $t(s)$ is the time of one complete swing. It is known that $l = kt^2$. If the length of the pendulum increases by $x\%$, x being small, find the corresponding increase in the time of the swing. **Ans:** $0.5x\%$
7. One side of a rectangle is three times the other. If the perimeter increases by 2% what is the percentage increase in the area? **Ans:** 4%
8. The base radius of the right circular cone increases and the volume changes by 2%. If the height remains constant, find the percentage increase in the circumference of the base. **(Unneb 2011) Ans:** 1%
9. An error of $2\frac{1}{2}\%$ is made in the measurement of the area of a circle. What percentage error results in the radius and circumference? **Ans:** 1.25% ; 1.25%
10. A group of sociologists has made a survey of unemployment in a certain London borough. The group claims that the number u of unemployment

in the borough at any time is related to the rate of inflation by the formula $u^3 = k/r^2$ (k is a constant). Using this formula, find the percentage change in u corresponding to a 6% increase in r . **Ans:** 4% decrease.

11. Given that $y = 2 + k/x$ (k is a constant). Find the percentage error in y due to a 3% error in x when;

i) $y = 0.5$	ii) $y = 1$	iii) $y = 1.5$
i) Ans: 9%	ii) Ans: 3%	iii) Ans: 1%

Acceleration and Velocity

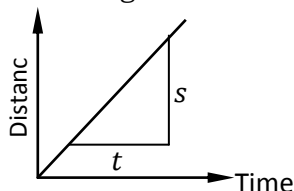
Constant Velocity

When a car moves a distance s (m) in a time t (s) along a straight road, the *velocity*, V

is constant.

$$V = \frac{s}{t} \text{ m/s}$$

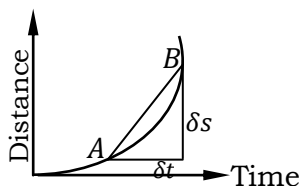
This is the gradient of the distance/time graph



$$\text{Here } V = \text{constant} = \frac{s}{t}$$

Variable Velocity

If however, the velocity of the car is not constant then the distance/time graph will not be a straight line.



For variable velocity we are concerned with *average velocity*

Definition

$$\text{Average velocity} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{\text{Increase in } s}{\text{Increase in } t}$$

$$\therefore \text{Average velocity over time } \delta t = \frac{\delta s}{\delta t}$$

Now as $\delta t \rightarrow 0$, the chord AB becomes a tangent, such that the velocity at point A is $V_A = \frac{ds}{dt}$

Velocity at an Instant

Thus, the velocity of a car at any instant is the gradient of the distance/time graph. Thus if the expression of distance is in terms of time t , then the velocity is obtained by differentiating the expression.

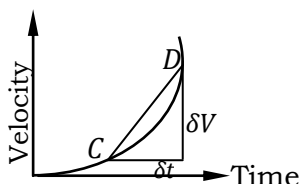
Definition:

Velocity at an instant is the limit of the average velocity for the interval following that instant, as the interval tends to zero. *i.e.*

$$V = \lim_{\delta t \rightarrow 0} (\delta s / \delta t) = ds/dt$$

Variable Acceleration

The acceleration of a car is defined as the rate of change of velocity.



If δV is the change in v , and δt is the corresponding change in time, then
 $a = \lim_{\delta t \rightarrow 0} (\delta V / \delta t)$, chord CD becomes a tangent at point C, the acceleration is given by $a = dV/dt$

Hence acceleration of the car at any instant is given by the gradient of velocity/time graph.

Now if the expression of velocity is known in terms of time t then the acceleration is obtained by differentiating the expression.

$$\therefore \text{Acceleration, } a = \frac{dV}{dt}$$

$$\text{However, } V = \frac{ds}{dt}$$

$$\text{Hence } a = \frac{d}{dt} \left(\frac{ds}{dt} \right) = \frac{d^2s}{dt^2}$$

Example

1. If a billiard ball is dropped from a height of 100 feet, its height, s at time t is given by the position function $s = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. Find the average velocity over the time intervals

- a) $1 \leq t \leq 2$ b) $1 \leq t \leq 1.5$ c) $1 \leq t \leq 1.1$

Solution

$$\begin{aligned} \text{a) } s_{t=1} &= -16 + 100 = 84 \\ s_{t=2} &= -64 + 100 = 36 \end{aligned} \quad \left. \vphantom{\begin{aligned} s_{t=1} \\ s_{t=2} \end{aligned}} \right\} \begin{array}{l} \text{For } 1 \leq t \leq 2, \text{ the object falls from a height} \\ \text{of } 84ft \text{ to a height of } 36ft \end{array}$$

$$\therefore \text{Average velocity} = \frac{36-84}{2-1} = -48ft/sec$$

$$\begin{aligned} \text{b) } s_{t=1} &= -16 + 100 = 84 \\ s_{t=1.5} &= -36 + 100 = 64 \end{aligned} \quad \left. \vphantom{\begin{aligned} s_{t=1} \\ s_{t=1.5} \end{aligned}} \right\} \begin{array}{l} \text{For } 1 \leq t \leq 1.5, \text{ the object falls from a} \\ \text{height} \end{array}$$

$$\therefore \text{Average velocity} = \frac{64-84}{1.5-1} = -40ft/sec$$

$$\text{c) For interval } 1 \leq t \leq 1.1, \text{ Average velocity} = \frac{80.64-84}{1.1-1} = -33.6ft/sec$$

NOTE:

Average velocity is *negative*, indicating that the object is moving downward.

2. At time $t = 0$, a diver jumps from a diving board that is 32 feet above the water the position of the diver is given by $s(t) = -16t^2 + 16t + 32$ where s is measured in feet and t is measured in seconds.

- a) When does the diver hit the water?
- b) What is the diver's velocity at impact?

Solution

- a) When the diver hits the water, $s = 0$

$$0 = -16t^2 + 16t + 32, \text{ solving } t = -1, 2$$

Taking $t = 2s$, the diver takes 2 seconds

- b) Velocity, $V = \frac{ds}{dt} = -32t + 16$

$$\text{Now when } t = 2, V = -64 + 16 = -48 \text{ ft/sec}$$

3. The distance $x(m)$ moved by a car in a time $t(s)$ is given by $x = 3t^3 - 2t^2 + 4t - 1$. Determine the velocity and acceleration when

- a) $t = 0$ and
- b) $t = 1.5s$

Solution

$$\text{Distance: } x = 3t^3 - 2t^2 + 4t - 1 \text{ m}$$

$$\text{Velocity: } V = \frac{dx}{dt} = 9t^2 - 4t + 4 \text{ m/s}$$

$$\text{Acceleration: } a = \frac{d^2x}{dt^2} = 18t - 4 \text{ m/s}^2$$

- a) When $t = 0$

$$\text{Velocity, } V = \frac{dx}{dt} = 9(0)^2 - 4(0) + 4 = 4 \text{ m/s}$$

$$\text{Acceleration, } a = \frac{d^2x}{dt^2} = 18(0) - 4 = -4 \text{ m/s}^2$$

- b) When time $t = 1.5s$

$$\text{Velocity, } V = \frac{dx}{dt} = 9(1.5)^2 - 4(1.5) + 4 = 18.25 \text{ m/s}$$

$$\text{Acceleration, } a = \frac{d^2x}{dt^2} = 18(1.5) - 4 = 23 \text{ m/s}^2$$

4. A particle moves along a straight line so that after $t(s)$, its distance from O affixed point on the line is $s(m)$ where $s = t^3 - 3t^2 + 2t$

- a) When is the particle at O?
- b) What is its velocity and acceleration at these times
- c) What is the average velocity after the first second?
- d) What is the average acceleration between $t = 0$ and $t = 2$?

Solution

Given: Distance: $s = t^3 - 3t^2 + 2t$ m

Velocity: $V = \frac{ds}{dt} = 3t^2 - 6t + 2$ m/s

Acceleration: $a = \frac{d^2s}{dt^2} = 6t - 6$ m/s²

- a) A particle is at O if $s = 0$
 $\therefore 0 = t^3 - 3t^2 + 2t \Rightarrow t(t-1)(t-2) = 0$
 $\therefore t = 0, 1, 2$
- b) At $t = 0, V = 3(0)^2 - 6(0) + 2 = 2$ m/s
 At $t = 0, a = 6(0) - 6 = -6$ m/s²
 At $t = 1, V = 3(1)^2 - 6(1) + 2 = -1$ m/s
 At $t = 1, a = 6(1) - 6 = 0$ m/s²
 At $t = 2, V = 3(2)^2 - 6(2) + 2 = 2$ m/s
 At $t = 2, a = 6(2) - 6 = 6$ m/s²
- c) Average Velocity = $\frac{s_{t=1} - s_{t=0}}{1-0} = \frac{(1)^3 - 3(1)^2 + 2(1) - 0}{1} = 0$ m/s
- d) Average Acceleration = $\frac{V_{t=1} - V_{t=0}}{2-0} = \frac{2-2}{2} = 0$ m/s²

Task

1. $x = \frac{20}{3}t^3 - \frac{23}{2}t^2 + 6t + 5$ represents the distance x metres, moved by a body in t seconds. Determine:
- The velocity and acceleration at the start
 - The velocity and acceleration when $t = 3$ s
 - The value of t when the body is at rest
 - The values of t when the acceleration is 37 m/s² and
- e) The distance travelled in the third second
- Ans:** a) 6m/s, -23m/s² b) 117m/s, 97 m/s² c) $t = \frac{3}{4}$ s or $\frac{2}{3}$ s
 d) $1\frac{1}{2}$ s e) $75\frac{1}{6}$ m
2. A missile fired from the ground level rises x (m) vertically upwards in t seconds and $x = 100t - \frac{25}{2}t^2$. Find:
- The initial velocity of the missile **Ans:** 100m/s
 - The time taken when the height of the missile is maximum. **Ans:** 4s
 - The maximum height reached. **Ans:** 200m
 - The velocity with which the missile strikes the ground **Ans:** -100m/s
3. At any time t seconds the distance x metres of a particle moving in a straight line from a fixed point is given by: $x = 4t + \ln(1-t)$. Determine
- The initial velocity and acceleration **Ans:** 3 m/s ; -1 m/s²
 - The velocity and acceleration after $t = 1.5$ s. **Ans:** 6m/s ; -4m/s²
 - The time taken when the velocity is zero. **Ans:** $\frac{3}{4}$ s

DIFFERENTIATING TRIGONOMETRIC FUNCTIONS

Differentiating Trigonometric Functions from First Principles

The following should be noted when differentiating trigonometric functions.

- 1) All angles of the trigonometric ratios must be in radians. If the angle is not in radians, use the conversion $1\pi = 180^\circ$
- 2) The knowledge of limits is needed. *i.e.*

For small angle θ in radians,
 $\sin \theta \simeq \theta$ or $\frac{\sin \theta}{\theta} \simeq 1$, $\cos \theta \simeq 1 - \frac{1}{2}\theta^2$, $\tan \theta \simeq \theta$

- 3) Factor formula is applied to simplify trigonometric expressions

1. Differentiate from first principles

- a) $\sin x$
- b) $\cos x$
- c) $\tan x$

Solution

- a) Let $y = \sin x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$y + \delta y = \sin(x + \delta x)$$

$$\delta y = \sin(x + \delta x) - y$$

$$= \sin(x + \delta x) - \sin x$$

$$= 2 \cos\left(x + \frac{\delta x}{2}\right) \cdot \sin\left(\frac{\delta x}{2}\right)$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = 2 \cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

Introducing limits on both sides

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) &= \lim_{\delta x \rightarrow 0} \left\{ 2 \cos\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x} \right\}, \text{ putting } \theta = \frac{\delta x}{2} \Rightarrow \delta x = 2\theta \\ &= \lim_{\theta \rightarrow 0} \left\{ 2 \cos(x + \theta) \cdot \frac{\sin(\theta)}{2\theta} \right\}, \text{ but } \frac{\sin \theta}{\theta} \simeq 1 \\ &= \cos x \end{aligned}$$

$$\therefore \frac{d}{dx}(\sin x) = \cos x$$

- b) Let $y = \cos x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$y + \delta y = \cos(x + \delta x)$$

$$\delta y = \cos(x + \delta x) - y$$

$$= \cos(x + \delta x) - \cos x$$

$$= -2 \sin\left(x + \frac{\delta x}{2}\right) \cdot \sin\left(\frac{\delta x}{2}\right)$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = 2 \sin\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

Introducing limits on both sides

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x}\right) &= \lim_{\delta x \rightarrow 0} \left\{ 2 \sin\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x} \right\}, \text{ putting } \theta = \frac{\delta x}{2} \Rightarrow \delta x = 2\theta \\ &= \lim_{\theta \rightarrow 0} \left\{ 2 \sin(x + \theta) \cdot \frac{\sin(\theta)}{2\theta} \right\}, \text{ but } \frac{\sin \theta}{\theta} \simeq 1 \\ &= \sin x \end{aligned}$$

$$\therefore \frac{d}{dx}(\cos x) = \sin x$$

c) Let $y = \tan x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$y + \delta y = \tan(x + \delta x)$$

$$\delta y = \tan(x + \delta x) - y$$

$$= \tan(x + \delta x) - \tan x$$

$$= \frac{\tan x + \tan \delta x}{1 - \tan x \cdot \tan \delta x} - \tan x$$

$$= \frac{\tan x + \tan \delta x - \tan x + \tan^2 x \cdot \tan \delta x}{1 - \tan x \cdot \tan \delta x}$$

$$= \frac{\tan \delta x + \tan^2 x \cdot \tan \delta x}{1 - \tan x \cdot \tan \delta x} = \frac{\tan \delta x (1 + \tan^2 x)}{1 - \tan x \cdot \tan \delta x}$$

$$= \frac{\tan \delta x (\sec^2 x)}{1 - \tan x \cdot \tan \delta x}$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = \frac{\frac{\tan \delta x (\sec^2 x)}{\delta x}}{1 - \tan x \cdot \tan \delta x}$$

Introducing limits on both sides

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x}\right) &= \lim_{\delta x \rightarrow 0} \left\{ \frac{\frac{\tan \delta x (\sec^2 x)}{\delta x}}{1 - \tan x \cdot \tan \delta x} \right\}, \text{ putting } \delta x = \theta \\ &= \lim_{\theta \rightarrow 0} \left\{ \frac{\frac{\tan \theta}{\theta} (\sec^2 x)}{1 - \tan x \cdot \tan \theta} \right\}, \text{ but } \frac{\tan \theta}{\theta} \simeq \frac{\theta}{\theta} = 1 \\ &= \sec^2 x \end{aligned}$$

$$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$$

Alternatively

$$y + \delta y = \tan(x + \delta x)$$

$$\delta y = \tan(x + \delta x) - y$$

$$= \tan(x + \delta x) - \tan x$$

$$= \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \sin x}{\cos(x + \delta x) \cos x}$$

$$= \frac{\sin(x + \delta x - x)}{\cos(x + \delta x) \cos x} = \frac{\sin(\delta x)}{\cos(x + \delta x) \cos x}$$

Divide through by δx

$$\frac{\delta y}{\delta x} = \frac{\frac{\sin(\delta x)}{\delta x}}{\cos(x + \delta x) \cos x}$$

Introducing limits on both sides

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \left\{ \frac{\frac{\sin(\delta x)}{\delta x}}{\cos(x + \delta x) \cos x} \right\}, \text{ putting } \delta x = \theta$$

$$\lim_{\theta \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\theta \rightarrow 0} \left\{ \frac{\frac{\sin(\theta)}{\theta}}{\cos(x + \theta) \cos x} \right\}, \text{ but } \frac{\sin \theta}{\theta} \simeq 1$$

$$\frac{dy}{dx} = \frac{1}{\cos x \cos x} = \sec^2 x$$

$$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$$

2. Differentiate from first principles

a) $y = \sin 3x$

b) $y = \cos 2x$ c) $y = \tan \frac{1}{2}x$

Solution

a) $y = \sin 3x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$y + \delta y = \sin 3(x + \delta x)$$

$$\delta y = \sin 3(x + \delta x) - y$$

$$= \sin 3(x + \delta x) - \sin 3x$$

$$= 2 \cos \left(3x + \frac{3\delta x}{2} \right) \cdot \sin \left(\frac{3\delta x}{2} \right)$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = 2 \cos \left(3x + \frac{3\delta x}{2} \right) \cdot \frac{\sin \left(\frac{3\delta x}{2} \right)}{\delta x}$$

Introducing limits on both sides

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) &= \lim_{\delta x \rightarrow 0} \left\{ 2 \cos \left(3x + \frac{3\delta x}{2} \right) \cdot \frac{\sin \left(\frac{3\delta x}{2} \right)}{\delta x} \right\}, \text{ putting } \theta = \frac{3\delta x}{2} \Rightarrow \delta x = \frac{2}{3}\theta \\ &= \lim_{\theta \rightarrow 0} \left\{ 2 \cos(3x + \theta) \cdot \frac{\sin(\theta)}{\frac{2}{3}\theta} \right\} = \lim_{\theta \rightarrow 0} \left\{ 3 \cos(3x + \theta) \cdot \frac{\sin(\theta)}{\theta} \right\}, \text{ but} \end{aligned}$$

$$\frac{\sin \theta}{\theta} \simeq 1$$

$$= 3 \cos 3x$$

$$\therefore \frac{d}{dx}(\sin 3x) = 3 \cos 3x$$

b) Left as an exercise. **Ans:** $-2 \sin 2x$

c) Left as an exercise. **Ans:** $\frac{1}{2} \sec^2 \frac{1}{2}x$

Differentiating Reciprocals of Cosine, Sine and Tangent

1. Differentiate from first principles

- a) $\sec x$
 b) $\operatorname{cosec} x$ c) $\cot x$

Solution

a) let $y = \sec x = \frac{1}{\cos x}$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$\begin{aligned} y + \delta y &= \frac{1}{\cos(x + \delta x)} \\ \delta y &= \frac{1}{\cos(x + \delta x)} - y \\ &= \frac{1}{\cos(x + \delta x)} - \frac{1}{\cos x} \\ &= \frac{\cos x - \cos(x + \delta x)}{\cos(x + \delta x) \cos x} = \frac{-2 \sin\left(x + \frac{\delta x}{2}\right) \sin\left(-\frac{\delta x}{2}\right)}{\cos(x + \delta x) \cos x} \\ &= \frac{2 \sin\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{\cos(x + \delta x) \cos x} \end{aligned}$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = \frac{2 \sin\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}}{\cos(x + \delta x) \cos x}$$

Introducing limits on both sides

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) &= \lim_{\delta x \rightarrow 0} \left\{ \frac{2 \sin\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{\cos(x + \delta x) \cos x} \right\}, \text{ putting } \theta = \frac{\delta x}{2} \Rightarrow \delta x = 2\theta \\ &= \lim_{\theta \rightarrow 0} \left\{ \frac{2 \sin(x + \theta) \cdot \frac{\sin(\theta)}{2\theta}}{\cos(x + \theta) \cos x} \right\} = \lim_{\theta \rightarrow 0} \left\{ \frac{\sin(x + \theta) \cdot \frac{\sin(\theta)}{\theta}}{\cos(x + \theta) \cos x} \right\}, \text{ but } \frac{\sin \theta}{\theta} \simeq 1 \\ &= \frac{\sin x}{\cos x \cos x} = \tan x \sec x \end{aligned}$$

$$\therefore \frac{d}{dx}(\sec x) = \tan x \sec x$$

- b) left as an exercise: **Ans:** $-\operatorname{cosec} x \cot x$
 c) left as an exercise: **Ans:** $-\operatorname{cosec}^2 x$

2. Differentiate from first principles

- a) $\operatorname{cosec} \frac{1}{2}x$
 b) $\sec 3x$ c) $\cot 4x$

Solution

a) Let $y = \operatorname{cosec} \frac{1}{2}x = \frac{1}{\sin\left(\frac{1}{2}x\right)}$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$y + \delta y = \frac{1}{\sin\left(\frac{x + \delta x}{2}\right)}$$

$$\begin{aligned}\delta y &= \frac{1}{\sin\left(\frac{x+\delta x}{2}\right)} - \frac{1}{\sin\left(\frac{1}{2}x\right)} \\ &= \frac{\sin\left(\frac{1}{2}x\right) - \sin\left(\frac{x+\delta x}{2}\right)}{\sin\left(\frac{x+\delta x}{2}\right) \cdot \sin\left(\frac{1}{2}x\right)} = \frac{2 \cos\left(\frac{x}{2} + \frac{\delta x}{4}\right) \cdot \sin\left(-\frac{\delta x}{4}\right)}{\sin\left(\frac{x+\delta x}{2}\right) \cdot \sin\left(\frac{1}{2}x\right)} = \frac{-2 \cos\left(\frac{x}{2} + \frac{\delta x}{4}\right) \cdot \sin\left(\frac{\delta x}{4}\right)}{\sin\left(\frac{x+\delta x}{2}\right) \cdot \sin\left(\frac{1}{2}x\right)}\end{aligned}$$

Divide through by δx

$$\frac{\delta y}{\delta x} = \frac{2 \cos\left(\frac{x}{2} + \frac{\delta x}{4}\right) \cdot \sin\left(-\frac{\delta x}{4}\right)}{\sin\left(\frac{x+\delta x}{2}\right) \cdot \sin\left(\frac{1}{2}x\right)}$$

Introducing limits on both sides

$$\begin{aligned}\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) &= \lim_{\delta x \rightarrow 0} \left\{ \frac{-2 \cos\left(\frac{x}{2} + \frac{\delta x}{4}\right) \cdot \sin\left(\frac{\delta x}{4}\right)}{\sin\left(\frac{x+\delta x}{2}\right) \cdot \sin\left(\frac{1}{2}x\right)} \right\}, \text{ putting } \frac{\delta x}{4} = \theta \Rightarrow \delta x = 4\theta \\ &= \lim_{\theta \rightarrow 0} \left\{ \frac{-2 \cos\left(\frac{x}{2} + \theta\right) \cdot \sin\left(\theta\right)}{\sin\left(\frac{x+4\theta}{2}\right) \cdot \sin\left(\frac{1}{2}x\right)} \right\} = \lim_{\theta \rightarrow 0} \left\{ \frac{-\cos\left(\frac{x}{2} + \theta\right) \cdot \sin\left(\theta\right)}{\sin\left(\frac{x}{2} + 2\theta\right) \cdot \sin\left(\frac{1}{2}x\right)} \right\}\end{aligned}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{2} \cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right) \cdot \sin\left(\frac{x}{2}\right)} = -\frac{1}{2} \cot \frac{x}{2} \operatorname{cosec} \frac{x}{2}$$

$$\therefore \frac{d}{dx} \left(\operatorname{cosec} \frac{1}{2} x \right) = -\frac{1}{2} \cot \frac{x}{2} \operatorname{cosec} \frac{x}{2}$$

b) Left as an exercise. **Ans:** $3 \sec 3x \tan 3x$

c) Left as an exercise. **Ans:** $-4 \operatorname{cosec}^2 3x$

Extension of Differentiating From First Principles

Differentiate from first principles

a) $y = \sin^2 x$

b) $y = \cos^2(x^2)$

c) $y = \tan^2(x^2 + 1)$

Solution

a) $y = \sin^2 x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$y + \delta y = \sin^2(x + \delta x)$$

$$\delta y = \sin^2(x + \delta x) - y$$

$$\delta y = \sin^2(x + \delta x) - \sin^2 x$$

$$= [\sin(x + \delta x) + \sin x] \cdot [\sin(x + \delta x) - \sin x]$$

$$= 2 \sin\left(x + \frac{\delta x}{2}\right) \cos\left(\frac{\delta x}{2}\right) \cdot 2 \cos\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)$$

$$= \left[2 \sin\left(x + \frac{\delta x}{2}\right) \cos\left(x + \frac{\delta x}{2}\right) \right] \cdot 2 \cos\left(\frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)$$

$$= \sin(2x + \delta x) \cdot \sin(\delta x)$$

Divide through by δx

$$\frac{\delta y}{\delta x} = \sin(2x + \delta x) \cdot \frac{\sin(\delta x)}{\delta x}$$

Introducing limits on both sides

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \left\{ \sin(2x + \delta x) \cdot \frac{\sin(\delta x)}{\delta x} \right\}, \text{ putting } \delta x = \theta$$

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\theta \rightarrow 0} \left\{ \sin(2x + \theta) \cdot \frac{\sin(\theta)}{\theta} \right\} = \sin 2x$$

$$\therefore \frac{d}{dx}(\sin^2 x) = \sin 2x$$

b) $y = \cos^2(x^2)$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$y + \delta y = \cos^2(x + \delta x)^2$$

$$\delta y = \cos^2(x + \delta x)^2 - y$$

$$\delta y = \cos^2(x + \delta x)^2 - \cos^2(x^2)$$

$$\begin{aligned} &= [\cos(x + \delta x)^2 + \cos x^2] \cdot [\cos(x + \delta x)^2 - \cos x^2] \\ &= 2 \cos \left[\frac{(x + \delta x)^2 + x^2}{2} \right] \cos \left[\frac{(x + \delta x)^2 - x^2}{2} \right] \cdot \left[-2 \left[\sin \left[\frac{(x + \delta x)^2 + x^2}{2} \right] \right] \sin \left[\frac{(x + \delta x)^2 - x^2}{2} \right] \right] \\ &= 2 \cos \left[\frac{(x + \delta x)^2 + x^2}{2} \right] \sin \left[\frac{(x + \delta x)^2 - x^2}{2} \right] \cdot -2 \cos \left[\frac{(x + \delta x)^2 - x^2}{2} \right] \sin \left[\frac{(x + \delta x)^2 + x^2}{2} \right] \\ &= \sin[(x + \delta x)^2 + x^2] [-\sin[(x + \delta x)^2 - x^2]] \\ &= \sin[(x + \delta x)^2 + x^2] [-\sin[(2x + \delta x)(\delta x)]] \\ &= -\sin[(x + \delta x)^2 + x^2] \cdot \sin[(\delta x)(2x + \delta x)] \end{aligned}$$

Divide through by δx

$$\frac{\delta y}{\delta x} = -\sin[(x + \delta x)^2 + x^2] \cdot \frac{\sin[(\delta x)(2x + \delta x)]}{\delta x}$$

Introducing limits on both sides

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) &= \lim_{\delta x \rightarrow 0} \left\{ -\sin[(x + \delta x)^2 + x^2] \cdot \frac{\sin[(\delta x)(2x + \delta x)]}{\delta x} \right\} \\ &= \lim_{\delta x \rightarrow 0} \{ -\sin[(x + \delta x)^2 + x^2] \} \cdot \lim_{\delta x \rightarrow 0} \left(\frac{\sin[(\delta x)(2x + \delta x)]}{\delta x} \right) \\ &= -\sin[x^2 + x^2] \cdot \lim_{\delta x \rightarrow 0} \left(\frac{(\delta x)(2x + \delta x)}{\delta x} \right) = -2x \cdot \sin(2x^2) \end{aligned}$$

$$\therefore \frac{d}{dx}[\cos^2(x^2)] = -2x \cdot \sin(2x^2)$$

c) Left as an exercise. **Ans:** $4x \tan(x^2 + 1) \cdot \sec^2(x^2 + 1)$

Differentiating Inverse of Trigonometric Functions from First Principles

Differentiate from first principles

a) $\sin^{-1} x$

b) $\cos^{-1} x$ c) $\tan^{-1} x$

Solution

a) Let $y = \sin^{-1} x \Rightarrow x = \sin y$

Let x change by a small amount δx and the corresponding change in y be δy then;

$$x + \delta x = \sin(y + \delta y)$$

Making δx the subject

$$\delta x = \sin(y + \delta y) - x$$

$$\delta x = \sin(y + \delta y) - \sin y$$

Dividing through by δy

$$\frac{\delta x}{\delta y} = 2 \cos \left(y + \frac{\delta y}{2} \right) \cdot \frac{\sin \left(\frac{\delta y}{2} \right)}{\delta y}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{1}{2 \cos\left(y + \frac{\delta y}{2}\right) \cdot \frac{\sin\left(\frac{\delta y}{2}\right)}{\delta y}}$$

Introducing limits on both sides

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \lim_{\delta y \rightarrow 0} \left(\frac{1}{2 \cos\left(y + \frac{\delta y}{2}\right) \cdot \frac{\sin\left(\frac{\delta y}{2}\right)}{\delta y}} \right)$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

b) Left as an exercise. **Ans:** $-\frac{1}{\sqrt{1-x^2}}$

c) Left as an exercise. **Ans:** $\frac{1}{1+x^2}$

Important General Deductions of Differentiating Trigonometric Functions

1. Differentiate $\sin[f(x)]$ w.r.t. x

Solution

Let $y = \sin[f(x)]$

$$u = f(x) \dots \dots \dots (1)$$

$$\Rightarrow y = \sin u \dots \dots \dots (2)$$

Writing Chain Rule from eqn. (1) and eqn. (2)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y = \sin u \Rightarrow \frac{dy}{du} = \cos u$$

$$u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$$

$$\therefore \frac{dy}{dx} = \cos u \cdot f'(x) = \cos f(x) \cdot f'(x)$$

$$\therefore \frac{d}{dx}\{\sin[f(x)]\} = f'(x) \cdot \cos[f(x)]$$

2. Differentiate $\cos[f(x)]$ w.r.t. x

Proceeding as above,

$$\therefore \frac{d}{dx}\{\cos[f(x)]\} = -f'(x) \cdot \sin[f(x)]$$

3. Differentiate $\tan[f(x)]$ w.r.t. x

It can be deduced as in (1) above that

$$\therefore \frac{d}{dx}\{\tan[f(x)]\} = f'(x) \cdot \sec^2[f(x)]$$

4. Differentiate $\sec[f(x)]$ w.r.t. x

Solution

Let $y = \sec[f(x)]$

$$u = f(x) \dots \dots \dots (1)$$

$$\Rightarrow y = \sec u \dots\dots\dots (2)$$

Writing Chain Rule from eqn. (1) and eqn. (1)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y = \sec u \Rightarrow \frac{dy}{du} = \sec u \cdot \tan u$$

$$u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$$

$$\therefore \frac{dy}{dx} = \sec u \cdot \tan u \cdot f'(x) = \sec[f(x)] \cdot \tan[f(x)] \cdot f'(x)$$

$$\therefore \frac{d}{dx}\{\sec[f(x)]\} = f'(x) \cdot \sec[f(x)] \cdot \tan[f(x)]$$

5. Differentiate $\sec[f(x)]$ w.r.t. x

Proceeding as in (4) above

$$\therefore \frac{d}{dx}\{\operatorname{cosec}[f(x)]\} = -f'(x) \cdot \operatorname{cosec}[f(x)] \cdot \cot[f(x)]$$

6. Differentiate $\cot[f(x)]$ w.r.t. x

Proceeding as in the above seductions

$$\therefore \frac{d}{dx}\{\cot[f(x)]\} = -f'(x) \cdot \operatorname{cosec}^2[f(x)]$$

Examples

1. Differentiate w.r.t. x

a) $y = \sin\left(4x - \frac{\pi}{6}\right)$

c) $y = \sec(3x^2 + 1)$

b) $y = \cos(4x^2 + 3x)$

d) $y = \sin[\tan 2x]$

e) $y = \cot \sqrt{\frac{x}{1+x^2}}$

Solution

a) $y = \sin\left(4x - \frac{\pi}{6}\right)$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{d}{dx}\left(4x - \frac{\pi}{6}\right) \cdot \cos\left(4x - \frac{\pi}{6}\right) \\ &= 4 \cos\left(4x - \frac{\pi}{6}\right) \end{aligned}$$

b) $y = \cos(4x^2 + 3x)$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{d}{dx}(4x^2 + 3x) \cdot [-\sin(4x^2 + 3x)] \\ &= -(8x + 3) \sin(4x^2 + 3x) \end{aligned}$$

c) $y = \sec(3x^2 + 1)$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{d}{dx}(3x^2 + 1) \cdot \sec(3x^2 + 1) \cdot \tan(3x^2 + 1) \\ &= 6x \cdot \sec(3x^2 + 1) \cdot \tan(3x^2 + 1) \end{aligned}$$

d) $y = \sin[\tan 2x]$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \frac{d}{dx}(\tan 2x) \cdot \cos[\tan 2x] \\ &= \frac{d}{dx}(2x) \cdot \sec^2 2x \cdot \cos[\tan 2x] \end{aligned}$$

$$= 2 \sec^2 2x \cdot \cos[\tan 2x]$$

e) $y = \cot \sqrt{\frac{x}{1+x^2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{x}{1+x^2}} \right) \cdot \left[-\operatorname{cosec}^2 \sqrt{\frac{x}{1+x^2}} \right]$$

Dealing with $\frac{d}{dx} \left(\sqrt{\frac{x}{1+x^2}} \right)$

$$\frac{d}{dx} \left(\sqrt{\frac{x}{1+x^2}} \right) = \frac{d}{dx} \left(\frac{x^{1/2}}{(1+x^2)^{1/2}} \right)$$

$$= \frac{(1+x^2)^{1/2} \cdot \frac{1}{2} x^{-1/2} - x^{1/2} \cdot \frac{1}{2} (1+x^2)^{-1/2} \cdot 2x}{[(1+x^2)^{1/2}]^2} = \frac{\frac{1}{2} x^{-1/2} \cdot (1+x^2)^{-1/2} [1+x^2-2x^2]}{1+x^2}$$

$$= \frac{\frac{1}{2} x^{-1/2} \cdot (1+x^2)^{-1/2} (1-x^2)}{1+x^2} = \frac{1-x^2}{2x^{1/2} \cdot (1+x^2)^{1/2} \cdot (1+x^2)}$$

$$= \frac{1-x^2}{2\sqrt{x} \cdot (1+x^2)^{3/2}} = \frac{1-x^2}{2\sqrt{x}(1+x^2)^3}$$

$$\therefore \frac{dy}{dx} = -\frac{1-x^2}{2\sqrt{x}(1+x^2)^3} \cdot \operatorname{cosec}^2 \sqrt{\frac{x}{1+x^2}} = \frac{x^2-1}{2\sqrt{x}(1+x^2)^3} \cdot \operatorname{cosec}^2 \sqrt{\frac{x}{1+x^2}}$$

2. Differentiate w.r.t.x

a) $6 \cos x - 8(x^2 + x)$

c) $4x^2 \sin x - 3x^2 \cos x$

e) $\frac{\sin x}{x^2 + \cos x}$

b) $x^2 \cdot \sin 2x$

d) $3x \sin x \cos x$

f) $\frac{x \cos x}{(x+1) \sin x}$

Solution

a) Let $y = 6 \cos x - 8(x^2 + x)$

$$y = 6 \cos x - 8x^2 - 8x$$

$$\Rightarrow \frac{dy}{dx} = 6 \frac{d}{dx} (\cos x) - 8 \frac{d}{dx} (x^2) - 8 \frac{d}{dx} (x)$$

$$\frac{dy}{dx} = -6 \sin x - 16x - 8$$

b) Let $y = x^2 \cdot \sin 2x$

By Product Rule

$$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx} (\sin 2x) + \sin 2x \cdot \frac{d}{dx} (x^2)$$

$$= 2x^2 \cos 2x + 2x \sin 2x$$

c) Let $y = 4x^2 \sin x - 3x^2 \cos x$

$$\frac{dy}{dx} = 4 \cdot \frac{d}{dx} (x^2 \sin x) - 3 \cdot \frac{d}{dx} (x^2 \cos x)$$

By Product Rule

$$\frac{dy}{dx} = 4x^2 \cdot \frac{d}{dx} (\sin x) + 4 \sin x \cdot \frac{d}{dx} (x^2) - \left[3x^2 \cdot \frac{d}{dx} (\cos x) + 3 \cos x \cdot \frac{d}{dx} (x^2) \right]$$

$$= x^2 \cos x + 8x \sin x + 3x^2 \sin x - 6x \cos x$$

$$= (x^2 - 6x) \cos x + (3x^2 + 8x) \sin x$$

d) Let $y = 3x \sin x \cos x$

By Product Rule

$$\frac{dy}{dx} = 3 \left[x \sin x \cdot \frac{d}{dx} (\cos x) + x \cos x \cdot \frac{d}{dx} (\sin x) + \sin x \cos x \cdot \frac{dx}{dx} \right]$$

$$\frac{dy}{dx} = 3[-x \sin x \sin x + x \cos x \cdot \cos x + \sin x \cos x]$$

$$\begin{aligned}
 &= 3[-x \sin^2 x + x \cos^2 x + \sin x \cos x] = 3[(\cos^2 x - \sin^2 x) + \frac{1}{2} \sin 2x] \\
 &= 3 \cos 2x + \frac{3}{2} \sin 2x
 \end{aligned}$$

NOTE

$\frac{d}{dx}(u \cdot v \cdot w) = vw \cdot \frac{du}{dx} + uw \cdot \frac{dv}{dx} + uv \cdot \frac{dw}{dx}$, is the extended product rule

e) Let $y = \frac{\sin x}{x^2 + \cos x}$; By Quotient Rule

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(x^2 + \cos x) \cdot \cos x - \sin x \cdot (2x - \sin x)}{(x^2 + \cos x)^2} = \frac{x^2 \cos x + \cos^2 x - 2x \sin x + \sin^2 x}{(x^2 + \cos x)^2} \\
 \therefore \frac{dy}{dx} &= \frac{x^2 \cos x - 2x \sin x + 1}{(x^2 + \cos x)^2}
 \end{aligned}$$

f) Let $y = \frac{x \cos x}{(x+1) \sin x}$, By Quotient Rule

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{[(x+1) \sin x] \cdot \frac{d}{dx}(x \cos x) - x \cos x \cdot \frac{d}{dx}[(x+1) \sin x]}{(x+1)^2 \sin^2 x} \\
 &= \frac{[(x+1) \sin x] \cdot (-x \sin x + \cos x) - x \cos x \cdot [\sin x + (x+1) \cos x]}{(x+1)^2 \sin^2 x} \\
 &= \frac{-x(x+1) \sin^2 x + (x+1) \cos x \sin x - x \cos x \sin x - x(x+1) \cos^2 x}{(x+1)^2 \sin^2 x} \\
 &= \frac{-x(x+1)[\sin^2 x + \cos^2 x] + \cos x \sin x(x+1-x)}{(x+1)^2 \sin^2 x} = \frac{-x(x+1) + \cos x \sin x}{(x+1)^2 \sin^2 x} \\
 &= \frac{\cos x \sin x - x(x+1)}{(x+1)^2 \sin^2 x}
 \end{aligned}$$

Task

Differentiate w.r.t. x

- a) $\frac{\sin x}{x^2 + \cos x}$ b) $\frac{x \cos x}{x+3}$ c) $\frac{x + \sin x}{1 + \cos x}$ d) $\frac{x}{x + \sin x}$
- e) $\frac{x \sin x}{\cos x + \sin x}$
- f) $\frac{x^2 \sin x}{(x+1)(x^2-1)}$ g) $8x^2(1 + \sin x)(1 + \cos x)$ h) $x \cos x + 3(x+1)(x-1)$

Answers

- a) $\frac{1+x(x \cos x - 2 \sin x)}{x^2 + \cos x}$ b) $\frac{3 \cos x - (x+3)x \sin x}{(x+3)^2}$ c) $\frac{3+2 \cos 2x + x \sin x}{(1+\cos x)^2}$ d) $\frac{\sin x - x \cos x}{(x+\sin x)^2}$
- e) $\frac{x + \sin x(\sin x + \cos x)}{(\cos x + \sin x)^2}$
- f) $\frac{\cos x(x^5 + x^4 - x^3 - x^2) - \sin x(x^4 + x^2 + 2x)}{(x+1)^2(x^2-1)^2}$ g) $16x(1 + \cos x)(1 + \sin x) + 8x^2(\cos x - \sin x)(\cos x + \sin x + 1)$
- h) $\cos x - x \sin x + 6x$

PROOFS INVOLVING THE ABOVE THEOREMS

1. If $y = \frac{\sin x}{x}$, show that $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + (x^2 + 2)y = 0$
2. If $y = \frac{\cos x}{x}$, show that $\frac{d^2 y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + y = 0$
3. If $y = \sin x$, show that $\frac{d^2 y}{dx^2} = -y$, $\frac{d^4 y}{dx^4} = y$
4. If $y = \sec \theta$, show that $\frac{d^2 y}{dx^2} - y(2y^2 - 1)$

Solution

1. $y = \frac{\sin x}{x} \Rightarrow y \cdot x^2 = \sin x$

Differentiating w.r.t x

$$x^2 \frac{dy}{dx} + 2xy = \cos x$$

Now differentiating w.r.t x

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + 2 \left(x \frac{dy}{dx} + y \right) = -\sin x$$

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = -\sin x = -y \cdot x^2$$

$$\therefore x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + (x^2 + 2)y = 0$$

2. Left as an exercise

3. $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

$$\Rightarrow \frac{d^2 y}{dx^2} = -\sin x = -y$$

Differentiating w.r.t x

$$\frac{d^3 y}{dx^3} = -\frac{dy}{dx}$$

Differentiating again w.r.t x

$$\frac{d^4 y}{dx^4} = -\frac{d^2 y}{dx^2} = -(-y) = y$$

$$\therefore \frac{d^4 y}{dx^4} = y$$

4. Left as an exercise

Differentiating $\sin^n[f(x)]$, $\cos^n[f(x)]$, $\tan^n[f(x)]$ w.r.t x

- a) Let $y = \sin^n[f(x)] = [\sin[f(x)]]^n$

$$u = \sin[f(x)] \Rightarrow \frac{du}{dx} = f'(x) \cdot \cos[f(x)]$$

$$\Rightarrow y = u^n \Rightarrow \frac{dy}{du} = nu^{n-1}$$

By Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= nu^{n-1} \cdot f'(x) \cdot \cos[f(x)]$$

$$= n \sin^{n-1}[f(x)] \cdot f'(x) \cdot \cos[f(x)]$$

$$\therefore \frac{d}{dx} \{\sin^n[f(x)]\} = n \sin^{n-1}[f(x)] \cdot f'(x) \cdot \cos[f(x)]$$

- b) Let $y = \cos^n[f(x)]$

Proceeding as in (a) above

$$\therefore \frac{d}{dx} \{\cos^n[f(x)]\} = -n \cos^{n-1}[f(x)] \cdot f'(x) \cdot \sin[f(x)]$$

c) Let $y = \tan^n[f(x)]$

Proceeding as in (a) above

$$\therefore \frac{d}{dx} \{\tan^n[f(x)]\} = n \tan^{n-1}[f(x)] \cdot f'(x) \cdot \sec^2[f(x)]$$

Examples

a) Differentiate w.r.t x

a) $y = \cos^4\left(2x + \frac{\pi}{3}\right)$

b) $y = \sqrt{\sin 4x}$

c)

$y = \tan^2(x^2 \tan x^2)$

d) $y = \sin^2 \sqrt{x}$

e) $y = \frac{1+\sin^2 x}{1-\sin^2 x}$

f) $y = \sqrt{4 + 3 \sin 2x}$

Solution

a) $y = \cos^4\left(2x + \frac{\pi}{3}\right) = \left[\cos\left(2x + \frac{\pi}{3}\right)\right]^4$

$$\begin{aligned} \frac{dy}{dx} &= 4 \left[\cos\left(2x + \frac{\pi}{3}\right)\right]^3 \cdot \frac{d}{dx} \left[\cos\left(2x + \frac{\pi}{3}\right)\right] \\ &= 4 \cos^3\left(2x + \frac{\pi}{3}\right) \cdot \frac{d}{dx} \left(2x + \frac{\pi}{3}\right) \cdot \left[-\sin\left(2x + \frac{\pi}{3}\right)\right] \\ &= 4 \cos^3\left(2x + \frac{\pi}{3}\right) \cdot \left[-2 \sin\left(2x + \frac{\pi}{3}\right)\right] \\ &= -8 \cos^3\left(2x + \frac{\pi}{3}\right) \sin\left(2x + \frac{\pi}{3}\right) \end{aligned}$$

b) $y = \sqrt{\sin 4x} = [\sin 4x]^{\frac{1}{2}}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} [\sin 4x]^{-\frac{1}{2}} \cdot \frac{d}{dx} (\sin 4x) \\ &= \frac{1}{2} [\sin 4x]^{-\frac{1}{2}} \cdot 4 \cdot \cos 4x \\ &= \frac{2 \cos 4x}{\sqrt{\sin 4x}} \end{aligned}$$

c) $y = \tan^2(x^2 \tan x^2) = [\tan(x^2 \tan x^2)]^2$

$$\begin{aligned} \frac{dy}{dx} &= 2[\tan(x^2 \tan x^2)] \cdot \frac{d}{dx} [\tan(x^2 \tan x^2)] \\ &= 2 \tan(x^2 \tan x^2) \cdot \frac{d}{dx} (x^2 \tan x^2) \cdot \sec^2(x^2 \tan x^2) \\ &= 2 \tan(x^2 \tan x^2) \cdot \sec^2(x^2 \tan x^2) [x^2 \cdot 2x \cdot \sec^2 x^2 + 2x \cdot \tan x^2] \\ &= 4x \cdot \tan(x^2 \tan x^2) \cdot \sec^2(x^2 \tan x^2) [x^2 \cdot \sec^2 x^2 + \tan x^2] \end{aligned}$$

d) $y = \sin^2 \sqrt{x} = [\sin x^{1/2}]^2$

$$\begin{aligned} \frac{dy}{dx} &= 2 \left[\sin x^{\frac{1}{2}}\right] \cdot \frac{d}{dx} (\sin x^{1/2}) \\ &= 2 \sin \sqrt{x} \cdot \frac{d}{dx} (x^{1/2}) \cdot \cos x^{1/2} \\ &= 2 \sin \sqrt{x} \cdot \cos \sqrt{x} \cdot \frac{1}{2} \cdot x^{-1/2} = \frac{\sin(2\sqrt{x})}{2\sqrt{x}} \end{aligned}$$

e) $y = \frac{1+\sin^2 x}{1-\sin^2 x}$

$$\frac{dy}{dx} = \frac{(1-\sin^2 x) \cdot \frac{d}{dx}(1+\sin^2 x) - (1+\sin^2 x) \cdot \frac{d}{dx}(1-\sin^2 x)}{(1-\sin^2 x)^2}$$

$$= \frac{(1-\sin^2 x) \cdot 2 \sin x \cos x - (1+\sin^2 x) \cdot (-2 \sin x \cos x)}{(1-\sin^2 x)^2} = \frac{2 \sin x \cos x (1-\sin^2 x + 1 + \sin^2 x)}{(\cos^2 x)^2}$$

$$= \frac{2 \sin x \cos x (2)}{\cos^4 x} = \frac{4 \sin x}{\cos^3 x} = 4 \tan x \sec^2 x$$

f) $y = \sqrt{4 + 3 \sin 2x} = [4 + 3 \sin 2x]^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2} [4 + 3 \sin 2x]^{-\frac{1}{2}} \cdot \frac{d}{dx} [4 + 3 \sin 2x]$$

$$= \frac{1}{2} (4 + 3 \sin 2x)^{-\frac{1}{2}} \cdot 6 \cos 2x = \frac{3 \cos 2x}{\sqrt{4 + 3 \sin 2x}}$$

Task

Differentiate w.r.t x

- a) $\sin^3 x$ b) $\tan^3(3x - 4)$ c) $\sin^3 x \tan 2x$
 d) $(1 - \cos^4 x)(4 \cos^4 x)$
 e) $\left(\frac{\cos x}{1 + \sin x}\right)^3$ f) $\frac{\sin^2 x}{2 + \sin^2 x}$ g) $\left(\frac{\sin 2x}{1 + \cos 2x}\right)^2$ h) $\sqrt{\frac{1 + \cos x}{1 - \cos x}}$

Answers

- a) $3 \sin^2 x \cos x$ b) $9 \tan^2(3x - 4) \sec^2(3x - 4)$
 c) $3 \sin^2 x \cos x \tan 2x + 2 \sin^3 x \sec^2 2x$
 d) $8 \cos^7 x \sin x$ e) $-\frac{3 \cos^2 x}{(1 + \sin x)^3}$ f) $\frac{4 \sin x \cos x}{(2 + \sin^2 x)^2}$
 g) $\frac{4 \sin 2x}{(1 + \cos x)^2}$ h) $-\frac{1}{1 - \cos x}$

PROOFS INVOLVING THE ABOVE THEOREMS

- a) If $y = (\tan x + \sec x)^m$ where $m > 0$, show that $\frac{dy}{dx} = my \sec x$
 b) If $y = \sqrt{\frac{1 + \sin x}{1 - \sin x}}$, show that $\frac{dy}{dx} = \frac{1}{1 - \sin x}$
 c) If $y = \tan x + \frac{1}{3} \tan^3 x$, show that $\frac{dy}{dx} = \sec^4 x$
 d) If $y = \tan^2 x$, prove that $\frac{d^2 y}{dx^2} = 2(1 + y)(1 + 3y)$
 e) If $y = \sqrt{1 + \sin x}$, show that $\frac{dy}{dx} = \frac{1}{2} \sqrt{1 - \sin x}$
 f) If $y = \frac{1 + \sin^2 x}{1 + \cos^2 x}$, show that $\frac{dy}{dx} = \frac{3 \sin 2x}{(1 + \cos^2 x)^2}$
 g) If $y = \sin m\theta$, show that $\frac{d^2 y}{d\theta^2} + m^2 y = 0$
 h) If $y = \cot^2 \theta$, show that $\frac{d^2 y}{d\theta^2} = 2(1 + y)(1 + 3y)$
 i) If $y = \tan x$, show that $\frac{d^2 y}{dx^2} = 2y(1 + y^2)$
 j) If $y = \sqrt{4 + 3 \sin x}$, prove that $2y \frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx}\right)^2 + y^2 = 4$

Solution

a) $y = (\tan x + \sec x)^m$

$$\begin{aligned}\frac{dy}{dx} &= m(\tan x + \sec x)^{m-1} \cdot (\sec^2 x + \tan x \sec x) \\ &= m \sec x (\tan x + \sec x)^{m-1} \cdot (\tan x + \sec x) \\ &= m \sec x (\tan x + \sec x)^m = m \sec x \cdot y\end{aligned}$$

b) $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$, squaring both sides to simplify

$$y^2 = \frac{1+\sin x}{1-\sin x}, \text{ differentiating w.r.t } x$$

$$2y \frac{dy}{dx} = \frac{(1-\sin x) \cdot \cos x - (1+\sin x) \cdot (-\cos x)}{(1-\sin x)^2}$$

$$2y \frac{dy}{dx} = \frac{\cos x (1-\sin x + 1 + \sin x)}{(1-\sin x)^2} = \frac{2 \cos x}{(1-\sin x)^2}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{\cos x}{(1-\sin x)^2} \cdot \frac{1}{y} = \frac{\cos x}{(1-\sin x)^2 \cdot \sqrt{\frac{1+\sin x}{1-\sin x}}} \\ &= \frac{\sqrt{1-\sin^2 x} \cdot \sqrt{1-\sin x}}{(1-\sin x)^2 \cdot \sqrt{1+\sin x}} = \frac{\sqrt{(1+\sin x)(1-\sin x)} \cdot \sqrt{1-\sin x}}{(1-\sin x)^2 \cdot \sqrt{1+\sin x}} \\ &= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}\end{aligned}$$

c) Left as an exercise

d) Left as an exercise

e) Left as an exercise

f) Left as an exercise

g) Left as an exercise

h) Left as an exercise

i) Left as an exercise

j) $y = \sqrt{4 + 3 \sin x}$, $2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 + y^2 = 4$

Squaring both sides to simplify

$$y^2 = 4 + 3 \sin x \dots\dots\dots (1)$$

Differentiating w.r.t x

$$2y \frac{dy}{dx} = 3 \cos x$$

Differentiating again w.r.t x

$$2y \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2 \frac{dy}{dx} = -3 \sin x$$

From eqn. (1), $-3 \sin x = 4 - y^2$

$$\therefore 2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 = 4 - y^2$$

$$\therefore 2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 + y^2 = 4$$

Differentiating $\sec^n[f(x)]$, $\operatorname{cosec}^n[f(x)]$ and $\cot^n[f(x)]$ w.r.t x

a) Let $y = \sec^n[f(x)] = [\sec[f(x)]]^n$

$$u = \sec[f(x)] \Rightarrow \frac{du}{dx} = f'(x) \cdot \sec[f(x)] \cdot \tan[f(x)]$$

$$\Rightarrow y = u^n \Rightarrow \frac{dy}{du} = nu^{n-1}$$

By Chain Rule

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= nu^{n-1} \cdot f'(x) \cdot \sec[f(x)] \cdot \tan[f(x)] \\ &= n \sec^{n-1}[f(x)] \cdot f'(x) \cdot \sec[f(x)] \cdot \tan[f(x)] \end{aligned}$$

$$\begin{aligned} \therefore \frac{d}{dx} \{\sec^n[f(x)]\} &= n \sec^{n-1}[f(x)] \cdot f'(x) \cdot \sec[f(x)] \cdot \tan[f(x)] \\ &= n \sec^n[f(x)] \cdot f'(x) \cdot \tan[f(x)] \end{aligned}$$

b) Proceeding as in (a) above

$$\begin{aligned} \therefore \frac{d}{dx} \{\operatorname{cosec}^n[f(x)]\} &= -n \operatorname{cosec}^{n-1}[f(x)] \cdot f'(x) \cdot \operatorname{cosec}[f(x)] \cdot \cot[f(x)] \\ &= -n \operatorname{cosec}^n[f(x)] \cdot f'(x) \cdot \cot[f(x)] \end{aligned}$$

c) Proceeding as above in a similar way

$$\therefore \frac{d}{dx} \{\cot^n[f(x)]\} = -n \cot^{n-1}[f(x)] \cdot f'(x) \cdot \operatorname{cosec}^2[f(x)]$$

Example

Differentiate w.r.t x

a) $\sqrt{\sec x}$

b) $\sin x \cdot \sec^2 x$ c) $\sec x (1 + \cot x)$

d) $\sec^3(2 \tan 3x)$

Solution

a) Let $y = \sqrt{\sec x} = (\sec x)^{1/2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} (\sec x)^{-1/2} \cdot \frac{d}{dx} (\sec x) \\ &= \frac{1}{2} (\sec x)^{-1/2} \cdot \sec x \cdot \tan x = \frac{1}{2} \sqrt{\sec x} \tan x \end{aligned}$$

b) Let $y = \sin x \cdot \sec^2 x$

$$\begin{aligned} \frac{dy}{dx} &= \sec^2 x \cdot \cos x + \sin x \cdot 2 \sec x \cdot \sec x \tan x \\ &= \sec^2 x (\cos x + 2 \sin x \cdot \tan x) \end{aligned}$$

c) Let $y = \sec x (1 + \cot x)$

$$\begin{aligned} \frac{dy}{dx} &= \sec x \cdot (-\operatorname{cosec}^2 x) + (1 + \cot x) \cdot \sec x \tan x \\ &= \sec x [-\operatorname{cosec}^2 x + \tan x + \cot x \cdot \tan x] = \sec x [-\operatorname{cosec}^2 x + \tan x + 1] \end{aligned}$$

d) Let $y = \sec^3(2 \tan 3x) = [\sec(2 \tan 3x)]^3$

$$\frac{dy}{dx} = 3[\sec(2 \tan 3x)]^2 \cdot \frac{d}{dx} [\sec(2 \tan 3x)]$$

$$\begin{aligned}
 &= 3 \sec^2(2 \tan 3x) \cdot \frac{d}{dx}(2 \tan 3x) \cdot \sec(2 \tan 3x) \tan(2 \tan 3x) \\
 &= 3 \sec^2(2 \tan 3x) \cdot (6 \sec^2 3x) \cdot \sec(2 \tan 3x) \tan(2 \tan 3x) \\
 &= 18 \sec^3(2 \tan 3x) \cdot \sec^2 3x \cdot \tan(2 \tan 3x)
 \end{aligned}$$

PROOFS INVOLVING THE ABOVE THEOREMS

- a) If $y^2 = \sec 2x$, prove that $\frac{d^2y}{dx^2} + y = 3y^5$
 b) If $y = \sec \theta$, show that $\frac{d^2y}{d\theta^2} = y(2y^2 - 1)$
 c) If $y = \cot^2 \theta$, prove that $\frac{d^2y}{d\theta^2} = 2(1 + y)(1 + 3y)$

Solution

- a) $y^2 = \sec 2x$, differentiating w.r.t x

$$2y \frac{dy}{dx} = 2 \sec 2x \tan 2x$$

$$2y \cdot \frac{dy}{dx} = 2y^2 \tan 2x \Rightarrow \frac{dy}{dx} = y \cdot \tan 2x \dots\dots\dots (1)$$

Differentiating again w.r.t

$$\frac{dy}{dx} = y \cdot 2 \sec^2 2x + \tan 2x \cdot \frac{dy}{dx}$$

$$= 2y \cdot y^4 + \tan 2x \cdot y \tan 2x \dots\dots\dots [\text{From eqn. (1)}]$$

$$= 2y^5 + y \cdot \tan^2 2x = 2y^5 + y \cdot (\sec^2 2x - 1)$$

$$= 2y^5 + y \cdot (y^4 - 1) = 3y^5 - y$$

$$\therefore \frac{d^2y}{dx^2} + y = 3y^5$$

- b) Left as an exercise

- c) $y = \cot^2 \theta$, differentiating w.r.t θ

$$\frac{dy}{d\theta} = 2 \cot \theta \cdot (-\operatorname{cosec}^2 \theta)$$

$$= 2 \cot \theta \cdot (1 + \cot^2 \theta) = -2 \cot \theta + 2 \cot^3 \theta$$

Differentiating w.r.t θ

$$\frac{dy}{d\theta} = -2(-\operatorname{cosec}^2 \theta) - 2 \cdot 3 \cot^2 \theta \cdot (-\operatorname{cosec}^2 \theta)$$

$$= 2 \operatorname{cosec}^2 \theta (1 + 3 \cot^2 \theta)$$

$$= 2(1 + \cot^2 \theta)(1 + 3 \cot^2 \theta) = 2(1 + y)(1 + 3y)$$

DIFFERENTIATING INVERSE OF TRIGONOMETRIC FUNCTIONS

- a) To differentiate $\cos^{-1}[f(x)]$ w.r.t x

$$\text{Let } y = \cos^{-1}[f(x)]$$

$$\cos y = f(x)$$

Differentiate w.r.t x

$$-\sin y \cdot \frac{dy}{dx} = f'(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{-\sin y}$$

$$\text{But from } \cos^2 y + \sin^2 y = 1 \Rightarrow \sin y = +\sqrt{1 - \cos^2 y}$$

$$\frac{dy}{dx} = \frac{f'(x)}{-\sqrt{1 - \cos^2 y}} = -\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$\therefore \frac{d}{dx} [\cos^{-1}[f(x)]] = -\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

- b) To differentiate $\sin^{-1}[f(x)]$ w.r.t x , proceed as above

$$\therefore \frac{d}{dx} [\sin^{-1}[f(x)]] = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

- c) To differentiate $\tan^{-1}[f(x)]$ w.r.t x , proceed as above

$$\therefore \frac{d}{dx} [\tan^{-1}[f(x)]] = \frac{f'(x)}{1 + [f(x)]^2}$$

- d) To differentiate $\sec^{-1}[f(x)]$ w.r.t x

$$\text{Let } y = \sec^{-1}[f(x)]$$

$$\sec y = f(x)$$

Differentiating w.r.t x

$$\sec y \tan y \frac{dy}{dx} = f'(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{\sec y \tan y}$$

$$\text{But from } \sec^2 y = 1 + \tan^2 y = 1 \Rightarrow \tan y = +\sqrt{\sec^2 y - 1} = \sqrt{[f(x)]^2 - 1}$$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$$

$$\frac{d}{dx} \{\sec^{-1}[f(x)]\} = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$$

- e) To differentiate $\operatorname{cosec}^{-1}[f(x)]$ w.r.t x , proceeding as in (f) above

$$\frac{d}{dx} \{\sec^{-1}[f(x)]\} = -\frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$$

f) To differentiate $\cot^{-1}[f(x)]$ w.r.t x , proceeding as in (f) above

$$\frac{d}{dx}\{\cot^{-1}[f(x)]\} = -\frac{f'(x)}{1+[f(x)]^2}$$

Example

Differentiate w.r.t x

a) $\sin^{-1}(2x)$

b) $\cos^{-1}\sqrt{x^3}$

c) $\tan^{-1}(5/\sqrt{x})$

d) $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

e) $\sin^{-1}(\cos x)$

f) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Solution

a) Let $y = \sin^{-1}(2x)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(2x)}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$$

b) Let $y = \cos^{-1}\sqrt{x^3} = \cos^{-1}x^{3/2}$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^{3/2})}{\sqrt{1-(x^{3/2})^2}} = \frac{\frac{3}{2}x^{1/2}}{\sqrt{1-x^3}} = -\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

c) Let $y = \tan^{-1}(5/\sqrt{x}) = \tan^{-1}(5x^{-1/2})$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}\left(5x^{-\frac{1}{2}}\right)}{1+\left(5x^{-\frac{1}{2}}\right)^2} = \frac{-\frac{5}{2}x^{-\frac{3}{2}}}{1+\left(5x^{-\frac{1}{2}}\right)^2} \\ &= \frac{-\frac{5}{2}x^{-\frac{3}{2}}}{1+25x^{-1}} = \frac{-\frac{5}{2}x^{-\frac{3}{2}}}{x+25} \\ &= \frac{-\frac{5}{2}x^{-\frac{1}{2}}}{x+25} = -\frac{5}{2(x+25)\sqrt{x}}\end{aligned}$$

d) Let $y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \sin^{-1}\left(\frac{x}{(1+x^2)^{1/2}}\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}\left(\frac{x}{(1+x^2)^{1/2}}\right)}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} = \frac{\frac{(1+x^2)^{1/2} \cdot 1 - x \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x}{1+x^2}}{\sqrt{1-\frac{x^2}{1+x^2}}} \\ &= \frac{(1+x^2)^{-1/2}(1+x^2-x^2)}{(1+x^2)\sqrt{\frac{1+x^2-x^2}{1+x^2}}} = \frac{(1+x^2)^{-1/2}}{(1+x^2)\sqrt{\frac{1}{1+x^2}}} \\ &= \frac{(1+x^2)^{-1/2} \cdot (1+x^2)^{1/2}}{1+x^2} = \frac{1}{1+x^2} \\ \therefore \frac{dy}{dx} &= \frac{1}{1+x^2}\end{aligned}$$

e) Let $y = \sin^{-1}(\cos x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)}{\sqrt{1-(\cos x)^2}} = \frac{-\sin x}{\sqrt{1-\cos^2 x}} \\ &= -\frac{\sin x}{\sin x} = -1 \\ \therefore \frac{dy}{dx} &= -1\end{aligned}$$

f) Left as an exercise. **Ans:** $\frac{1}{\sqrt{1-x^2}}$

Task

1. Differentiate w.r.t x

- a) $\sin^{-1} \sqrt{x}$ b) $\tan^{-1} x^{5/2}$ c) $\cos^{-1} 2/\sqrt{x}$ d) $\sin^{-1}(2x+3)/7$
 e) $\cos^{-1}(3x-2)/4$
 f) $\tan^{-1}(x-5)/2$ g) $\sin^{-1}(2x+7)/\sqrt{2}$ h) $\cos^{-1}(2x+7)/\sqrt{2}$ i) $\tan^{-1}(2x+7)/\sqrt{2}$
 j) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ k) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ l) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Answers

- a) $\frac{1}{2\sqrt{x(1-x)}}$ b) $\frac{5x^{3/2}}{2(1+x^5)}$ c) $\frac{1}{x\sqrt{x-4}}$ d) $\frac{1}{\sqrt{10-3x-x^2}}$
 e) $-\frac{3}{\sqrt{12+12x-9x^2}}$ f) $\frac{2}{x^2-10x-29}$ g) $\frac{2}{\sqrt{-47-28x-4x^2}}$
 h) $\frac{-2}{\sqrt{-47-28x-4x^2}}$ i) $\frac{2\sqrt{2}}{\sqrt{51+28x+4x^2}}$ j) $\frac{2}{1+x^2}$
 k) $\frac{2}{1+x^2}$ l) $\frac{2}{1+x^2}$

2. a) If $\sqrt{y} = \tan^{-1} x$, show that $(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2$

b) If $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, show that $\frac{dy}{dx} = -\frac{2}{1+x^2}$

c) If $y = \sin^{-1}(3x-4x^3)$, show that $\sqrt{1-x^2} \frac{dy}{dx} = 3$

d) If $u = \tan^{-1} \theta$, show that $(1+\theta^2) \frac{d^2u}{d\theta^2} + 2\theta \frac{du}{d\theta} = 0$

e) Show that:

i) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

ii) $\frac{d}{dx}\left(\cot^{-1}\left(\frac{x}{a}\right)\right) = -\frac{1}{\sqrt{a^2-x^2}}$

iii) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

iv) If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $\frac{dy}{dx} = \frac{2}{1+x^2}$

v) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, $(1-x^2) \frac{dy}{dx} = xy + 1$

vi) If $y = (\tan^{-1} x)^2$, $\frac{d}{dx}\left\{(1+x^2) \cdot \frac{dy}{dx}\right\} = \frac{2}{1+x^2}$

vii) If $y = \cos^{-1}(\sin 2x)$, $\frac{dy}{dx} = -2$

viii) If $y = \tan(m \tan^{-1} x)$, $(1+x^2) \frac{d^2y}{dx^2} = 2(my-x) \frac{dy}{dx}$

ix) If $y = \sin(m \sin^{-1} x)$, $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0$

x) If $y = \sqrt{1-x^2} \sin^{-1} x$, $(1-x^2) \frac{dy}{dx} + xy = 1-x^2$

Solution

a) $\sqrt{y} = \tan^{-1} x$; $(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2$

Squaring both sides

$$y = (\tan^{-1} x)^2$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{2 \tan^{-1} x}{1+x^2}$$

$$(1+x^2) \cdot \frac{dy}{dx} = 2 \tan^{-1} x$$

Differentiating w.r.t x

$$(1+x^2) \cdot \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = \frac{2}{1+x^2}, \text{ multiplying through by } (1+x^2)$$

$$\therefore (1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2$$

b) Left as an exercise

$$\text{c) } y = \sin^{-1}(3x - 4x^3) ; \sqrt{1-x^2} \frac{dy}{dx} = 3$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(3-12x^2)}{\sqrt{1-(3x-4x^3)^2}} = \frac{3(1-4x^2)}{\sqrt{(1+3x-4x^3)(1-3x+4x^3)}} \\ &= \frac{3(1-2x)(1+2x)}{\sqrt{(x-1)(1+2x)(1+2x)[-(x+1)(2x-1)(2x-1)]}} = \frac{3(1-2x)(1+2x)}{(1-2x)(1+2x)\sqrt{(x-1)[-(x+1)]}} \\ &= \frac{3}{1-x^2} \end{aligned}$$

$$\therefore \sqrt{1-x^2} \frac{dy}{dx} = 3$$

d) Left as an exercise

e) (i) up to (vii), (ix), (x) are left as an exercise

$$\text{viii) } y = \tan(m \tan^{-1} x), (1+x^2) \frac{d^2y}{dx^2} = 2(my-x) \frac{dy}{dx}$$

Differentiating w.r.t x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(m \tan^{-1} x) \cdot \sec^2(m \tan^{-1} x) \\ &= \frac{m}{1+x^2} \cdot (1 + \tan^2(m \tan^{-1} x)) = \frac{m}{1+x^2} \cdot (1 + y^2) \end{aligned}$$

$$\Rightarrow (1+x^2) \frac{dy}{dx} = m(1+y^2)$$

Differentiating w.r.t x

$$(1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = m \cdot 2y \frac{dy}{dx}$$

$$(1+x^2) \frac{d^2y}{dx^2} = m \cdot 2y \frac{dy}{dx} - 2x \frac{dy}{dx}$$

$$\therefore (1+x^2) \frac{d^2y}{dx^2} = 2(my-x) \frac{dy}{dx}$$

DIFFERENTIATION INVOLVING PARAMETRIC EQUATIONS

Given two parametric equations

$$y = f(\theta) \text{ and } x = g(\theta)$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solution

By chain Rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

By chain Rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \times \frac{d\theta}{dx}$$

Examples

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of the parameter when
 - a) $y = \cos 4\theta$, $x = \sin 2\theta$
 - b) $y = \theta - \sin \theta$, $x = 2 \cos \theta$
 - c) $x = a \cos^3 \theta$, $y = a \sin^3 \theta$
 - d) $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$
 - e) $x = a(\theta \sin \theta + \cos \theta - 1)$, $y = a(\sin \theta - \theta \cos \theta)$
 - f) $x = 4 \cos \theta + 3 \sin \theta + 2$, $y = 3 \cos \theta - 4 \sin \theta - 1$ at the point where $\theta = \pi/2$
 - g) $x = 3 \cos \theta - \cos^3 \theta$, $y = 3 \sin \theta - \sin^3 \theta$
 - h) $x = b \cos^3 \theta$, $y = b \sin^3 \theta$
 - i) $x = a(2 \cos \theta + \cos 2\theta)$, $y = a(2 \sin \theta - \sin 2\theta)$
 - j) $x = 4b \cos \theta - b \cos 4\theta$, $y = 4b \sin \theta - b \sin 4\theta$

Solution

$$\text{a) } y = \cos 4\theta \Rightarrow \frac{dy}{d\theta} = -4 \sin 4\theta$$

$$x = \sin 2\theta \Rightarrow \frac{dx}{d\theta} = 2 \cos 2\theta$$

$$\begin{aligned} \text{By Chain Rule, } \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= -4 \sin 4\theta \times \frac{1}{2 \cos 2\theta} = \frac{-4 \sin 4\theta}{2 \cos 2\theta} \\ &= \frac{-2 \cdot 2 \sin 2\theta \cos 2\theta}{\cos 2\theta} = -4 \sin 2\theta \end{aligned}$$

$$\therefore \frac{dy}{dx} = -4 \sin 2\theta$$

$$\begin{aligned} \text{By Chain Rule, } \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \times \frac{d\theta}{dx} \\ \frac{d^2y}{dx^2} &= -8 \cos 2\theta \cdot \frac{d\theta}{dx} = -8 \cos 2\theta \cdot \frac{1}{2 \cos 2\theta} = -4 \end{aligned}$$

b) $y = \theta - \sin \theta \Rightarrow \frac{dy}{d\theta} = 1 - \cos \theta$

$x = 2 \cos \theta \Rightarrow \frac{dx}{d\theta} = -2 \sin \theta$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\therefore \frac{dy}{dx} = -\frac{1 - \cos \theta}{2 \sin \theta}$

By Chain Rule, $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \times \frac{d\theta}{dx}$

$\frac{d^2y}{dx^2} = - \left[\frac{2 \sin \theta \cdot \sin \theta - (1 - \cos \theta) \cdot 2 \cos \theta}{4 \sin^2 \theta} \right] \cdot \frac{1}{-2 \sin \theta}$

$= \frac{\sin^2 \theta - \cos \theta + \cos^2 \theta}{4 \sin^3 \theta} = \frac{1 - \cos \theta}{4 \sin^3 \theta}$

$= \frac{1}{4} (\operatorname{cosec}^3 \theta - \cot \theta \operatorname{cosec}^2 \theta)$

c) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\tan \theta$, $\frac{d^2y}{dx^2} = \frac{\sec^4 \theta \operatorname{cosec} \theta}{3a}$

d) Left as an exercise. **Ans:** $\frac{dy}{dx} = \tan \frac{\theta}{2}$, $\frac{d^2y}{dx^2} = \frac{1}{4a} \sec^4 \frac{\theta}{2}$

e) Left as an exercise. **Ans:** $\frac{dy}{dx} = \tan \theta$, $\frac{d^2y}{dx^2} = \frac{\sec^3 \theta}{a\theta}$

f) Left as an exercise. **Ans:** $3/4$

g) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\cot^3 \theta$, $\frac{d^2y}{dx^2} = -\cot^2 \theta \operatorname{cosec}^5 \theta$

h) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\tan \theta$, $\frac{d^2y}{dx^2} = \frac{\sec^4 \theta \operatorname{cosec} \theta}{3b}$

i) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\tan \frac{\theta}{2}$, $\frac{d^2y}{dx^2} = \frac{1}{8a} \operatorname{cosec}^3 \frac{\theta}{2} \sec^3 \frac{\theta}{2}$

j) Left as an exercise. **Ans:** $\frac{dy}{dx} = \frac{1}{b} \tan \frac{5\theta}{2}$, $\frac{d^2y}{dx^2} = \frac{5}{16b} \sec^5 \frac{\theta}{2} \operatorname{cosec}^3 \frac{\theta}{2}$

2. a) Prove that if $\sin \theta = 2 \sin \varphi$ and $x = \cos \theta - 2 \cos \varphi$ then $\frac{dx}{d\theta} = x \tan \varphi$

b) if $x = a(2 \cos \theta + \cos 2\theta)$, $y = a(2 \sin \theta - \sin 2\theta)$, prove that $8a \frac{d^2y}{dx^2} = \operatorname{cosec}^3 \frac{\theta}{2} \cdot \sec^3 \frac{\theta}{2}$

c) Given that $x = 4b \cos \theta - b \cos 4\theta$, $y = 4b \sin \theta - b \sin 4\theta$, prove that

$\frac{d^2y}{dx^2} = \frac{5}{16b} \sec^3 \frac{5\theta}{2} \operatorname{cosec}^3 \frac{3\theta}{2}$

d) Given that $b x = a(\theta - \sin \theta)$ and $y = a(1 - \cos \theta)$, prove that $\frac{dy}{dx} = \cot \frac{\theta}{2}$

and $\frac{d^2y}{dx^2} = -\frac{1}{4a} \operatorname{cosec}^3 \frac{\theta}{2}$

e) If $y = \sin n\theta \operatorname{cosec} \theta$ and $x = \cos \theta$, prove that

i) $(1 - x^2) \frac{dy}{dx} - xy + n \cos n\theta = 0$

ii) $(1 - x^2) \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + (n^2 - 1)y = 0$

Solution

a) $\sin \theta = 2 \sin \varphi$, $x = \cos \theta - 2 \cos \varphi$

Differentiating w.r.t θ

$$\begin{aligned}\frac{dx}{d\theta} &= -\sin \theta + 2 \sin \varphi \frac{d\varphi}{d\theta} \\ &= -2 \sin \varphi + 2 \sin \varphi \frac{d\varphi}{d\theta}\end{aligned}$$

Also $\cos \theta = 2 \cos \varphi \frac{d\varphi}{d\theta} \Rightarrow \frac{d\varphi}{d\theta} = \frac{\cos \theta}{2 \cos \varphi}$

$$\begin{aligned}\frac{dx}{d\theta} &= -2 \sin \varphi + 2 \sin \varphi \cdot \frac{\cos \theta}{2 \cos \varphi} = -2 \sin \varphi + \tan \varphi \cos \theta \\ &= -2 \sin \varphi + \tan \varphi \cos \theta = \tan \varphi \left[\cos \theta - \frac{2 \sin \varphi}{\tan \varphi} \right] \\ &= \tan \varphi \left[\cos \theta - \frac{2 \sin \varphi \cos \varphi}{\sin \varphi} \right] = \tan \varphi [\cos \theta - 2 \cos \varphi]\end{aligned}$$

$$\therefore \frac{dx}{d\theta} = x \tan \varphi$$

b) $x = a(2 \cos \theta + \cos 2\theta)$, $y = a(2 \sin \theta - \sin 2\theta)$

$$\frac{dx}{d\theta} = a(-2 \sin \theta - 2 \sin 2\theta) = -2a(\sin \theta + \sin 2\theta)$$

$$\frac{dy}{d\theta} = a(2 \cos \theta + 2 \cos 2\theta) = 2a(\cos \theta + \cos 2\theta)$$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{2a(\cos \theta + \cos 2\theta)}{-2a(\sin \theta + \sin 2\theta)} = \frac{-2 \sin \frac{3}{2}\theta \sin \left(-\frac{1}{2}\theta\right)}{-2 \sin \frac{3}{2}\theta \cos \left(-\frac{1}{2}\theta\right)} \\ &= \frac{\sin \left(\frac{1}{2}\theta\right)}{-\cos \left(\frac{1}{2}\theta\right)} = -\tan \frac{\theta}{2}\end{aligned}$$

By Chain Rule, $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \times \frac{d\theta}{dx}$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \times \frac{d\theta}{dx} = \frac{d}{d\theta} \left(-\tan \frac{\theta}{2} \right) \times \frac{1}{-2a(\sin \theta + \sin 2\theta)} \\ &= -\frac{1}{2} \sec^2 \frac{\theta}{2} \cdot \frac{1}{-2a \left[2 \sin \frac{3}{2}\theta \cos \left(-\frac{1}{2}\theta\right) \right]} = -\frac{1}{8a} \sec^2 \frac{\theta}{2} \cdot \operatorname{cosec} \frac{3\theta}{2} \cdot \sec \frac{\theta}{2}\end{aligned}$$

$$\therefore 8a \frac{d^2y}{dx^2} = \operatorname{cosec} \frac{3}{2}\theta \cdot \sec^3 \frac{\theta}{2}$$

c) Left as an exercise

d) Left as an exercise

LOGARITHMIC DIFFERENTIATION

Differentiating $\log_a x$ from first principles

Let $y = \log_a x$

$$y + \delta y = \log_a (x + \delta x)$$

$$\delta y = \log_a (x + \delta x) - y = \log_a (x + \delta x) - \log_a x$$

$$= \log_a \left(\frac{x + \delta x}{x} \right) = \log_a \left(1 + \frac{\delta x}{x} \right)$$

From Maclaurin's Expansion

$$\ln(1 + x) = x - \frac{x^2}{2} + \dots \dots \dots$$

$$\log_a \left(1 + \frac{\delta x}{x} \right) = \frac{\ln \left(1 + \frac{\delta x}{x} \right)}{\ln a}$$

[Changing base to e]

$$\therefore \delta y = \frac{1}{\ln a} \left[\frac{\delta x}{x} - \frac{1}{2} \left(\frac{\delta x}{x} \right)^2 \right]$$

[Higher power terms neglected]

Dividing through by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\ln a} \left[\frac{1}{x} - \frac{\delta x}{2x} \right]$$

Introducing limits

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\ln a} \left[\frac{1}{x} - \frac{\delta x}{2x} \right] \right)$$

$$\frac{dy}{dx} = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$$

$$\therefore \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a}$$

Corrary

If the base $a = e$ i.e. $\log_a x = \log_e x = \ln x$, the derivative becomes

$$\therefore \frac{d}{dx} (\log_e x) = \frac{d}{dx} (\ln x) = \frac{1}{x} \quad \because \ln e = \log_e e = 1$$

Differentiating $\log_e [f(x)]$

Let $y = \ln[f(x)]$

$$u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$$

$$y = \ln u \Rightarrow \frac{dy}{du} = \frac{1}{u}$$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{u} \times f'(x) = \frac{f'(x)}{f(x)}$$

$$\therefore \frac{d}{dx} (\ln[f(x)]) = \frac{f'(x)}{f(x)}$$

Example

1. Find $\frac{dy}{dx}$ if

a) $y = \ln 2x$

b) $y = \ln \sqrt{x^2 - 1}$

c) $y = x \ln x - x$

d) $y = \ln \left(\frac{1-x}{1+x} \right)$

e) $y = \frac{\ln x}{x}$

f) $y = \ln(\sec x + \tan x)$

g) $y = \ln \left(\frac{\cos x}{1+\sin x} \right)$

Solution

a) $y = \ln 2x \Rightarrow \frac{dy}{dx} = \frac{\frac{d}{dx}(2x)}{2x} = \frac{1}{x}$

b) $y = \ln \sqrt{x^2 - 1} = \frac{1}{2} \ln(x^2 - 1)$

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{x^2 - 1} = \frac{x}{x^2 - 1}$$

c) $y = x \ln x - x$

$$\begin{aligned} \frac{dy}{dx} &= x \cdot \frac{d}{dx}(\ln x) + \ln x \cdot \frac{d}{dx}(x) - 1 \\ &= x \cdot \frac{1}{x} + \ln x - 1 = \ln x \end{aligned}$$

d) $y = \ln \left(\frac{1-x}{1+x} \right) = \ln(1-x) - \ln(1+x)$

$$\frac{dy}{dx} = -\frac{1}{1-x} - \frac{1}{1+x} = \frac{-1-x-1+x}{1-x^2} = -\frac{2}{1-x^2}$$

e) $y = \frac{\ln x}{x} \Rightarrow \frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\ln x) - \ln x \cdot \frac{d}{dx}x}{x^2}$

$$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

f) $y = \ln(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

2. Logarithmic Differentiation of Complicated Products or Quotients or Functions of the form $[f(x)]^{g(x)}$

Differentiate w.r.t x

a) $y = \frac{x(1+x^2)^3}{\sqrt[3]{1+x^3}}$

c) $y = (\cos x)^{\sqrt{\sin 2x}}$

b) $y = \sqrt{\left(\frac{a^2 + ax + x^2}{a^2 - ax + x^2} \right)}$

d) $y = 2x \cdot (\sin x)^x$

Solution

a) $y = \frac{x(1+x^2)^3}{\sqrt[3]{1+x^3}}$

Introducing $\ln$ on both sides

$$\ln y = \ln x + 3 \ln(1+x^2) - \frac{1}{3} \ln(1+x^3)$$

Differentiating w.r.t x

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{6x}{1+x^2} - \frac{x^2}{1+x^3}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+x^2)(1+x^3)+6x^2(1+x^3)-x^3(1+x^2)}{x(1+x^2)(1+x^3)} \cdot y \\ \frac{dy}{dx} &= \frac{1+x^2+x^3+x^5+6x^2+6x^5-x^3-x^5}{x(1+x^2)(1+x^3)} \cdot y \\ &= \frac{1+7x^2+6x^5}{x(1+x^2)(1+x^3)} \cdot y \\ &= \frac{(1+7x^2+6x^5)x \cdot (1+x^2)^3}{x(1+x^2)(1+x^3) \cdot \sqrt[3]{1+x^3}} = \frac{(1+7x^2+6x^5)(1+x^2)^2}{(1+x^3)^{4/3}}\end{aligned}$$

b) $y = \sqrt{\frac{a^2+ax+x^2}{a^2-ax+x^2}}$

Introducing $\ln$ on both sides

$$\ln y = \frac{1}{2} \ln(a^2 + ax + x^2) - \frac{1}{2} \ln(a^2 - ax + x^2)$$

Differentiating w.r.t x

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{a+2x}{2(a^2+ax+x^2)} - \frac{-a+2x}{2(a^2-ax+x^2)} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2} \left[\frac{(a+2x)(a^2-ax+x^2) - (-a+2x)(a^2+ax+x^2)}{(a^2+ax+x^2)(a^2-ax+x^2)} \right] \cdot y \\ &= \frac{1}{2} \left[\frac{2a(a^2+x^2) - 4ax^2}{(a^2+ax+x^2)(a^2-ax+x^2)} \right] \cdot y = \left[\frac{a(a^2+x^2) - 2ax^2}{(a^2+ax+x^2)(a^2-ax+x^2)} \right] \cdot y \\ &= \frac{a(a^2-x^2)}{\sqrt{\{(a^2+ax+x^2)(a^2-ax+x^2)\}^3}}\end{aligned}$$

c) $y = (\cos x)^{\sqrt{\sin 2x}}$

Introducing $\ln$ on both sides

$$\ln y = \sqrt{\sin 2x} \cdot \ln \cos x$$

Differentiating w.r.t x

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \ln \cos x \cdot \frac{1}{2} \cdot (\sin 2x)^{-1/2} \cdot 2 \cos 2x + \sqrt{\sin 2x} \cdot \left(-\frac{\sin x}{\cos x} \right) \\ &= \frac{\cos 2x \ln \cos x}{\sqrt{\sin 2x}} + \sqrt{\sin 2x} \cdot (-\tan x) \\ &= \frac{\cos 2x \ln \cos x - \sin 2x \tan x}{\sqrt{\sin 2x}} \\ \therefore \frac{dy}{dx} &= \left[\frac{\cos 2x \ln \cos x - \sin 2x \tan x}{\sqrt{\sin 2x}} \right] \cdot (\cos x)^{\sqrt{\sin 2x}}\end{aligned}$$

d) $y = 2x \cdot (\sin x)^x$

Introducing $\ln$ on both sides

$$\ln y = \ln 2x + x \ln \sin x$$

Differentiating w.r.t x

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} + x \cdot \frac{\cos x}{\sin x} + \ln \sin 2x \\ \frac{dy}{dx} &= \frac{\sin x + x^2 \cos x + x \sin x \ln \sin 2x}{x \sin x} \cdot y \\ &= \frac{\sin x + x^2 \cos x + x \sin x \ln \sin 2x}{x \sin x} \cdot 2x \cdot (\sin x)^x \\ &= (\sin x + x^2 \cos x + x \sin x \ln \sin 2x)(\sin x)^{x-1}\end{aligned}$$

Task

1. Differentiate w.r.t x

- a) $\frac{e^{x^2}\sqrt{\sin x}}{(2x+1)^3}$ **Ans:** $\frac{e^{x^2}}{2(2x+1)^3\sqrt{\sin x}}\{(8x^2 + 4x - 12) \sin x + (2x + 1) \cos x\}$
- b) $\sqrt[3]{\frac{x+1}{x-1}}$ **Ans:** $-\frac{2}{3}(x+1)^{-2/3}(x-1)^{-4/3}$
- c) $\frac{\sqrt{x^2+1}}{(2x-1)^2}$ **Ans:** $-\frac{2x^2+x+4}{(2x-1)^3\sqrt{x^2+1}}$
- d) $\frac{x^2e^x}{(x-1)^3}$ **Ans:** $\frac{xe^x(x^2-2x-2)}{(x-1)^4}$
- e) $3x^x$ **Ans:** $(\ln x + 1)(3x^x)$
- f) $x^{\cos x}$ **Ans:** $(-2x \sin 2x \ln x + \cos 2x)(x^{\cos x-1})$
- g) $2x^x + 3x^{e^x}$ **Ans:** $(\ln x + 1)(2x^x) + 3x^{e^{2x-1}}(1 + x \ln x)$
- h) $(\sin 2x)^x + x^{\sin^{-1} 2x}$ **Ans:** $\frac{\sqrt{1-4x^2} \cdot \sin^{-1} 2x + 2x \ln x}{\sqrt{1-4x^2}} + (2x \cot 2x + \ln \sin 2x)(\sin 2x)^x$
- i) $x \cdot 5^{1+\sin^2 x}$ **Ans:** $\left(\frac{1+x \sin 2x \ln 5}{x}\right)(x \cdot 5^{1+\sin^2 x})$
- j) $\sqrt{\left\{\frac{(x+2)^3}{1-5x}\right\}}$ **Ans:** $\frac{1}{2}(13-10x)\sqrt{\frac{x+2}{(1-5x)^3}}$
- k) $4^{\sin^{-1} x}$ **Ans:** $\frac{4^{\sin^{-1} x} \ln 4}{\sqrt{1-x^2}}$
- l) 4^{x+x^2} **Ans:** $4^{x+x^2} \ln 4 (1 + 2x)$
- m) $2^{\sin^{-1} x} + 4^{\cos^{-1} x}$ **Ans:** $\frac{2^{\sin^{-1} x} \ln 2 \cdot -4^{\cos^{-1} x} \ln 4}{\sqrt{1-x^2}}$
- n) 5^{e^x} **Ans:** $5^{e^x} \cdot e^x \ln 5$
- o) $4ax^2 + 5b^{x^2+1}$ **Ans:** $8xa^{x^2} \ln a + 10xb^{x^2+1} \ln b$
2. (i) If $y = \cos(\ln x)$, show that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$
- ii) If $y = \ln(x^2 - 5)$, show that $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2e^{-y}$
- iii) If $y = \sin 2x \cdot \ln \tan x$, show that $\frac{d^2y}{dx^2} + 4y = 4 \cot 2x$
3. If $y = a \cos(\ln bx) + b \sin(\ln ax)$, show that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

DIFFERENTIATING EXPONENTIAL FUNCTIONS

1. Differential from first principles $y = a^{bx}$

Solution

$$y + \delta y = a^{b(x+\delta x)} = a^{bx} \cdot a^{b\delta x}$$

$$\begin{aligned}\delta y &= a^{bx} \cdot a^{b\delta x} - y \\ &= a^{bx} \cdot a^{b\delta x} - a^{bx} \\ &= a^{bx} (a^{b\delta x} - 1)\end{aligned}$$

Divide through by δx

$$\frac{\delta y}{\delta x} = a^{bx} \left(\frac{a^{b\delta x} - 1}{\delta x} \right)$$

Introducing limits

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \left[a^{bx} \left(\frac{a^{b\delta x} - 1}{\delta x} \right) \right]$$

$$\frac{dy}{dx} = a^{bx} \lim_{\delta x \rightarrow 0} \left(\frac{a^{b\delta x} - 1}{\delta x} \right)$$

$$\therefore \frac{d}{dx} (a^{bx}) = a^{bx} \lim_{\delta x \rightarrow 0} \left(\frac{a^{b\delta x} - 1}{\delta x} \right)$$

Note

It can be shown that $\lim_{\delta x \rightarrow 0} \left(\frac{a^{b\delta x} - 1}{\delta x} \right) = b \ln a$

$$\therefore \frac{d}{dx} (a^{bx}) = b \cdot a^{bx} \ln a$$

2. Differentiate from first principles

i) $y = e^{ax}$

ii) $y = e^{x^2}$

Solution

i) $y = e^{ax}$

$$y + \delta y = e^{a(x+\delta x)} = e^{ax} \cdot e^{a\delta x}$$

$$\begin{aligned}\delta y &= e^{ax} \cdot e^{a\delta x} - y \\ &= e^{ax} \cdot e^{a\delta x} - e^{ax} \\ &= e^{ax} (e^{a\delta x} - 1)\end{aligned}$$

Expanding $e^{a\delta x}$ using Maclaurin's Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$\therefore e^{a\delta x} = 1 + a\delta x$, taking only the first two terms and ignoring higher power terms

$$\begin{aligned}\therefore \delta y &= e^{ax} (1 + a\delta x - 1) \\ &= a\delta x \cdot e^{ax}\end{aligned}$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = a e^{ax}$$

Introducing limits

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} (ae^{ax})$$

$$\therefore \frac{dy}{dx} = ae^{ax}$$

$$\therefore \frac{d}{dx}(e^{ax}) = ae^{ax}$$

NOTE

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

ii) $y = e^{x^2}$

$$y + \delta y = e^{(x+\delta x)^2} = e^{x^2+2x\delta x+(\delta x)^2} = e^{x^2} \cdot e^{2x\delta x+(\delta x)^2}$$

$$\begin{aligned} \delta y &= e^{x^2} \cdot e^{2x\delta x+(\delta x)^2} - y \\ &= e^{x^2} \cdot e^{2x\delta x+(\delta x)^2} - e^{x^2} \\ &= e^{x^2}(e^{2x\delta x+(\delta x)^2} - 1) \end{aligned}$$

Expanding $e^{2x\delta x+(\delta x)^2}$ using Maclaurin's Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$\therefore e^{2x\delta x+(\delta x)^2} = 1 + 2x\delta x + (\delta x)^2$, taking only the first two terms and ignoring higher power terms

$$\therefore \delta y = e^{x^2}(1 + 2x\delta x + (\delta x)^2 - 1) = e^{x^2}(2x\delta x + (\delta x)^2)$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = e^{x^2}(2x + \delta x)$$

Introducing limits

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} (e^{x^2}(2x + \delta x))$$

$$\therefore \frac{dy}{dx} = 2xe^{x^2}$$

$$\therefore \frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$$

Differentiating $e^{f(x)}$

Let $y = e^{f(x)}$

$$u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$$

$$y = e^u \Rightarrow \frac{dy}{du} = e^u$$

By Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = e^u \cdot f'(x) = e^{f(x)} \cdot f'(x)$$

$$\boxed{\frac{d}{dx}(e^{f(x)}) = e^{f(x)} \cdot f'(x)}$$

Examples

Differentiate w.r.t x

a) $y = e^{x^3}$

c) $y = 1/e^{x \cdot \cos^{-1} x^2}$

b) $y = e^{ax^2 \cdot \sec bx}$

d) $y = x/e^{\cos x}$

Solution

a) $y = e^{x^3} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^3) \cdot e^{x^3} = 3x^2 e^{x^3}$

b) $y = e^{ax^2 \cdot \sec bx} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(ax^2 \cdot \sec bx) \cdot e^{ax^2 \cdot \sec bx}$
 $\frac{dy}{dx} = (\sec bx \cdot 2ax + ax^2 \cdot b \sec bx \tan bx) e^{ax^2 \cdot \sec bx}$
 $= a \sec bx (2x + bx^2 \tan bx) e^{ax^2 \cdot \sec bx}$

c) $y = \frac{1}{e^{x \cdot \cos^{-1} x^2}} = e^{-x \cdot \cos^{-1} x^2}$
 $\frac{dy}{dx} = \frac{d}{dx}(-x \cdot \cos^{-1} x^2) \cdot e^{-x \cdot \cos^{-1} x^2}$
 $= \left[-x \cdot \frac{-2x}{\sqrt{1-x^4}} + \cos^{-1} x^2 \cdot (-1) \right] \cdot e^{-x \cdot \cos^{-1} x^2}$
 $= [2x^2 - \sqrt{1-x^4} \cos^{-1} x^2] \cdot \frac{e^{-x \cdot \cos^{-1} x^2}}{\sqrt{1-x^4}}$
 $= \frac{e^{-x \cdot \cos^{-1} x^2}}{\sqrt{1-x^4}} \cdot [2x^2 - \sqrt{1-x^4} \cos^{-1} x^2]$

d) $y = \frac{x}{e^{\cos x}} = x e^{-\cos x}$
 $\frac{dy}{dx} = x \cdot [-(-\sin x)] e^{-\cos x} + e^{-\cos x}$
 $= (x \sin x + 1) \cdot e^{-\cos x}$

NOTE

Logarithmic differentiation can also be used in handling such type of functions

PROOFS INVOLVING EXPONENTIAL FUNCTIONS

- a) If $y = Ae^{-x} \cos(x + \alpha)$, where A and α are constants prove that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
- b) If $y = e^{-2x} \cos 4x$, prove that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 20y = 0$
- c) If $y = e^x \sin x$, show that $\frac{dy}{dx} = \sqrt{2}e^x \sin(x + \pi/2)$ and $\frac{d^2y}{dx^2} = 2e^x \sin\left(x + \frac{\pi}{2}\right)$
- d) If $y = e^{-t} \cos(t + \beta)$, show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
- e) If $y = e^{4x} \cos 3x$, prove that $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 25y = 0$

Solution

- a) $y = Ae^{-x} \cos(x + \alpha)$, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
- $$\begin{aligned} \frac{dy}{dx} &= A[e^{-x}[-\sin(x + \alpha)] - e^{-x} \cos(x + \alpha)] \\ &= -Ae^{-x} \sin(x + \alpha) - Ae^{-x} \cos(x + \alpha) \\ &= -Ae^{-x} \sin(x + \alpha) - y \dots \dots \dots (1) \end{aligned}$$

Differentiating w.r.t x

$$\begin{aligned} \frac{d^2y}{dx^2} &= -A[e^{-x} \cos(x + \alpha) - e^{-x} \sin(x + \alpha)] - \frac{dy}{dx} \\ &= -Ae^{-x} \cos(x + \alpha) + Ae^{-x} \sin(x + \alpha) - \frac{dy}{dx} \end{aligned}$$

From eqn. (1), $Ae^{-x} \sin(x + \alpha) = -y - \frac{dy}{dx}$

$$\therefore \frac{d^2y}{dx^2} = -y - y - \frac{dy}{dx} - \frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

- b) , (c) , (d) and (e) are left as an exercise

INTEGRATION

Definition and notation

The process which deals with finding the function when its derivative is given .i.e..

If $\frac{d}{dx}\varphi(x) = f(x)$, then in integral calculus we find $\varphi(x)$ when $f(x)$ is given.

In general, integration is the *inverse* of differentiation.

NOTE

When $\varphi(x)$ is found, it is called the *integral* of $f(x)$ with respect to x
Mathematically

If $\frac{d}{dx}\varphi(x) = f(x)$ then $\varphi(x) = \int f(x) dx$.

The symbol $\int f(x) dx$ is read as integral of $f(x)$ w.r.t x

NOTE

When the function $\varphi(x)$ is obtained, it is said that a differential equation is solved. i.e. $\varphi(x) = \int f(x) dx$

Now the process of determining an integral is called *Integration* and the function to be integrated is called the *Integrand*.

$$\Rightarrow \frac{d}{dx} \int f(x) dx = \int \left[\frac{d}{dx} f(x) \right] dx = f(x)$$

Addition of an arbitrary constant

$$\frac{d}{dx}\varphi(x) = f(x)$$

$$\Rightarrow \int f(x) dx = \varphi(x) + c, \text{ where } c \text{ is an arbitrary constant}$$

Proof

$$\begin{aligned} \frac{d}{dx} [\varphi(x) + c] &= \varphi(x) + 0 \left[\because \frac{d}{dx}(c) = 0 \right] \\ &= \frac{d}{dx}\varphi(x) = f(x), \text{ given.} \end{aligned}$$

$$\therefore \int f(x) dx = \varphi(x) + c$$

NOTE

$\varphi(x) + C$ is called an *indefinite integral* of $f(x)$, because the constant c is indefinite.

Table of Standard Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int \sec^2 x dx = \tan x + c$$

Relate these with differentiation. *i.e.*

$$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c \text{ or } -\sin^{-1} x + c$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c \text{ or } -\cot^{-1} x + c$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$$

Differentiation Formula

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx} kx = k$$

Integration Formula

$$\int 0 dx = c$$

$$\int k dx = kx + c$$

Integrating Functions of the Form x^n

Integrate

a) $\int (2x^3 + x^2) dx$

c) $\int (ax^2 + bx + c) dx$

b) $\int (\sqrt{x} + 1)(2\sqrt{x} - 1) dx$

d) $\int \frac{x^2+2x+3}{\sqrt{x}} dx$

Solution

a) $\int (2x^3 + x^2) dx = \frac{2x^{3+1}}{3+1} + C = \frac{x^4}{4} + \frac{x^3}{3} + c$

b) $\int (\sqrt{x} + 1)(2\sqrt{x} - 1) dx = \int (2x - \sqrt{x} + 2\sqrt{x} - 1) dx$
 $= \int (2x + \sqrt{x} - 1) dx = \frac{x^2}{2} + \frac{x^{3/2}}{3/2} - x + c$
 $= x^2 + \frac{2}{3}x^{3/2} - x + c$

c) $\int (ax^2 + bx + c) dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$

d) $\int \frac{x^2+2x+3}{\sqrt{x}} dx = \int (x^{2-1/2} + 2x^{1-1/2} + 3x^{-1/2}) dx = \int (x^{3/2} + 2x^{1/2} + 3x^{-1/2}) dx$
 $= \frac{x^{5/2}}{5/2} + \frac{2x^{3/2}}{3/2} + \frac{3x^{1/2}}{1/2} + c = \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 6x^{1/2} + c$
 $= \frac{2}{5}\sqrt{x^5} + \frac{4}{3}\sqrt{x^3} + 6\sqrt{x} + c$

Task

a) $\int (2x^3 - 5x^2 - 3x + 4)\sqrt{x} dx$

Ans: $\frac{4}{9}x^{9/2} - \frac{10}{7}x^{7/2} - \frac{6}{5}x^{5/2} + \frac{8}{3}x^{3/2} + c$

b) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^3 dx$

Ans: $\frac{2}{5}x^{5/2} - 2x^{3/2} + 6x^{1/2} + 2x^{-1/2} + c$

Definite Integrals

Definition: If $\int f(x) dx = \varphi(x)$, then $\varphi(b) - \varphi(a)$ is called the *definite integral of $f(x)$ between the limits a and b* and is written as;

$$\int_a^b f(x) dx \quad [\text{Read as integral from } a \text{ to } b, \text{ of } f(x) \text{ w.r.t } x]$$

$\therefore \int_a^b f(x) dx = \varphi(b) - \varphi(a)$, this gives the Fundamental Theorem of Calculus.

Here a is called the *Lower Limit*, b is the *Upper Limit* of Integration. The interval (a, b) is the *Range of Integration*.

Note

- 1) Since $\int_c^d f(x) dx$ has a definite value $\varphi(b) - \varphi(a)$ then the name “Definite Integral”
- 2) The difference $\varphi(b) - \varphi(a)$ is also written as $[\varphi(x)]_a^b = \varphi(b) - \varphi(a)$

Rule to write the Definite Integral $\int_a^b f(x) dx$

- a) Integrate $\int f(x) dx$ and let the indefinite integral be $\varphi(x)$
- b) Substitute in $\varphi(x)$ the first upper limit and then the lower limit
- c) Subtract the second result from the first

NOTE

Definition of Two special Definite Integral

- a) $\int_a^a f(x) dx = 0$
- b) $\int_b^a f(x) dx = -\int_a^b f(x) dx$, where f integrable on $[a, b]$

Other properties include

- 1) $\int_a^b kf(x) dx = k \int_a^b f(x) dx$
- 2) $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$, where f and g are integrable on $[a, b]$ and k is a constant

Example

Evaluate the definite integrals

- a) $\int_3^0 (x+3) dx$ b) $\int_1^3 (-x^2 + 4x - 3) dx$ c) $\int_{-1}^1 (2x^2 - x^3) dx$

Solution

- a) $\int_3^0 (x+3) dx = -\int_0^3 (x+3) dx$
 $= \left[\frac{x^2}{2} + 3x \right]_0^3 = -\left(\frac{9}{2} + 9 - 0 \right) = -\frac{27}{2}$
- b) $\int_1^3 (-x^2 + 4x - 3) dx = \left[-\frac{x^3}{3} + 2x^2 - 3x \right]_1^3$
 $= \left(-\frac{27}{3} + 18 - 9 \right) - \left(-\frac{1}{3} + 2 - 3 \right) = \frac{4}{3}$
- c) $\int_{-1}^1 (2x^2 - x^3) dx = \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_{-1}^1 = \left(\frac{2}{3} - \frac{1}{4} \right) - \left(-\frac{2}{3} + \frac{1}{4} \right) = \frac{4}{3}$

INTEGRATION BY SUBSTITUTION

Introduction:

From the previous presentations, integrals can be obtained with the help of standard results of differentiation by mere inspection. However the integrand $f(x)$ may not be easily guessed by mere inspection, the function of which $f(x)$ is the derivative.

In such cases, a suitable substitution like $x = \varphi(z)$ may change $f(x)$ in to a function of z which can be integrated by mere inspection.

The procedure of evaluating integrals is called *Integration by Substitution*.

Review:

Differentiation formula

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[kx] = k$$

$$\frac{d}{dx}[k f(x)] = k f'(x)$$

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$\int g(x) dx$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cot x$$

Integration formula

$$\int 0 dx = c$$

$$\int k dx = kx + c$$

$$\int k f(x) dx = k \int f(x) dx$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

$$\int \cos x dx = \sin x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + c$$

$$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$$

For easy understanding the discussion has been split in to two parts. *i.e.*

- a) Pattern Recognition
- b) Change of variable

Both techniques involve a *u*-substitution

Pattern Substitution

Here perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of chain rule in differentiation. *i.e.*

Given that $y = f[g(x)]$, then

$$\text{Let } u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$$

$$y = f(u) \Rightarrow \frac{dy}{du} = f'(u)$$

By Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{dx} = f'(u) \cdot g'(x) = f'[g(x)] \cdot g'(x)$$

$$\therefore \frac{d}{dx}\{f[g(x)]\} = f'[g(x)] \cdot g'(x)$$

Now integrating both sides

$$\int \frac{d}{dx}\{f[g(x)]\} dx = \int f'[g(x)] \cdot g'(x) dx$$

$$= f[g(x)] + c$$

$$\text{Or } \int f(u) \cdot du = F(u) + c$$

Generally

$$\int \overset{\text{Outside}}{f[g(x)]} \cdot \overset{\text{Derivative of outside}}{g'(x)} dx = \overset{\text{Inside}}{F(g(x))} + c$$

Recognizing the $f[g(x)] \cdot g'(x)$

Evaluate $\int 2x(x^2 + 1)^2 dx$

Solution

Letting $g(x) = x^2 + 1$, $g'(x) = 2x$ and $f[g(x)] = [g(x)]^2$

From this you can recognize that the integrand follows $f[g(x)] \cdot g'(x)$ pattern

Using the Power Rule .i.e

$$\int [g(x)]^n \cdot g'(x) dx = \frac{[g(x)]^{n+1}}{n+1} + c, \quad n \neq -1$$

Equivalently if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + c, \quad n \neq -1$$

$$[g(x)]^2 \cdot g'(x)$$

$$\Rightarrow \int (x^2 + 1)^2 \cdot (2x) \cdot dx = \frac{(x^2+1)^3}{3} + c$$

NOTE

Many integrands contain the essential part (the Variable Part) of $g'(x)$ but are missing a constant multiple. In such cases you can multiply and divide by the necessary constant multiple.

Evaluate $\int x(x^2 + 1)^2 dx$

Solution

$$\int x(x^2 + 1)^2 dx = \int (x^2 + 1)^2 \cdot (2x) \cdot \frac{1}{2} dx \dots\dots\dots \text{Multiply and Divide by 2}$$

$$= \frac{1}{2} \int (x^2 + 1)^2 \cdot (2x) \cdot dx \equiv \frac{1}{2} \int [g(x)]^2 \cdot g'(x) \cdot dx$$

$$= \frac{1}{2} (x^2 + 1)^3 + c \quad \equiv \frac{1}{2} [g(x)]^3 + c$$

Change of Variable

Sometimes the change of the variable x in $\int f(x) dx$, by means of a substitution $x = \varphi(z)$ changes the integrand to a function of z which is either a standard form or a form easier to deal with.

$$\text{Let } I = \int f(x) dx$$

$$\frac{dI}{dx} = f(x)$$

$$\text{But } x = \varphi(z) \therefore \frac{dx}{dz} = \varphi'(z)$$

$$\text{Now } \frac{dI}{dz} = f(x) \cdot \varphi'(z)$$

$\Rightarrow I = \int f(\varphi(z)) \cdot \varphi'(z) dz$, which is a new integral in z which is more simpler/easier to deal with.

General Approach

$$\text{If } I = \int f(x) dx,$$

let $x = \varphi(z)$, $dx = \varphi'(z) \cdot dz$ are the key substitutions

$$\Rightarrow I = \int f(x) dx = \int f(\varphi(z)) \cdot \varphi'(z) dz, \text{ which is as before}$$

NOTE

Sometimes the substitution is of the form $F(x) = \varphi(z)$

$$\Rightarrow F'(x)dx = \varphi'(z)dz$$

, which eliminates the integrand, transforming to a new integrand in terms of z .

Example

Evaluate

$$\text{a) } \int \sqrt{2x-1} dx$$

$$\text{b) } \int x\sqrt{2x-1} dx$$

Solution

$$\text{a) } \int \sqrt{2x-1} dx, \text{ let } u = 2x-1 \Rightarrow du = 2dx \therefore dx = \frac{du}{2}$$

$$\begin{aligned} \int \sqrt{2x-1} dx &= \int \sqrt{u} \cdot \frac{du}{2} \\ &= \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \left(\frac{u^{3/2}}{3/2} \right) + c \\ &= \frac{1}{3} u^{3/2} + c = \frac{1}{3} (2x-1)^{3/2} + c \end{aligned}$$

$$\text{b) } \int x\sqrt{2x-1} dx; \text{ let } 2x-1 \Rightarrow du = 2dx \therefore dx = \frac{du}{2}$$

Now since the integrand has a factor of x , you must also solve for x in terms of u . i.e.

$$x = \frac{u+1}{2}$$

$$\begin{aligned} \Rightarrow \int x\sqrt{2x-1} dx &= \int \left(\frac{u+1}{2} \right) \cdot \sqrt{u} \cdot \frac{du}{2} \\ &= \frac{1}{4} \int (u^{3/2} + u^{1/2}) du = \frac{1}{4} \left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right) + c \\ &= \frac{1}{10} (2x-1)^{5/2} + \frac{1}{6} (2x-1)^{3/2} + c \end{aligned}$$

Other examples on Substitution and the general Power Rule

$$\begin{aligned}
 \text{a) } \int 3 \cdot (3x-1)^4 dx &= \int \overbrace{(3x-1)^4}^u \cdot \overbrace{(3)}^{du} dx = \frac{u^5/5}{5} + c \\
 \text{(b) } \int (2x+1) \cdot (2x^2+x) dx &= \int \overbrace{(2x+1)}^u \cdot \overbrace{(2x^2+x)}^{du} dx = \frac{u^2/2}{\frac{(x^2+x)^2}{2}} + c \\
 \text{(c) } \int 3x^2 \cdot \sqrt{x^3-2} dx &= \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}} \cdot \overbrace{(3x^2)}^{du} dx = \frac{\frac{u^{3/2}}{3/2}}{\frac{(x^3-2)^{3/2}}{2}} + c \\
 \text{(d) } \int -\frac{4x}{(1-2x^2)^2} dx &= \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}} \cdot \overbrace{(-4x)}^{du} dx = \frac{\frac{u^{-1}}{-1}}{-1} + c
 \end{aligned}$$

How to Identify the Proper Substitution

There is no general rule for finding a proper substitution, however the following suggestions will prove useful.

1. If a part of the integrand, in any form whatsoever that its differential coefficient is a factor of the integrand, then

$$\text{Part} = u$$

Example: $\int x \cdot \cos x^2 dx$

Here we note that the differential coefficient of x^2 is $2x$, which contains the factor x (Neglecting the constant factor). Hence proper substitution is

$$x^2 = u$$

$$\Rightarrow 2x dx = du \text{ or } x = \frac{1}{2} du$$

$$\therefore \int x \cdot \cos x^2 dx = \frac{1}{2} \cos u \cdot du = \frac{1}{2} \sin u + c = \frac{1}{2} \sin x^2 + c$$

2. If the integrand contains "Trigonometrical ratio, a "Logarithmic function" or "an exponential" in which the index is a "a function of x" then put this

$$\text{Function} = u$$

Example: $\int e^{\sqrt{x}} dx$

Put $u = \sqrt{x}$ or $u^2 = x \Rightarrow 2u du = dx$

$$\therefore \int e^{\sqrt{x}} dx = \int e^u \cdot 2u du = 2 \int u \cdot e^u du, \text{ which can be integrated by parts}$$

3. If the integrand contains $\log_e x$, it may be helpful to use

$$u = \log_e x \text{ or } x = e^u$$

4. If the integrand contains an inverse trigonometrical function of x like $\sin^{-1} x$, this ratio may be put as

Inverse of Trigonometrical Function = u

Example: $\int \frac{\tan^{-1} x}{1+x^2} dx$, $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}} dx$

If the trigonometrical ratio is that of a “function of x ” then the trigonometrical ratio whole (and not merely the function) must be put as **Inverse of Trigonometrical Function = u** , unless there is a better way to remove the inverseness.

Thus if $I = \int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$, put $\sin^{-1} \sqrt{\frac{x}{a+x}} = u \Rightarrow \sin u = \sqrt{\frac{x}{a+x}}$
 $\Rightarrow x = (a+x) \sin^2 u$ or $x = \frac{a \sin^2 u}{1-\sin^2 u}$

5. If you integrate an **odd** power of $\sin x$, put **$u = \cos x$**
 6. If you integrate an **odd** power of $\cos x$, put **$u = \sin x$**
 7. If you integrate the product of functions of sine and cosine with power of **sine** being **odd**, put **$u = \cos x$** if the power of **cosine** is **even**, and put **$u = \sin x$** if the power of cosine is odd and that of sine is even.

Example:

- i) $\int \sin^2 x \cos^5 x dx$, let $u = \sin x$
 ii) $\int \sin^3 x \cos^4 x dx$, let $u = \cos x$

8. If the integral are in the form;
 a) $\int \sec^m x \tan^n x dx$, $m = \text{even}$, use $u = \tan x$, and if $n = \text{odd}$, use $u = \sec x$
 b) $\int \sin^m x \cos^n x dx$, $m, n = \text{odd}$, use a function with the *highest power*
 Now if $m = \text{even}, n = \text{odd}$, use $u = \sin x$
 $m = \text{odd}, n = \text{even}$, use $u = \cos x$
 c) $\int \tan^m x dx$, $m = \text{even}$, $u = \tan x$
9. If the integrand is a fraction whose numerator is (constant of a function of $\tan x$) and the denominator a homogeneous function of second degree in $\sin x$ or $\cos x$, divide the numerator and denominator by $\cos^2 x$ and put $u = \tan x$ (or divide by $\sin^2 x$) and put $u = \cot x$.
 If the degree of denominator be fourth, divide by $\cos^4 x$ or $\sin^4 x$ and make the same substitution.
 For some of the latter cases, a substitution of **$u = \tan^2 x$** may be effective which is evidently better.

Example

Let $I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

Divide the numerator and denominator by $\cos^2 x$

$I = \int \frac{\sec^2 x dx}{a^2 \tan^2 x + b^2} = \int \frac{du}{a^2 u^2 + b^2}$, after putting $u = \tan x$ and $du = \sec^2 x dx$

Again if the denominator be not homogeneous it may be possible to make it so

$\int \frac{dx}{1+3 \sin^2 x} = \int \frac{dx}{(\cos^2 x + \sin^2 x) + 3 \sin^2 x} = \int \frac{dx}{\cos^2 x + 4 \sin^2 x}$

- 10.** When the integrand is a rational function of the trigonometrical ratios of x , the substitution $t = \tan \frac{x}{2}$ is often useful. This transforms the integrand to a rational function of t because;

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}, \text{ and } \frac{1}{2} \sec^2 x dx = dt \therefore dx = \frac{2dt}{1+t^2}$$

- 11.** If the integrand is a rational function of e^x put $u = e^x$ such that

$$e^x dx = du \text{ or } dx = \frac{du}{e^x} = \frac{du}{u}$$

- 12.** If the integrand contains a surd, it is better to first rationalize it by proper substitution.

The following substitutions are very useful

Form of Surd

$$\sqrt{ax + b}$$

$$(ax + b)^{r/s}$$

$$\sqrt{x + b}$$

$$\sqrt{a^2 - x^2}$$

$$\sqrt{a^2 + x^2}$$

$$\sqrt{x^2 - a^2}$$

$$\sqrt{ax - x^2}$$

$$\sqrt{2ax - x^2}$$

$$\sqrt{\left(\frac{a-x}{a+x}\right)}$$

$$\sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)}$$

Substitution

$$ax + b = u^2$$

$$ax + b = u^s$$

$$x + b = u^2, \text{ or } x = \tan^2 \theta$$

$$x = a \sin \theta, \text{ or } x = a \cos \theta$$

$$x = a \tan \theta$$

$$x = a \sec \theta$$

$$x = a \sin^2 \theta$$

$$x = 2a \sin^2 \theta \text{ or } a(1 - \cos \theta)$$

$$x = a \cos 2\theta$$

$$x^2 = a^2 \cos 2\theta$$

General standard formula for surds

- $\int (ax + b)^n \cdot a \cdot dx = \frac{(ax+b)^{n+1}}{(n+1)} + c$
- $\int (ax + b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

8-Standard integrals to be used

1. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) \quad \text{or} \quad -\cos^{-1} \left(\frac{x}{a} \right)$
2. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) \quad \text{or} \quad -\frac{1}{a} \cot^{-1} \left(\frac{x}{a} \right)$
3. $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) \quad \text{or} \quad -\frac{1}{a} \operatorname{cosec}^{-1} \left(\frac{x}{a} \right)$
4. $\int \sqrt{a^2-x^2} dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right)$
5. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log_e \left(\frac{x+\sqrt{a^2+x^2}}{a} \right)$
6. $\int \sqrt{a^2+x^2} dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \ln \left(\frac{x+\sqrt{a^2+x^2}}{2} \right)$
7. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln \left(\frac{x+\sqrt{x^2-a^2}}{a} \right)$
8. $\int \sqrt{x^2-a^2} dx = \frac{x\sqrt{x^2-a^2}}{2} - \ln \left(\frac{x+\sqrt{x^2-a^2}}{a} \right)$

Other Four Standard Integrals

Show that

- a) $\int \cot x dx = \ln \sin x$
- b) $\int \tan x dx = -\ln \cos x \quad \text{or} \quad \ln \sec x$
- c) $\int \operatorname{cosec} x dx = \ln \tan \frac{1}{2} x \quad \text{or} \quad \ln(\operatorname{cosec} x - \cot x)$
- d) $\int \sec x dx = \ln \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \quad \text{or} \quad \ln(\sec x + \tan x)$

Change of Variable for Definite Integrals

From $\int_a^b f(g(x)) \cdot g'(x) dx$, let $u = g(x) \Rightarrow du = g'(x) dx$

$$\Rightarrow \int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Example

a) Evaluate $\int_0^1 x(x^2 + 1)^3 dx$

Solution

Let $u = x^2 + 1 \Rightarrow du = 2x dx$

Before substituting, determine the new upper and lower limits of integration

Lower limit

When $x = 0$, $u = 0^2 + 1 = 1$

Upper limit

When $x = 1$, $u = 1^2 + 1 = 2$

Now substituting

$$\begin{aligned} \int_0^1 x(x^2 + 1)^3 dx &= \frac{1}{2} \int_1^2 (x^2 + 1)^3 \cdot (2x) dx \\ &= \frac{1}{2} \int_1^2 (u)^3 du = \frac{1}{2} \left[\frac{u^4}{4} \right]_1^2 \\ &= \frac{1}{2} \left(4 - \frac{1}{4} \right) = \frac{15}{8} \end{aligned}$$

b) Evaluate $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$

Solution

Let $u = \sqrt{2x-1} \Rightarrow u^2 = 2x-1 \therefore x = \frac{u^2+1}{2}$

$dx = u \cdot du$

Before substituting, determine the new upper and lower limits of integration

Lower limit

When $x = 1$, $u = \sqrt{2-1} = 1$

Upper limit

When $x = 5$, $u = \sqrt{10-1} = 3$

Now substituting to obtain

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}} dx &= \frac{1}{2} \int_1^3 \left(\frac{u^2+1}{u} \right) u \cdot du \\ &= \frac{1}{2} \int_1^3 (u^2 + 1) du = \frac{1}{2} \left[\frac{u^3}{3} + u \right]_1^3 \\ &= \frac{1}{2} \left(9 + 3 - \frac{1}{3} - 1 \right) = \frac{16}{3} \end{aligned}$$

Task

1. Evaluate

a) $\int (1 + 2x)^4 \cdot 2dx$

Ans: $\frac{(1+2x)^5}{5} + c$

b) $\int \sqrt{9-x^2} \cdot (-2x) dx$

Ans: $\frac{2}{9} (9-x^2)^{3/2} + c$

c) $\int (5x)^3 \sqrt{1-x^2} dx$

Ans: $-\frac{15}{8} (1-x^2)^{4/3} + c$

d) $\int \frac{x^2}{(1+x^3)^2} dx$

Ans: $-\frac{1}{3(1+x^3)} + c$

e) $\int \left(1 + \frac{1}{t}\right)^3 \cdot \left(\frac{1}{t^2}\right) dt$

Ans: $-\frac{1}{4} \left(1 + \frac{1}{t}\right)^4 + c$

f) $\int \frac{1}{\theta^2} \cos\left(\frac{1}{\theta}\right) d\theta$

Ans: $-\sin\left(\frac{1}{\theta}\right) + c$

g) $\int \frac{\operatorname{cosec}^2 x}{\cot^3 x} dx$

Ans: $\frac{1}{2} \tan^2 x + c$

h) $\int \tan^4 x \sec^2 x dx$

Ans: $\frac{1}{5} \tan^5 x + c$

i) $\int \frac{x^2-1}{\sqrt{2x-1}} dx$

Ans: $\frac{\sqrt{2x-1}}{15} (3x^2 + 2x - 13) + c$

j) $\int \frac{-x}{(x+1)\sqrt{x+1}} dx$

Ans: $-x - 1 - 2\sqrt{x+1} + c$

2. Evaluate the definite integral

a) $\int_{-1}^1 x(x^2 + 1)^3 dx$

Ans: 0

b) $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

Ans: $\frac{1}{2}$

c) $\int_0^7 x^3 \sqrt{x+1} dx$

Ans: $\frac{1209}{28}$

d) $\int_1^2 (x-1)\sqrt{2-x} dx$

Ans: $\frac{4}{15}$

Four important integrals

To integrate

i) $\int \frac{1}{x^2+a^2} dx$

iii) $\int \frac{1}{x\sqrt{x^2-a^2}} dx$

ii) $\int \frac{1}{\sqrt{a^2-x^2}} dx$

Solution

i) Put $x = a \tan \theta \Rightarrow dx = a \sec^2 \theta d\theta$

$$\begin{aligned} \int \frac{1}{x^2+a^2} dx &= \int \frac{1}{a^2 \tan^2 \theta + a^2} \cdot a \sec^2 \theta d\theta \\ &= \int \frac{a \sec^2 \theta d\theta}{a^2 (1+\tan^2 \theta)} = \frac{1}{a} \int d\theta \\ &= \frac{1}{a} \theta + c \\ &= \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c \end{aligned}$$

$$\therefore \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

ii) Put $x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta$

$$\begin{aligned} \int \frac{1}{\sqrt{a^2-x^2}} dx &= \int \frac{1}{\sqrt{a^2-a^2 \sin^2 \theta}} \cdot a \cos \theta d\theta \\ &= \int \frac{a \cos \theta}{a \cos \theta} d\theta = \theta + c \end{aligned}$$

$$\therefore \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

iii) Put $x = a \sec \theta \Rightarrow dx = a \sec \theta \tan \theta d\theta$

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2-a^2}} dx &= \int \frac{1}{a \sec \theta \sqrt{a^2 \sec^2 \theta - a^2}} \cdot a \sec \theta \tan \theta d\theta \\ &= \int \frac{\tan \theta}{a \sqrt{\sec^2 \theta - 1}} d\theta = \theta + c \end{aligned}$$

$$\therefore \int \frac{1}{x\sqrt{x^2-a^2}} dx = \sec^{-1} \left(\frac{x}{a} \right) + c$$

Illustrative examples

Evaluate

i) $\int \sin^3 x \cos x \, dx$

ii) $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

iii) $\int \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} dx$

Solution

i) Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$$\begin{aligned} \int \sin^3 x \cos x \, dx &= \int \sin^2 x \cdot u \cdot \frac{du}{-\sin x} \\ &= -\int \sin^2 x \cdot u \cdot du = -\int (1 - \cos^2 x) \cdot u \cdot du \\ &= -\int (u - u^3) du = -\frac{u^2}{2} + \frac{u^4}{4} + c \\ &= -\frac{\cos^2 x}{2} + \frac{\cos^4 x}{4} + c \end{aligned}$$

ii) The presence of $\frac{1}{1+x^2} \cdot dx$ suggests the substitution of $u = \tan^{-1} x$

$$\Rightarrow du = \frac{dx}{1+x^2}$$

$$\begin{aligned} \therefore \int \frac{e^{\tan^{-1} x}}{1+x^2} dx &= \int e^u du = e^u + c \\ &= e^{\tan^{-1} x} + c \end{aligned}$$

iii) Let $t = 1 + \sin^2 x \Rightarrow dt = 2 \sin x \cos x$

$$\begin{aligned} \int \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \int t^{-1/2} dt \\ &= \sqrt{t} + c = \sqrt{1 + \sin^2 x} + c \end{aligned}$$

3. Integrate

a) $\frac{x^2 \tan^{-1}(x^3)}{1+x^6}$

c) $\frac{\cos \sqrt{x}-3}{\sqrt{x}}$

b) $\frac{1}{x} \cos(\ln x)$

d) $\frac{1}{x^{2/3}(1+x^{2/3})}$

Solution

a) Let $u = \tan^{-1}(x^3)$ because of $\frac{x^2}{1+x^6} dx$ being present

$$\begin{aligned} \Rightarrow du &= \frac{3x^2}{1+x^6} dx \\ \therefore \int \frac{x^2 \tan^{-1}(x^3)}{1+x^6} dx &= \frac{1}{3} \int \frac{3x^2 \cdot \tan^{-1}(x^3)}{1+x^6} \cdot dx \\ &= \frac{1}{3} \int u \, du = \frac{1}{6} u^2 + c \\ &= \frac{1}{6} (\tan^{-1}(x^3))^2 + c \end{aligned}$$

b) The presence of $\frac{dx}{x}$ suggests that $u = \ln x \Rightarrow du = \frac{dx}{x}$ or $dx = x \, du$

$$\begin{aligned} \therefore \int \frac{1}{x} \cos(\ln x) \, dx &= \int \cos u \, du = \sin u + c \\ &= \sin \ln x + c \end{aligned}$$

c) $\int \frac{\cos \sqrt{x}-3}{\sqrt{x}} dx$ putting $t = \sqrt{x} \Rightarrow dt = \frac{dx}{2\sqrt{x}}$ or $dx = 2\sqrt{x} dt$

$$\begin{aligned} \therefore \int \frac{\cos \sqrt{x}-3}{\sqrt{x}} dx &= \int (\cos t - 3) \cdot 2dt \\ &= 2(\sin t - 3t) + c = 2 \sin \sqrt{x} - 6\sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \text{d) } \int \frac{1}{x^{2/3}(1+x^{2/3})} dx, \text{ putting } t = x^{1/3} &\Rightarrow dt = \frac{1}{3}x^{-2/3}dx \text{ or } dx = 3x^{2/3}dt \\ \int \frac{1}{x^{2/3}(1+x^{2/3})} dx &= \int \frac{3dt}{1+t^2} \\ &= 3 \tan^{-1} t + c = 3 \tan^{-1}(x^{1/3}) + c \end{aligned}$$

4. Integrate

$$\text{a) } \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$

$$\text{b) } \int \frac{\sin^2 x \cos x}{1 - \sin x} dx$$

$$\text{c) } \int \frac{1}{x\sqrt{1-x}} dx$$

Solution

$$\begin{aligned} \text{a) } \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx, \text{ putting } t^2 = \tan x &\Rightarrow 2t dt = \sec^2 x dx \text{ or } dx = \frac{2tdt}{\sec^2 x} \\ \therefore \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx &= \int \frac{t}{\sin x \cos x} \cdot \frac{2tdt}{\sec^2 x} \\ &= \int \frac{2t^2}{\tan x} dt = \int \frac{2t^2}{t^2} dt \\ &= 2t + c = 2\sqrt{\tan x} + c \end{aligned}$$

$$\begin{aligned} \text{b) } \int \frac{\sin^2 x \cos x}{1 - \sin x} dx, \text{ putting } t = \sin x &\Rightarrow dt = \cos x dx \text{ or } dx = \frac{dt}{\cos x} \\ \therefore \int \frac{\sin^2 x \cos x}{1 - \sin x} dx &= \int \frac{t^2}{1-t} dt = \int \frac{-t(1-t)+t}{1-t} dt \\ &= \int \left(-t + \frac{t}{1-t}\right) dt = \int \left(-t + \frac{-(1-t)+1}{1-t}\right) dt \quad (\text{Apply partial fractions}) \\ &= \int \left(-t - 1 + \frac{1}{1-t}\right) dt = -\frac{t^2}{2} - t - \ln(1-t) + c \\ &= -\left(\sin x + \frac{\sin^2 x}{2} + \ln(1 - \sin x)\right) + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int \frac{1}{x\sqrt{1-x}} dx, \text{ putting } t^2 = 1-x &\Rightarrow 2t dt = -1 dx \text{ or } dx = -2t dt \\ \therefore \int \frac{1}{x\sqrt{1-x}} dx &= \int \frac{1}{(1-t^2)t} \cdot -2t dt = \int \frac{2}{1-t^2} dt \\ &= \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt \quad (\text{Apply partial fractions}) \\ &= \ln(t-1) - \ln(t+1) + c \\ &= \ln\left(\frac{t-1}{t+1}\right) + c = \ln\left[\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right] + c \end{aligned}$$

Integrals involving Inverse of Trigonometrical Functions: completing squares

The derivatives of six inverse trigonometrical functions occur in three pairs. In each pair the derivative of one function is the negative of the other. *e.g.*

$$\frac{d}{dx} [\arcsin x] = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{d}{dx} [\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$

When listing the *anti derivative* that corresponds to each of the inverse of trigonometric functions, you need to use only one member of each pair.

It is conventional to use $\arcsin x$ as the anti derivative of $\frac{1}{\sqrt{1-x^2}}$, rather than $-\arccos x$

Three important integrals

To integrate

$$\text{i) } \int \frac{1}{x^2+a^2} dx$$

$$\text{iii) } \int \frac{1}{x\sqrt{x^2-a^2}} dx$$

$$\text{ii) } \int \frac{1}{\sqrt{a^2-x^2}} dx, \text{ see the four important integrals presented above.}$$

Solutions

$$\text{i) } \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\text{ii) } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\text{iii) } \int \frac{1}{x\sqrt{x^2-a^2}} dx = \sec^{-1} \left(\frac{x}{a} \right) + c$$

NOTE

The above results are useful. The reader should commit them to memory.

Examples

Integration with Inverse Trigonometrical Functions

$$\text{a) } \int \frac{1}{\sqrt{4-x^2}} dx$$

$$\text{c) } \int \frac{1}{x\sqrt{3x^2-1}} dx$$

$$\text{b) } \int \frac{1}{2+9x^2} dx$$

$$\text{d) } \int \frac{1}{\sqrt{2-x^2}} dx$$

Solution

$$\text{a) } \int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{2^2-x^2}} dx = \arcsin \frac{x}{2} + c$$

$$\begin{aligned} \text{b) } \int \frac{1}{2+9x^2} dx &= \frac{1}{3} \int \frac{3 dx}{(3x)^2 + (\sqrt{2})^2}, \quad u = 3x, a = \sqrt{2} \\ &= \frac{1}{3} \cdot \frac{1}{\sqrt{2}} \cdot \arctan \frac{3x}{\sqrt{2}} + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int \frac{1}{x\sqrt{3x^2-1}} dx &= \int \frac{1}{x\sqrt{3(x^2-1/3)}} dx \\ &= \frac{1}{\sqrt{3}} \int \frac{1}{x\sqrt{(x^2-1/3)}} dx = \frac{1}{\sqrt{3}} \int \frac{1}{x\sqrt{(x^2-(1/\sqrt{3})^2)}} dx \\ &= \frac{1}{\sqrt{3}} \cdot \frac{1}{1/\sqrt{3}} \operatorname{arcsec} \left(\frac{x}{1/\sqrt{3}} \right) + c \\ &= \operatorname{arcsec}(\sqrt{3}x) + c \end{aligned}$$

Alternatively

$$\begin{aligned} \int \frac{1}{x\sqrt{3x^2-1}} dx &= \int \frac{1}{x\sqrt{(\sqrt{3}x)^2-(1)^2}} dx \\ &= \int \frac{\sqrt{3}}{(\sqrt{3}x)\sqrt{(\sqrt{3}x)^2-(1)^2}} dx, \quad u = \sqrt{3}x, a = 1 \\ &= \operatorname{arcsec} \left(\frac{\sqrt{3}x}{1} \right) + c = \operatorname{arcsec}(\sqrt{3}x) + c \end{aligned}$$

NOTE

The above examples portray the application of the three important integration formulas above.

Disguised Inverse Trigonometric Integration Formulae
Integration by Substitution

Evaluate

a) $\int \frac{dx}{\sqrt{e^{2x}-1}}$

c) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx$

e) $\int_0^{\pi/4} \frac{\cos x}{1+\sin^2 x} dx$

b) $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$

d) $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

f) $\int \frac{e^{2x}}{\sqrt{1-e^{2x}}} dx$

Solution

a) Let $u = e^x \Rightarrow du = e^x \cdot dx$ or $dx = \frac{du}{e^x} = \frac{du}{u}$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{du}{u\sqrt{(e^x)^2-1}} \\ &= \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}\left(\frac{u}{1}\right) + c \\ &= \operatorname{arcsec}(e^x) + c \end{aligned}$$

b) $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}} dx$

Let $u = e^x \Rightarrow du = e^x \cdot dx$ or $dx = \frac{du}{e^x} = \frac{du}{u}$

$$\begin{aligned} \therefore \int \frac{e^x}{\sqrt{1-e^{2x}}} dx &= \int \frac{du}{u\sqrt{(e^x)^2-1}} \\ &= \int \frac{u}{\sqrt{1-u^2}} \cdot \frac{du}{u} = \sin^{-1}\left(\frac{u}{1}\right) + c \\ &= \arcsin(e^x) + c \end{aligned}$$

c) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx$, let $u = \sin x \Rightarrow du = \cos x \cdot dx$ or $dx = \frac{du}{\cos x}$

$$\begin{aligned} \therefore \int \frac{\cos x}{\sqrt{4-\sin^2 x}} dx &= \int \frac{\cos x}{\sqrt{4-u^2}} \cdot \frac{du}{\cos x} = \int \frac{1}{\sqrt{4-u^2}} du \\ &= \int \frac{1}{\sqrt{2^2-u^2}} du, a = 2 \\ &= \arcsin\left(\frac{u}{2}\right) + c = \arcsin\left(\frac{\sin x}{2}\right) + c \end{aligned}$$

d) $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x} dx$, let $u = \cos x \Rightarrow du = -\sin x \cdot dx$ or $dx = \frac{du}{-\sin x}$

Changing limits

When $x = \pi/2$, $u = \cos \pi/2 = 0$ and when $x = \pi$, $u = \cos \pi = -1$

$$\begin{aligned} \int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x} dx &= \int_0^{-1} \frac{\sin x}{1+u^2} \cdot \frac{du}{-\sin x} = \int_0^{-1} \frac{1}{1+u^2} du \\ &= [\tan^{-1} u]_{-1}^0 = \tan^{-1}(0) - \tan^{-1}(-1) = \frac{\pi}{4} \end{aligned}$$

e) Left as an exercise.

Ans: $\tan^{-1}(1/\sqrt{2})$

f) Left as an exercise

Ans: $\frac{1}{4} \arctan \frac{e^{2x}}{2} + c$

Rewriting as the Sum of Two Quotients

Evaluate

a) $\int \frac{x+2}{\sqrt{4-x^2}} dx$

b) $\int \frac{x+1}{x^2+1} dx$

Solution

$$\begin{aligned} \text{a) } \int \frac{x+2}{\sqrt{4-x^2}} dx &= \int \frac{x}{\sqrt{4-x^2}} dx + \int \frac{2}{\sqrt{4-x^2}} dx \\ &= -\frac{1}{2} \int (-2x) \cdot (4-x^2)^{-1/2} dx + 2 \int \frac{1}{\sqrt{(4)^2-x^2}} dx \\ &= -\frac{1}{2} \left[\frac{(4-x^2)^{-1/2}}{1/2} \right] + 2 \arcsin\left(\frac{x}{2}\right) + c \\ &= -\sqrt{4-x^2} + 2 \arcsin\left(\frac{x}{2}\right) + c \\ \text{b) } \int \frac{x+1}{x^2+1} dx &= \int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \ln(x^2+1) + \arctan x + c \end{aligned}$$

NOTE

The above integrals when the integrand is split in to two, one yields an integral to be evaluated by the Power Rule and the second part yields an Inverse Trigonometric Function

Integrating an Improper Rational Function

Evaluate

a) $\int \frac{3x^3-2}{x^2+4} dx$

b) $\int \frac{x^3}{x^2+1} dx$

Solution

- a) The integrand is an improper rational function because the degree of the numerator is greater than the degree of the denominator.

Use long division to rewrite the integrand.

$$\begin{array}{r} 3x \\ x^2 + 4 \overline{) 3x^3} \\ \underline{3x^3 + 12x} \\ -12x - 2 \end{array}$$

$$\begin{aligned} \Rightarrow \int \frac{3x^3-2}{x^2+4} dx &= \int \left(3x - \frac{12x+2}{x^2+4} \right) dx \\ &= \int 3x dx - 6 \int \frac{2x}{x^2+4} dx - 2 \int \frac{dx}{x^2+4} \\ &= 3x^2 - 6 \ln(x^2+4) - \arctan \frac{x}{2} + c \end{aligned}$$

- b) Left as an exercise.

Ans: $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + c$

Completing squares

Evaluate

a) $\int \frac{dx}{x^2-4x+7}$

b) $\int \frac{dx}{x^2+x+1}$

Solution

a) $x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7$

$$\begin{aligned}
 &= (x-2)^2 + 3 \equiv u^2 + a^2 \\
 \int \frac{dx}{x^2-4x+7} &= \int \frac{dx}{(x-2)^2 + (\sqrt{3})^2} \\
 &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x-2}{\sqrt{3}} \right) + c \\
 \text{b) } x^2 + x + 1 &\equiv \left(x^2 + x + \frac{1}{4} \right) - \frac{1}{4} + 1 \\
 &= \left(x + \frac{1}{2} \right)^2 + \frac{3}{4} \equiv u^2 + a^2 \\
 \therefore \int \frac{dx}{x^2+x+1} &= \frac{1}{\sqrt{3}/2} \arctan \left(\frac{x+\frac{1}{2}}{\sqrt{3}/2} \right) + c \\
 &= \frac{2}{\sqrt{3}} \arctan \left(\frac{2x+1}{\sqrt{3}} \right) + c
 \end{aligned}$$

NOTE

If the leading coefficient is not 1, it helps to factor before completing squares. *e.g*

$$\begin{aligned}
 ax^2 + bx + c &= a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) \\
 &= a \left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \right) \\
 &= a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]
 \end{aligned}$$

Completing the squares (Leading coefficient not 1)

Evaluate

a) $\int \frac{dx}{2x^2-8x+10}$

b) $\int \frac{dx}{2x^2-3x+2}$

Solution

$$\begin{aligned}
 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\
 &= 2(x^2 - 4x + 4 - 4 + 5) \\
 &= 2[(x-2)^2 + 1] \\
 \int \frac{dx}{2x^2-8x+10} &= \frac{1}{2} \int \frac{dx}{(x-2)^2+1} \\
 &= \frac{1}{2} \arctan(x-2) + c
 \end{aligned}$$

Completing the squares (Negative Leading Coefficient)

Evaluate

a) $\int \frac{dx}{\sqrt{3x-x^2}}$

c) $\int \frac{dx}{\sqrt{2+3x-2x^2}}$

b) $\int \frac{dx}{\sqrt{-x^2-4x}}$

d) $\int \frac{3x-1}{\sqrt{1-x-2x^2}}$

Solution

a) $3x - x^2 = -(x^2 - 3x)$

$$\begin{aligned}
 &= - \left[x^2 - 3x + \left(-\frac{3}{2} \right)^2 - \left(-\frac{3}{2} \right)^2 \right] \\
 &= - \left[\left(x - \frac{3}{2} \right)^2 - \left(-\frac{3}{2} \right)^2 \right] \\
 &= \left(\frac{3}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2
 \end{aligned}$$

$$\begin{aligned}\Rightarrow \int \frac{dx}{\sqrt{3x-x^2}} &= \int \frac{dx}{\sqrt{(3/2)^2 - (x-3/2)^2}} \\ &= \arcsin \frac{x-3/2}{3/2} + c \\ &= \arcsin \frac{2x-3}{3} + c\end{aligned}$$

b) Left as an exercise **Ans:** $\arcsin \frac{x+2}{2} + c$

$$\begin{aligned}\text{c) } 2 + 3x - 2x^2 &= -2 \left(x^2 - \frac{3}{2}x - 1 \right) \\ &= -2 \left[x^2 - \frac{3}{2}x + \left(-\frac{3}{4} \right)^2 - \left(-\frac{3}{4} \right)^2 - 1 \right] \\ &= -2 \left[\left(x - \frac{3}{4} \right)^2 - \frac{25}{16} \right] \\ &= -2 \left[\left(x - \frac{3}{4} \right)^2 - \left(\frac{5}{4} \right)^2 \right] \\ &= 2 \left[\left(\frac{5}{4} \right)^2 - \left(x - \frac{3}{4} \right)^2 \right]\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{2+3x-2x^2}} &= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(5/4)^2 - (x-3/4)^2}} \\ &= \frac{1}{\sqrt{2}} \arcsin \left(\frac{x-3/4}{5/4} \right) + c \\ &= \frac{1}{\sqrt{2}} \arcsin \left(\frac{4x-3}{5} \right) + c\end{aligned}$$

Task

a) $\int \frac{1}{(x-1)\sqrt{x^2-2x}} dx$

b) $\int \frac{x+2}{\sqrt{-x^2-4x}} dx$

c) $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx$

Ans: $-\sqrt{x^2-4x} + c$

Ans: $4 - 2\sqrt{3} + \pi/6 \approx 1.059$

More Exploration of the Basic Rules of Integration

a) $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + c$

b) $\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + c$

c) $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + c$

These rules involve the sum or difference of two squares:

$$a^2 - [f(x)]^2, \quad a^2 + [f(x)]^2, \quad [f(x)]^2 - a^2$$

With such an expression, consider the substitution $u = f(x)$

Example

Evaluate

a) $\int \frac{x^2}{\sqrt{16-x^6}} dx$

b) $\int \frac{dx}{x^2+2x+5}$

Solution

a) Because the radical in the denominator can be written in the form $\sqrt{a^2 - u^2} = \sqrt{4^2 - (x^3)^2}$, try the substitution $u = x^3 \Rightarrow du = 3x^2 dx$

$$\int \frac{x^2}{\sqrt{16-x^6}} dx = \frac{1}{3} \int \frac{x^2}{\sqrt{4^2-(x^3)^2}} dx = \frac{1}{3} \int \frac{du}{\sqrt{4^2-(x^3)^2}}$$

$$\begin{aligned}
 &= \frac{1}{3} \arcsin \frac{u}{4} + c \\
 &= \frac{1}{3} \arcsin \frac{x^3}{4} + c \\
 \text{b) } \int \frac{dx}{x^2+2x+5} &= \int \frac{dx}{(x+1)^2+2^2} \\
 &= \frac{1}{2} \arctan \frac{x+1}{2} + c
 \end{aligned}$$

Two Important Forms of Integrals

$$\text{A) } \int \frac{f'(x)}{f(x)} dx = \ln[f(x)] + c$$

$$\text{B) } \int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c ; n \neq 1 \quad (\text{Power Rule})$$

For (A) put $u = f(x) \Rightarrow du = f'(x) dx$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \int \frac{du}{u} = \ln u + c = \ln[f(x)] + c$$

$$\text{Thus } \int \frac{f'(x)}{f(x)} dx = \ln[f(x)] + c$$

Thus the integral of a fraction whose numerator is the derivative of the denominator is equal to the logarithm of the denominator.

For (B) put $u = f(x) \Rightarrow du = f'(x) dx$

$$\therefore \int [f(x)]^n \cdot f'(x) dx = \int u^n du = \frac{u^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c ; n \neq 1$$

$$\text{Thus } \int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c ; n \neq 1$$

Thus if the integrand is the product of a power of a function $f(x)$ and its derivative v , then the integral is obtained by increasing the index by unity and dividing the result by the increased index.

Example

1. Integrate w.r.t x

$$\text{a) } \frac{2x+1}{x^2+x+1}$$

$$\text{b) } \frac{1}{(1+x^2) \tan^{-1} x}$$

$$\text{c) } \frac{x^{e-1} - e^{x-1}}{x^e - e^x}$$

Solution

a) Since the numerator is the derivative of the denominator

$$\int \frac{2x+1}{x^2+x+1} dx = \ln(x^2 + x + 1) + c$$

$$\text{b) } \int \frac{1}{(1+x^2) \tan^{-1} x} dx = \int \frac{\frac{1}{1+x^2}}{\tan^{-1} x} dx = \ln \tan^{-1} x + c$$

$$\text{c) } \int \frac{x^{e-1} - e^{x-1}}{x^e - e^x} dx = \frac{1}{e} \int \frac{e \cdot x^{e-1} - e^x}{x^e - e^x} dx = \frac{1}{e} \ln(x^e - e^x) + c$$

2. Evaluate

$$\text{a) } \int \frac{x+1}{\sqrt{x^2+2x+3}} dx$$

$$\text{c) } \int \frac{1}{x(1+\ln x)^3} dx$$

$$\text{b) } \int \sqrt{\sin x} \cdot \cos x dx$$

$$\text{d) } \int \cot^2 x \operatorname{cosec}^4 x dx$$

Solution

$$\text{a) } \int \frac{x+1}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} \int (x^2 + 2x + 3)^{-1/2} (2x + 2) dx$$

Here $f(x) = x^2 + 2x + 3$, $f'(x) = 2x + 2$, $n = 1/2$

$$\Rightarrow \frac{1}{2} \int (x^2 + 2x + 3)^{-1/2} (2x + 2) dx = \frac{1}{2} \frac{(x^2 + 2x + 3)^{-\frac{1}{2} + 1}}{-\frac{1}{2} + 1} + c$$

$$\therefore \int \frac{x+1}{\sqrt{x^2 + 2x + 3}} dx = \sqrt{x^2 + 2x + 3} + c$$

$$\begin{aligned} \text{b) } \int \sqrt{\sin x} \cdot \cos x dx &= \int (\sin x)^{\frac{1}{2}} \cdot \cos x dx \\ &= \frac{(\sin x)^{\frac{1}{2} + 1}}{\frac{1}{2}} + c = \frac{2}{3} (\sin x)^{\frac{3}{2}} + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int \frac{1}{x(1 + \ln x)^3} dx &= \int (1 + \ln x)^{-3} \cdot \frac{1}{x} dx \\ &= \frac{(1 + \ln x)^{-3 + 1}}{-3 + 1} + c = -\frac{1}{2(1 + \ln x)^2} + c \end{aligned}$$

$$\begin{aligned} \text{d) } \int \cot^2 x \operatorname{cosec}^4 x dx &= \int \cot^2 x \cdot \operatorname{cosec}^2 x \cdot \operatorname{cosec}^2 x dx \\ &= \int \cot^2 x \cdot (1 + \cot^2 x) \cdot \operatorname{cosec}^2 x dx \\ &= \int \cot^2 x \cdot \operatorname{cosec}^2 x + \int \cot^4 x \operatorname{cosec}^2 x dx \\ &= -\int \cot^2 x \cdot \frac{d}{dx} \cot x dx + \int \cot^4 x \cdot \frac{d}{dx} \cot x dx \\ &= -\frac{\cot^3 x}{3} - \frac{\cot^5 x}{5} + c \end{aligned}$$

Disguised Integrals

$$\text{a) } \int \frac{1}{1 + e^x} dx \qquad \text{b) } \int \cot x [\ln \sin x] dx$$

Solution

a) This integral does not appear to fit any of the basic rules. However the quotient form suggests the logarithm rule.

Let $u = 1 + e^x \Rightarrow du = e^x \cdot dx$

To get the required du , add and subtract e^x in the numerator

$$\begin{aligned} \int \frac{1}{1 + e^x} dx &= \int \frac{1 + e^x - e^x}{1 + e^x} dx \\ &= \left(\int \frac{1 + e^x}{1 + e^x} - \frac{e^x}{1 + e^x} \right) dx = \int dx - \int \frac{e^x}{1 + e^x} dx \\ &= x - \ln(1 + e^x) + c \end{aligned}$$

Note

This integral can still be handled by multiplying numerator / denominator by e^{-x}

$$\text{b) } \int \cot x [\ln \sin x] dx, \text{ let } \ln \sin x \Rightarrow du = \frac{\cos x}{\sin x} dx$$

$$\begin{aligned} \int \cot x [\ln \sin x] dx &= \int [\ln \sin x] \cdot \cot x dx \\ &= \int u du = \frac{u^2}{2} + c \end{aligned}$$

$$\therefore \int \cot x [\ln \sin x] dx = \frac{1}{2} [\ln \sin x]^2 + c$$

Procedures for Fitting Integrands to Basic Rules

Technique

Expand (Numerator)

Separate Numerator

Complete the square

Divide improper fraction

Add and subtract terms

In the numerator

Multiply and divide by the

Pythagorean conjugate

Example

$$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$$

$$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$$

$$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$$

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

$$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$$

$$\begin{aligned} \frac{1}{1+\sin x} &= \left(\frac{1}{1+\sin x} \right) \left(\frac{1-\sin x}{1-\sin x} \right) \\ &= \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x} \end{aligned}$$

Further Application of Trigonometric Substitutions

Integrals involving radicals,

$$\sqrt{a^2 - u^2}, \sqrt{a^2 + u^2}, \sqrt{u^2 - a^2}$$

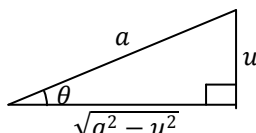
, can be evaluated using *Trigonometric Substitution*.

The objective with trigonometric substitution is to eliminate the radical in the integrand using Pythagorean identities. i.e.

$$\cos^2 \theta = 1 - \sin^2 \theta, \sec^2 \theta = 1 + \tan^2 \theta, \tan^2 \theta = \sec^2 \theta - 1$$

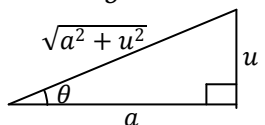
Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let $u = a \sin \theta$



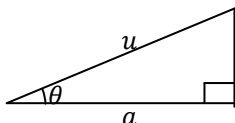
$$\text{Then } \sqrt{a^2 - u^2} = a \cos \theta, \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

2. For integrals involving $\sqrt{a^2 + u^2}$, let $u = a \tan \theta$



$$\text{Then } \sqrt{a^2 + u^2} = a \sec \theta, \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

3. For integrals involving $\sqrt{u^2 - a^2}$, let $u = a \sec \theta$



$$\text{Then } \sqrt{u^2 - a^2} = \pm a \tan \theta, \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

If $u > a$, use the positive value.

Examples

Evaluate

a) $\int \frac{dx}{x^2 \sqrt{9-x^2}}$

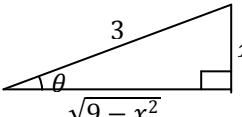
b) $\int \frac{dx}{\sqrt{4x^2+1}}$

Solution

- a) Since $\sqrt{9-x^2}$ is of the form $\sqrt{a^2-u^2}$, use the substitution

$$x = a \sin \theta = 3 \sin \theta$$

$$\Rightarrow dx = 3 \cos \theta d\theta, \sqrt{9-x^2} = 3 \cos \theta \text{ and } x^2 = 9 \sin^2 \theta$$



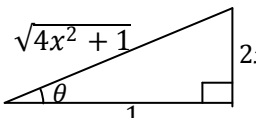
$$\int \frac{dx}{x^2 \sqrt{9-x^2}} = \int \frac{3 \cos \theta d\theta}{(9 \sin^2 \theta)(3 \cos \theta)}$$

$$= \frac{1}{9} \int \frac{d\theta}{\sin^2 \theta} = \frac{1}{9} \int \operatorname{cosec}^2 \theta d\theta$$

$$\begin{aligned} \therefore \int \frac{dx}{x^2 \sqrt{9-x^2}} &= -\frac{1}{9} \cot \theta + c \\ &= -\frac{1}{9} \left(\frac{\sqrt{9-x^2}}{x} \right) + c = -\frac{\sqrt{9-x^2}}{9x} + c \end{aligned}$$

- b) $\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{dx}{\sqrt{(2x)^2+1}}$, let $2x = \tan \theta$

$$\Rightarrow 2dx = \sec^2 \theta d\theta \text{ or } dx = \frac{1}{2} \sec^2 \theta, \text{ and } \sqrt{4x^2+1} = \sec \theta$$



$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2} \int \frac{\sec^2 \theta d\theta}{\sec \theta}$$

$$= \frac{1}{2} \int \sec \theta d\theta$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{4x^2+1}} &= \frac{1}{2} \ln |\sec \theta + \tan \theta| + c \\ &= -\frac{1}{2} \ln |\sqrt{4x^2+1} + 2x| + c \end{aligned}$$

NOTE

Extension of the use of trigonometric substitution to cover integrals involving expressions such as

$$(a^2 - u^2)^{n/2}$$

, by writing the expression

$$(a^2 - u^2)^{n/2} = (\sqrt{a^2 - u^2})^n$$

Example

Evaluate

a) $\int \frac{1}{(25-x^2)^{3/2}} dx$

c) $\int \frac{1}{x^2 \sqrt{25-x^2}} dx$

b) $\int \frac{\sqrt{25-x^2}}{x} dx$

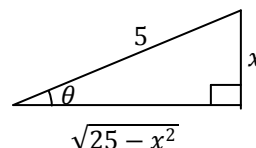
d) $\int \frac{x^2}{\sqrt{25-x^2}} dx$

Solution

a) $\int \frac{1}{(25-x^2)^{3/2}} dx = \int \frac{1}{\sqrt{(25-x^2)^3}} dx = \int \frac{1}{\sqrt{(5^2-x^2)^3}} dx$

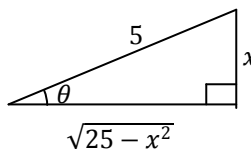
$$x = 5 \sin \theta \Rightarrow dx = 5 \cos \theta d\theta, \sqrt{25-x^2} = 5 \cos \theta$$

$$\begin{aligned} \therefore \int \frac{1}{(25-x^2)^{3/2}} dx &= \int \frac{5 \cos \theta d\theta}{125 \cos^3 \theta} \\ &= \frac{1}{25} \int \sec^2 \theta d\theta = \frac{1}{25} \tan \theta + c \\ &= \frac{1}{25} \cdot \frac{x}{\sqrt{25-x^2}} + c \end{aligned}$$



b) $\int \frac{\sqrt{25-x^2}}{x} dx$, let $x = 5 \sin \theta \Rightarrow dx = 5 \cos \theta d\theta, \sqrt{25-x^2} = 5 \cos \theta$

$$\begin{aligned}
 \int \frac{\sqrt{25-x^2}}{x} dx &= \int \frac{5 \cos \theta \cdot 5 \cos \theta d\theta}{\sin \theta} \\
 &= 5 \int \frac{\cos^2 \theta}{\sin \theta} d\theta = 5 \int \frac{1-\sin^2 \theta}{\sin \theta} d\theta \\
 &= 5 \int (\operatorname{cosec} \theta - \sin \theta) d\theta \\
 &= 5 [\ln(\operatorname{cosec} \theta - \cot \theta) + \cos \theta] + c \\
 &= 5 \left[\ln \left| \frac{5}{x} - \frac{\sqrt{25-x^2}}{x} \right| \right] + \sqrt{25-x^2} + c \\
 &= 5 \left[\ln \left| \frac{5-\sqrt{25-x^2}}{x} \right| \right] + \sqrt{25-x^2} + c
 \end{aligned}$$



- c) Left as an exercise
d) Left as an exercise

Trigonometric Substitution: Rational Powers

Evaluate

a) $\int \frac{dx}{(x^2+1)^{3/2}}$

c) $\int \frac{dx}{(x^2+3)^{3/2}}$

b) $\int \frac{dx}{(x^2+1)^2}$

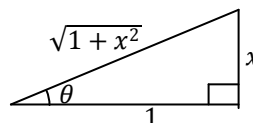
d) $\int \frac{dx}{(x^2-4)^{3/2}}$

Solution

a) $\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{dx}{(\sqrt{x^2+1})^3}$

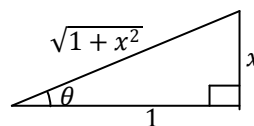
Let $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$, $\sqrt{1+x^2} = \sec \theta$

$$\begin{aligned}
 \Rightarrow \int \frac{dx}{(x^2+1)^{3/2}} &= \int \frac{dx}{(\sqrt{x^2+1})^3} = \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta} \\
 &= \int \frac{d\theta}{\sec \theta} = \int \cos \theta d\theta \\
 &= \sin \theta + c = \frac{x}{\sqrt{1+x^2}} + c
 \end{aligned}$$



b) $\int \frac{dx}{(x^2+1)^2}$, let $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$, $\sqrt{1+x^2} = \sec \theta$, $\sec^2 \theta = 1+x^2$

$$\begin{aligned}
 \therefore \int \frac{dx}{(x^2+1)^2} &= \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} = \int \cos^2 \theta d\theta \\
 &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta = \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right] + c \\
 &= \frac{1}{2} [\theta + \sin \theta \cos \theta] + c \\
 &= \frac{1}{2} \left[\theta + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \right] + c = \frac{1}{2} \left[\arctan x + \frac{x}{1+x^2} \right] + c
 \end{aligned}$$



- c) Left as an exercise
d) Left as an exercise

Ans: $-\frac{x}{4\sqrt{x^2-4}}$

Completing Squares

Evaluate

a) $\int \frac{dx}{(x^2-4x)^{3/2}}$, $x > 4$

b) $\int \frac{x^2}{\sqrt{2x-x^2}} dx$

c) $\int \frac{x^2}{\sqrt{x^2+10x+9}} dx$

Solution

a) Completing squares

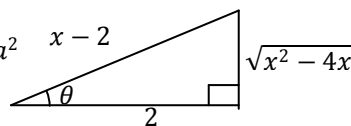
$$x^2 - 4x = x^2 - 4x + 4 - 4 = (x - 2)^2 - 2^2 \equiv u^2 - a^2 \quad x - 2$$

Let $a = 2$ and $u = x - 2 = 2 \sec \theta$

$$\Rightarrow dx = \sec \theta \tan \theta d\theta$$

$$, \text{ and } \sqrt{(x - 2)^2 - 2^2} = 2 \tan \theta$$

$$\begin{aligned} \therefore \int \frac{dx}{(x^2 - 4x)^{3/2}} &= \int \frac{\sec \theta \tan \theta d\theta}{(2 \tan \theta)^3} \\ &= \frac{2}{8} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{4} \int (\sin \theta)^{-2} (\cos \theta) d\theta \\ &= -\frac{1}{4} \operatorname{cosec} \theta + c = -\frac{x-2}{4\sqrt{x^2-4x}} + c \end{aligned}$$



b) Completing squares

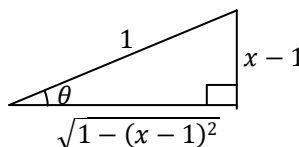
$$2x - x^2 = -(x^2 - 2x + 1 - 1) = 1 - (x - 1)^2$$

$$\int \frac{x^2}{\sqrt{2x - x^2}} dx = \int \frac{x^2}{\sqrt{1 - (x-1)^2}} dx$$

Let $x - 1 = \sin \theta \Rightarrow dx = \cos \theta d\theta$

, and $x^2 = (1 + \sin \theta)^2$

$$\begin{aligned} \therefore \int \frac{x^2}{\sqrt{2x - x^2}} dx &= \int \frac{(1 + \sin \theta)^2}{\cos \theta} \cdot \cos \theta d\theta \\ &= \int (1 + 2 \sin \theta + \sin^2 \theta) d\theta \\ &= \theta - 2 \cos \theta + \frac{1}{2} \int (1 - \cos 2\theta) d\theta \\ &= \theta - 2 \cos \theta + \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta + c \\ &= \frac{3}{2} \theta - 2 \cos \theta - \frac{1}{2} \sin \theta \cos \theta + c \\ &= \frac{3}{2} \sin^{-1}(x - 1) - 2\sqrt{2x - x^2} - \frac{1}{2}(x - 1)\sqrt{2x - x^2} + c \\ &= \frac{3}{2} \sin^{-1}(x - 1) - 2\sqrt{2x - x^2} \cdot \left(1 + \frac{1}{4}(x - 1)\right) + c \\ &= \frac{3}{2} \arcsin(x - 1) - \frac{1}{2}(x + 3)\sqrt{2x - x^2} + c \end{aligned}$$



Integrals of Trigonometric Functions

Integrals of $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$

These are handled using the logarithmic rule

a) $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$

Knowing that $\frac{d}{dx} \cos x = -\sin x$, let $u = \cos x$

$$\begin{aligned} \Rightarrow \int \tan x dx &= -\int \frac{-\sin x}{\cos x} dx \equiv -\int \frac{u'}{u} dx \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + c = \ln|\sec x| + c \end{aligned}$$

$$\therefore \int \tan x dx = \ln|\sec x| + c$$

b) $\int \cot x dx = \int \frac{\cos x}{\sin x} dx$, let $u = \sin x \Rightarrow du = \cos x dx$

$$= \int \frac{du}{u} = \ln|u| + c$$

$$\therefore \int \cot x \, dx = \ln|\sin x| + c$$

c) $\int \sec x \, dx$

Rewriting $\sec x$ as:

$$\sec x = \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$$

$$\int \sec x \, dx = \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \right) dx$$

$$, \text{ let } u = \sec x + \tan x \Rightarrow du = (\sec^2 x + \sec x \tan x) dx$$

$$\Rightarrow \int \sec x \, dx = \int \frac{du}{u} = \ln|u| + c$$

$$\therefore \int \sec x \, dx = \ln|\sec x + \tan x| + c$$

d) $\int \operatorname{cosec} x \, dx$

Rewriting $\operatorname{cosec} x$ as:

$$\operatorname{cosec} x = \operatorname{cosec} x \cdot \left(\frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x - \cot x} \right) = \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x}$$

$$\int \operatorname{cosec} x \, dx = \int \left(\frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x} \right) dx$$

$$, \text{ let } u = \operatorname{cosec} x - \cot x \Rightarrow du = (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) dx$$

$$\Rightarrow \int \operatorname{cosec} x \, dx = \int \frac{du}{u} = \ln|u| + c$$

$$\therefore \int \operatorname{cosec} x \, dx = \ln|\operatorname{cosec} x - \cot x| + c$$

NOTE

$$\begin{aligned} \operatorname{cosec} x - \cot x &= \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1 - \cos x}{\sin x} \\ &= \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \tan \frac{x}{2} \end{aligned}$$

$$\text{Also } \int \operatorname{cosec} x \, dx = \ln \left| \tan \frac{x}{2} \right| + c$$

Integration of Positive Even Integral powers of $\sec x$ or $\operatorname{cosec} x$

Rule to integrate $\int \sec^n x \, dx$ or $\int \operatorname{cosec}^n x \, dx$, where n is a positive even integer

- i) If the index of $\sec x$ is a positive even integer, put the substitution

$$u = \tan x$$

- ii) If the index of $\operatorname{cosec} x$ is a positive even integer, put the substitution

$$u = \cot x$$

NOTE

When n an odd positive integer, Reduction formula is used.

Example

i) $\int \sec^4 2x \, dx$

ii) $\int \operatorname{cosec}^4 2x \, dx$

Solution

i) Let $u = \tan 2x \Rightarrow du = 2 \sec^2 2x \, dx$ or $dx = \frac{du}{2 \sec^2 2x}$

$$\begin{aligned} \int \sec^4 2x \, dx &= \int \sec^2 2x \cdot \frac{du}{2 \sec^2 2x} \\ &= \frac{1}{2} \int \sec^2 2x \, du = \frac{1}{2} \int (1 + \tan^2 2x) \, du \end{aligned}$$

$$= \frac{1}{2} \int (1 + u^2) du = \frac{1}{2} \left[u + \frac{1}{3} u^3 \right] + c$$

$$\therefore \int \sec^4 2x dx = \frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + c$$

ii) Let $u = \cot 2x \Rightarrow du = -2 \operatorname{cosec}^2 2x dx$ or $dx = -\frac{du}{2 \operatorname{cosec}^2 2x}$

$$\begin{aligned} \int \operatorname{cosec}^4 2x dx &= \int \operatorname{cosec}^2 2x \cdot \frac{du}{-2 \operatorname{cosec}^2 2x} \\ &= -\frac{1}{2} \int \operatorname{cosec}^2 2x du = -\frac{1}{2} \int (1 + \cot^2 2x) du \\ &= -\frac{1}{2} \int (1 + u^2) du = -\frac{1}{2} \left[u + \frac{1}{3} u^3 \right] + c \end{aligned}$$

$$\therefore \int \sec^4 2x dx = -\frac{1}{2} \cot 2x - \frac{1}{6} \cot^3 2x + c$$

Integration of positive Even Integral Powers of $\cot x$ and $\tan x$

Rule to integrate $\int \cot^n x dx$ or $\int \tan^n x dx$, where n is a positive even integer

i) If the index of $\cot x$ is a positive even integer, put the substitution

$$\boxed{u = \cot x}, \text{ then apply } 1 + \cot^2 x = \operatorname{cosec}^2 x$$

ii) If the index of $\tan x$ is a positive even integer, put the substitution

$$\boxed{u = \tan x}, \text{ then apply } 1 + \tan^2 x = \sec^2 x$$

Example

i) $\int \tan^4 x dx$

ii) $\int \cot^4 x dx$

Solution

i) Let $u = \tan x \Rightarrow u = \tan^{-1} x$

$$\Rightarrow du = \sec^2 x dx \text{ or } dx = \frac{du}{\sec^2 x} = \frac{du}{1 + \tan^2 x} = \frac{du}{1 + u^2}$$

$$\int \tan^4 x dx = \int u^4 \cdot \frac{du}{1 + u^2}$$

By long division

$$\begin{aligned} u^2 + 1 \overline{) \begin{array}{r} u^4 - 1 \\ u^4 + u^2 \\ \hline -u^2 - 1 \\ -u^2 - 1 \\ \hline 1 \end{array}} & \quad \therefore \int \tan^4 x dx = \int \frac{u^4 du}{1 + u^2} = \int \left(u^2 - 1 + \frac{1}{u^2 + 1} \right) du \\ &= \frac{1}{3} u^3 - u + \tan^{-1} u + c \\ &= \frac{1}{3} \tan^3 x - \tan x + x + c \end{aligned}$$

ii) Let $u = \cot x \Rightarrow u = \cot^{-1} x$

$$\Rightarrow du = -\operatorname{cosec}^2 x dx \text{ or } dx = -\frac{du}{\operatorname{cosec}^2 x} = -\frac{du}{1 + \cot^2 x} = -\frac{du}{1 + u^2}$$

$$\int \cot^4 x dx = -\int u^4 \cdot \frac{du}{1 + u^2}$$

By long division

$$\begin{array}{r}
 u^2 - 1 \\
 u^2 + 1 \overline{) u^4} \\
 \underline{u^4 + u^2} \\
 -u^2 \\
 \underline{-u^2 - 1} \\
 1
 \end{array}
 \quad \therefore \int \cot^4 x \, dx = - \int \frac{u^4 du}{1+u^2} = - \int \left(u^2 - 1 + \frac{1}{u^2+1} \right) du$$

$$\begin{aligned}
 &= - \left[\frac{1}{3} u^3 - u + \cot^{-1} u \right] + c \\
 &= - \frac{1}{3} \cot^3 x + \cot x - x + c
 \end{aligned}$$

NOTE

If the index in integrals $\int \cot^n x \, dx$ or $\int \tan^n x \, dx$ is *odd* use the substitutions

a) $\int \cot^n x \, dx$, $u = \operatorname{cosec} x$

b) $\int \tan^n x \, dx$, $u = \sec x$

Example

a) $\int \cot^3 2x \, dx$

b) $\int_0^{\pi/4} \tan^3 x \, dx$

Solution

a) Let $u = \operatorname{cosec} x \Rightarrow du = -2 \operatorname{cosec} 2x \cot 2x \, dx$ or $dx = -\frac{du}{2 \operatorname{cosec} 2x \cot 2x}$

$$\begin{aligned}
 \int \cot^3 2x \, dx &= \int -\frac{\cot^3 2x du}{2 \operatorname{cosec} 2x \cot 2x} \\
 &= -\frac{1}{2} \int \frac{\cot^2 2x}{\operatorname{cosec} 2x} \cdot du = -\frac{1}{2} \int \frac{\operatorname{cosec}^2 2x - 1}{\operatorname{cosec} 2x} \cdot du \\
 &= -\frac{1}{2} \int \frac{u^2 - 1}{u} du = -\frac{1}{2} \int \left(u - \frac{1}{u} \right) du \\
 &= -\frac{1}{2} \left[\frac{u^2}{2} - \ln u \right] + c \\
 &= \frac{1}{2} \left[\ln u - \frac{u^2}{2} \right] + c = \frac{1}{4} [2 \ln u - u^2] + c \\
 &= \frac{1}{4} [\ln |\operatorname{cosec}^2 x| - \operatorname{cosec}^2 x] + c \\
 &= \frac{1}{4} [\ln |u|^2 - u^2] + c
 \end{aligned}$$

b) Let $u = \sec x \Rightarrow du = \sec x \tan x \, dx$ or $dx = \frac{du}{\sec x \tan x}$

Changing limits

When $x = 0$, $u = 1$ and when $x = \frac{\pi}{4}$, $u = \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

$$\begin{aligned}
 \int_0^{\pi/4} \tan^3 x \, dx &= \int_1^{\sqrt{2}} \tan^3 x \cdot \frac{du}{\sec x \tan x} \\
 &= \int_1^{\sqrt{2}} \tan^2 x \cdot \frac{du}{\sec x} = \int_1^{\sqrt{2}} (\sec^2 x - 1) \cdot \frac{du}{\sec x} \\
 &= \int_1^{\sqrt{2}} (u^2 - 1) \cdot \frac{du}{u} = \int_1^{\sqrt{2}} \left(u - \frac{1}{u} \right) du \\
 &= \left[\frac{u^2}{2} - \ln |u| \right]_1^{\sqrt{2}} = \left(\frac{(\sqrt{2})^2}{2} - \ln \sqrt{2} \right) - \left(\frac{1^2}{2} - \ln 1 \right) \\
 &= 1 - \frac{1}{2} \ln 2 - \frac{1}{2} = \frac{1}{2} (1 - \ln 2)
 \end{aligned}$$

Integrals of the form $\int \cos^n x \, dx$ and $\int \sin^n x \, dx$

Case I: n is positive and odd

- a) $\int \cos^n x \, dx$, use the substitution $u = \sin x$
 b) $\int \sin^n x \, dx$, use the substitution $u = \cos x$
 , then apply the identity $\cos^2 x + \sin^2 x = 1$

Example

a) $\int_0^{\pi/2} \cos^3 x \, dx$

b) $\int \sin^5 2x \, dx$

Solution

- a) Let $u = \sin x \Rightarrow du = \cos x \, dx$

$$\begin{aligned} \int_0^{\pi/2} \cos^3 x \, dx &= \int_0^{\pi/2} \cos^2 x \cdot \cos x \, dx \\ &= \int_0^{\pi/2} (1 - \sin^2 x) \cdot \cos x \, dx \\ &= \int_0^1 (1 - u^2) \cdot du \\ &= \left[u - \frac{u^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

Changing limits

When $x = 0$, $u = \sin 0 = 0$

When $x = \frac{\pi}{2}$, $u = \sin \frac{\pi}{2} = 1$

- b) Let $u = \cos 2x \Rightarrow du = -2 \sin 2x \, dx$

$$\begin{aligned} \int \sin^5 2x \, dx &= -\frac{1}{2} \int \sin^4 2x \cdot (-2 \sin 2x \, dx) \\ &= -\frac{1}{2} \int (\sin^2 2x)^2 \cdot du = -\frac{1}{2} \int (1 - \cos^2 2x)^2 \cdot du \\ &= -\frac{1}{2} \int (1 - u^2)^2 \cdot du = -\frac{1}{2} \int (1 - 2u^2 + u^4) \, du \\ &= \frac{1}{2} \left[u - \frac{2}{3} u^3 + \frac{u^5}{5} \right] + c = \frac{1}{30} [15u - 20u^3 + 3u^5] + c \\ &= \frac{1}{30} [3 \cos^5 2x - 20 \cos^3 2x + 15 \cos 2x] + c \end{aligned}$$

Case II: n is positive and even

- a) $\int \cos^n x \, dx$, use the substitution $\cos^2 x = \frac{1 + \cos 2x}{2}$
 b) $\int \sin^n x \, dx$, use the substitution $\sin^2 x = \frac{1 - \cos 2x}{2}$

NOTE: The above identities are for double angles

Example

a) $\int \cos^4 x \, dx$

b) $\int_0^{\pi/2} \sin^4 x \, dx$

Solution

$$\begin{aligned} \text{a) } \cos^4 x &= [\cos^2 x]^2 = \left[\frac{1 + \cos 2x}{2} \right]^2 \\ &= \frac{1}{4} (1 + 2 \cos 2x + \cos^2 2x) \\ &= \frac{1}{4} \left(1 + 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) = \frac{1}{8} (\cos 4x + 4 \cos 2x + 3) \\ \therefore \int \cos^4 x \, dx &= \frac{1}{8} \int (\cos 4x + 4 \cos 2x + 3) \, dx \\ &= \frac{1}{8} \left[\frac{1}{4} \sin 4x + 4 \cdot \frac{1}{2} \sin 2x + 3x \right] + c \end{aligned}$$

$$= \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

$$\begin{aligned} \text{b) } \sin^4 x &= [\sin^2 x]^2 = \left[\frac{1 - \cos 2x}{2} \right]^2 \\ &= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \\ &= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right) = \frac{1}{8}(2 - 4\cos 2x + 1 + \cos 4x) \\ &= \frac{1}{8}(3 - 4\cos 2x + \cos 4x) \end{aligned}$$

$$\begin{aligned} \therefore \int_0^{\pi/2} \sin^4 x \, dx &= \frac{1}{8} \int_0^{\pi/2} (3 - 4\cos 2x + \cos 4x) \, dx \\ &= \frac{1}{8} \left[3x - 2\sin 2x + \frac{1}{4}\sin 4x \right]_0^{\pi/2} \\ &= \frac{1}{8} \left[\frac{3\pi}{2} - 2\sin \pi + \frac{1}{4}\sin 2\pi \right] - 0 = \frac{3\pi}{16} \end{aligned}$$

Integrals involving Sine-Cosine Products with same/different angle

These integrals are evaluated using the product-to-sum identities (Factor Formula).

$$\begin{aligned} \sin mx \sin nx &= \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] \\ \sin mx \cos nx &= \frac{1}{2} [\sin(m-n)x + \sin(m+n)x] \\ \cos mx \cos nx &= \frac{1}{2} [\cos(m-n)x + \cos(m+n)x] \end{aligned}$$

Example

$$\text{a) } \int \sin 5x \cos 4x \, dx$$

$$\text{b) } \int_0^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta$$

Solution

$$\begin{aligned} \text{a) } \int \sin 5x \cos 4x \, dx &= \frac{1}{2} \int 2 \sin 5x \cos 4x \, dx \\ &= \frac{1}{2} \int (\sin 9x + \sin x) \, dx \\ &= \frac{1}{2} \left[-\frac{\cos 9x}{9} - \cos x \right] + c = -\frac{\cos 9x}{18} - \frac{\cos x}{2} + c \\ \text{b) } \int_0^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta &= \frac{1}{2} \int_0^{\pi/4} 2 \sin 2\theta \sin 3\theta \, d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} (\cos(-\theta) - \cos 5\theta) \, d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} (\cos \theta - \cos 5\theta) \, d\theta \\ &= \frac{1}{2} \left[\sin \theta - \frac{\sin 5\theta}{5} \right]_0^{\pi/4} = \frac{1}{2} \left[\sin \frac{\pi}{4} - \frac{\sin 5\pi/4}{5} \right] - 0 \\ &= \frac{1}{2} \left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{10} \right) \right] = \frac{1}{2} \left(\frac{5\sqrt{2} + \sqrt{2}}{10} \right) \\ &= \frac{3\sqrt{2}}{10} \end{aligned}$$

Integration of $\sin^m x \cos^n x$ (at least one index is a positive integer)

The integral $\int \sin^m x \cos^n x dx$ can easily be evaluated when either m or n is a positive *odd integer*, for all values of the other index (including zero) Different cases which arise are:

- i) $\int \sin^n x dx$ and $\int \cos^n x dx$ where n is a positive **odd** integer.
- ii) $\int \sin^m x \cos^n x dx$ where either m or n or both are positive *odd* integers

The rule to integrate these is as follows.

- a) If the index of **sin** x is **odd**, put $u = \cos x$
- b) If the index of **cos** x is **odd**, put $u = \sin x$

Example

- a) $\int \sin^5 x \cos^4 x dx$
- b) $\int \sin^2 x \cos^3 x dx$
- c) $\int \sin^5 x \cos^3 x dx$

Solution

- a) The index of $\sin x$ being odd, let $t = \cos x \Rightarrow dt = -\sin x dx$

$$\begin{aligned} \therefore \int \sin^5 x \cos^4 x dx &= \int \sin^4 x \cos^4 x \cdot \sin x dx \\ &= \int \cos^4 x \cdot (1 - \cos^2 x)^2 \cdot \sin x dx \\ &= -\int t^4 \cdot (1 - t^2)^2 \cdot dt = -\int (t^4 - 2t^6 + t^8) \cdot dt \\ &= -\frac{t^5}{5} + \frac{2t^7}{7} - \frac{t^9}{9} + c \\ &= \frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + c \end{aligned}$$

- b) The index of $\cos x$ being odd, let $t = \sin x \Rightarrow dt = \cos x dx$

$$\begin{aligned} \therefore \int \sin^2 x \cos^3 x dx &= \int \sin^2 x \cos^2 x \cdot \cos x dx \\ &= \int \sin^2 x (1 - \sin^2 x) \cdot \cos x dx \\ &= \int t^2 (1 - t^2) \cdot dt = \int (t^2 - t^4) dt \\ &= \frac{t^3}{3} - \frac{t^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c \end{aligned}$$

- c) Since both the indices are odd on sine and cosine we put either $u = \sin x$ or $u = \cos x$

In practice it is convenient to take the smaller index as odd and use the corresponding substitution.

Since both indices are odd but the index of $\cos x$ is smaller, we put

$$u = \sin x \Rightarrow du = \cos x dx$$

$$\begin{aligned} \therefore \int \sin^5 x \cos^3 x dx &= \int \sin^5 x \cos^2 x \cdot \cos x dx \\ &= \int \sin^5 x (1 - \sin^2 x) \cdot \cos x dx \\ &= \int u^5 (1 - u^2) \cdot du = \frac{u^6}{6} - \frac{u^8}{8} + c \\ &= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c \end{aligned}$$

NOTE

If the integral $\int \sin^m x \cos^n x dx$ where m and n are even, express each of the functions using double angle identities as:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Evaluate $\int \sin^2 x \cos^4 x \, dx$

Solution

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^4 x = \frac{(1 + \cos 2x)^2}{4} = \frac{1 + 2 \cos 2x + \cos^2 2x}{4} = \frac{1 + 2 \cos 2x + \frac{1 + \cos 4x}{2}}{4} = \frac{3 + 4 \cos 2x + \cos 4x}{8}$$

$$\begin{aligned} \text{Now } \sin^2 x \cos^4 x &= \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{3 + 4 \cos 2x + \cos 4x}{8} \right) \\ &= \frac{3 + 4 \cos 2x + \cos 4x - 3 \cos 2x - 4 \cos^2 2x - \cos 4x \cos 2x}{16} \\ &= \frac{3 + \cos 2x + \cos 4x - 4 \left(\frac{1 + \cos 2x}{2} \right) - \cos 4x \cos 2x}{16} \\ &= \frac{1 + \cos 2x + \cos 4x - \cos 4x \cos 2x}{16} \end{aligned}$$

$$\begin{aligned} \therefore \int \sin^2 x \cos^4 x \, dx &= \frac{1}{16} \int (1 + \cos 2x + \cos 4x - \cos 4x \cos 2x) \, dx \\ &= \frac{1}{16} \int \left(1 + \cos 2x + \cos 4x - \frac{1}{2} (\cos 6x + \cos 2x) \right) \, dx \\ &= \frac{1}{16} \left[x + \frac{\sin 2x}{2} - \frac{\sin 4x}{4} - \frac{\sin 6x}{12} - \frac{\sin 2x}{4} \right] + c \end{aligned}$$

Integrals Involving Powers of Secant and Tangent

Integrals of the form $\int \sec^m x \tan^n x \, dx$

Case I: Power of the secant is even. i.e. $m = 2k$

Rewrite the integrand as:

$$\begin{aligned} \int \sec^{2k} x \tan^n x \, dx &= \int (\sec^2 x)^{k-1} \cdot \sec^2 x \cdot \tan^n x \, dx \\ &= \int (1 + \tan^2 x)^{k-1} \cdot \sec^2 x \cdot \tan^n x \, dx \end{aligned}$$

Save a secant squared factor
Rest of the terms converted to tangents

Now let $u = \tan x$

Case II: power of the tangent is odd and positive. i.e. $n = 2k + 1$

Rewrite the integrand as:

$$\begin{aligned} \int \sec^m x \tan^{2k+1} x \, dx &= \int \sec^{m-1} x \cdot \sec x \cdot (\tan^2 x)^k \cdot \tan x \, dx \\ &= \int \sec^{m-1} x \cdot (\tan^2 x)^k \cdot (\sec x \tan x) \, dx \\ &= \int \sec^{m-1} x \cdot (\sec^2 x - 1)^k \cdot (\sec x \tan x) \, dx \end{aligned}$$

Save a secant squared factor
Rest of the terms converted to secants

Now let $u = \sec x$

Example

a) $\int \sec^4 3x \tan^3 3x \, dx$

b) $\int \sec^3 x \tan x \, dx$

Solution

a) Since the index on secant is even, let $u = \tan 3x \Rightarrow du = 3 \sec^2 3x \, dx$

$$\begin{aligned} \therefore \int \sec^4 3x \tan^3 3x \, dx &= \frac{1}{3} \int \sec^2 3x \cdot \tan^3 3x \cdot (3 \sec^2 3x \, dx) \\ &= \frac{1}{3} \int (1 + \tan^2 3x) \cdot \tan^3 3x \cdot (3 \sec^2 3x \, dx) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{3} \int (\tan^3 3x + \tan^5 3x) \cdot (3 \sec^2 3x \, dx) \\
 &= \frac{1}{3} \int (u^3 + u^5) \, du = \frac{1}{3} \left[\frac{u^4}{4} + \frac{u^6}{6} \right] \\
 &= \frac{\tan^4 x}{12} + \frac{\tan^6 x}{18} + c
 \end{aligned}$$

- b) Since the index on tangent is odd, let $u = \sec x \Rightarrow du = \sec x \tan x \, dx$
 $\therefore \int \sec^3 x \tan x \, dx = \int \sec^2 x \cdot (\sec x \tan x) \, dx$
 $= \int u^2 \, du = \frac{u^3}{3} + c = \frac{\sec^3 x}{3} + c$

INTEGRATION OF ALGEBRAIC RATIONAL FRACTIONS

General Method for Integrating a Rational Fraction $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$

- 1) If the degree of $f(x)$ be equal to or greater than that of $g(x)$, divide $f(x)$ until the degree of remainder is less than that of $g(x)$.i.e.

$$f(x) = Q(x) + \frac{R(x)}{g(x)}$$

, where $Q(x)$ is the quotient of the polynomial, $R(x)$ is the remainder whose degree is less than that of $g(x)$.

The part $Q(x)$ can be integrated easily by the formula $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

Now how do we integrate $\frac{R(x)}{g(x)}$ where the degree of $R(x)$ is smaller than that of $g(x)$?

Cases to consider

- a) By algebra, if $g(x)$ can be resolved in to factors of first and second degree. i.e.

$$g(x) = A(a_1x + b_1)^{p_1}(a_1x + b_1)^{p_2} \dots \dots \dots \text{or}$$

$$g(x) = (A_1x + B_1x + C_1)^{q_1}(A_2x + B_2x + C_2)^{q_2} \dots \dots \dots$$

, where $p_1, p_2, \dots$ and $q_1, q_2, \dots$ are positive integers denoting the number of times each factor is repeated.

Since some/all of the integers $p_1, p_2, \dots, q_1, q_2, \dots$ may be repeated, then $g(x)$ has four major factors;

- i) Linear Non Repeated
- ii) Linear Repeated
- iii) Quadratic Non Repeated
- iv) Quadratic Repeated

Hence the fraction $\frac{R(x)}{g(x)}$ may have partial fractions of the form, where n is positive

$$\text{i)} \quad \frac{1}{ax+b}$$

$$\text{ii)} \quad \frac{1}{(ax+b)^n}$$

$$\text{iii)} \quad \frac{lx+m}{Ax^2+Bx+C}$$

$$\text{iv)} \quad \frac{lx+m}{(Ax^2+Bx+C)^2}$$

, respectively.

Integration of (i) and (ii) is quite easy

i) $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + c$

ii) $\int \frac{1}{(ax+b)^n} dx = \int (ax+b)^{-n} dx = \frac{1}{a(-n+1)} \cdot \frac{1}{(ax+b)^{n-1}} + c$

Now let us discuss the integration of (iii) and (iv).i.e.

Integration of $\int \frac{lx+m}{Ax^2+Bx+C} dx$, where $Ax^2 + Bx + C$ cannot be resolved in to real linear factors. i.e. $Ax^2 + Bx + C$ is non factorisable.

Two cases arise

Case I: $l = 0$. i.e. $\int \frac{m}{Ax^2+Bx+C} dx$

First express $Ax^2 + Bx + C$ in terms of completing squares. i.e.

$$\begin{aligned} Ax^2 + Bx + C &= A \left(x^2 + \frac{B}{A}x + \frac{C}{A} \right) \\ &= A \left[x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} + \left(\frac{C}{A} - \frac{B^2}{4A^2} \right) \right] \\ &= A \left[\left(x + \frac{B}{2A} \right)^2 + \left(\frac{\sqrt{4AC-B^2}}{2A} \right)^2 \right] \\ &= A[(x+h)^2 + k^2], \text{ where } h = \frac{B}{2A} \text{ and } k = \frac{\sqrt{4AC-B^2}}{2A} \end{aligned}$$

$$\therefore \int \frac{m}{Ax^2+Bx+C} dx = \frac{m}{A} \int \frac{dx}{(x+h)^2+k^2} = \frac{m}{Ak} \tan^{-1} \left(\frac{x+h}{h} \right) + c_1$$

Case II: $l \neq 0$. i.e. $\int \frac{lx+m}{Ax^2+Bx+C} dx$

Here split the numerator

$$\begin{aligned} \text{Let } lx + m &\equiv \lambda \frac{d}{dx} (Ax^2 + Bx + C) + \mu \\ &\equiv \lambda(2Ax + B) + \mu \end{aligned}$$

From which values of λ and μ can be obtained.

$$\begin{aligned} \therefore \int \frac{lx+m}{Ax^2+Bx+C} dx &= \lambda \int \frac{2Ax+B}{Ax^2+Bx+C} dx + \mu \int \frac{1}{Ax^2+Bx+C} dx \\ \therefore \int \frac{lx+m}{Ax^2+Bx+C} dx &= \lambda \ln|Ax^2 + Bx + C| + \frac{\mu}{Ak} \tan^{-1} \left(\frac{x+h}{h} \right) + c_2 \end{aligned}$$

NOTE

- 1) In $Ax^2 + Bx + C$, if $B^2 - 4AC$ is positive, then $Ax^2 + Bx + C$ can be resolved in to two integrals of the form $\int \frac{l}{ax+b} dx$, already discussed.
- 2) In $Ax^2 + Bx + C$, if $B^2 - 4AC$ is negative or $4AC - B^2$ is positive, then $Ax^2 + Bx + C$ cannot be resolved in to linear factors. Hence complete squares.

Integration of $\int \frac{lx+m}{(Ax^2+Bx+C)^n} dx$ **where** n **is a positive integer greater than 1**

Put $lx + m \equiv \lambda(2Ax + B) + \mu$, so that

$$\frac{l}{2A} = \lambda \text{ and } \mu = \frac{2Am - Bl}{2A}$$

$$\begin{aligned} \therefore \int \frac{lx+m}{(Ax^2+Bx+C)^n} dx &= \lambda \int \frac{2Ax+B}{(Ax^2+Bx+C)^n} dx + \mu \int \frac{1}{(Ax^2+Bx+C)^n} dx \\ &= \frac{\lambda}{-n+1} \cdot \frac{1}{(Ax^2+Bx+C)^{n-1}} + \mu \int \frac{dx}{(Ax^2+Bx+C)^n} \end{aligned}$$

Now to evaluate the integral on R.H.S.,

$$\text{Let } Ax^2 + Bx + C = A[(x+h)^2 + k^2], \text{ where } h = \frac{B}{2A}, \quad k = \frac{\sqrt{4AC-B^2}}{2A}$$

$$\Rightarrow \int \frac{dx}{(Ax^2+Bx+C)^n} = \int \frac{dx}{[(x+h)^2 - k^2]^n}$$

, which can be evaluated in to integrals

$$\text{i) } \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c; \text{ Substitution is } x = a \sec \theta$$

$$\text{ii) } \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c; \text{ Resolve in to partial fractions}$$

Example

$$\text{a) } \int \frac{x+1}{x^2+4x+8} dx$$

$$\text{b) } \int \frac{2-5x}{\sqrt{3-4x-4x^2}} dx$$

Solution

- a) Since $B^2 - 4AC = 16 - 32 = -16 < 0$, $x^2 + 4x + 8$ has no real linear factors

By completing squares

$$\begin{aligned} x^2 + 4x + 8 &= x^2 + 4x + 4 + 4 \\ &= (x+2)^2 + 4 \end{aligned}$$

Also splitting the numerator

$$x + 1 = A(2x + 4) + B$$

$$1 = 2A, \quad A = \frac{1}{2}$$

$$1 = 4A + B, \quad B = 1 - 2 = -1$$

$$\begin{aligned} \therefore \int \frac{x+1}{x^2+4x+8} dx &= \int \frac{\frac{1}{2}(2x+4)}{x^2+4x+8} dx - \int \frac{dx}{x^2+4x+8} \\ &= \frac{1}{2} \ln|x^2 + 4x + 8| - \int \frac{dx}{(x+2)^2 + 2^2} \\ &= \frac{1}{2} \ln|x^2 + 4x + 8| - \frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) \end{aligned}$$

- b) Let $2 - 5x = A(-4 - 8x) + B$

$$-5 = -8A, \quad A = 5/8$$

$$2 = -4A + B, \quad B = 2 + \frac{5}{2} = \frac{9}{2}$$

$$\begin{aligned} \int \frac{2-5x}{\sqrt{3-4x-4x^2}} dx &= \int \frac{A(-4-8x)}{\sqrt{3-4x-4x^2}} dx + B \int \frac{dx}{\sqrt{3-4x-4x^2}} \\ &= A \int (-4 - 8x) \cdot (3 - 4x - 4x^2)^{-1/2} + B \int \frac{dx}{\sqrt{3-4x-4x^2}} \end{aligned}$$

Now dealing with the R.H.S integral

$$-4x^2 - 4x + 3 = -4 \left(x^2 + x - \frac{3}{4} \right)$$

Completing squares

$$\begin{aligned}
 &= -4 \left[x^2 + x + \frac{1}{4} - \frac{1}{4} - \frac{3}{4} \right] \\
 &= -4[(x + 1/2)^2 - 1] = 4[1 - (x + 1/2)^2] \\
 \int \frac{dx}{\sqrt{3-4x-4x^2}} &= \frac{1}{4} \int \frac{dx}{\sqrt{1-(x+1/2)^2}} \\
 &= \frac{1}{4} \sin^{-1} \left(\frac{x+1/2}{1} \right) + c_1 \\
 &= \frac{1}{4} \sin^{-1} \left(\frac{2x+1}{2} \right) + c_1 \\
 \therefore \int \frac{2-5x}{\sqrt{3-4x-4x^2}} dx &= A \int (-4-8x). (3-4x-4x^2)^{-1/2} + B \int \frac{dx}{\sqrt{3-4x-4x^2}} \\
 &= \frac{5}{4\sqrt{3-4x-4x^2}} + \frac{9}{4} \sin^{-1} \left(\frac{2x+1}{2} \right) + c
 \end{aligned}$$

Task

$$\begin{aligned}
 \text{a) } \int \frac{3x-1}{\sqrt{5x^2-2x-3}} dx & \quad \text{Ans: } \frac{3}{5} \sqrt{5x^2-2x-3} - \frac{2}{5\sqrt{5}} \ln \left| x-5 + \sqrt{x^2 - \frac{2}{5}x - \frac{3}{5}} \right| + c \\
 \text{b) } \int \frac{3x+5}{x^2+6x+11} dx & \quad \text{Ans: } \frac{3}{2} \ln|x^2+6x+11| - 2\sqrt{2} \tan^{-1} \frac{x+3}{\sqrt{2}} + c
 \end{aligned}$$

Integration of Irrational Algebraic Functions

- a) **Integration of rational functions of x^n and $(ax+b)^{\frac{1}{n}}$, n is a positive integer.**

$$\begin{aligned}
 \text{i) } & \int x^n \cdot (ax+b)^{\frac{1}{n}} dx \\
 \text{ii) } & \int \frac{x^n}{(ax+b)^{\frac{1}{n}}} dx
 \end{aligned}$$

These are solved using the substitution

$$u = (ax+b)^{\frac{1}{n}} \quad \text{or} \quad u^n = ax+b$$

Rational functions of x^n , $(ax+b)^{\frac{1}{n}}$ and $(ax+b)^{\frac{1}{m}}$ can similarly be integrated by the substitution

$$u^p = ax+b, \text{ where } p \text{ is the L.C.M of } m \text{ and } n$$

Example

$$\begin{aligned}
 \text{a) } \int \frac{x^2}{\sqrt[3]{x+2}} dx & \quad \text{b) } \int \frac{1+x^{1/2}}{1+x^{1/3}} dx & \quad \text{c) } \int \frac{dx}{(1+x)^{1/3} + (1+x)^{1/2}}
 \end{aligned}$$

Solution

$$\text{a) Let } x+2 = u^3 \Rightarrow dx = 3u^2 du$$

$$\begin{aligned}
 \int \frac{x^2}{\sqrt[3]{x+2}} dx &= \int \frac{(u^3-2)^2}{u} \cdot 3u^2 du \\
 &= 3 \int (u^7 - 4u^4 + 4u) du = 3 \left[\frac{u^8}{8} - \frac{4}{5}u^5 + 2u^2 \right] + c \\
 &= \frac{3}{40} u^2 [5u^6 - 32u^3 + 80] + c \\
 &= \frac{3}{40} (x+2)^{\frac{2}{3}} [5(x+2)^2 - 32(x+2) + 80] + c
 \end{aligned}$$

$$= \frac{3}{40} (x+2)^{\frac{2}{3}} (5x^2 - 12x + 36) + c$$

b) $\int \frac{1+x^{1/2}}{1+x^{1/3}} dx$, L.C.M of 2 and 3 is 6

$$\text{Let } x = u^6 \Rightarrow dx = 6u^5 du$$

$$\int \frac{1+x^{1/2}}{1+x^{1/3}} dx = \int \frac{1+u^3}{1+u^2} \cdot 6u^5 du = 6 \int \frac{u^5+u^8}{1+u^2} du$$

By long division

$ \begin{array}{r} u^6 - u^4 + u^3 + u^2 - u - 1 \\ u^2 + 1 \overline{) u^8 + u^5} \\ \underline{u^8 + u^5} \\ -u^6 + u^5 \\ \underline{-u^6 - u^4} \\ u^5 + u^4 \\ \underline{u^5 + u^3} \\ u^4 - u^3 \\ \underline{u^4 + u^2} \\ -u^3 - u^2 \\ \underline{-u^3 - u} \\ -u^2 - u \\ \underline{-u^2 - 1} \\ -u + 1 \end{array} $	$ \begin{aligned} &6 \int \frac{u^5+u^8}{1+u^2} du \\ &= \frac{6}{7} u^7 - \frac{6}{5} u^5 + \frac{3}{2} u^4 + 2u^2 - 3u^2 - \\ &\quad 6u + 3 \ln u^2 + 1 + \\ &\quad 6 \tan^{-1} u + c \\ &= \frac{6}{7} x^{\frac{7}{6}} - \frac{6}{5} x^{\frac{5}{6}} + \frac{3}{2} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{\frac{1}{2}} - \\ &\quad 6x^{\frac{1}{6}} + 3 \ln \left x^{\frac{1}{3}} + 1 \right + \\ &\quad 6 \tan^{-1} x^{\frac{1}{6}} + c \end{aligned} $

c) L.C.M of 3 and 2 being 6

$$\text{Let } 1+x = u^6 \Rightarrow dx = 6u^5 du$$

$$\begin{aligned}
 \int \frac{dx}{(1+x)^{1/3} + (1+x)^{1/2}} dx &= \int \frac{6u^5 du}{u^2+u^3} = 6 \int \frac{u^3}{1+u} du \\
 &= 6 \int \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\
 &= 6 \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1| \right] + c \\
 &= 2(1+x)^{\frac{1}{2}} - 3(1+x)^{\frac{1}{3}} + 6(1+x)^{\frac{1}{6}} - 6 \ln \left| (1+x)^{\frac{1}{6}} + 1 \right| + c
 \end{aligned}$$

NOTE

A rational function of $\left(\frac{ax+b}{cx+d} \right)^{1/n}$ can also be integrated by the substitution

$$u^n = \frac{ax+b}{cx+d}$$

Example

a) $\int \frac{1}{(1+x)^{3/2} + (1+x)^{1/2}} dx$

b) $\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx$

Solution

a) Let $u = (1+x)^{1/2} \Rightarrow u^2 = 1+x$ or $dx = 2u du$

$$\begin{aligned}\therefore \int \frac{1}{(1+x)^{3/2} + (1+x)^{1/2}} dx &= \int \frac{2u du}{u^3 + u} = 2 \int \frac{du}{u^2 + 1} \\ &= 2 \tan^{-1} u + c = 2 \tan^{-1} (1+x)^{1/2} + c\end{aligned}$$

b) Let $y^2 = \frac{1-x}{1+x} \Rightarrow x = \frac{1-y^2}{1+y^2}$, $dx = \frac{4y dy}{(1+y^2)^2}$

$$\begin{aligned}\therefore \int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx &= -4 \int \frac{y^2}{(1-y^2)(1+y^2)} dy \\ &= -2 \int \left(\frac{1}{1-y^2} - \frac{1}{1+y^2} \right) dy = -2 \int \left[\frac{1}{2} \left(\frac{1}{1-y} + \frac{1}{1+y} \right) - \frac{1}{1+y^2} \right] dy \\ &= -2 \left[\frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| - \tan^{-1} y \right] + c \\ &= 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} - \ln \left| \frac{1 + \sqrt{\frac{1-x}{1+x}}}{1 - \sqrt{\frac{1-x}{1+x}}} \right| + c \\ &= 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} - \ln \left| \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}} \right| + c\end{aligned}$$

NOTE

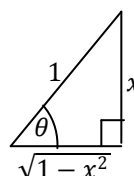
- 1) For $\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx$, $x = \cos 2\theta$ or $x = \cos \theta$ can evaluate the integral
- 2) Also $\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx$ can be handled by Rationalization of the numerator. i.e.

$$\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx = \int \frac{1-x}{x\sqrt{1-x^2}} dx, \text{ let } x = \sin \theta \text{ or } dx = \cos \theta d\theta$$

$$\begin{aligned}\therefore \int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx &= \int \frac{1-\sin \theta}{\sin \theta \sqrt{1-\sin^2 \theta}} \cos \theta d\theta \\ &= \int \frac{1-\sin \theta}{\sin \theta} d\theta = \int (\operatorname{cosec} \theta - 1) d\theta \\ &= \ln |\operatorname{cosec} \theta - \cot \theta| - \theta + c\end{aligned}$$

But $\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{1}{x}$, $\cot \theta = \frac{1}{\tan \theta} = \frac{\sqrt{1-x^2}}{x}$

$$\begin{aligned}\therefore \int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx &= \ln \left| \frac{1}{x} - \frac{\sqrt{1-x^2}}{x} \right| - \sin^{-1} x + c \\ &= \ln \left| \frac{1-\sqrt{1-x^2}}{x} \right| - \sin^{-1} x + c\end{aligned}$$



Example

Integrate

a) $\int \sqrt{\frac{1+x}{1-x}} dx$

b) $\int \sqrt{\frac{x-1}{2-x}} dx$

c) $\int \sqrt{\frac{4-3x}{5x+2}} dx$

d) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$

e) $\int \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} dx$

Ans: $\tan^{-1} \sqrt{\frac{3}{5}} \sqrt{\frac{4-3x}{5x+2}} + c$ use $t^2 = \frac{4-3x}{5x+2}$

Ans: $\sqrt{x-x^2} - 2\sqrt{1-x} - \sin^{-1} \sqrt{x} + c$

Hint: use $\sqrt{x} = \cos \theta$

- f) $\int \sqrt{\frac{1-x}{1+x}} dx$ **Ans:** $\sin^{-1} x + \sqrt{1-x^2} + c$
- g) $\int \sqrt{\frac{x}{a-x}} dx$
- h) $\int \frac{\sqrt{x}}{\sqrt{x}+2} dx$ **Ans:** $x - 4\sqrt{x} + 8 \ln|\sqrt{x} + 2| + c$
- i) $\int_0^2 \sqrt{\frac{2+x}{2-x}} dx$ **Ans:** $+2$
- j) $\int \sqrt{\frac{5-x}{x-2}} dx$ **Ans:** $\sqrt{7x-x^2-10} + 3 \sin^{-1} x + \sqrt{1-x^2} + c$
Use $t = \sqrt{x-2}$
- k) $\int \sqrt{\frac{x-2}{5-x}} dx$ **Ans:** $-\left[\sqrt{7x-x^2-10} + 3 \sin^{-1} x + \sqrt{1-x^2}\right] + c$
Use $t = \sqrt{5-x}$
- l) $\int \frac{dx}{\sqrt{1+x}+\sqrt{x}}$ **Ans:** $\frac{2}{3} \sqrt{(1+x)^3} - \frac{2}{3} \sqrt{x^3} + c$; Rationalize
- m) $\int_0^1 x^3 \sqrt{\frac{1+x^2}{1-x^2}} dx$ **Ans:** $\frac{\pi}{8} \ln 2$; use $x^2 = \cos \theta$
- n) Using the substitution $x = 3 \sin \theta$, evaluate $\int_0^3 \sqrt{\frac{3+x}{3-x}} dx$ **(Unib 2001 16(c))**

More techniques of handling $\int \sqrt{\frac{1-x}{1+x}} dx$

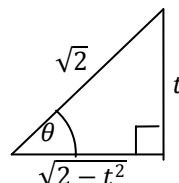
Alternative I

Let $t^2 = 1+x \Rightarrow 1-x = 2-t^2$ and $dx = 2t dt$

$$\int \sqrt{\frac{1-x}{1+x}} dx = \int \sqrt{\frac{2-t^2}{t^2}} \cdot 2t dt = 2 \int \sqrt{2-t^2} dt$$

Let $t = \sqrt{2} \sin \theta \Rightarrow dt = \sqrt{2} \cos \theta d\theta$

$$\begin{aligned} 2 \int \sqrt{2-t^2} dt &= 2 \int \sqrt{2-2\sin^2 \theta} \cdot \sqrt{2} \cos \theta d\theta \\ &= 4 \int \cos^2 \theta d\theta = 2 \int (1 + \cos 2\theta) d\theta \\ &= 2 \left[\theta + \frac{1}{2} \sin 2\theta \right] + c = 2\theta + \sin 2\theta + c \\ &= 2\theta + 2 \sin \theta \cos \theta + c \\ &= 2 \sin^{-1} \left(\frac{t}{\sqrt{2}} \right) + \frac{2t}{\sqrt{2}} \cdot \frac{\sqrt{2-t^2}}{\sqrt{2}} + c = 2 \sin^{-1} \left(\frac{t}{\sqrt{2}} \right) + t\sqrt{2-t^2} + c \end{aligned}$$



But $t = \sqrt{1+x}$, $1-x = 2-t^2$

$$\begin{aligned} \therefore \int \sqrt{\frac{1-x}{1+x}} dx &= 2 \sin^{-1} \left(\frac{\sqrt{1+x}}{\sqrt{2}} \right) + \sqrt{1+x} \cdot \sqrt{1-x} + c \\ &= 2 \sin^{-1} \left(\sqrt{\frac{1+x}{2}} \right) + \sqrt{1-x^2} + c \end{aligned}$$

Simplifying using trigonometry

$$\text{Let } y = 2 \sin^{-1} \left(\sqrt{\frac{1+x}{2}} \right), \theta = \sin^{-1} \left(\sqrt{\frac{1+x}{2}} \right) \Rightarrow y = 2\theta$$

$$\sin y = \sin 2\theta = 2 \sin \theta \cos \theta = 2 \sin \theta \sqrt{1-\sin^2 \theta}$$

$$= 2\sqrt{\frac{1+x}{2}} \cdot \sqrt{1 - \frac{1+x}{2}} = 2\sqrt{\frac{1-x^2}{4}} = \sqrt{1-x^2}$$

$$\therefore \int \frac{1-x}{1+x} dx = \sin^{-1} \sqrt{1-x^2} + \sqrt{1-x^2} + c$$

Alternative II

Let $t^2 = \frac{1-x}{1+x} \Rightarrow t^2 + t^2 x = 1-x \quad \therefore x = \frac{1-t^2}{1+t^2}$

Now $2t dt = \frac{-(1+x)-(1-x)}{(1+x)^2} dx = -\frac{2}{(1+x)^2} \quad \therefore dx = -t(1+x)^2 = -t\left(1 + \frac{1-t^2}{1+t^2}\right)$

$$\therefore \int \frac{1-x}{1+x} dx = - \int t \cdot t \left(1 + \frac{1-t^2}{1+t^2}\right) dt$$

$$= - \int \frac{4t^2}{(1+t^2)^2} dt$$

Let $t = \tan \theta \Rightarrow dt = \sec^2 \theta d\theta$

$$\therefore \int \frac{1-x}{1+x} dx = - \int \frac{4t^2}{(1+t^2)^2} dt = \int \frac{-4 \tan^2 \theta}{(1+\tan^2 \theta)^2} \cdot \sec^2 \theta d\theta$$

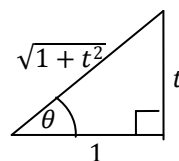
$$= \int \frac{-4 \tan^2 \theta}{\sec^2 \theta} d\theta = \int \frac{-4 \sin^2 \theta}{\cos^2 \theta} \cdot \cos^2 \theta d\theta$$

$$= -4 \int \sin^2 \theta d\theta = -4 \int \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$= -2 \left[\theta - \frac{1}{2} \sin 2\theta \right] + c = -2\theta + \sin 2\theta + c$$

$$= -2\theta + 2 \sin \theta \cos \theta + c$$

$$= -2 \tan^{-1} t + \frac{2t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} + c$$



$$= -2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} + \frac{2\sqrt{\frac{1-x}{1+x}}}{1+\frac{1-x}{1+x}} + c$$

$$= -2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} + \frac{2\sqrt{1-x}\sqrt{1+x}}{2} + c$$

$$= -2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} + \sqrt{1-x^2} + c$$

$$\therefore \int \frac{1-x}{1+x} dx = -\sin^{-1} \sqrt{1-x^2} + \sqrt{1-x^2} + c$$

By Trigonometry

Let $y = \tan^{-1} \sqrt{\frac{1-x}{1+x}}, z = 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}}$

$$\tan y = \sqrt{\frac{1-x}{1+x}}$$

$$z = 2y \Rightarrow \sin z = \sin 2y = \frac{2 \tan y}{1+\tan^2 y}$$

$$\sin z = 2 \frac{\sqrt{\frac{1-x}{1+x}}}{1+\frac{1-x}{1+x}} = \sqrt{1-x^2}$$

Alternative III

By Rationalization

$$\int \frac{1-x}{1+x} dx = \int \frac{1-x}{\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x + \frac{1}{2} \int (-2x) \cdot (1-x^2)^{-1/2} dx$$

$$= \sin^{-1} x + \frac{1}{2} \cdot \frac{(1-x^2)^{1/2}}{1/2} + c$$

$$\therefore \int \frac{1-x}{1+x} dx = \sin^{-1} \sqrt{1-x^2} + \sqrt{1-x^2} + c$$

Alternative IV

$$\int \sqrt{\frac{1-x}{1+x}} dx, \text{ let } x = \cos 2\theta \Rightarrow dx = -2 \sin 2\theta \, d\theta$$

$$\begin{aligned} \int \sqrt{\frac{1-x}{1+x}} dx &= \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \cdot -2 \sin 2\theta \, d\theta \\ &= -\int \sqrt{\tan^2 \theta} \cdot -2 \cdot (2 \sin \theta \cos \theta) d\theta = -\int \sqrt{\tan^2 \theta} \cdot -2 \cdot (2 \sin \theta \cos \theta) d\theta \\ &= -\int \frac{\sin \theta}{\cos \theta} \cdot -2 \cdot (2 \sin \theta \cos \theta) d\theta = -2 \int 2 \sin^2 \theta \, d\theta \\ &= -2 \int (1 - \cos 2\theta) d\theta \\ &= -2 \left[\theta - \frac{1}{2} \sin 2\theta \right] + c = -2\theta + \sin 2\theta + c \\ &= -2 \left[\frac{\cos^{-1} x}{2} - \frac{1}{2} \sqrt{1 - \cos^2 2\theta} \right] + c \\ &= -\cos^{-1} x + \sqrt{1 - x^2} + c \end{aligned}$$

From trigonometry

$$\text{Let } y = \cos^{-1} x \Rightarrow \cos y = x \text{ and } z = -\cos^{-1} x \quad \therefore z = -y$$

$$\sin z = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$$

$$\therefore \int \sqrt{\frac{1-x}{1+x}} dx = -\sin^{-1} \sqrt{1 - x^2} + \sqrt{1 - x^2} + c$$

Comparing the above approaches, Rationalization leads us to the answer easily.

Integrals of the type $\int \frac{dx}{X\sqrt{Y}}$, where X and Y are Linear or Quadratic

Functions of x

Four cases are to be considered

Case I: Rule to integrate $\int \frac{1}{\text{Linear} \cdot \sqrt{\text{Linear}}} dx$

$$\text{Put } \sqrt{\text{Linear}} = t \text{ or } \text{Linear} = t^2$$

Case II: Rule to integrate $\int \frac{1}{\text{Quadratic} \cdot \sqrt{\text{Linear}}} dx$

$$\text{Put } \sqrt{\text{Linear}} = t \text{ or } \text{Linear} = t^2$$

Case III: Rule to integrate $\int \frac{1}{\text{Linear} \cdot \sqrt{\text{Quadratic}}} dx$

$$\text{Put } \text{Linear} = \frac{1}{t}$$

NOTE

In all the above cases if instead of *unity* in the numerator there is a polynomial in x whose degree is equal or greater than that of the rational factor of the denominator, first divide the numerator by this rational factor, and then use the rule, where necessary.

Case IV: Rule to integrate $\int \frac{1}{\text{Pure Quadratic} \cdot \sqrt{\text{Pure Quadratic}}} dx$

$$\text{Put } x = \frac{1}{t} \text{ and then in the resulting integral put } u = \sqrt{\text{Quadratic}}$$

Example

a) $\int \frac{dx}{(x+2)\sqrt{x+3}}$

b) $\int \frac{x^2-x+1}{(x+1)\sqrt{x-2}} dx$

Solution

a) Put $x + 3 = t^2 \Rightarrow x = t^2 - 3, dx = 2t dt$

$$\begin{aligned} \int \frac{dx}{(x+2)\sqrt{x+3}} &= \int \frac{2t dt}{(t^2-1)t} = \int \frac{2}{(t+1)(t-1)} dt \\ &= \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \ln|t-1| - \ln|t+1| + c \\ &= \ln \left| \frac{t-1}{t+1} \right| + c = \ln \left| \frac{\sqrt{x+3}-1}{\sqrt{x+3}+1} \right| + c \end{aligned}$$

b) Here the degree of the numerator $x^2 - x + 1$ is greater than that of the rational factor $(x + 1)$ of the denominator. Thus we divide $x^2 - x + 1$ by $x + 1$

$$\frac{x^2-x+1}{x+1} = x - 2 + \frac{3}{x+1}$$

$$\begin{aligned} \therefore \int \frac{x^2-x+1}{(x+1)\sqrt{x-2}} dx &= \int \left(x - 2 + \frac{3}{x+1} \right) \cdot \frac{1}{\sqrt{x-2}} dx \\ &= \int \sqrt{x-2} dx + 3 \int \frac{1}{(x+1)\sqrt{x-2}} dx = \frac{(x-2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 3 \int \frac{1}{(x+1)\sqrt{x-2}} dx \end{aligned}$$

For the integral $\int \frac{1}{(x+1)\sqrt{x-2}} dx$, put $x - 2 = t^2 \Rightarrow dx = 2t dt$

$$\begin{aligned} \therefore \int \frac{1}{(x+1)\sqrt{x-2}} dx &= \int \frac{1}{(t^2+3)t} \cdot 2t dt = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x-2}{\sqrt{3}} \right) \\ \therefore \int \frac{x^2-x+1}{(x+1)\sqrt{x-2}} dx &= \frac{2}{3} (x-2)^{3/2} + 3 \cdot \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x-2}{\sqrt{3}} \right) + c \\ \therefore \int \frac{x^2-x+1}{(x+1)\sqrt{x-2}} dx &= \frac{2}{3} (x-2)^{3/2} + 2\sqrt{3} \tan^{-1} \left(\frac{x-2}{\sqrt{3}} \right) + c \end{aligned}$$

Task

a) $\int \frac{1}{x\sqrt{1-x}} dx$

Ans: $\ln \left| \frac{\sqrt{1-x}-1}{\sqrt{1-x}+1} \right| + c$

b) $\int \frac{\sqrt{x}}{1+x} dx$

Ans: $2(\sqrt{x} - \tan^{-1} \sqrt{x}) + c$

c) $\int \frac{x+3}{\sqrt{x-3}} dx$

Ans:

d) $\int_8^{15} \int \frac{1}{(x-3)\sqrt{x+1}} dx$

Ans: $\frac{\ln 5 - \ln 3}{2}$

e) $\int_3^8 \int \frac{2-3x}{x\sqrt{x+1}} dx$

Ans: $2 \ln(3/2e^3)$

f) $\int \frac{dx}{(x-3)\sqrt{x+1}}$

Ans: $\frac{1}{2} \ln \left| \frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} \right| + c$

g) $\int_0^1 \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Ans: $\frac{\pi}{3}$

h) $\int \frac{dx}{(x+1)\sqrt{x^2+1}}$

Ans: $\sqrt{\frac{x-1}{x+1}} + c$

Integration of Trigonometric Functions

Integration of $\int \frac{dx}{a \sin x + b \cos x}$

$$\begin{aligned} \text{Let } a \sin x + b \cos x &= R \sin(x + \alpha) \\ &= R \sin x \cos \alpha + R \cos x \sin \alpha \end{aligned}$$

Comparing

$$\text{For } \sin x: a = R \cos \alpha \dots\dots\dots (1)$$

$$\cos x: a = R \sin \alpha \dots\dots\dots (2)$$

Solving for R and α

$$\text{Squaring the equations and adding gives } R = \sqrt{a^2 + b^2}$$

$$\text{Dividing eqn. (2) by eqn. (1) gives } \tan \alpha = \frac{b}{a} \text{ or } \alpha = \tan^{-1} \left(\frac{b}{a} \right)$$

$$\begin{aligned} \therefore \int \frac{dx}{a \sin x + b \cos x} &= \frac{1}{R} \int \frac{dx}{\sin(x + \alpha)} = \frac{1}{R} \int \operatorname{cosec}(x + \alpha) dx \\ &= \frac{1}{R} \ln \left| \tan \frac{x + \alpha}{2} \right| + c = \frac{1}{R} \ln \left| \tan \frac{1}{2} \left(x + \tan^{-1} \left(\frac{b}{a} \right) \right) \right| + c \end{aligned}$$

$$\therefore \int \frac{dx}{a \sin x + b \cos x} = \frac{1}{R} \ln \left| \tan \frac{1}{2} \left(x + \tan^{-1} \left(\frac{b}{a} \right) \right) \right| + c$$

Example

$$\text{a) } \int \frac{dx}{2 \sin x - \cos x}$$

$$\text{b) } \int \frac{dx}{5 \sin x + 12 \cos x}$$

Solution

$$\text{a) Let } 2 \sin x - \cos x = R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$$

Comparing

$$2 = R \cos \alpha, -1 = R \sin \alpha$$

$$\text{Solving } R = \sqrt{5}, \tan \alpha = -\frac{1}{2} \text{ or } \alpha = \tan^{-1}(-1/2) = -\tan^{-1}(1/2)$$

$$\begin{aligned} \therefore \int \frac{dx}{2 \sin x - \cos x} &= \frac{1}{\sqrt{5}} \int \frac{dx}{\sin(x + \tan^{-1}(-1/2))} \\ &= \frac{1}{\sqrt{5}} \int \operatorname{cosec}(x + \tan^{-1}(-1/2)) dx = \frac{1}{\sqrt{5}} \ln \left| \tan \left(\frac{x - \tan^{-1}(1/2)}{2} \right) \right| + c \end{aligned}$$

$$\text{b) Left as an exercise.} \quad \text{Ans: } \frac{1}{13} \ln \left| \tan \left(\frac{x + \tan^{-1}(12/5)}{2} \right) \right| + c$$

Alternative Approach

Using the t - substitution $t = \tan \frac{x}{2}$

$$\int \frac{dx}{5 \sin x + 12 \cos x} = \int \frac{dx}{10 \sin \frac{x}{2} \cos \frac{x}{2} + 12 (\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2})}$$

Divide the numerator and denominator by $\cos^2 \frac{x}{2}$

$$\int \frac{dx}{5 \sin x + 12 \cos x} = \int \frac{\sec^2 \frac{x}{2} dx}{10 \tan \frac{x}{2} + 12 - \tan^2 \frac{x}{2}}, \text{ let } t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$$

$$\therefore \int \frac{dx}{5 \sin x + 12 \cos x} = \int \frac{2 dt}{10t + 12 - t^2} = \int \frac{dt}{6 + 5t - t^2}$$

Since $B^2 - 4AC < 0$, completing squares

$$6 + 5t - 6t^2 \equiv -6 \left(t^2 - \frac{5}{6}t - 1 \right) = -6 \left(\left(t - \frac{5}{12} \right)^2 - \frac{25}{144} - 1 \right) = 6 \left(\frac{169}{144} - \left(t - \frac{5}{12} \right)^2 \right)$$

$$\int \frac{dx}{5 \sin x + 12 \cos x} = \frac{1}{6} \int \frac{dt}{\frac{169}{144} - \left(t - \frac{5}{12} \right)^2}, \text{ taking } a^2 = \frac{169}{144}, u^2 = \left(t - \frac{5}{12} \right)^2$$

$$\therefore \int \frac{dx}{5 \sin x + 12 \cos x} = \frac{1}{6} \int \frac{du}{a^2 - u^2} = \frac{1}{6} \int \frac{du}{(a+u)(a-u)}$$

By partial fractions

$$\frac{1}{(a+u)(a-u)} = \frac{A}{a+u} + \frac{B}{a-u}$$

$$1 = A(a-u) + B(a+u), \text{ solving for A and B}$$

$$1 = Aa + Ba \dots \dots \dots (1)$$

$$0 = -A + B \dots \dots \dots (2)$$

$$\text{Adding the equations gives } \frac{1}{a} = 2B, B = \frac{1}{2a} \text{ also } A = B = \frac{1}{2a}$$

$$\therefore \int \frac{du}{a^2 - u^2} = \int \frac{du}{(a+u)(a-u)} = \frac{1}{2a} \int \left(\frac{A}{a+u} + \frac{B}{a-u} \right) du$$

$$= \frac{1}{2a} \ln|a+u| - \ln|a-u| = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + c$$

$$\therefore \int \frac{dx}{5 \sin x + 12 \cos x} = \frac{1}{6} \cdot \frac{1}{2(13/12)} \ln \left| \frac{\frac{13}{12} + t - \frac{5}{12}}{\frac{13}{12} - t + \frac{5}{12}} \right| + c$$

$$= \frac{1}{13} \ln \left| \frac{8+12t}{18-12t} \right| = \frac{1}{13} \ln \left| \frac{4+6t}{9-6t} \right| + c$$

$$\therefore \int \frac{dx}{5 \sin x + 12 \cos x} = \frac{1}{13} \ln \left| \frac{2/3+t}{3/2-t} \right| + c = \frac{1}{13} \ln \left| \tan \left(\frac{x + \tan^{-1}(12/5)}{2} \right) \right| + c$$

Integrals of Rational Functions of $\sin x$ and $\cos x$

To integrate $\int \frac{dx}{a+b \cos x}$

$$\int \frac{dx}{a+b \cos x} = \int \frac{dx}{a+b \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)} = \int \frac{\sec \frac{x}{2} dx}{a \sec \frac{x}{2} + b(1 - \tan \frac{x}{2})} \int \frac{\sec \frac{x}{2} dx}{a + a \tan \frac{x}{2} + b \tan \frac{x}{2}}$$

$$\text{Let } t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$$

$$\therefore \int \frac{dx}{a+b \cos x} = \int \frac{2 dt}{(a+b)+(a-b)t^2} \\ = \frac{2}{a-b} \int \frac{dt}{\frac{a+b}{a-b} + t^2}$$

Case I: $a > b$

In this case $\frac{a+b}{a-b}$ is positive

$$\therefore \frac{2}{a-b} \int \frac{dt}{\frac{a+b}{a-b} + t^2} = \frac{2}{a-b} \cdot \frac{1}{\sqrt{\frac{a+b}{a-b}}} \tan^{-1} \left(\frac{t}{\sqrt{\frac{a+b}{a-b}}} \right) + c$$

$$\therefore \int \frac{dx}{a+b \cos x} = \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1} \left(\sqrt{\frac{a+b}{a-b}} \cdot \tan \frac{x}{2} \right) + c$$

NOTE

$$\int_0^\pi \frac{dx}{a+b \cos x} = \frac{2}{\sqrt{a^2-b^2}} \cdot \left[\tan^{-1} \left(\sqrt{\frac{a+b}{a-b}} \cdot \tan \frac{x}{2} \right) \right]_0^\pi \\ = \frac{2}{\sqrt{a^2-b^2}} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{\pi}{2} \\ = \frac{\pi}{\sqrt{a^2-b^2}}$$

Case II: $a < b$

In this case $\frac{a+b}{a-b}$ is negative

$$\begin{aligned}\therefore \int \frac{dx}{a+b \cos x} &= \int \frac{2 dt}{(a+b)-(b-a)t^2} \\ &= \frac{2}{b-a} \int \frac{dt}{\frac{a+b}{b-a}-t^2} \\ &= \frac{2}{b-a} \cdot \frac{1}{2 \cdot \sqrt{\frac{a+b}{b-a}}} \ln \left| \frac{\sqrt{\frac{a+b}{b-a}}+t}{\sqrt{\frac{a+b}{b-a}}-t} \right| + c \\ &= \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{\sqrt{a+b}+\sqrt{b-a} \cdot t}{\sqrt{a+b}-\sqrt{b-a} \cdot t} \right| + c \\ &= \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{\sqrt{a+b}+\sqrt{b-a} \cdot \tan \frac{x}{2}}{\sqrt{a+b}-\sqrt{b-a} \cdot \tan \frac{x}{2}} \right| + c\end{aligned}$$

Thus we get $\int \frac{dx}{a+b \cos x}$

$$\begin{aligned}\therefore \int \frac{dx}{a+b \cos x} &= \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1} \left(\sqrt{\frac{a+b}{a-b}} \cdot \tan \frac{x}{2} \right) + c \quad (\text{where } a > b) \\ &= \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{\sqrt{a+b}+\sqrt{b-a} \cdot \tan \frac{x}{2}}{\sqrt{a+b}-\sqrt{b-a} \cdot \tan \frac{x}{2}} \right| + c \quad (\text{where } a > b)\end{aligned}$$

To integrate $\int \frac{dx}{a+b \sin x}$

Putting $t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$, $\sin x = \frac{2t}{1+t^2}$

$$\begin{aligned}\int \frac{dx}{a+b \sin x} &= \int \frac{1}{a+b \left(\frac{2t}{1+t^2} \right)} \cdot \frac{2 dt}{1+t^2} \\ &= 2 \int \frac{dt}{at^2+2bt+a} = \frac{2}{a} \int \frac{dt}{t^2+\frac{2b}{a}t+1} \\ &= \frac{2}{a} \int \frac{dt}{\left(t+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}}\end{aligned}$$

Case I: $a^2 > b^2$

$$\begin{aligned}\int \frac{dx}{a+b \sin x} &= \frac{2}{a} \int \frac{dt}{\left(t+\frac{b}{a}\right)^2+\left(\frac{\sqrt{a^2-b^2}}{a}\right)^2} \\ &= \frac{2}{a} \cdot \frac{1}{\sqrt{a^2-b^2}/a} \tan^{-1} \left(\frac{t+b/a}{\sqrt{a^2-b^2}/a} \right) + c \\ \therefore \int \frac{dx}{a+b \sin x} &= \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1} \frac{a \tan \frac{x}{2}+b}{\sqrt{a^2-b^2}} + c\end{aligned}$$

Case II: $a^2 < b^2$

$$\begin{aligned}\int \frac{dx}{a+b \sin x} &= \frac{2}{a} \int \frac{dt}{\left(t+\frac{b}{a}\right)^2-\left(\frac{a^2-b^2}{a^2}\right)} \\ &= \frac{2}{a} \int \frac{dt}{\left(t+\frac{b}{a}\right)^2-\left(\frac{b^2-a^2}{a^2}\right)}\end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{a} \int \frac{dt}{\left(t + \frac{b}{a}\right)^2 - \left(\frac{\sqrt{b^2 - a^2}}{a}\right)^2} \\
 &= \frac{2}{a} \cdot \frac{1}{2\sqrt{b^2 - a^2}} \cdot \ln \left| \frac{t + \frac{b}{a} + \frac{\sqrt{b^2 - a^2}}{a}}{t + \frac{b}{a} - \frac{\sqrt{b^2 - a^2}}{a}} \right| + c \\
 &= \frac{1}{\sqrt{b^2 - a^2}} \cdot \ln \left| \frac{a \tan \frac{x}{2} + b + \sqrt{b^2 - a^2}}{\tan \frac{x}{2} + b + \sqrt{b^2 - a^2}} \right| + c
 \end{aligned}$$

Evaluate

Evaluate

a) $\int \frac{dx}{5+4 \cos x}$ b) $\int_0^{\pi/2} \frac{dx}{4+5 \cos x}$ c) $\int \frac{dx}{5+4 \sin x}$

Solution

a) $\int \frac{dx}{5+4 \cos x}$, put $t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$, $\cos x = \frac{1-t^2}{1+t^2}$

$$\begin{aligned}
 \Rightarrow \int \frac{dx}{5+4 \cos x} &= \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2 dt}{1+t^2} \\
 &= 2 \int \frac{dt}{9+t^2} = 2 \cdot \frac{1}{3} \tan^{-1} \left(\frac{t}{3} \right) + c \\
 &= \frac{2}{3} \tan^{-1} \left(\frac{\tan x/2}{3} \right) + c
 \end{aligned}$$

NOTE:

$\int_0^{\pi/2} \frac{dx}{5+4 \cos x} = \left| \frac{2}{3} \tan^{-1} \left(\frac{\tan x/2}{3} \right) \right|_0^{\pi/2} = \left(\frac{2}{3} \tan^{-1} \left(\frac{1}{3} \right) - \tan^{-1}(0) \right) = \frac{2}{3} \tan^{-1} \left(\frac{1}{3} \right)$

b) $\int_0^{\pi/2} \frac{dx}{4+5 \cos x} = \int_0^{\pi/2} \frac{dx}{4+5\left(\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)}$

$$= \int_0^{\pi/2} \frac{(1+\tan^2 \frac{x}{2}) dx}{4+4 \tan^2 \frac{x}{2} + 5 - 5 \tan^2 \frac{x}{2}} = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2} dx}{9 - \tan^2 \frac{x}{2}}$$

Let $t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx \therefore 2 dt = \sec^2 \frac{x}{2} dx$

Changing limits

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = 1$

$$\begin{aligned}
 \therefore \int_0^{\pi/2} \frac{dx}{4+5 \cos x} &= \int_0^1 \frac{2 dt}{9-t^2} = \int_0^1 \frac{2 dt}{(3+t)(3-t)} \\
 &= 2 \cdot \frac{1}{3 \cdot 2} \ln \left| \frac{3+t}{3-t} \right|_0^1 = \frac{1}{3} \left(\log \frac{4}{2} - \log 1 \right) \\
 &= \frac{1}{3} \ln 2
 \end{aligned}$$

c) Let $t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx \therefore \frac{2 dt}{1+t^2} = dx$ and $\sin x = \frac{2t}{1+t^2}$

$$\begin{aligned}
 \int \frac{dx}{5+4 \sin x} &= \int \frac{1}{5+4\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2 dt}{1+t^2} \\
 &= 2 \int \frac{dt}{5t^2+8t+5} = \frac{2}{5} \int \frac{dt}{(t+4/5)^2 + (3/5)^2} \\
 &= \frac{2}{5} \cdot \frac{1}{3/5} \tan^{-1} \left[\frac{5 \tan x/2 + 4}{3} \right]
 \end{aligned}$$

Task

$$\int_{\pi/3}^{\pi/2} \frac{dx}{4-5 \cos x}$$

$$\text{Ans: } \frac{1}{3} \ln \left| \frac{\sqrt{3}+1}{2(\sqrt{3}-1)} \right|$$

Example

Evaluate

$$\text{a) } \int \frac{dx}{3 \cos x - 4 \sin x + 5}$$

$$\text{b) } \int \frac{dx}{3 \cos x + 4 \sin x + 6}$$

Solution

$$\text{a) Putting } t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx \therefore dx = \frac{2 dt}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2} \text{ and } \sin x = \frac{2t}{1+t^2}$$

$$\begin{aligned} \int \frac{dx}{3 \cos x - 4 \sin x + 5} &= \int \frac{1}{3 \left(\frac{1-t^2}{1+t^2} \right) - 4 \left(\frac{2t}{1+t^2} \right) + 5} \cdot \frac{2 dt}{1+t^2} \\ &= \int \frac{dt}{t^2 - t + 4} = \int \frac{dt}{(t-2)^2} \\ &= \int (t-2)^{-2} dt = \frac{(t-2)^{-2+1}}{-2+1} = -\frac{1}{t-2} \end{aligned}$$

$$\therefore \int \frac{dx}{3 \cos x - 4 \sin x + 5} = -\frac{1}{\tan \frac{x}{2} - 2} + c = \frac{1}{2 - \tan \frac{x}{2}} + c$$

$$\begin{aligned} \text{b) } 3 \cos x + 4 \sin x + 6 &= 3 \left(\frac{1-t^2}{1+t^2} \right) + \frac{8t}{1+t^2} + 6 \\ &= \frac{3-3+8t+6+6t^2}{1+t^2} = \frac{3t^2+8t+9}{1+t^2} \end{aligned}$$

$$\text{Putting } t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx \therefore dx = \frac{2 dt}{1+t^2}$$

$$\begin{aligned} \int \frac{dx}{3 \cos x + 4 \sin x + 6} &= \int \frac{2 dt}{3t^2+8t+9} \\ &= \frac{2}{3} \int \frac{dt}{t^2+(8/3)t+3} = \frac{2}{3} \int \frac{dt}{(t+4/3)^2+11/9} \\ &= \frac{2}{3} \times \frac{1}{\sqrt{11/3}} \tan^{-1} \left(\frac{t+4/3}{\sqrt{11/3}} \right) + c \\ &= \frac{2}{\sqrt{11}} \tan^{-1} \left(\frac{t+4/3}{\sqrt{11}} \right) + c \end{aligned}$$

$$\therefore \int \frac{dx}{3 \cos x + 4 \sin x + 6} = \frac{2}{\sqrt{11}} \tan^{-1} \left(\frac{3 \tan (x/2) + 4}{3\sqrt{11}} \right) + c$$

To integrate $\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx$

Here the Numerator = l (Denominator) + m (Diff. coeff. of Denominator) where l and m are the constants to be found

$$\begin{aligned} \Rightarrow a \cos x + b \sin x &= l(c \cos x + d \sin x) + m(-c \sin x + d \cos x) \\ &= (cl + dm) \cos x + (dl - cm) \sin x \end{aligned}$$

Equating coefficients

$$\text{For } \cos x: a = cl + dm \dots \dots \dots (1)$$

$$\sin x: b = dl - cm \dots \dots \dots (2)$$

Solving eqn. (1) and eqn. (2)

$$l = \frac{ac+bd}{c^2+d^2}, \quad m = \frac{ad-bc}{c^2+d^2}$$

$$\begin{aligned} \therefore \int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx &= \int \frac{l(c \cos x + d \sin x) + m(-c \sin x + d \cos x)}{c \cos x + d \sin x} dx \\ &= \int \left(l + m \cdot \frac{-c \sin x + d \cos x}{c \cos x + d \sin x} \right) dx \end{aligned}$$

$$\therefore \int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx = lx + m \ln |c \cos x + d \sin x| + c$$

Example

Evaluate $\int \frac{26 \sin x - 13 \cos x}{4 \sin x - 7 \cos x} dx$

Solution

Let $26 \sin x - 13 \cos x = l(4 \sin x - 7 \cos x) + m(4 \cos x + 7 \sin x)$
 $= (4l + 7m) \sin x + (4m - 7l) \cos x$

Equating coefficients

For $\sin x$: $26 = 4l + 7m$

$\cos x$: $-13 = -7 + 4m$

Solving equations: $l = 3$, $m = 2$

$$\therefore \int \frac{26 \sin x - 13 \cos x}{4 \sin x - 7 \cos x} dx = \int \frac{l(4 \sin x - 7 \cos x) + m(4 \cos x + 7 \sin x)}{4 \sin x - 7 \cos x} dx$$

$$= \int \left(l + m \cdot \frac{4 \cos x + 7 \sin x}{4 \sin x - 7 \cos x} \right) dx$$

$$\therefore \int \frac{26 \sin x - 13 \cos x}{4 \sin x - 7 \cos x} dx = lx + m \ln|4 \sin x - 7 \cos x| + c$$

To integrate $\int \frac{1}{a^2 \cos^2 x + b \sin^2 x} dx$

Divide the Numerator and Denominator by $\cos^2 x$

$$\int \frac{1}{a^2 \cos^2 x + b \sin^2 x} dx = \int \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x}$$

Now put $t = \tan x \Rightarrow dt = \sec^2 x dx$

$$\int \frac{1}{a^2 \cos^2 x + b \sin^2 x} dx = \int \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2} \int \frac{dt}{t^2 + \frac{a^2}{b^2}}$$

$$= \frac{1}{b^2} \cdot \frac{1}{a/b} \tan^{-1} \left(\frac{t}{a/b} \right) + c = \frac{1}{ab} \tan^{-1} \left(\frac{b}{a} \tan x \right) + c$$

$$\therefore \int \frac{1}{a^2 \cos^2 x + b \sin^2 x} dx = \frac{1}{ab} \tan^{-1} \left(\frac{b}{a} \tan x \right) + c$$

Example

a) $\int \frac{dx}{1+3 \sin^2 x}$

c) $\int \frac{dx}{9 \sin^2 x + 4 \cos^2 x}$

b) $\int \frac{dx}{1+\cos^2 x}$

d) $\int_0^{\pi/2} \frac{dx}{9+16 \cos^2 x}$

Solution

a) $\int \frac{dx}{1+3 \sin^2 x} = \int \frac{\sec^2 x dx}{\sec^2 x + 3 \tan^2 x} = \int \frac{\sec^2 x dx}{1+4 \tan^2 x}$

Let $t = \tan x \Rightarrow dt = \sec^2 x dx$

$$\int \frac{dx}{1+3 \sin^2 x} = \int \frac{dt}{1+4t^2}$$

$$= \frac{1}{4} \int \frac{dt}{t^2 + 1/4} = \frac{1}{4} \int \frac{dt}{t^2 + (1/2)^2}$$

$$= \frac{1}{4} \cdot \frac{1}{1/2} \tan^{-1} \left(\frac{t}{1/2} \right) + c$$

$$\therefore \int \frac{dx}{1+3 \sin^2 x} = \frac{1}{2} \tan^{-1}(2 \tan x) + c$$

b) Left as an exercise

Ans: $\frac{1}{2} \tan^{-1} \left[\frac{1}{\sqrt{2}} \tan x \right]$

c) Left as an exercise

Ans: $\frac{1}{6} \tan^{-1} \left[\frac{3}{2} \tan x \right]$

d) Left as an exercise

Ans: $\frac{\pi}{30}$

Integration by Parts

This technique applies to a wide variety of functions and is particularly useful for integrands involving a *product* of algebraic and functions involving trigonometry. E.g. $\int x^n \ln x \, dx$, $\int x^n e^{ax} \, dx$ and $\int x^n \sin mx \, dx$

Statement

If u and v are functions of x , then

$$\int u \cdot v \, dx = u \int v \, dx - \int \frac{du}{dx} \cdot (\int v \, dx) \, dx \text{ or } \int u \, dv = uv - \int v \, dx$$

Proof

Integration by parts is based on the formula for the derivative of a product.

$$\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Integrating both sides

$$\int \frac{d}{dx}(u \cdot v) \cdot dx = \int u \frac{dv}{dx} \cdot dx + \int v \frac{du}{dx} \cdot dx$$

$$\therefore u \cdot v = \int u \cdot dv + \int v \cdot du$$

Rewriting

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

NOTE

$$\text{From } \int \frac{d}{dx}(u \cdot v) \cdot dx = \int u \frac{dv}{dx} \cdot dx + \int v \frac{du}{dx} \cdot dx,$$

$$\text{Let } w = \frac{dv}{dx} \Rightarrow v = \int w \, dx$$

$$\therefore \int u \cdot w \cdot dx = u \int w \, dx - \int (\int w \, dx) \cdot \frac{du}{dx} \cdot dx$$

Guidelines for Integration by Parts

The choices of u and dv in the integration by parts, because it easens the evaluation of second integral than the original one.

- 1) Try letting dv be the most complicated portion of the integrand that fits the basic integration rule. Then u be the remaining factors of the integrand.
- 2) Try letting u be the portion of the integrand whose derivative is a simpler function than dv then dv will be the remaining factors of the integrand.

Example

To integrate $\int x^n e^{ax} \, dx$, $\int x^n \sin ax \, dx$ or $\int x^n \cos ax \, dx$

Let $u = x^n$ and $dv = e^{ax} \, dx, \sin ax, \cos ax$

Integrate

a) $\int x^2 \cdot e^{2x} \, dx$

b) $\int x^2 \cdot \sin x \, dx$

c) $\int_0^{\pi/2} x \cos x \, dx$

Solution

a) Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$, $\frac{dv}{dx} = e^{2x}$ or $v = \int e^{2x} \, dx = \frac{e^{2x}}{2}$

$$\begin{aligned}\therefore \int x^2 \cdot e^{2x} dx &= \frac{x^2 e^{2x}}{2} - \int x \cdot e^{2x} dx \\ &= \frac{x^2 e^{2x}}{2} - \left[\frac{x e^{2x}}{2} - \frac{1}{2} \int e^{2x} dx \right] \\ \therefore \int x^2 \cdot e^{2x} dx &= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4} + c = \frac{e^{2x}}{4} (2x^2 - 2x + 1) + c\end{aligned}$$

b) Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$, $\frac{dv}{dx} = \sin x$ or $v = \int \sin x dx = -\cos x$

$$\begin{aligned}\therefore \int x^2 \cdot \sin x dx &= -x^2 \cos x - \int -\cos x \cdot 2x dx \\ &= -x^2 \cos x + 2 \int x \cdot \cos x dx \\ &= -x^2 \cos x + 2[x \sin x - \int \sin x dx] \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c\end{aligned}$$

c) Let $u = x \Rightarrow \frac{du}{dx} = 1$, $\frac{dv}{dx} = \cos x$ or $v = \int \cos x dx = \sin x$

$$\begin{aligned}\int_0^{\pi/2} x \cos x dx &= [x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx \\ &= \left[\frac{\pi}{2} \sin \frac{\pi}{2} - 0 \right] + [\cos x]_0^{\pi/2} \\ &= \frac{\pi}{2} + \cos \frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1\end{aligned}$$

NOTE

Tabular Method

For problems involving repeated application of integration by parts, tabular method can help organize work.

It works best on integrals of the form

$$\int x^n \sin ax dx, \int x^n \cos ax dx \text{ and } \int x^n e^{ax} dx$$

Example

Evaluate $\int x^2 \sin 4x dx$

Solution

Begin as usual by letting $u = x^2$ and $\frac{dv}{dx} = \sin 4x$

Next create a table consisting of three columns

Alternate signs	u and its derivatives	$\frac{dv}{dx}$ and its anti derivatives
+ —	$\longrightarrow x^2$	$\sin 4x$
- —	$\longrightarrow 2x$	$\longrightarrow -\frac{\cos 4x}{4}$
+ —	$\longrightarrow 2$	$\longrightarrow -\frac{\sin 4x}{16}$
- —	$\longrightarrow 0$	$\longrightarrow \frac{\cos 4x}{64}$

↑
Differentiate until you obtain 0 as a derivative

Finally the solution is given by adding the *signed* products of the diagonal entries to obtain

$$\therefore \int x^2 \sin 4x \, dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32} \cos 4x + c$$

To integrate $\int x^n \ln x \, dx$, $\int x^n \arcsin ax \, dx$ or $\int x^n \arctan ax \, dx$

Let $u = \ln x$, $\arcsin ax$ or $\arctan ax$ and $dv = x^n \, dx$

Evaluate

a) $\int x^2 \ln x \, dx$ b) $\int_0^1 \arcsin x \, dx$ c) $\int x^2 \tan^{-1} x \, dx$

Solution

a) In this case x^2 is more easily integrated than $\ln x$.

Furthermore, the derivative of $\ln x$ is simpler than $\ln x$

$$dv = x^2 \, dx \Rightarrow v = \int x^2 \, dx = \frac{1}{3}x^3, u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \text{ or } du = \frac{dx}{x}$$

$$\begin{aligned} \therefore \int x^2 \ln x \, dx &= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} \cdot dx \\ &= \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^2 \, dx = \frac{x^3 \ln x}{3} - \frac{1}{9}x^3 + c \end{aligned}$$

b) Let $dv = dx \Rightarrow v = \int dx = x$, $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx$

$$\begin{aligned} \therefore \int \arcsin x \, dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx \\ &= x \arcsin x - \frac{1}{2} \int (1-x^2)^{-1/2} (-2x) \, dx \\ &= x \arcsin x - \sqrt{1-x^2} + c \end{aligned}$$

Now evaluating a definite integral

$$\begin{aligned} \int_0^1 \arcsin x \, dx &= [x \arcsin x - \sqrt{1-x^2} + c]_0^1 \\ &= \frac{\pi}{2} - 1 \simeq 0.571 \end{aligned}$$

c) Let $dv = x^2 \Rightarrow v = \int x^2 \, dx = \frac{1}{3}x^3$, $u = \tan^{-1} x \Rightarrow du = \frac{1}{1+x^2} dx$

$$\begin{aligned} \int x^2 \tan^{-1} x \, dx &= \frac{x^3 \tan^{-1} x}{3} - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} \, dx \\ &= \frac{x^3 \tan^{-1} x}{3} - \int \frac{x(1+x^2)-x}{1+x^2} \, dx \\ &= \frac{x^3 \tan^{-1} x}{3} - \int \left(x - \frac{x}{1+x^2} \right) \, dx = \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \left[\frac{x^2}{2} - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \right] \\ &= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{6}x^2 + \frac{1}{6} \ln|x^2 + 1| + c \end{aligned}$$

To integrate $\int e^{ax} \sin bx \, dx$ or $\int e^{ax} \cos bx \, dx$

Let $u = \sin bx$ or $\cos bx$ and $dv = e^{ax} \, dx$

These require repeated use of integration by parts

1. Evaluate

a) $\int e^x \cos 2x \, dx$

b) $\int e^{2x} \cos x \, dx$

Solution

a) Let $dv = e^x \, dx \Rightarrow v = \int e^x \, dx = e^x$, $u = \cos 2x \Rightarrow du = -2 \sin 2x \, dx$
 $\therefore \int e^x \cos 2x \, dx = e^x \cos 2x + 2 \int e^x \sin 2x \, dx \dots\dots\dots (1)$

Dealing with $\int e^x \sin 2x \, dx$

Let $dv = e^x \, dx \Rightarrow v = \int e^x \, dx = e^x$, $u = \sin 2x \Rightarrow du = 2 \cos 2x \, dx$
 $\therefore \int e^x \sin 2x \, dx = e^x \sin 2x - 2 \int e^x \cos 2x \, dx \dots\dots\dots (2)$

Eqn. (2) in to eqn. (1)

$$\therefore \int e^x \cos 2x \, dx = e^x \cos 2x + 2[e^x \sin 2x - 2 \int e^x \cos 2x \, dx]$$

$$= e^x \cos 2x + 2e^x \sin 2x - 4 \int e^x \cos 2x \, dx$$

Because the R.H.S integral is a constant multiple of original integral, you can add it to both sides of the equation

$$\therefore 5 \int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x$$

$$\therefore \int e^x \cos 2x \, dx = \frac{1}{5} e^x \cos 2x + \frac{2}{5} e^x \sin 2x + c$$

2. Show that

a) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$

b) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$

Solution

a) Let $dv = e^{ax} \, dx \Rightarrow v = \int e^{ax} \, dx = \frac{1}{a} e^{ax}$, $u = \sin bx \Rightarrow du = b \cos bx \, dx$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax} \sin bx}{a} - \frac{1}{a} \int b e^{ax} \cos bx \, dx$$

$$= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx \, dx$$

Integrating again

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \left[\frac{e^{ax} \cos bx}{a} - \frac{1}{a} \int -b e^{ax} \sin bx \, dx \right]$$

$$= \frac{e^{ax} \sin bx}{a} - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx$$

Transposing the integral on the R.H.S to the L.H.S

$$\left(1 + \frac{b^2}{a^2} \right) e^{ax} \sin bx = \frac{e^{ax}}{a^2} [a \sin bx - b \cos bx]$$

$$\therefore \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

b) Left as an exercise

NOTE

From integration of trigonometric functions of secants .i.e. $\int \sec^n(mx) \, dx$, when n is *odd*, use integration by parts and for $n = \text{even}$, use $t = \tan mx$

Example

$$\int \sec^3 x \, dx$$

Since the power on secant is odd, use $u = \sec x \Rightarrow du = \sec x \tan x \, dx$ and $dv = \sec^2 x \Rightarrow v = \int \sec^2 x \, dx = \tan x$

$$\begin{aligned} \int \sec^3 x \, dx &= \int \sec^2 x \sec x \, dx = \sec x \tan x - \int \tan x \cdot \sec x \tan x \, dx \\ &= \sec x \tan x - \int \tan^2 x \cdot \sec x \, dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \cdot \sec x \, dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \end{aligned}$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\therefore \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + c$$

Task

Show that

$$\text{a) } \int_0^1 x^2 \tan^{-1} x \, dx = \frac{\pi}{12} - \frac{1}{16} (1 - \ln 2)$$

$$\text{b) } \int_0^1 x \sin^{-1} x \, dx = \frac{\pi}{8}$$

$$\text{c) } \int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + c$$

$$\text{d) } \int x^2 \sin^{-1} x \, dx = \frac{x^3}{3} \sin^{-1} x - \frac{1}{9} (1 - x^2)^{3/2} + \frac{1}{4} \sqrt{1 - x^2} + c$$

$$\text{e) } \int x^{-n} \ln x \, dx = \frac{x^{1-n}}{1-n} \left[\ln x - \frac{1}{1-n} \right] + c$$

$$\text{f) } \int \ln(x^2 + 2) \, dx = x \ln|x^2 + 2| - 2x + 2\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + c$$

$$\text{g) } \int (\ln x)^2 \, dx = x(\ln x)^2 - 2x \ln x + 2x + c$$

$$\text{h) } \int_1^2 x \ln(x^2 + 1) \, dx = \frac{5}{2} \ln 5 - \ln 2 - \frac{3}{2}$$

Disguised Integration by Parts

Evaluate

$$\text{a) } \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx \quad \text{Ans: } -\sin^{-1} x \sqrt{1-x^2} + x + c \text{ use } [u = \sin^{-1} x]$$

$$\text{b) } \int \frac{x^2 \tan^{-1} x}{1+x^2} \, dx \quad \text{Ans: } x \tan^{-1} x - \frac{1}{2} \ln|x^2 + 1| - \frac{1}{2} (\tan^{-1} x)^2 + c$$

Use $[u = \tan^{-1} x]$

$$\text{c) } \int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} \, dx \quad \text{Ans: } \frac{x - \tan^{-1} x}{\sqrt{1+x^2}}$$

$$\text{d) } \int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx \quad \text{Ans:}$$

$$\text{e) } \int \cos \sqrt{x} \, dx \quad \text{Ans:}$$

$$\text{f) } \int \frac{x^3 \sin^{-1} x}{\sqrt{1-x^2}} \, dx \quad \text{Ans:}$$

$$\text{g) } \int_0^{\pi/2} \frac{x}{x + \sin x} \, dx \quad \text{Ans:}$$

$$\text{h) } \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} \, dx \quad \text{Ans: } \frac{\pi}{4} - \frac{1}{2} \ln 2$$

Other numbers on Integration by Parts

a) $\int \sqrt{9-x^2} dx$ b) $\int \frac{x \cos x}{\sin^2 x} dx$

Solution

a) This can be evaluated using $x = 3 \sin \theta$

By integration by parts

$$\begin{aligned} \int \sqrt{9-x^2} dx &= \int 1 \times \sqrt{9-x^2} dx \\ &= \sqrt{9-x^2} \cdot \int 1 dx - \int \frac{d}{dx} \sqrt{9-x^2} \cdot \int 1 dx \\ &= x\sqrt{9-x^2} - \int -\frac{x^2}{\sqrt{9-x^2}} dx \\ &= x\sqrt{9-x^2} - \int \frac{9-x^2-9}{\sqrt{9-x^2}} dx \\ &= x\sqrt{9-x^2} - \int \frac{9-x^2}{\sqrt{9-x^2}} dx + 9 \int \frac{dx}{\sqrt{9-x^2}} \\ &= x\sqrt{9-x^2} - \int \sqrt{9-x^2} dx + 9 \sin^{-1} \frac{x}{3} \end{aligned}$$

$$2 \int \sqrt{9-x^2} dx = x\sqrt{9-x^2} + 9 \sin^{-1} \frac{x}{3}$$

$$\therefore \int \sqrt{9-x^2} dx = \frac{1}{2} x\sqrt{9-x^2} + \frac{9}{2} \sin^{-1} \frac{x}{3} + c$$

Integrals of Exponential Functions

a) **Integrating an exponential function to another base other than e**

$$\int a^x dx$$

To evaluate this integral, differentiate the same function a^x

Let $u = a^x \Rightarrow \ln u = x \ln a$, differentiating w.r.t x

$$\frac{1}{u} \frac{du}{dx} = \ln a \cdot \frac{d}{dx}(x) = \ln a$$

$$\therefore \frac{d}{dx}(a^x) = a^x \ln a \Rightarrow a^x = \frac{1}{\ln a} \cdot \frac{d}{dx}(a^x)$$

Now integrating both sides

$$\Rightarrow \int a^x dx = \frac{1}{\ln a} \int \frac{d}{dx}(a^x) \cdot dx = \frac{a^x}{\ln a}$$

$$\therefore \int a^x dx = \frac{a^x}{\ln a} + c$$

Corrary: If $a = e$, then

$$\int e^x dx = \frac{e^x}{\ln e} + c = e^x + c$$

$$\therefore \int e^x dx = e^x + c$$

Example

Evaluate

a) $\int 2^x dx$

b) $\int 5^{x+3} dx$

c) $\int 4^{-x} dx$

Solution

a) Let $u = 2^x \Rightarrow \ln u = x \ln 2$

$$\frac{1}{u} \frac{du}{dx} = \ln 2 \quad \Rightarrow \quad du = u \ln 2 \, dx = 2^x \ln 2 \, dx \quad \therefore \frac{du}{\ln 2} = 2^x \, dx$$

$$\therefore \int 2^x \, dx = \int \frac{du}{\ln 2} = \frac{u}{\ln 2} + c$$

$$= \frac{2^x}{\ln 2} + c$$

b) Let $u = 5^{x+3} \Rightarrow \ln u = (x+3) \ln 5$

$$\frac{1}{u} \frac{du}{dx} = \ln 5 \, dx \quad \Rightarrow \quad du = u \ln 5 = 5^{x+3} \ln 5 \, dx \quad \therefore \frac{du}{\ln 5} = 5^{x+3} \, dx$$

$$\therefore \int 5^{x+3} \, dx = \int \frac{du}{\ln 5} = \frac{u}{\ln 5} + c$$

$$= \frac{5^{x+3}}{\ln 5} + c$$

c) Left as an exercise. **Ans:** $-\frac{4^{-x}}{\ln 4} + c$

b) Integrating an Exponential Function of the form $\int f'(x).a^{f(x)} dx$

Here let $u = f(x)$

$$\Rightarrow du = f'(x)dx$$

$$\therefore \int f'(x).a^{f(x)} dx = \int a^u du = \frac{a^u}{\ln a} + c$$

$$\therefore \int f'(x).a^{f(x)} dx + c = \frac{a^{f(x)}}{\ln a} + c$$

Corrory: If $a = e$, then

$$\therefore \int f'(x).e^{f(x)} dx + c = \frac{e^{f(x)}}{\ln e} + c = e^{f(x)} + c$$

$$\therefore \int f'(x).e^{f(x)} dx = e^{f(x)} + c$$

Example

Evaluate

a) $\int 5xe^{-x^2} dx$

c) $\int \cos \pi x e^{\sin \pi x} dx$

b) $\int \frac{e^x}{x^2} dx$

d) $\int x \cdot 5^{-x^2} dx$

e) $\int (3-x) \cdot 7^{(3-x)^2} dx$

Solution

a) $\int 5xe^{-x^2} dx = -\frac{5}{2} \int (-2x) \cdot e^{-x^2} dx$

Let $u = -x^2 \Rightarrow du = -2x dx$

$$\begin{aligned} \therefore \int 5xe^{-x^2} dx &= -\frac{5}{2} \int e^u du = -\frac{5}{2} e^u + c \\ &= -\frac{5}{2} e^{-x^2} + c \end{aligned}$$

b) $\int \frac{e^x}{x^2} dx = -\int e^x \cdot \left(-\frac{1}{x^2}\right) dx$

Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$

$$\therefore \int \frac{e^x}{x^2} dx = -\int e^u du = -e^u + c$$

c) $\int \cos \pi x e^{\sin \pi x} dx$

Let $u = \sin \pi x \Rightarrow du = \pi \cos \pi x dx$

$$\begin{aligned} \int \cos \pi x e^{\sin \pi x} dx &= \frac{1}{\pi} \int e^u du = \frac{e^u}{\pi} + c \\ &= \frac{e^{\sin \pi x}}{\pi} + c \end{aligned}$$

d) $\int x \cdot 5^{-x^2} dx$

Let $u = -x^2 \Rightarrow du = -2x dx$

$$\begin{aligned} \int x \cdot 5^{-x^2} dx &= -\frac{1}{2} \int (5^{-x^2}) (-2x) dx \\ &= -\frac{1}{2} \int 5^u du = -\frac{1}{2} \cdot \frac{5^u}{\ln 5} + c \\ &= -\frac{5^{-x^2}}{2 \ln 5} + c \end{aligned}$$

e) $\int (3-x) \cdot 7^{(3-x)^2} dx$

Let $u = (3-x)^2 \Rightarrow du = -2(3-x) dx$

$$\begin{aligned} \int (3-x) \cdot 7^{(3-x)^2} dx &= -\frac{1}{2} \int [-2(3-x)] \cdot 7^{(3-x)^2} dx \\ &= -\frac{1}{2} \int 7^u du = -\frac{1}{2} \cdot \frac{7^u}{\ln 7} + c \\ &= -\frac{7^{(3-x)^2}}{\ln 7} + c \end{aligned}$$

Integration by Partial Fractions

A function of the form $\frac{F(x)}{g(x)}$ where $F(x)$ and $g(x)$ are polynomials is called a Rational Function.

Decomposition of $\frac{F(x)}{g(x)}$ into Partial Fractions

1. Divide if improper

When the degree of the numerator is less than that of the denominator, the fraction is known as *proper* otherwise *improper*.

- i) If the fraction is proper, we resolve it into simple partial fractions and integrate each partial fraction
- ii) In case the fraction is improper *i.e.* degree of the numerator is not less than that of the denominator then we divide the numerator by the denominator till the remainder is of degree less than that of the denominator. Thus an improper fraction can be expressed in terms of a polynomial and a proper fraction. E.g.

$$\frac{2x^3 - 3x^2 + 5}{x^2 - 3x + 2} = 2x + 3 + \frac{5x - 1}{x^2 - 3x + 2}$$

It is easy to integrate a polynomial. To integrate a proper fraction we resolve in to partial fractions.

For resolving proper fractions into partial fractions, we break the denominator in to prime factors which may be linear/quadratic.

2. Factor Denominator

Factorizing the denominator can give factors of the form $(px + q)^n$ or $(ax^2 + bx + c)^n$, where $(ax^2 + bx + c)^n$ is irreducible.

3. Linear Factors

To each linear factor (non repeated) factor $(px + q)$ there is a partial fraction of the form $\frac{A}{px + q}$ where A is a constant.

Further more for each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions

$$\frac{A_1}{px + q} + \frac{A_1}{(px + q)^2} + \dots + \frac{A_m}{(px + q)^m}$$

4. Quadratic Factors

To each non repeated quadratic factor $(ax^2 + bx + c)$ there is a partial fraction of the form $\frac{Ax + B}{ax^2 + bx + c}$

Further more for each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of $n -$ factors

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \dots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Four major cases are illustrated below

Examples

a) Linear Factors (Non Repeated) only, in the Denominator

1. Resolve $f(x) = \frac{x+1}{x^2-5x+6}$ into partial fractions

Solution

$$f(x) = \frac{x+1}{x^2-5x+6} = \frac{x+1}{(x-2)(x-3)}$$

$$\frac{x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

Multiply this equation by the L.C.M of denominator $(x-2)(x-3)$ leads to a *basic equation*

$$x+1 = A(x-3) + B(x-2) \quad \text{(Basic equation)}$$

Because this equation is to be true for all x you substitute any *convenient* values for x to obtain equation in A and B. the most convenient values are the ones that make particular factors equal to zero.

To solve for B, let $x = 3$

$$4 = A(3-3) + B(3-2) \quad \therefore B = 4$$

To solve for A, let $x = 2$

$$3 = A(2-3) + B(2-2) \quad \therefore A = -3$$

$$\therefore f(x) = \frac{x+1}{x^2-5x+6} = -\frac{3}{x-2} + \frac{4}{x-3}$$

2. Integrate

a) $\frac{3x}{(x-1)(x-2)(x-3)}$

b) $\frac{x^2}{x^2+7x+10}$

Solution

a) Let $\frac{3x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$$3x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$\text{Put } x = 1 \Rightarrow 3 = A(-1)(-2) \quad \therefore A = 3/2$$

$$\text{Put } x = 2 \Rightarrow 6 = B(1)(-1) \quad \therefore B = -6$$

$$\text{Put } x = 3 \Rightarrow 9 = C(2)(1) \quad \therefore C = 9/2$$

$$\therefore \frac{3x}{(x-1)(x-2)(x-3)} = \frac{3}{2(x-1)} - \frac{6}{x-2} + \frac{9}{2(x-3)}$$

$$\begin{aligned} \therefore \int \frac{3x}{(x-1)(x-2)(x-3)} &= \frac{3}{2} \int \frac{1}{(x-1)} dx - 6 \int \frac{1}{x-2} dx + \frac{9}{2} \int \frac{1}{(x-3)} dx \\ &= \frac{3}{2} \ln|x-1| - 6 \ln|x-2| + \frac{9}{2} \ln|x-3| + c \end{aligned}$$

b) Since the degree of the numerator is the same as that of the denominator, we divide x^2 by $x^2 + 7x + 10$

$$\frac{x^2}{x^2+7x+10} = 1 - \frac{7x+10}{x^2+7x+10}$$

$$\text{Now let } \frac{7x+10}{x^2+7x+10} = \frac{7x+10}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5}$$

$$\therefore 7x+10 = A(x+5) + B(x+2)$$

$$\text{Put } x = -2 \Rightarrow 7(-2)+10 = A(-2+5) \quad \therefore A = -4/3$$

$$\text{Put } x = -5 \Rightarrow 7(-5)+10 = B(-5+2) \quad \therefore B = 25/3$$

$$\therefore \frac{x^2}{x^2+7x+10} = 1 + \frac{4}{3(x+2)} - \frac{25}{3(x+5)}$$

$$\begin{aligned}\therefore \int \frac{x^2}{x^2+7x+10} dx &= \int \left[1 + \frac{4}{3(x+2)} - \frac{25}{3(x+5)} \right] dx \\ &= x + \frac{4}{3} \ln|x+2| + \frac{25}{3} \ln|x+5| + c\end{aligned}$$

Task

Evaluate

a) $\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} dx$ **Ans:** $x + \ln \left| \frac{(x-3)^3(x-6)^{30}}{(x-5)^{24}} \right| + c$

b) $\int \frac{x^3}{x(x^2-1)} dx$ **Ans:** $x - \ln|x| + \ln|x-1| + c$

3. Two Important Integrals

i) $\int \frac{dx}{x^2-a^2} \quad (x > a)$

ii) $\int \frac{dx}{a^2-x^2} \quad (x < a)$

Solution

i) $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$

Here $A = \frac{1}{2a}$, $B = -\frac{1}{2a}$

$$\begin{aligned}\therefore \int \frac{dx}{x^2-a^2} &= \frac{1}{2a} \left[\int \frac{1}{x-a} dx + \int \frac{1}{x+a} dx \right] \\ &= \frac{1}{2a} [\ln|x-a| - \ln|x+a|] + c\end{aligned}$$

Hence $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c \quad (x > a)$

ii) $\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{A}{x-a} + \frac{B}{x+a}$

Here $A = \frac{1}{2a}$, $B = \frac{1}{2a}$

$$\begin{aligned}\therefore \int \frac{dx}{a^2-x^2} &= \frac{1}{2a} \left[\int \frac{1}{a+x} dx + \int \frac{1}{a-x} dx \right] \\ &= \frac{1}{2a} [\ln|a+x| - \ln|a-x|] + c\end{aligned}$$

Hence $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c \quad (x < a)$

4. Evaluate

a) $\int \frac{dx}{2x^2-3}$

b) $\int \frac{dx}{x^2-25}$

Solution

a) $\int \frac{dx}{2x^2-3} = \frac{1}{2} \int \frac{dx}{x^2-3/2} = \frac{1}{2} \int \frac{dx}{x^2-(\sqrt{3/2})^2}$

$$= \frac{1}{2} \cdot \frac{1}{2 \cdot \sqrt{3/2}} \ln \left| \frac{x-\sqrt{3/2}}{x+\sqrt{3/2}} \right| + c = \frac{1}{4\sqrt{3}} \ln \left| \frac{x-\sqrt{3/2}}{x+\sqrt{3/2}} \right| + c$$

b) $\int \frac{dx}{x^2-25} = \int \frac{dx}{x^2-5^2}$

$$= \frac{1}{2(5)} \ln \left| \frac{x-5}{x+5} \right| + c = \frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| + c$$

b) Linear Repeated and Non Repeated Factors in the Denominator

Evaluate

a) $\int \frac{5x^2+20x+6}{x^3+2x^2+x} dx$

c) $\int \frac{x^2}{(x-1)^3(x+1)} dx$

b) $\int \frac{3x+2}{(x+1)^2(x-2)} dx$

d) $\int \frac{x^2}{(x+2)^2(x+1)} dx$

Solution

$$a) \int \frac{5x^2+20x+6}{x^3+2x^2+x} dx = \int \frac{5x^2+20x+6}{x(x^2+2x+1)} dx = \int \frac{5x^2+20x+6}{x(x+1)^2} dx$$

$$\text{Let } \frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Multiplying through by the L.C.M $x(x+1)^2$

$$5x^2 + 20x + 6 = A(x+1)^2 + B(x+1) + Cx$$

$$\text{Put } x = 0, \Rightarrow 6 = A + 0 + 0 \therefore A = 6$$

$$\text{Put } x = -1, \Rightarrow 5 - 20 + 6 = 0 + 0 - C = A + 0 + 0 \therefore C = 9$$

The most convenient choices of values of x have been used, so to find the value of B you can use *any other value* of x along with the calculated values of A and C .

Using $x = 1, A = 6$ and $C = 9$

$$5 + 20 + 6 = A(4) + B(2) + C$$

$$31 = 6(4) + B(2) + 9 \therefore B = -1$$

$$\begin{aligned} \therefore \int \frac{5x^2+20x+6}{x^3+2x^2+x} dx &= \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2} \right) dx = \int \left(\frac{6}{x} - \frac{1}{x+1} + 9(x+1)^{-2} \right) dx \\ &= 6 \ln|x| - \ln|x+1| - \frac{9}{x+1} + c \end{aligned}$$

$$b) \text{ Left as an exercise. } \quad \textbf{Ans: } -\frac{1}{3(x+1)} + \frac{8}{9} \ln \left| \frac{x-2}{x+1} \right| + c$$

$$c) \text{ Left as an exercise. } \quad \textbf{Ans: } -\frac{1}{4(x-1)^2} - \frac{3}{4(x-1)} + \frac{1}{8} \ln \left| \frac{x-1}{x+1} \right| + c$$

$$d) \text{ Left as an exercise. } \quad \textbf{Ans: } \ln|x+1| + \frac{4}{x+2} + c$$

c) Linear and Quadratic Factors (Non Repeated) in the Denominator

Evaluate

$$a) \int \frac{x}{(x-1)(x^2+4)} dx$$

$$b) \int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)} dx$$

Solution

$$a) \text{ Let } \frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

Multiplying both sides by $(x-1)(x^2+4)$

$$x = A(x^2+4) + (Bx+C)(x-1)$$

$$\text{Put } x = 1, \Rightarrow 1 = A(1^2+4) + 0 \therefore A = 1/5$$

$$\text{Put } x = 0, \Rightarrow 0 = 4A - C, \text{ but } A = 1/5 \therefore C = 4/5$$

Using the other convenient value of x to obtain B

Putting $x = -1$, then using $A = 1/5$ and $C = 4/5$

$$\Rightarrow -1 = A(1+4) + (-B+C)(-1-1)$$

$$\Rightarrow -1 = \frac{1}{5}(5) - 2\left(-B + \frac{4}{5}\right) \therefore -1 = 1 + 2B - \frac{8}{5} \Rightarrow B = -\frac{1}{5}$$

$$\begin{aligned} \therefore \int \frac{x}{(x-1)(x^2+4)} dx &= \frac{1}{5} \int \frac{1}{x-1} dx + \int \frac{-\frac{1}{5}x + \frac{4}{5}}{x^2+4} dx \\ &= \frac{1}{5} \ln|x-1| - \frac{1}{5} \int \frac{x-4}{x^2+4} dx \\ &= \frac{1}{5} \ln|x-1| - \frac{1}{5} \int \frac{x}{x^2+4} dx + \frac{1}{5} \int \frac{4}{x^2+4} dx \\ &= \frac{1}{5} \ln|x-1| - \frac{1}{5(2)} \int \frac{2x}{x^2+4} dx + \frac{4}{5} \int \frac{1}{x^2+2^2} dx \end{aligned}$$

$$= \frac{1}{5} \ln|x-1| - \frac{1}{10} \ln|x^2+4| dx + \frac{4}{5} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + c$$

$$= \frac{1}{5} \ln|x-1| - \frac{1}{10} \ln|x^2+4| dx + \frac{2}{5} \tan^{-1} \frac{x}{2} + c$$

b) Because $(x^2-x)(x^2+4) = x(x-1)(x^2+4)$

$$\text{Let } \frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

Multiplying through by the least common denominator $x(x-1)(x^2+4)$ yields the *basic equation*

$$2x^3 - 4x - 8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

$$\text{Putting } x = 0 \Rightarrow -8 = -4A + 0 + 0 \quad \therefore A = 2$$

$$\text{Putting } x = 1 \Rightarrow -10 = 0 + 5B + 0 \quad \therefore B = -2$$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values of x and solving the resulting system of linear equations.

If $x = -1$, then using $A = 2$ and $B = -2$

$$-6 = (2)(-2)(5) + (-2)(-1)(5) + (-C+D)(-1)(-2) \quad \therefore 2 = -C + D$$

If $x = 2$, then using $A = 2$ and $B = -2$

$$0 = (2)(1)(8) + (-2)(2)(8) + (2C+D)(2)(1) \quad \therefore 8 = 2C + D$$

Solving the linear system of equations

$$\therefore C = 2, D = 4$$

$$\therefore \int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)} dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4} \right) dx$$

$$= 2 \ln|x| - 2 \ln|x-1| + \ln|x^2+4| - 2 \tan^{-1} \frac{x}{2} + c$$

Repeated Quadratic Factors

Evaluate

$$\text{a) } \int \frac{8x^3+13x}{(x^2+2)^2} dx$$

$$\text{b) } \int \frac{x^2}{(x^2+1)^2} dx$$

Solution

$$\text{a) Let } \frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

Multiplying through by the least common denominator $(x^2+2)^2$ yields the *basic equation*.

$$8x^3 + 13x = (Ax+B)(x^2+2) + Cx + D$$

Expanding the equation and collecting the like terms

$$8x^3 + 13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

$$= Ax^3 + Bx^2 + (2A+C)x + (2B+D)$$

Now you can equate coefficients of like terms on opposite sides of the equation

$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A+C)x + (2B+D)$$

Using the known values $A = 8$ and $B = 0$

$$13 = 2A + C = 2(8) + C \quad \therefore C = -3$$

$$0 = 2B + D = 2(0) + D \quad \therefore D = 0$$

$$\therefore \int \frac{8x^3+13x}{(x^2+2)^2} dx = \int \left(\frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2} \right) dx = 4 \ln|x^2+2| + \frac{3}{2(x^2+2)} + c$$

b) Let $\frac{x^2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+2)^2}$

Multiplying through by the least common denominator $(x^2+1)^2$ yields the *basic equation*.

$$\begin{aligned} x^2 &= (Ax+B)(x^2+1) + Cx + D \\ &= Ax^3 + Bx^2 + (A+C)x + (B+D) \end{aligned}$$

Equating coefficients you obtain

$$A = 0, B = 1, C = 0, D = -1$$

$$\begin{aligned} \therefore \int \frac{x^2}{(x^2+1)^2} dx &= \int \left[\frac{1}{x^2+1} - \frac{1}{(x^2+2)^2} \right] dx \\ &= \arctan x - \frac{1}{2} \left(\arctan x + \frac{x}{x^2+1} \right) + c \\ &= \frac{1}{2} \left(\arctan x + \frac{x}{x^2+1} \right) + c \end{aligned}$$

Task

Evaluate

a) $\int \frac{x^4-x^3-x-1}{x^3-x^2} dx$

Ans: $\frac{x^2}{2} + 2 \ln|x| - \frac{1}{x} - 2 \ln|x-1| + c$

b) $\int \frac{x}{(1+x)(1+x^2)} dx$

Ans: $-\frac{1}{2} \ln|1+x| + \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \tan^{-1} x + c$

c) $\int \frac{x^2}{x^4-2x^2-8} dx$

Ans: $\frac{1}{6} \left[\ln \left| \frac{x-2}{x+2} \right| + \sqrt{2} \arctan \left(\frac{x}{\sqrt{2}} \right) \right] + c$

d) $\int \frac{x}{16x^4-1} dx$

Ans: $\frac{1}{16} \ln \left| \frac{4x^2-1}{4x^2+1} \right| + c$

Guidelines for Solving the Basic Equation

Linear factors

1. Substitute the roots of the distinct linear factors in to the basic equation
2. For repeated linear factors, use the coefficients determined in part (1) to rewrite the basic equation. Then substitute *other* convenient values of x and solve for the remaining coefficients

Quadratic factors

1. Expand the basic equation
2. Collect terms according to powers of x
3. Equate the coefficient of like terms to obtain a system of linear equations involving A, B, C etc
4. Solve the system of linear equations

NOTE

When integrating rational expressions, keep in mind that for *improper* rational expressions such as;

$$\frac{F(x)}{g(x)} = \frac{2x^3+x^2-7x+7}{x^2+x-2}. \text{ You must first divide to obtain;}$$

$$\frac{F(x)}{g(x)} = 2x - 1 + \frac{-2x+5}{x^2+x-2}$$

The proper rational expression is then decomposed in to its partial fractions.

AREA OF PLANE CURVES

DISC METHOD

a) Areas of plane curves. Rectangular coordinates

Consider the curve $y = f(x)$

The area between the curve, the x – axis and the ordinates at $x = a$ and $x = b$ is given by

$$\boxed{\text{Area} = \int_a^b y \, dx} \quad \dots\dots\dots (1)$$

The value of y in terms of x is substituted in to the above equation. It is obtained from the equation of the curve.

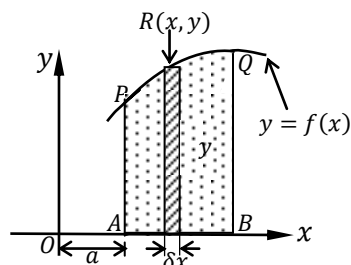
Proof

Considering a vertical rectangle element of base δx and altitude y

Area of the rectangle, $\delta A = y \cdot \delta x$

Total area is obtained as the limit of sum of all areas of these elements between the ordinates AP and BQ

$$A = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} y \cdot \delta x = \int_a^b y \, dx$$



Similarly the area bounded by the curve $x = \varphi(y)$, the y –axis and the horizontal lines $y = c$ and $y = d$ is given by

$$\boxed{\text{Area} = \int_c^d x \, dy} \quad \dots\dots\dots (2)$$

The value of x in terms of y is substituted from the equation of the curve. It is obtained from the equation of the curve.

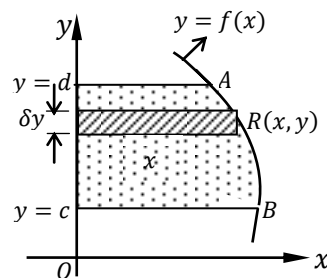
Proof

Considering a horizontal rectangle element of base δy and altitude x

Area of the rectangle, $\delta A = x \cdot \delta y$

Total area is obtained as the limit of sum of all areas of these elements between the ordinates $y = c$ and $y = d$

$$A = \lim_{\delta y \rightarrow 0} \sum_{y=c}^{y=d} x \cdot \delta y = \int_c^d x \cdot dy$$



NOTE

1. It is always necessary to trace the curve with reference to which the region whose area is to be determined is given. The shape of the curve will enable the area to be expressed as a definite integral or a suitable combination of integrals.

2. If the area to be determined lies *above* the x – axis, the area formula $\int_a^b y \, dx$ gives a *positive value* as in fig (i) and a *negative value* if the area is *below* the x –axis as in fig (ii).

If however, the curve AEC crosses the x –axis at say, at E, the area formula $\int_a^b y \, dx$ *does not* give the sum of the area of the two portions AEC and EDC but their *difference*.

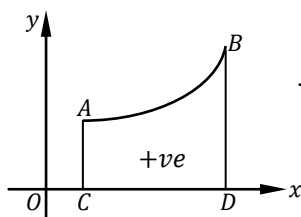


Fig. (i)

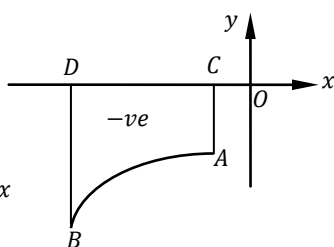


Fig. (ii)

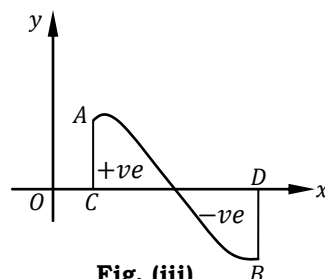


Fig. (iii)

Area = Area ACE – Area BDE , when the formula of area $\int_a^b y \, dx$ is used

NOTE

This area can be obtained by taking separate numerical values of ACE and BDE, then adding, i.e

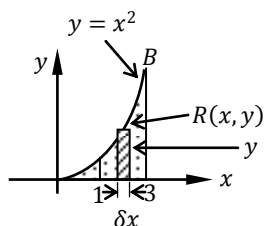
Area = Area ACE (Numerically) + Area BDE (Numerically)

Example

1. Find the area bounded by the curve $y = x^2$, the x –axis and the ordinates $x = 1$ and $x = 3$

Solution

$$\begin{aligned} \text{Required area} &= \int_1^3 y \, dx \\ &= \int_1^3 x^2 \, dx \\ &= \left[\frac{x^3}{3} \right]_1^3 = 9 - \frac{1}{3} \\ &= \frac{26}{3} \text{ sq. units} \end{aligned}$$

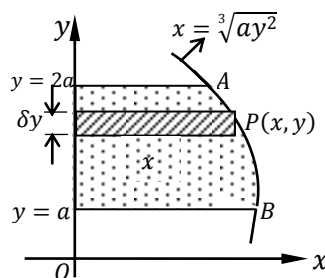


P

2. Find the area bounded by the curve $ay^2 = x^3$, the y – axis and the lines.

Solution

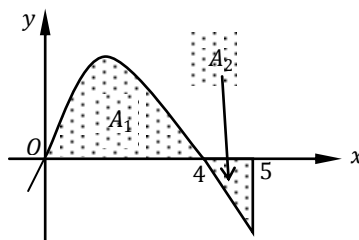
$$\begin{aligned} \text{Area required} &= \int_a^{2a} x \cdot dy \\ &= \int_a^{2a} a^{1/3} \cdot y^{2/3} dy \\ &= \frac{3}{5} a^{1/3} [y^{5/3}]_a^{2a} \\ &= \frac{3}{5} a^2 (\sqrt[3]{32} - 1) \\ &= 1.304a^2 \end{aligned}$$



3. Find the area between the curve $y = x(4 - x)$ and the x – axis from $x = 0$ to $x = 5$

Solution

First, make a sketch of the curve $y = x(4 - x)$. The sketch shows that the required area is in two parts. One part lies above the x -axis and therefore its area is *ve* and the other part lies below the x -axis and has a *-ve* area.



Using $\int_a^b y \, dx$, calculating the areas separately

$$\begin{aligned} A_1 &= \int_0^4 x(4 - x) \, dx \\ &= \left[2x^2 - \frac{x^3}{3} \right]_0^4 \\ &= \left(32 - \frac{64}{3} \right) - 0 \\ &= +\frac{32}{3} = +10\frac{2}{3} \end{aligned}$$

$$\begin{aligned} A_2 &= \int_4^5 x(4 - x) \, dx \\ &= \left[2x^2 - \frac{x^3}{3} \right]_4^5 \\ &= \left(50 - \frac{125}{3} \right) - \left(32 - \frac{64}{3} \right) \\ &= -2\frac{1}{3} \end{aligned}$$

Thus the total area under the curve between $x = 0$ and $x = 5$ is given by the sum of the *numerical* values of these two parts.

$$\text{Required area} = 10\frac{2}{3} + 2\frac{1}{3} = 13 \text{ sq. units}$$

NOTE

1. Integrating $\int_0^5 x(4 - x) \, dx$ does not give that required area.
2. Required area $= A_1 + A_2$ (Taking areas with their signs)
 $= 10\frac{2}{3} - 2\frac{1}{3} = 13 \text{ sq. units}$

Area of a Region between Two Curves

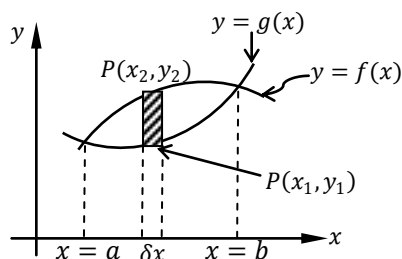
Let $y = f(x)$ and $y = g(x)$ be the two equations of the two curves. Solve these two equations simultaneously for x by eliminating y and thus find the x -coordinates of the point of intersection.

Let these curves intersect at $x = a$ and $x = b$.

Taking the elementary strip of base/width δx and altitude/height [value of y at the upper boundary] - [value of y at the lower boundary]

$$\text{i.e. } [f(x) - g(x)]$$

$$\text{Area of the strip} = [f(x) - g(x)]\delta x$$



$$\therefore \text{Required Area} = \int_a^b [f(x) - g(x)] \, dx$$

Thus area of the region between $y = f(x)$ and $y = g(x)$ is:

(Area of Region under $y = f(x)$) - (Area of region under $y = g(x)$) or

$$\text{Area} = \int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$

NOTE

The above formula $\text{Area} = \int_a^b [f(x) - g(x)] dx$ on the interval $[a, b]$ is used provided $g(x) \leq f(x)$.

Vertical elements (of width δx) implies integration w.r.t x , where as a horizontal element (of width δy) implies integration w.r.t y .

Example

1. Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$

Solution

Obtaining x - coordinates of points of intersection

$$4x = (2x - 4)^2 \Rightarrow 4x = 4x^2 - 16x + 16 \Rightarrow 6$$

$$\Rightarrow 4x^2 - 20x + 16 = 0 \text{ or } x^2 - 5x + 4 = 0$$

$$\Rightarrow (x - 1)(x - 4) = 0 \therefore x = 1, 4$$

$$\text{For } x = 1, y = 2(1) - 4 = -2; \quad x = 4, y = 2(4) - 4 = 4$$

$$\therefore (1, -2), (4, 4)$$

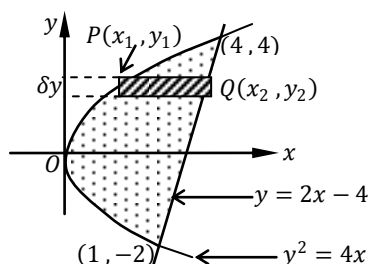
Sketching curves

Considering a horizontal strip of width δy and height $(x_2 - x_1)$, area of strip is

$$\begin{aligned} \delta A &= (x_2 - x_1) \delta y \\ &= \left[\frac{y+4}{2} - \frac{y^2}{4} \right] \delta y \end{aligned}$$

$$\text{Area of the region} = \int_{-2}^4 \left(2 + \frac{y}{2} - \frac{y^2}{4} \right) dy$$

$$= \left[2y + \frac{y^2}{4} - \frac{y^3}{12} \right]_{-2}^4 = 9 \text{ sq. units}$$



NOTE

When a horizontal strip is drawn, this strip runs from the parabola to the line at any point drawn, hence one definite integral is formulated.

Alternatively

Considering a vertical strip, one strip can run from the line to the parabola and another runs from one branch of parabola to the other branch.

Dividing the area by the line $x = 1$.

For the approximating rectangle to the right, the width is δx , the height is:

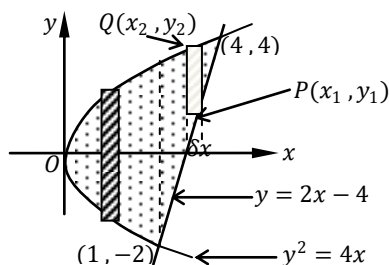
$$2\sqrt{x} - (2x - 4) = 2\sqrt{x} - 2x + 4, \text{ and area is:}$$

$$\delta A = (2\sqrt{x} - 2x + 4) \delta x$$

For the approximating rectangle to the left of this line the width is,

δx , the height (Making use of symmetry) is $2y = 4\sqrt{x}$, and the area is $4\sqrt{x} \cdot \delta x$

$$\text{Required area} = \int_0^1 4\sqrt{x} dx + \int_1^4 (2\sqrt{x} - 2x + 4) dx$$



$$= \left[\frac{8}{3} x^{3/2} \right]_0^1 + \left[\frac{4}{3} x^{3/2} - x^2 + 4x \right]_1^4$$

$$= \frac{8}{3} + \frac{19}{3} = 9 \text{ sq. units}$$

NOTE

When finding area between two curves. i.e. a curve and a line, choose a suitable element which gives only one *definite integral*. For the above case, a vertical element gives two integrals where as a horizontal element gives one definite integral.

2. Find the area of the region bounded by the graphs

a) $y = x^2 + 2, y = -x, x = 0$ and $x = 1$

b) $y = 2 - x^2, y = x$

c) $x = 3 - y^2$ and $y = x - 1$

Solution

a) Points of intersection

$$y = x^2 + 2, y = -x \Rightarrow -x = x^2 + 2 \text{ or } x^2 + x + 2 = 0$$

This equation has no real values of x , hence no points of intersection

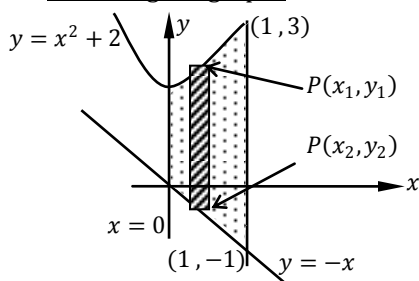
$$y = x^2 + 2, x = 0 \Rightarrow (0, 2)$$

$$y = -x, x = 0 \Rightarrow (0, 0)$$

$$y = x^2 + 2, x = 1 \Rightarrow (1, 3)$$

$$y = -x, x = 1 \Rightarrow (1, -1)$$

Sketching the graphs



Taking a vertical element of width δx and a height $(y_2 - y_1)$

Its area, $\delta A = (y_2 - y_1)\delta x$

$$\begin{aligned} \text{Required area} &= \int_0^1 (x^2 + 2 - (-x)) dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1 = \frac{1}{3} + \frac{1}{2} + 2 \\ &= \frac{17}{6} \text{ sq. units} \end{aligned}$$

b) See example (a) above.

Ans: $\frac{9}{2}$

c) See example (a) above.

Ans: $\frac{9}{2}$

3. Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$

Solution

Obtaining the points of intersection (x -coordinates)

$$3x^3 - x^2 - 10x = -x^2 + 2x$$

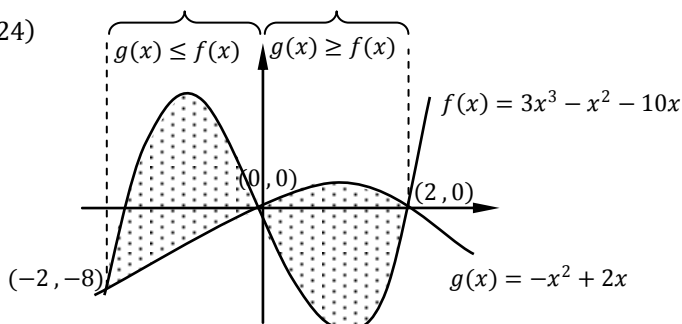
$$3x^3 - 12x = 0 \Rightarrow 3x(x^2 - 4) = 0 \therefore x = 0, \pm 2$$

Sketch the curves to get the curve below, from which,

$$\begin{aligned} \text{Required area, } A &= \int_{-2}^0 [f(x) - g(x)] dx + \int_0^2 [g(x) - f(x)] dx \\ &= \int_{-2}^0 (3x^3 - 12x) dx + \int_0^2 (-3x^3 + 12x) dx \\ &= \left[\frac{3x^4}{4} - 6x^2 \right]_{-2}^0 + \left[-\frac{3x^4}{4} + 6x^2 \right]_0^2 \end{aligned}$$

$$= -(12 - 24) + (-12 + 24)$$

$$= 24 \text{ sq. units}$$



NOTE

In general, to determine the area between two curves you can use

$$A = \int_{x_1}^{x_2} [(Top \ Curve) - (Bottom \ Curve)] dx \quad [\text{Vertical rectangular strip used}]$$

$$A = \int_{y_1}^{y_2} [(Right \ Curve) - (Left \ Curve)] dx \quad [\text{Horizontal rectangular strip used}]$$

, where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

4. Sketch the region bounded by the graphs and find the area of the region

a) $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq 2\pi$

b) $y = 2 \sin x$ and $y = \tan x$ for $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$

c) $y = 2 \sin x + \sin 2x$ and $y = 0$ for $0 \leq x \leq \pi$

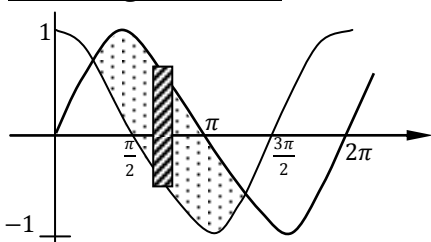
Solution

a) Points of intersection

$$\sin x = \cos x \Rightarrow \frac{\sin x}{\cos x} = 1 \text{ or } \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\text{Thus } a = \frac{\pi}{4} \text{ and } x = \frac{5\pi}{4}$$

Sketching the curves



Since $\sin x \geq \cos x$ on the interval

$\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$, the area is given by

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx$$

$$= [-\cos x - \sin x]_{\pi/4}^{5\pi/4} = 2\sqrt{2}$$

b) Left as an exercise.

Ans: $A = 2(1 - \ln 2) \approx 0.614$

c) Left as an exercise.

Ans: $A = 2(1 - \ln 2) \approx 0.614$

Task

a) Find the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.

Ans: $\frac{1}{6}(3\pi + 2)$

b) Find the area common to the parabolas $y^2 = 5x + 6$ and the circle $x^2 = y$.

Ans: $\frac{27}{5}$

- c) Find the area common to the parabolas $x^2 = 9y$ and the circle $y^2 = 9x$.

Ans: 27

5. Find the area of the region bounded by graphs $x = 3 - y^2$ and $y = x - 1$

Solution

Points of intersection

$$y + 1 = 3 - y^2 \Rightarrow y^2 + y - 2 = 0 \quad \therefore y = 1, -2 \text{ points are: } (-1, -2), (2, 1)$$

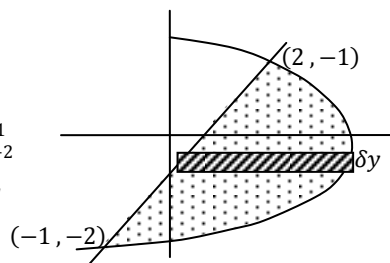
Let $f(y) = y + 1, g(y) = 3 - y^2$; since $f(y) \leq g(y)$ on the interval $[-2, 1]$ then

$$\delta A = [g(y) - f(y)]\delta y = [(3 - y^2 - (y + 1))]\delta y$$

$$\text{Hence area, } A = \int_{-2}^1 (3 - y^2 - (y + 1)) dy$$

$$= \int_{-2}^1 (-y^2 - y + 2) dy = \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y \right]_{-2}^1$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) = \frac{9}{2} \text{ sq. units}$$



Alternatively

Let $f(x) = x - 1, g(x) = \pm\sqrt{3 - x}$. For the

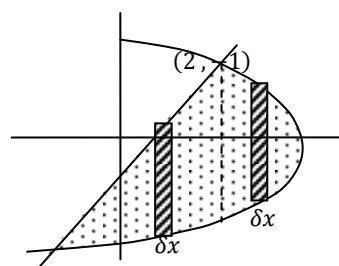
interval $x = -1$ to $x = 2, f(x) \geq -g(x)$

and for the interval $x = -1$ to $x = 2,$

$g(x) \leq -g(x)$

Total area, $A = \int_{-1}^2 [f(x) - g(x)] dx +$

$$\int_{-1}^2 [g(x) - (-g(x))] dx \quad (-1, -2)$$



$$\text{Area, } A = \int_{-1}^2 [(x - 1) + \sqrt{3 - x}] dx + \int_{-1}^2 [\sqrt{3 - x} - (-\sqrt{3 - x})] dx$$

$$= \left[\frac{x^2}{2} - x - \frac{(3 - x)^{3/2}}{3/2} \right]_{-1}^2 + 2 \cdot \left[\frac{(3 - x)^{3/2}}{3/2} \right]_{-1}^2$$

$$= \left(\frac{2^2}{2} - 2 - \frac{(3 - 2)^{3/2}}{3/2} - \left(\frac{(-1)^2}{2} + 1 - \frac{(3 + 1)^{3/2}}{3/2} \right) \right) + 2 \left(\frac{(3 - 2)^{3/2}}{3/2} - \frac{(3 - 3)^{3/2}}{3/2} \right)$$

$$= -\frac{2}{3} - \frac{3}{2} + \frac{16}{3} + \frac{4}{3} = \frac{9}{2}$$

VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION

Solids of Revolution

A solid generated by revolving a plane area about a line in the plane is called a *solid of revolution*. The line is called the *axis of revolution*

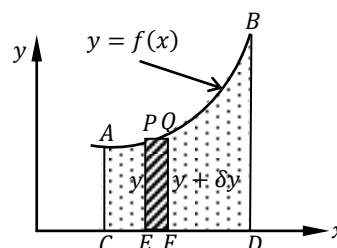
Revolution about x – axis

The area bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and the x – axis, revolves about the x – axis. Find the volume of the solid generated.

Solution

Let AB be the curve $f = f(x)$, CA and DB be ordinates $x = a$ and $x = b$ respectively.

Let $P(x, y)$ be any point on the curve and $Q(x + \delta x, y + \delta y)$ be the neighboring point



PE and QF be ordinates of P and Q so that $EP = y$ and $FQ = y + \delta y$ and $EF = \delta x$

When that area ACDB revolves about the x –axis, the strip PEQF (rectangle) forms a disc of volume $\delta V = \pi y^2 \delta x$ (Approximately)

Summing up the volumes of all such discs generated by the different strips from $x = a$ and $x = b$, the required volume of revolution, V is given by

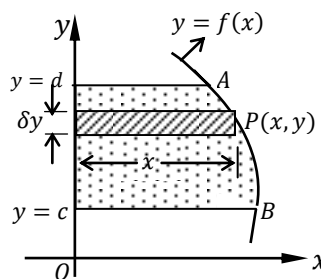
$$V = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} \pi y^2 \delta x = \pi \int_a^b y^2 dx$$

$$\therefore V = \pi \int_a^b y^2 dx$$

Revolution about y – axis

Proceeding as above, the volume of solid generated by revolution of the area bounded by the curve $x = \phi(y)$, the line $y = c$ and $y = d$ and y – axis, about y – axis is given by:

$$\therefore V = \pi \int_c^d x^2 dy$$



NOTE

1. The above method of finding volume of a solid of revolution about the axis is known as *Disc Method*. Its major application involves a plane bounded by the graph $y = f(x)$ or $x = \phi(y)$ and the x –axis or y – axis respectively.

If the axis of revolution is x –axis, then the radius is simply $f(x)$ and if the radius is y –axis, the radius is $\phi(y)$

2. In general to find the volume of a solid of revolution with the disc method, use the formula

Horizontal axis of revolution

$$\text{Volume} = \pi \int_a^b [R(x)]^2 dx$$

Vertical axis of revolution

$$\therefore V = \pi \int_c^d [R(y)]^2 dy$$

Example

1. Find the volume of a sphere of radius a .

Solution

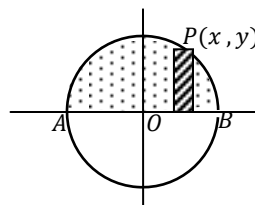
$\therefore V = \pi \int_c^d x^2 dy$ Let the sphere be formed by the revolution of the circle $x^2 + y^2 = a^2$ about the x -axis

Consider a vertical element of width δx and height y .

Revolving the strip gives the volume of a disc

$$\delta V = \pi y^2 \delta x$$

$$\begin{aligned} \text{Required volume} &= \int_{-a}^a \pi y^2 dx = \int_{-a}^a \pi (a^2 - x^2) dx \\ &= 2\pi \int_0^a \pi (a^2 - x^2) dx \quad (\because a^2 - x^2 \text{ is an even function}) \\ &= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a = 2\pi \left(a^3 - \frac{a^3}{3} \right) = \frac{4}{3} \pi a^3 \end{aligned}$$



NOTE

1. The sphere is formed by revolution of a semi circular area
2. The same sphere is formed by revolution of the above circle about y -axis
2. Find the volume generated by revolving
 - i) The area in the first quadrant bounded by the parabola $y^2 = 8x$ and its latus rectum about x -axis
 - ii) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x -axis

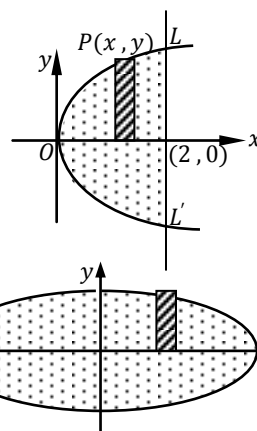
Solution

- i) The latus rectum of a parabola $y^2 = 8x$ is $x = 2$

Required is the volume generated by revolving the area in the first quadrant bounded by the parabola $y^2 = 8x$, the line $x = 2$ and x -axis.

$$\begin{aligned} \text{Using } V &= \pi \int_0^2 y^2 dx = \pi \int_0^2 8x dx \\ &= 8\pi \left[\frac{x^2}{2} \right]_0^2 = 16\pi \text{ cubic units} \end{aligned}$$

- ii) $V = \int_{-a}^a \pi y^2 dx = 2\pi \int_0^a b^2 \left(1 - \frac{x^2}{a^2} \right) dx$
- $$\left(\because \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \right)$$
- $$= 2\pi \left[x - \frac{x^3}{3a^2} \right]_0^a = 2\pi b^2 \left(a - \frac{a^3}{3a^2} \right) = \frac{4\pi b^2 a}{3}$$



3. Find the volume of the solid generated by revolving the region bounded by

a) $y = 2 - x^2$ and $y = 1$ about the line $y = 1$

b) $y^2 = 8x$ and its latus rectum about the latus rectum

Solution

a) Points of intersection.

$$1 = 2 - x^2 \Rightarrow x^2 = 1 \therefore x = \pm 1$$

The radius is $y_2 - y_1$

$$\therefore R(x) = 2 - x^2 - 1 \\ = 1 - x^2$$

Finally, integrate between -1 and 1 to find the volume

$$V = \pi \int_a^b [R(x)]^2 dx = \pi \int_{-1}^1 (1 - x^2)^2 dx = \pi \int_{-1}^1 [1 - 2x^2 + x^4] dx \\ = \pi \left[x - \frac{2}{3}x^3 + \frac{x^5}{5} \right]_{-1}^1 = \frac{16\pi}{15} \text{ cubic units}$$

b) Taking a strip to be horizontal

$$R(x) = 2 - x$$

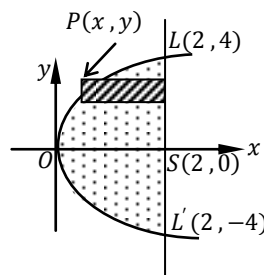
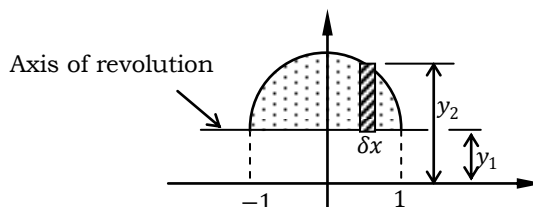
$$\therefore \text{Required volume} = \pi \int_{-4}^4 (2 - x)^2 dy$$

$$= \pi \int_{-4}^4 \left(2 - \frac{y^2}{8} \right)^2 dy \quad (\because y^2 = 8x, x = \frac{y^2}{8})$$

$$= 2\pi \int_0^4 \left(4 + \frac{y^4}{64} - \frac{y^2}{2} \right) dy \quad (\text{Integrand is even})$$

$$= 2\pi \left[4y + \frac{1}{64} \cdot \frac{y^5}{5} - \frac{1}{2} \cdot \frac{y^3}{3} \right]_0^4$$

$$= 2\pi \left[4(4) + \frac{1}{64} \cdot \frac{(4)^5}{5} - \frac{1}{2} \cdot \frac{(4)^3}{3} \right] = \frac{256\pi}{15}$$



NOTE

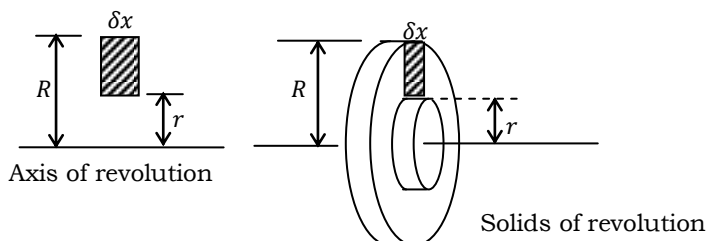
The disc method is very useful if the axis of revolution is one of the boundaries of the curve to be rotated.

THE WASHER METHOD

The washer is formed by revolving a rectangle about an axis.

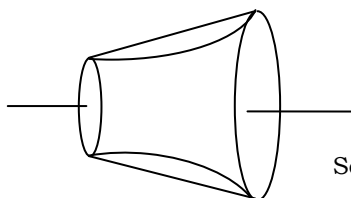
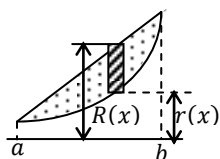
If r and R are the inner and outer radii of the washer and δx is the width of the washer, then the volume is given by

$$\text{Volume of washer} = \pi(R^2 - r^2) \cdot \delta x$$



Mathematics for an A-Level Student Volume II

Consider a region bounded by an *outer radius* $R(x)$ and an *inner radius* $r(x)$ as below.



Solid of revolution with a hole

If the region is revolved about its axis of revolution, then the volume of the resulting solid is given by;

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx$$

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx$$

NOTE

The axis of revolution is not the boundary of the area being rotated.

Example

1. a) Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about the y -axis
- b) Find the volume generated by revolving about the x -axis the area cut off from the parabola $9y = 4(9 - x^2)$ by the line $4x + 3y = 12$
- c) Find the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x -axis

Solution

a)

The revolution of the strip about the y -axis generates a washer whose volume
 =(Volume generated by revolving Rectangle ABCD)

Minus

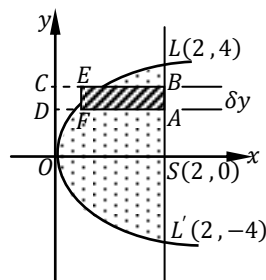
(Volume generated by revolving Rectangle DPEC)

$$= \pi(2)^2 \delta y - \pi x^2 \delta y$$

$$= \pi(4 - x^2) \delta y$$

$$\text{Required volume} = \int_{-4}^4 \pi(4 - x^2) dy = 2\pi \int_{-4}^4 \left(4 - \frac{y^4}{64}\right) dy$$

$$= 2\pi \left[4y - \frac{y^5}{320}\right] = \frac{128\pi}{5}$$



- b) Parabola is symmetrical about y - axis and its intercepts are: $C(-3, 0)$ and $B(3, 0)$

Points of intersection

$$9y = 4(9 - x^2), \quad 4x + 3y = 12 \quad \text{or} \quad y = \left(\frac{12-4x}{3}\right)$$

$$\Rightarrow 9\left(\frac{12-4x}{3}\right) = 4(9 - x^2), \quad \text{solving } x = 0, 3$$

$\therefore$ Points are: $A(0, 4)$ $B(3, 0)$

When the strip PQRS revolves about

x - axis it generates a washer with volume

$$\delta V = \pi(y_1^2 - y_2^2)\delta x$$

Hence Required volume $= \pi \int_0^3 (y_1^2 - y_2^2) dx$

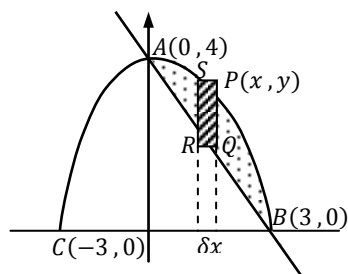
$$= \pi \int_0^3 \left[\left(\frac{4(9-x^2)}{9}\right)^2 - \left(\frac{12-4x}{3}\right)^2 \right] dx$$

$$= \pi \int_0^3 \left[\left(\frac{16}{81}(81 + x^4 - 18x^2)\right) - \frac{16}{81}(9 + x^2 - 6x) \right] dx$$

$$= \frac{16\pi}{81} \int_0^3 (x^4 - 27x^2 + 54x) dx$$

$$= \frac{16\pi}{81} \left[\frac{x^5}{5} - 9x^3 + 27x^2 \right]_0^3$$

$$= \frac{16\pi}{81} \left[\frac{243}{5} - 243 + 243 \right] = \frac{48\pi}{5} \text{ cubic units}$$



- c) Points of intersection

$$y = \sqrt{x}, \quad y = x^2 \Rightarrow x^4 = x$$

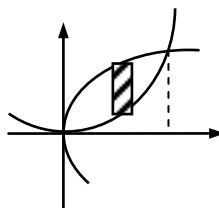
$$\Rightarrow x(x^3 - 1) = 0 \quad \therefore x = 0, 1$$

Points are; $(0, 0)$, $(1, 1)$

$$\therefore V = \int_a^b ([R(x)]^2 - [r(x)]^2) dx$$

$$= \pi \int_0^1 (x - x^4) dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$$

$$= \frac{3\pi}{10} \text{ cubic units}$$



NOTE

For all the above formulae on solids of revolution the angle of rotation is

$360^\circ = 2\pi$. However if the angle is θ , the formulae are multiplied by the

constant $\frac{\theta}{360^\circ}$ if θ is in degrees or $\frac{\theta}{2\pi}$ if θ is in radians.

Example

If rotation is through 90° about the x - axis, the formula will be:

$$V = \frac{\pi}{4} \int_a^b y^2 dx \quad \text{or} \quad V = \frac{\pi}{4} \int_a^b ([R(x)]^2 - [r(x)]^2) dx$$

VOLUME

Using the disc method, the volume of a solid having a circular section whose area is $A = \pi R^2$ can be obtained.

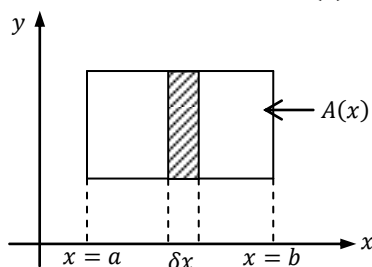
In general, solids of any shape, as long as you know a formula for the area of an arbitrary cross section, volume can be obtained.

Some of the common cross sections are:

- Squares
- Rectangles
- Triangles
- Semi circles
- Trapezoids

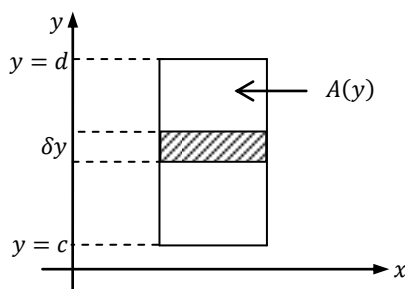
Formula used

1. For cross sections of area $A(x)$ taken perpendicular to the x – axis



$$\text{Volume} = \int_a^b A(x) dx$$

2. For cross sections of area $A(y)$ taken perpendicular to the y – axis



$$\text{Volume} = \int_c^d A(y) dy$$

Example

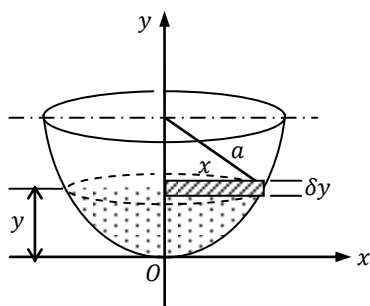
1. Water is poured into a hemispherical bowl whose axis is vertical and radius a . Show that at the instant when the depth measured from the water's surface to the lowest point of bowl is x , the area of the water's surface is given by:

$$S(x) = \pi x(2a - x)$$

Find by integration the volume of water at this instant

Solution

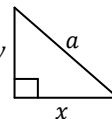
Consider a cross section of area πx^2 at a height y from its horizontal base.



$$\begin{aligned}
 x^2 &= a^2 - (a - y)^2 \\
 &= a^2 - (a^2 - 2ay + y^2) \quad a - y \\
 &= 2ay - y^2
 \end{aligned}$$

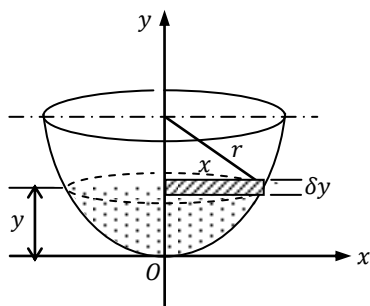
But $x = y$ as per the question

$$\begin{aligned}
 \therefore S(y) &= \pi(2ay - y^2) \\
 \therefore S(x) &= \pi(2ax - x^2) \\
 \therefore \text{Volume} &= \int_0^x S(y) dy \\
 &= \int_0^x \pi(2ay - y^2) dy \\
 &= \pi \left[ay^2 - \frac{y^3}{3} \right]_0^x = \pi \left(ax^2 - \frac{x^3}{3} \right)
 \end{aligned}$$



2. A hemispherical bowl of internal radius r is fixed with its rim horizontal and contains liquid to a depth h , show by integration that the volume of the liquid in the bowl is $\frac{1}{3}\pi h^2(3r - h)$

Solution



Consider a cross section $A(y)$, y units above x - axis

$$\begin{aligned}
 A(y) &= \pi x^2 \\
 r - y & \quad r \\
 r^2 &= x^2 + (r - y)^2 \\
 &= x^2 + (r^2 - 2ry + y^2) \\
 x^2 &= 2ry - y^2 \\
 \therefore A(y) &= \pi(2ry - y^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{Volume} &= \int_0^h \pi(2ry - y^2) dy \\
 &= \pi \left[ry^2 - \frac{y^3}{3} \right]_0^h = \pi \left(rh^2 - \frac{h^3}{3} \right) = \frac{1}{3}\pi h^2(3r - h)
 \end{aligned}$$

3. A hemispherical bowl of internal radius 13 cm contains water to a maximum depth of 8 cm. Find the volume of the water.

Solution

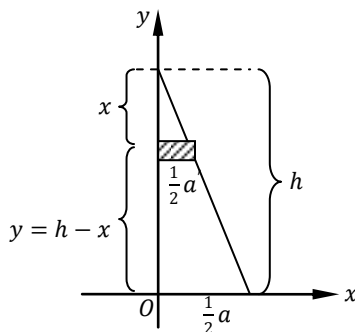
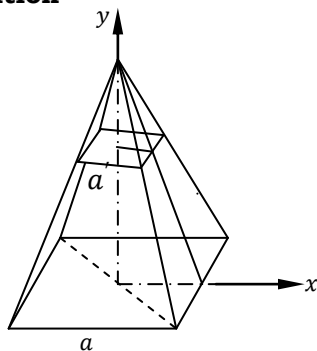
Drawing the above diagrams in question (2) above, $r = 13$, $h = 8$

$$\therefore \text{Volume} = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi(8)^2(3(13) - 8) = 661\frac{1}{3}\pi \text{ cm}^3$$

An Application to Geometry

4. A regular pyramid has a square base of side a and its height is h . Show that the area of a plane section at a depth x below the vertex is $\left(\frac{ax}{h}\right)^2$. Prove also that the volume of the pyramid is $\frac{a^2 h}{3}$.

Solution



Using similar triangles

$$\frac{a'}{a} = \frac{x}{h} \Rightarrow a' = \left(\frac{ax}{h}\right)$$

Area of the cross section $A(y) = a'^2$

$$\Rightarrow A(y) = \left(\frac{ax}{h}\right)^2$$

$$\text{Volume} = \int_0^h A(y) dy = \int_0^h \left(\frac{ax}{h}\right)^2 dy, \text{ but } x = y - h$$

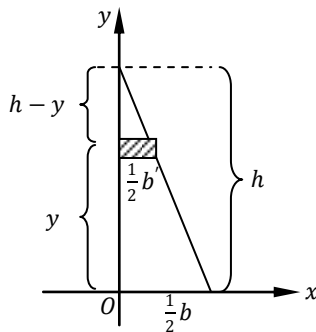
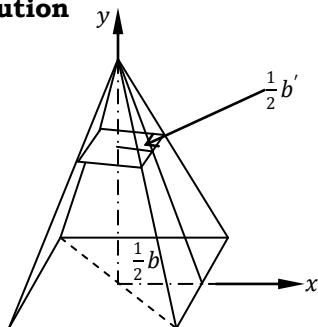
$$= \int_0^h \frac{a^2}{h^2} (y - h)^2 dy$$

$$= \frac{a^2}{h^2} \int_0^h (y - h)^2 dy$$

$$= \frac{a^2}{h^2} \left[\frac{(y-h)^3}{3} \right]_0^h = \frac{a^2}{h^2} \left[0 - \frac{(-h)^3}{3} \right] = \frac{a^2 h}{3}$$

5. Prove that the volume of a pyramid with a square base is $V = \frac{1}{3} h \cdot B$ where h is the height of the pyramid, and B is the area of the base.

Solution



Using similar triangles

$\frac{b'}{b} = \frac{h-y}{h} \Rightarrow b' = \frac{b}{h}(h-y)$, where b is the length of the sides of the base of the pyramid

Area of the cross section $A(y) = b'^2$

$$\Rightarrow A(y) = \frac{b^2}{h^2}(h-y)^2$$

$$\text{Volume} = \int_0^h A(y) dy = \int_0^h \frac{b^2}{h^2}(h-y)^2 dy$$

$$= \frac{b^2}{h^2} \int_0^h (h-y)^2 dy$$

$$= \frac{b^2}{h^2} \left[\frac{(h-y)^3}{-3} \right]_0^h = -\frac{b^2}{h^2} \left[0 - \frac{(h)^3}{3} \right]$$

$$= \frac{b^2 h^3}{3h^2} = \frac{1}{3} b^2 h$$

$$\therefore V = \frac{1}{3} h.B$$

Task

1. A gold fish bowl is a glass sphere of inside diameter 20 cm. Calculate the volume of the water it contains when the maximum depth is 18 cm.

Ans: 1296π

2. A wall vessel has a plane face and its volume is equivalent to that generated by rotating area enclosed by $x = \frac{y^3}{64} + 1$, the y -axis and $y = 8$ through two right angles about y -axis, and units being cm. calculate the volume. **Ans:** $57\frac{6}{7} \text{ cm}^3$

FIRST ORDER DIFFERENTIAL EQUATIONS

Definition:

A *differential equation* is an equation showing the relationship between an independent variable x , a dependent variable, y and more or one differential coefficients of y w.r.t x .

e.g. $x^2 \frac{dy}{dx} + y \sin x = 0$, $xy \frac{d^2y}{dx^2} + y \frac{dy}{dx} = e^{-3x}$

if an equation has only *one* independent variable, the equation is called an *Ordinary Differential Equation*.

e.g. $\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} = 2y$

Order of a Differential Equation

The order of a differential equation is given by the highest derivative involved in the equation.

Classifying Differential Equation

<u>Equation</u>	<u>Order</u>
a) $\frac{d^3y}{dx^3} + 4y = 2$	3
b) $\frac{d^2s}{dt^2} = -32$	2
c) $\left(\frac{dy}{dx}\right)^2 - 3y = e^x$	1
d) $\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = x^2$	2

Formation of a Differential Equation

Differential equations may be formed from:

- Considering the physical problems to which they refer
- Mathematically eliminating arbitrary constants

Forming a Differential Equation from Eliminating Arbitrary Constants

Form a differential equation, and state the order if:

- $y = A \sin x + B \cos x$
- $y = x + \frac{A}{x}$
- $y = Ax^2 + Bx$
- $y = e^{3x}(A \cos 3x + B \sin 3x)$

Solution

NOTE

When forming a differential equation from eliminating arbitrary constants,

- 1) An n^{th} order differential equation is derived from the function having n – arbitrary constants
- 2) The number of arbitrary constants in an equation determines the number of times you are to differentiate the equation

a) $y = A \sin x + B \cos x \dots\dots\dots (1)$

$$\frac{dy}{dx} = A \cos x - B \sin x \dots\dots\dots (2)$$

$$\frac{d^2y}{dx^2} = -A \sin x - B \cos x = -(A \sin x + B \cos x) \dots\dots\dots (3)$$

Solving the three equations simultaneously

Eqn. (1) in to eqn. (3)

$$\Rightarrow \frac{d^2y}{dx^2} = -y \quad \therefore \frac{d^2y}{dx^2} + y = 0 \text{ is an equation of order 2}$$

b) $y = x + \frac{A}{x} \dots\dots\dots (1)$

$$\frac{dy}{dx} = 1 - \frac{A}{x^2} \dots\dots\dots (2)$$

Eliminating A from eqn. (1) and eqn. (2)

$$\text{eqn. (1)} \times \frac{1}{x} + \text{eqn. (2)}$$

$$\Rightarrow \frac{y}{x} = 1 + \frac{A}{x^2} \dots\dots\dots (3)$$

eqn. (2) + eqn. (3)

$$\frac{dy}{dx} + \frac{y}{x} = 2 \Rightarrow x \frac{dy}{dx} + y = 2x, \text{ which is an equation of first order}$$

c) $y = Ax^2 + Bx \dots\dots\dots (1)$

$$\frac{dy}{dx} = 2Ax + B \dots\dots\dots (2)$$

$$\frac{d^2y}{dx^2} = 2A \dots\dots\dots (3)$$

Since eqn. (3) does not contain, eliminating B from eqn. (1) and eqn.

(2)

Eqn. (2) $\times x$ - eqn. (1)

$$x \frac{dy}{dx} - y = 2Ax^2 + Bx - (2Ax^2 + Bx) = Ax^2$$

$$\therefore x \frac{dy}{dx} - y = Ax^2 \dots\dots\dots (4)$$

Now eqn. (4) $\div$ eqn. (3) to eliminate A

$$\Rightarrow \frac{x \frac{dy}{dx} - y}{\frac{d^2y}{dx^2}} = \frac{Ax^2}{2A} = \frac{x^2}{2}$$

$$\therefore x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0, \text{ which is an equation of order 2}$$

Methods of Solving Differential Equation

1. BY INTEGRATION

An equation of the form $\frac{dy}{dx} = f(x)$ can be solved by Direct Integration

Example

Solve the differential equation

a) $\frac{dy}{dx} = 3x^2 - 6x + 4$

b) $\cos x \frac{dy}{dx} = 1$

c) $e^{2x} \frac{dy}{dx} = 4$

Solution

- a) $\frac{dy}{dx} = 3x^2 - 6x + 4$, integrating w.r.t x
 $\int \frac{dy}{dx} \cdot dx = \int (3x^2 - 6x + 4) dx$
 $\therefore y = x^3 - 3x^2 + 4x + c$
- b) $\cos x \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos x} = \sec x$, integrating w.r.t x
 $\int \frac{dy}{dx} \cdot dx = \int \sec x dx$
 $\therefore y = \ln|\sec x + \tan x| + c$
- c) Left as an exercise. **Ans:** $y = 2e^{-2x} + c$

NOTE

1. An arbitrary constant c is included in the solution which we always get when solving a 1st order differential equation.
2. The arbitrary constant's value can only be got when further information about the function is given
3. The solution containing an arbitrary constant is called a *General Solution/Primitive Solution* of the equation. e.g. $y = \ln|\sec x + \tan x| + c$ is a general solution.
4. If we are told the value of y for a given value of x , then c can be evaluated, and the resultant solution is known as a *Particular Solution*.

Example

Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$ given that when $x = 0, y = 3$

Solution

$$e^x \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = 4e^{-x}$$

$$\Rightarrow \int \frac{dy}{dx} \cdot dx = \int 4e^{-x} dx$$

$$y = -4e^{-x} + c$$

Put $x = 0, y = 3$ in to the equation

$$\Rightarrow 3 = -4e^0 + c \therefore c = 7$$

$$y = -4e^{-x} + 7$$

2. BY SEPARATION OF VARIABLES

An equation of the form

- $\frac{dy}{dx} = M(x) \cdot N(y)$ or
- $\frac{dy}{dx} = \frac{M(x)}{N(y)}$ or
- $M(x) + N(y) \cdot \frac{dy}{dx} = 0$

, can be solved by this method.

For this equation, all x – terms can be collected with dx and y – terms with dy , and the solution can be obtained by integration.

Example

Solve:

a) $(x^2 + 4) \frac{dy}{dx} = xy$

b) $\frac{dy}{dx} = \frac{y^2 + xy^2}{x^2 y - x^2}$

c) $y \tan x \frac{dy}{dx} = (4 + y^2) \sec^2 x$

, and simplify your answer.

Solution

a) $(x^2 + 4) \frac{dy}{dx} = xy$

Separating variables

$\Rightarrow \frac{dy}{y} = \frac{x}{x^2 + 4}, \text{ integrating}$

$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4} dx \Rightarrow \ln|y| = \frac{1}{2} \ln|x^2 + 4| + c_1$

Simplifying using logarithms

$\ln|y| - \frac{1}{2} \ln|\sqrt{x^2 + 4}| = c_1 \Rightarrow \ln \left| \frac{y}{\sqrt{x^2 + 4}} \right| = c_1$

$e^{c_1} = \frac{y}{\sqrt{x^2 + 4}} \therefore y = e^{c_1} \sqrt{x^2 + 4}$

Since e^{c_1} is a constant, replacing it with a general constant c

$\therefore y = c \sqrt{x^2 + 4}$

b) $\frac{dy}{dx} = \frac{y^2 + xy^2}{x^2 y - x^2} = \frac{y^2}{x^2} \left(\frac{1+x}{y-1} \right)$

Separating variables

$\Rightarrow \frac{y-1}{y^2} = \frac{x+1}{x^2}, \text{ integrating}$

$\int \left(\frac{y}{y^2} - y^{-2} \right) dy = \int \left(\frac{1}{x^2} + \frac{x}{x^2} \right) dx$

$\Rightarrow \int \left(\frac{1}{y} - y^{-2} \right) dy = \int \left(x^{-2} + \frac{1}{x} \right) dx$

$\Rightarrow \ln|y| + \frac{1}{y} = -\frac{1}{x} + \ln|x| + c$

c) $y \tan x \frac{dy}{dx} = (4 + y^2) \sec^2 x$

Separating variables

$\Rightarrow \frac{y \cdot dy}{4 + y^2} = \frac{\sec^2 x}{\tan x} dx, \text{ integrating}$

$\int \frac{y}{4 + y^2} dy = \int \frac{\sec^2 x}{\tan x} dx \Rightarrow \frac{1}{2} \ln|4 + y^2| = \ln \tan x + c_1$

$\therefore \ln|4 + y^2| = 2 \ln \tan x + 2c_1 \text{ let } 2c_1 = \ln c$

$\Rightarrow \ln|4 + y^2| = 2 \ln \tan x + \ln c$

$\Rightarrow \ln|4 + y^2| = \ln|\tan^2 x| + \ln|c| = \ln|c \tan^2 x|$

$\Rightarrow 4 + y^2 = c \tan^2 x \therefore y^2 = c \tan^2 x - 4$

Task

Find the general solution to the equation

a) $\frac{dy}{dx} = \frac{y}{x}$

Ans: $y = A \cdot x$

b) $\cos^2 x \frac{dy}{dx} = y + 3$

Ans: $\ln y = \frac{x^2}{2} - x + c$

c) $\frac{\sin x}{1+y} \frac{dy}{dx} = \cos x$

Ans: $y = A \sin x - 1$

3. LINEAR EQUATIONS: Use Integrating Factor

This solves the equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = f(x), Q = g(x) \text{ or } P, Q \text{ are constants}$$

To solve this equation, we multiply the equation by an integrating factor (I.F) which is always

$$\text{I.F} = e^{\int p dx}$$

This converts the L.H.S in to a complete differential coefficient.

NOTE

When evaluating $\text{I.F} = e^{\int p dx}$, we don't include a constant of integration. This omission is for convenience because if the constant is added on both sides of the equation, they would subsequently cancel.

Rule to Solve a Linear Differential Equation

$$\frac{dy}{dx} + Py = Q, \text{ where } P = f(x), Q = g(x) \text{ or } P, Q \text{ are constants}$$

Multiply both sides by an Integrating Factor.

The general solution to Linear Differential equation is

$$y \cdot \text{I.F} = \int Q \cdot \text{I.F} dx$$

Example

Solve the differential equation

a) $\frac{dy}{dx} - y = x$

b) $x \frac{dy}{dx} + y = x^3$

c) $(1 - x^2) \frac{dy}{dx} = xy + 1$

Solution

a) Comparing $\frac{dy}{dx} - y = x$ with $\frac{dy}{dx} + Py = Q$

$$\Rightarrow P = -1, Q = x$$

$$\Rightarrow \text{I.F} = e^{\int p dx} = e^{\int -1 dx} = e^{-x}$$

Multiplying through by e^{-x}

$$e^{-x} \cdot \frac{dy}{dx} - (y \cdot e^{-x}) = x \cdot e^{-x}$$

$$\Rightarrow \frac{d}{dx}(y \cdot e^{-x}) = x \cdot e^{-x}, \text{ integrating}$$

$$\Rightarrow \int \frac{d}{dx}(y \cdot e^{-x}) dx = \int x \cdot e^{-x} dx$$

Integrating $\int x \cdot e^{-x} dx$

$$\text{Let } u = x \Rightarrow du = dx, dv = e^{-x} dx \Rightarrow v = -e^{-x}$$

$$\begin{aligned} \therefore \int x \cdot e^{-x} dx &= -x \cdot e^{-x} - \int -e^{-x} dx \\ &= -x \cdot e^{-x} - e^{-x} + c \end{aligned}$$

$$\therefore y \cdot e^{-x} = -x \cdot e^{-x} - e^{-x} + c$$

$$\therefore y = -x - 1 + ce^x$$

b) $x \frac{dy}{dx} + y = x^3$, rewriting, divide through by x

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2$$

Comparing $\frac{dy}{dx} + \frac{y}{x} = x^2$ with $\frac{dy}{dx} + Py = Q$

$$\Rightarrow P = \frac{1}{x}, Q = x^2$$

$$\Rightarrow I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Multiplying through by x

$$x \cdot \frac{dy}{dx} - \left(\frac{y}{x} \cdot x \right) = x^3$$

$$\Rightarrow \frac{d}{dx}(y \cdot x) = x^3, \text{ integrating}$$

$$\Rightarrow \int \frac{d}{dx}(y \cdot x) dx = \int x^3 dx$$

$$\Rightarrow y \cdot x = \frac{x^4}{4} + c \quad \text{or} \quad y = \frac{x^3}{4} + \frac{c}{x}$$

c) $(1 - x^2) \frac{dy}{dx} = xy + 1$

Re arranging

$$(1 - x^2) \frac{dy}{dx} - xy = 1, \text{ dividing through by } (1 - x^2)$$

$$\Rightarrow \frac{dy}{dx} - y \cdot \frac{x}{1-x^2} = \frac{1}{1-x^2}$$

Comparing $\frac{dy}{dx} - y \cdot \frac{x}{1-x^2} = \frac{1}{1-x^2}$ with $\frac{dy}{dx} + Py = Q$

$$\Rightarrow P = -\frac{x}{1-x^2}, Q = \frac{1}{1-x^2}$$

$$\Rightarrow I.F = e^{\int P dx} = e^{\int \frac{-x}{1-x^2} dx} = e^{\frac{1}{2} \ln |1-x^2|} = \sqrt{1-x^2}$$

Using $y \cdot I.F = \int Q \cdot I.F dx$

$$\Rightarrow y \cdot \sqrt{1-x^2} = \int \frac{\sqrt{1-x^2}}{1-x^2} dx = \int \frac{1}{\sqrt{1-x^2}} dx$$

$$\Rightarrow y \cdot \sqrt{1-x^2} = \sin^{-1} x + c$$

$$\therefore y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$$

Task

Solve the differential equation

a) $\frac{dy}{dx} + 3y = e^{4x}$

Ans: $y = \frac{e^{4x}}{7} + ce^{-3x}; I.F = e^{3x}$

b) $x \frac{dy}{dx} + y = x \sin x$

Ans: $xy = \sin x - x \cos x + c \quad I.F = \sin x$

c) $(x^2 - 1) \frac{dy}{dx} + 2xy = x$

Ans: $y(x^2 - 1) = \frac{x^2}{2} + c$

d) $x \frac{dy}{dx} - 2y = x^3 \cos x$

Ans: $y = x^2(\sin x + c)$

e) $(y^2 - x)dx + 2ydy = 0$

Ans: $y^2 - x + 1 = ce^{-x}$

HOMOGENEOUS EQUATIONS

By substitution $y = vx$

A homogeneous differential equation is an equation of the form

$$M(x, y)dx + N(x, y)dy = 0, \text{ where } M \text{ and } N \text{ are homogeneous functions of } x \text{ and } y \text{ of same degree}$$

NOTE

Example

i) $(x^2 + xy)dx + y^2 dy = 0$ or $y^2 \frac{dy}{dx} = x^2 + xy$ Degree 2^0

ii) $\frac{dy}{dx} = \frac{x+2y}{2x}$ Degree 1^0

iii) $x^3 + y^3 = 3xy^2 \frac{dy}{dx}$ Degree 3^0

To solve the equation use:

a) $y = vx$ (1)

b) Differentiating eqn. (1) w.r.t x

$$\frac{dy}{dx} = x \frac{dv}{dx} + v \text{ (2)}$$

Example

1. Solve the differential equation

a) $\frac{dy}{dx} = \frac{x+2y}{2x}$

b) $(x^2 + y^2)dx = xy \cdot dy$

Solution

a) $\frac{dy}{dx} = \frac{x+2y}{2x}$

Let $y = vx$ (1)

$$\Rightarrow \frac{dy}{dx} = \frac{x+3.vx}{2x} = \frac{1+3v}{2}$$

Differentiating eqn. (1) w.r.t x

$$\frac{dy}{dx} = x \frac{dv}{dx} + v \text{ (2)}$$

$$\therefore x \frac{dv}{dx} + v = \frac{1+3v}{2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1+3v}{2} - v = \frac{1+v}{2} \quad \therefore x \frac{dv}{dx} = \frac{1+v}{2}$$

Separating variables

$$\frac{2}{1+v} dv = -\frac{1}{x} dx, \text{ integrating}$$

$$\int \frac{2}{1+v} dv = \int -\frac{1}{x} dx$$

$$2 \ln|1+v| = \ln|x| + c_1 \Rightarrow \ln \left| \frac{(1+v)^2}{x} \right| = c_1$$

$$\Rightarrow e^{c_1} = \frac{(1+v)^2}{x}$$

$$\Rightarrow xe^{c_1} = (1+v)^2, \text{ but } v = \frac{y}{x}$$

$$\Rightarrow xe^{c_1} = \left(1 + \frac{y}{x}\right)^2 = \frac{(x+y)^2}{x^2}$$

$$\Rightarrow x^3 e^{c_1} = (x+y)^2 \quad \therefore x^3 A = (x+y)^2 ; A = e^{c_1}$$

b) $(x^2 + y^2)dx = xy \cdot dy$

Let $y = vx$ (1)

$$\Rightarrow \frac{dy}{dx} = \frac{x^2(1+v^2)}{x^2v} = \frac{1+v^2}{v} \dots\dots\dots (*)$$

Differentiating eqn. (1) w.r.t x

$$\frac{dy}{dx} = x \frac{dv}{dx} + v \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (*)

$$\therefore x \frac{dv}{dx} + v = \frac{1+v^2}{v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1+v^2}{v} - v = \frac{1}{v}, \text{ separating variables}$$

$$v \, dv = \frac{dx}{x}, \text{ integrating}$$

$$\int v \, dv = \int \frac{dx}{x} \Rightarrow \frac{v^2}{2} = \ln|x| + c, \text{ but } v = \frac{y}{x}$$

$$\Rightarrow \frac{y^2}{2x^2} = \ln|x| + c \quad \therefore y^2 = 2x^2 \ln|x| + 2x^2c$$

Solution

Solve the differential equation

a) $(x - y) \frac{dy}{dx} = x + y$ **Ans:** $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2} \ln \left| \frac{x^2 + y^2}{x^2} \right| + \ln|x| + \ln|A|$

b) $2x^2 dy = (x^2 + y^2) dx$ **Ans:** $\frac{2x}{x-y} = \ln|x| + c$

c) $x^3 + y^3 = 3xy^2 \frac{dy}{dx}$ **Ans:** $x^3 - 2y^3 = 2Ax$

d) $(2y - x) \frac{dy}{dx} = 2x + y$ given that when $x = 2, y = 3$ **Ans:** $y^2 - xy - x^2 + 1 = 0$

e) $xy + y^2 + (x^2 - xy) \frac{dy}{dx} = 0$ **Ans:** $xy = Ae^{y/x}$

Transformations

Using the substitutions given, solve the differential equation

a) $(3x + 3y - 4) \frac{dy}{dx} = -(x + y)$, use $v = x + y$ **Ans:** $x + 3y + 2 \ln|x + y - 2| = A$

b) $(x - y - 1) + (4y + x - 1) \frac{dy}{dx} = 0$, use $v = x - 1$

Ans: $\ln|4y^2 + (x - 1)^2| + \tan^{-1}\left(\frac{2y}{x-1}\right) = A$

APPLICATION OF DIFFERENTIAL EQUATIONS

Some of the applications of differential equations are:

- Radioactivity Decay
- Population Growth
- Chemical Reactions
- Newton's Law of Cooling
- Gravitational Force
- Tangents

a) **Newton's Law of Cooling**

This states that "Rate of decrease of the difference in temperature of a body is proportional to the difference between the temperature of the body and that of the medium"

If θ is the temperature of the body at time t and θ_0 the constant temperature of the medium, then

$$\frac{d\theta}{dt} \propto (\theta - \theta_0) \text{ or } \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\Rightarrow \frac{d\theta}{\theta - \theta_0} = -k dt, \text{ integrating}$$

$$\int \frac{d\theta}{\theta - \theta_0} = - \int k dt \Rightarrow \ln|\theta - \theta_0| = -k t + c$$

If initially θ_1 is the temperature of the body at $t = 0$, then

$$\ln|\theta - \theta_0| = c$$

$$\therefore \ln|\theta - \theta_0| - \ln|\theta_1 - \theta_0| = -k t$$

$$\ln \left| \frac{\theta - \theta_0}{\theta_1 - \theta_0} \right| = -k t \Rightarrow e^{-kt} = \frac{\theta - \theta_0}{\theta_1 - \theta_0}$$

$$\therefore \theta = \theta_0 + (\theta_1 - \theta_0)e^{-kt}$$

Example

1. Water at a temperature of 100°C cools in 10 minutes to 88°C in a room of temperature 25°C . Find the temperature of water after 20 minutes.

Solution

Using the above result

$$\theta = \theta_0 + (\theta_1 - \theta_0)e^{-kt}$$

$$\Rightarrow \theta_0 = 25, t = 10, \theta_1 = 100, \theta = 88$$

$$\therefore 88 = 25 + (100 - 25)e^{-kt} \Rightarrow \frac{21}{25} = e^{-10t} \dots\dots\dots (1)$$

We want to find θ after 20 minutes

$$\therefore \theta = 25 + (100 - 25)e^{-20t}$$

$$\theta = 25 + 75e^{-20t} \dots\dots\dots (2)$$

Eqn. (1) in to eqn. (2)

$$\therefore \theta = 25 + 75 \left(\frac{21}{25} \right)^2 = 77.92^\circ\text{C}$$

2. The rate at which the body cools is proportional to the difference between the temperature of the body and that of the surrounding air.

If a body in air at 25°C will cool from 100°C to 75°C in 1 minute, find its temperature at the end of 3 minutes.

Solution

Let the temperature of the body at any time t be θ .

$$\frac{d\theta}{dt} = -k(\theta - 25) \Rightarrow \frac{d\theta}{\theta - 25} = -k dt, \text{ integrating}$$

$$\int \frac{d\theta}{\theta - 25} = - \int k dt \Rightarrow \ln|\theta - 25| = -k t + c \dots\dots\dots (1)$$

When $t = 0$, $\theta = 100$

$$\therefore c = \ln(100 - 25) = \ln 75$$

$$\Rightarrow \ln|\theta - 25| = -k t + \ln 75 \dots\dots\dots (2)$$

When $t = 1$, $\theta = 75$

$$\Rightarrow \ln|75 - 25| = -k + \ln 75 \quad \therefore -k = \ln\left(\frac{50}{75}\right) = \ln\left(\frac{2}{3}\right)$$

$$\Rightarrow e^{-k} = \frac{2}{3}$$

From eqn. (2)

$$\ln|\theta - 25| = -k t + \ln 75$$

$$\Rightarrow \ln\left|\frac{\theta - 25}{75}\right| = -k t \quad \therefore \frac{\theta - 25}{75} = e^{-kt}$$

$$\therefore \theta = 25 + 75e^{-kt}$$

When $t = 3$, then $\theta = ?$

$$\therefore \theta = 25 + 75e^{-3k} = 25 + 75\left(\frac{2}{3}\right)^3 = 47.22^\circ\text{C}$$

- 3.** *The rate at which ice melts is proportional to the amount of ice at that instant. Find the amount of ice left after 2 hours if half the quantity melts in 30 minutes.*

Solution

Let m = amount of ice at any instant time t

$$\frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = k$$

$$\Rightarrow \frac{dm}{m} = k dt, \text{ integrating}$$

$$\ln m = k t + c \dots\dots\dots (1)$$

$$\text{At } t = 0, m = M$$

$$\therefore \ln M = c$$

$$\ln m = k t + \ln M \dots\dots\dots (2)$$

$$\text{Now when } t = \frac{1}{2} \text{ hrs} = 30 \text{ minutes}, m = \frac{M}{2}$$

$$\Rightarrow \ln\left(\frac{M}{2}\right) = \frac{k}{2} + \ln M$$

$$\Rightarrow \frac{k}{2} = \ln\left(\frac{M}{2}\right) - \ln M = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\therefore k = -2 \ln 2$$

From eqn. (2)

$$\ln m = -2 \ln 2 t + \ln M \dots\dots\dots (3)$$

On putting $t = 2$ hours into eqn. (3)

$$\ln\left(\frac{m}{M}\right) = -4 \ln 2 = \ln\left(\frac{1}{16}\right)$$

$$\frac{m}{M} = \frac{1}{16} \quad \therefore m = \frac{M}{16}$$

After 2 hours, amount of ice left is $\frac{1}{16}$ of the amount of ice at the beginning.

- 4.** *According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air.*
- a) *If the temperature of the air is 300k and the substance cools from 370k to 340k in 15 minutes, find when the temperature will be 310k*

- b) Suppose a room is kept at a constant temperature of 70°C , and an object cools from 350°C to 150°C in 45 minutes. How long will it take for the object to cool to a temperature of 80°C ?

Solution

let T be the temperature of the substance at time t min

$$\Rightarrow \frac{dT}{dt} = -k(T - 300) \text{ or } \frac{dT}{T-300} = -k dt$$

NOTE

The use of $-k$ here is optional. It will be found that k is $+ve$, but if $+k$ is used it will be found that k is equally negative.

Integrating between limits

$$t = 0, T = 370 \text{ and } t = 15, T = 340$$

$$\int_{370}^{340} \frac{dT}{T-300} = \int_0^{15} -k dt$$

$$\Rightarrow \ln 40 - \ln 70 = -15k = \ln\left(\frac{4}{7}\right) \text{ and } 15k = \ln\frac{7}{4} = 0.56$$

Integrating between limits

$$t = 0, T = 370 \text{ and } t = t, T = 310$$

$$\int_{370}^{310} \frac{dT}{T-300} = \int_0^t -k dt$$

$$\Rightarrow \ln 10 - \ln 70 = -kt = \ln\left(\frac{4}{7}\right) \text{ and } 15kt = 15 \ln 7$$

$$\therefore t = \frac{15 \ln 7}{0.56} = 52 \text{ min}$$

$$t = 119.7 \text{ min}$$

Task

1. A hot body at a temperature of 100°C is placed in a hot room of temperature of 20°C . 10 minutes later its temperature is 60°C .
 - i) Write down a differential equation to represent the rate of change of temperature θ of the body with time t .
 - ii) Determine the temperature of the body after another 10 minutes
(Uneb 2005)

Answers

- i) $\frac{d\theta}{dt} = -k(\theta - R)$, R = room temperature = 20°C
 - ii) $\theta = 40^{\circ}\text{C}$ at $t = 10 + 10 = 20 \text{ min}$
2. At 3.00 pm, the temperature of a hot metal was 80°C and that of the surroundings 20°C . At 3:03 pm the temperature of the metal had dropped to 42°C . The rate of cooling of the metal was directly proportional to the difference between its temperature θ and that of the surroundings.
 - a) (i) write a differential equation to represent rate of cooling of the metal.
 - b) Find the temperature of the metal at 3:05 pm (Uneb 2012)
 3. It is observed that the rate at which a body cools is proportional to the amount by which its temperature exceeds that of its surroundings. A body at 78°C is placed in a room at 20°C and after 5 minutes the body

has cooled to 65°C , what will be the temperature after a further 5 minutes? **(Uneb 2011)** **Ans:**

4. According to Newton's law of cooling, the rate of cooling of a body in air is proportional to the difference between the temperature T of the body and the temperature T_0 of the air. If the air temperature is kept constant at 20°C and the body cools from 100°C to 60°C in 20 minutes, in what further time will the body cool to 30°C ? **Ans:** 40 minute
5. A body is placed in a room which is kept at a constant temperature. The temperature of the body falls at a rate $k\theta$ C per minute; where k is a constant and θ is the difference between the temperature of the body and that of the room at time t .
Express this information in form of a differential equation and hence show that $\theta = \theta_0 e^{-kt}$, where θ_0 is the temperature difference at time $t = 0$. The temperature of the body falls 5°C in the first minute and 4°C in the second minute. Show that the fall of temperature in the third minute is 3.2°C . **Ans:** $\frac{d\theta}{dt} = -k\theta$
6. The rate at which a heated body cools in air is proportional to the difference between the temperature of the body and that of the surrounding air. A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of air being 40°C . What will be the temperature of the body after 40 minutes from the original?

Population Growth

1. If the population of a country doubles in 50 years, in how many years will it treble, assuming that the rate of increase is proportional to the number on inhabitants?

Solution

Let t = time in years

y = Population after t – years

p = Original population (when $t = 0$)

Rate of increase $\propto$ population

$$\Rightarrow \frac{dy}{dt} = k y, \text{ where } t = \text{constant}$$

$$\frac{dy}{y} = \int k dt \Rightarrow \ln|y| = kt + c$$

When $t = 0, y = p$

When $t = 50, y = 2p$

$$\Rightarrow \ln|p| = c \therefore \ln|y| = kt + \ln|p|$$

$$\Rightarrow \ln|2p| = 50k + \ln|p|$$

$$\Rightarrow \ln\left|\frac{2p}{p}\right| = 50k \therefore k = \frac{1}{50} \ln 2$$

Now when $y = 3p, t = ?$

$$\Rightarrow \ln|y| = \frac{t}{50} \ln 2 + \ln|p|$$

$$\Rightarrow \ln \left| \frac{y}{p} \right| = \frac{t}{50} \ln 2$$

$$\therefore \ln \left| \frac{3p}{p} \right| = \frac{t}{50} \ln 2 \Rightarrow \frac{50 \ln 3}{\ln 2} = t \quad \therefore t = 79 \text{ years}$$

2. The rate of growth of a population of fruit flies is proportional to the size of the population at any instant. If there are 180 flies after the 2nd day of the experiment and 300 after the 4th day, how many flies were there in the original population?

Solution

Let the original population = p_0 at any time and p = population at any time.

$$\frac{dp}{dt} = k p$$

$$\int \frac{dp}{p} = \int k dt \Rightarrow \ln p = kt + c$$

$$\text{At } t = 0, p = p_0 \Rightarrow \ln p_0 = c$$

$$\therefore \ln p = kt + \ln p_0 \quad \text{or} \quad \ln \left| \frac{p}{p_0} \right| = kt$$

$$\text{Now when } t = 2, p = 180$$

$$t = 4, p = 300$$

$$\Rightarrow \ln \left| \frac{180}{p_0} \right| = 2k \dots \dots \dots (1)$$

$$\Rightarrow \ln \left| \frac{300}{p_0} \right| = 4k \dots \dots \dots (4)$$

$$\text{Eqn. (2)} \div \text{eqn. (1)}$$

$$\Rightarrow \frac{\ln \left| \frac{300}{p_0} \right|}{\ln \left| \frac{180}{p_0} \right|} = \frac{4k}{2k} = 2$$

$$\Rightarrow \ln \left| \frac{300}{p_0} \right| = 2 \ln \left| \frac{180}{p_0} \right| = \ln \left| \frac{32400}{p_0^2} \right|$$

$$\Rightarrow \frac{300}{p_0} = \frac{32400}{p_0^2} \quad \therefore p_0 = \frac{32400}{300} = 108$$

$$\therefore \text{Original population} = 108$$

3. In a certain culture of bacteria the rate of increase is proportional to the number present.

- a) If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?
- b) If there are 10^4 at the end of 3 hours and 4×10^4 at the end of 5 hours, how many were there at the beginning?

Solution

Let x denote the number of bacteria at time t hours

$$\Rightarrow \frac{dx}{dt} = k x \quad \text{or} \quad \frac{dx}{x} = k dt$$

- a) Integrating

$$\ln|x| = kt + c$$

$$\text{Assuming that } x = x_0 \text{ at time } t = 0, \quad c = \ln x_0$$

$$\therefore \ln|x| = kt + \ln|x_0| \dots \dots \dots (1)$$

$$\text{At } t = 4, \quad x = 2x_0$$

$$\ln|2x_0| = 4k + \ln x_0$$

$$4k = \ln \left| \frac{2x_0}{x_0} \right| = \ln 2 \quad \therefore k = \frac{\ln 2}{4}$$

$$\therefore \ln|x| = \frac{\ln 2}{4}t + \ln x_0$$

Now at $t = 12$, $x = ?$

$$\ln|x| = \ln|x_0| + \frac{12 \ln 2}{4} \Rightarrow \ln|x| = \ln|x_0| + \ln 8$$

$$\ln x = \ln 8x_0 \quad \therefore x = 8x_0$$

$\therefore$ There are 8 times the original numbers

b) When $t = 3$, $x = 10^4$

From eqn. (1)

$$\ln|x| = \ln|x_0| + k t \Rightarrow \ln \left| \frac{x}{x_0} \right| = k t$$

$$\frac{x}{x_0} = e^{kt} \Rightarrow x = x_0 e^{kt}$$

$$\Rightarrow 10^4 = x_0 e^{3k} \dots\dots\dots (1)$$

When $t = 5$, $x = 4 \times 10^4$

$$\Rightarrow 4 \times 10^4 = x_0 e^{5k} \dots\dots\dots (2)$$

Eqn. (2) $\div$ eqn. (1)

$$\Rightarrow \frac{4 \times 10^4}{10^4} = \frac{x_0 e^{5k}}{x_0 e^{3k}} = e^{2k}$$

$$\therefore e^{2k} = 4$$

From eqn. (1)

$$\Rightarrow 10^4 = x_0 e^{3k} = x_0 (e^k)^3$$

$$\Rightarrow 10^4 = x_0 (4)^3 \quad \therefore x_0 = \frac{10^4}{8} \text{ bacteria.}$$

Radioactive Decay

1. Uranium disintegrates at a rate proportional to the amount present at any instant. If M_1 and M_2 grams of uranium are present at a time T_1 and T_2 respectively, show that the half life of uranium is

$$\frac{(T_2 - T_1) \ln 2}{\ln(M_1/M_2)}$$

Solution

Let m = Amount present at any time t

$$\frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = -k m$$

$$\Rightarrow \int \frac{dm}{dt} = \int -k dt$$

$$\Rightarrow \ln|m| = -kt + c$$

For $m = M_1$, $t = T_1$; $m = M_2$, $t = T_2$

$$\Rightarrow \ln|M_1| = -kT_1 + c \dots\dots\dots (1)$$

$$\Rightarrow \ln|M_2| = -kT_2 + c \dots\dots\dots (2)$$

Eqn. (1) $-$ eqn. (2)

$$\Rightarrow \ln \left| \frac{M_1}{M_2} \right| = -k(T_1 - T_2) = k(T_2 - T_1) \dots\dots\dots (3)$$

From $\ln|m| = -kt + c$

If the initial mass is M_0 at $t = 0$

$$\ln|M_0| = c$$

$$\therefore \ln|m| = -kt + \ln|M_0|$$

$$\ln\left|\frac{m}{M_0}\right| = -kt$$

$$\text{Now for } m = \frac{M_0}{2}, t = T = ?$$

$$\Rightarrow \ln\left|\frac{M_0}{2M_0}\right| = -kT$$

$$\Rightarrow \ln\frac{1}{2} = -kT, \text{ from eqn. (3) } \ln\left|\frac{M_1}{M_2}\right| = -k(T_1 - T_2) = k(T_2 - T_1) \Rightarrow k = \frac{\ln\left|\frac{M_1}{M_2}\right|}{(T_2 - T_1)}$$

$$\therefore T = \frac{\ln 2}{k} = \frac{\ln 2}{\frac{\ln\left|\frac{M_1}{M_2}\right|}{(T_2 - T_1)}} = \frac{(T_2 - T_1) \ln 2}{\ln\left|\frac{M_1}{M_2}\right|}$$

$$\therefore T = \frac{(T_2 - T_1) \ln 2}{\ln\left|\frac{M_1}{M_2}\right|}$$

2. The rate of decomposition of radioactive radium is proportional to the amount present at a given instant. Find the percentage of a present amount that remains after 25 years if the half life of radioactive radium is 1600 years.

Solution

Let the amounts present at any instant be m

$$\Rightarrow \frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = -k m$$

$$\Rightarrow \int \frac{dm}{m} = \int -k dt \text{ or } \ln|m| = -kt + c$$

Let the present amount be m_0 at $t = 0$

$$\Rightarrow \ln|m_0| = c$$

$$\ln|m| = -kt + \ln|m_0| \dots\dots\dots (1)$$

$$\text{Now for } m = \frac{m_0}{2}, t = 1600$$

$$\Rightarrow \ln\left|\frac{m_0}{2}\right| = -1600k + \ln|m_0| \quad \therefore k = \frac{\ln 2}{1600} \dots\dots\dots (2)$$

$$\text{Now for } t = 25, m = M$$

$$\ln|M| = -25k + \ln|m_0| \Rightarrow \ln\left|\frac{m_0}{M}\right| = 25k = \frac{25 \ln 2}{1600}$$

$$\ln\left|\frac{m_0}{M}\right| = \frac{\ln 2}{64} \Rightarrow \frac{m_0}{M} = 2^{1/64} = 1.010889$$

$$\therefore \frac{M}{m_0} = 0.9892$$

$$\%age \text{ of original amount} = 98.92\%$$

Task

A substance loses mass at a rate proportional to the amount M present at any time t .

- Form a differential equation connecting M , t and the constant of proportionality k .
- If initially the mass of the substance is M_0 show that $M = M_0 e^{-kt}$

- c) Given that half life of the substance is lost in 1600 years,
determine the number of years 15g of the substance would
reduce to 13.6g

Chemical Reactions

- 1) Under certain conditions, cane sugar is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If out of 75grams of sugar at $t = 0$, 8 grams are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.

Solution

Let M be the amount of cane sugar initially,

m be the amount of cane sugar converted in to dextrose

$$\Rightarrow \frac{dm}{dt} = k(M - m) \text{ or } \frac{dm}{dt} + k m = k M$$

$$I.F = e^{\int k dt} = e^{kt}$$

$$\text{Using I.F. } m = \int k M e^{kt} dt$$

$$\Rightarrow e^{kt} \cdot m = \int k M e^{kt} dt = M e^{kt} + c$$

$$\therefore m = M + c e^{-kt} \dots\dots\dots (1)$$

- i) At $t = 0, m = 0, M = 75$

Eqn. (1) becomes

$$0 = 75 + c e^{-k(0)} \Rightarrow c = -75$$

$$\therefore m = 75 - 75e^{-kt}$$

- ii) At $t = 30, m = 8, M = 75$

$$\therefore 8 = 75 - 75e^{-30t} \Rightarrow 67 = 75e^{-30t}$$

Now at $t = 1\frac{1}{2}$ hours = 30 minutes, $m = ?$

$$m = 75 - 75e^{-90k}$$

$$= 75 - 75 \left(\frac{67}{75} \right)^3 = 21.55$$

Alternatively

Let m be the amount unconverted

M be the initial amount

$$\text{Given: } \frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = -k m$$

[$-ve$ is used because when cane sugar is converted, the amount unconverted decrease]

$$\Rightarrow \int \frac{dm}{m} = \int -k dt \text{ or } \ln|m| = -kt + c$$

$$\text{Now at } t = 0, m = 75 \therefore c = \ln 75$$

$$\Rightarrow \ln|m| = -kt + \ln 75 \dots\dots\dots (1)$$

$$\text{Now at } t = 30, m = 75 - 8 = 67 \therefore c = \ln 75$$

$$\Rightarrow \ln 67 = -30k + \ln 75$$

$$\Rightarrow \ln \left| \frac{67}{75} \right| = -30k \text{ or } e^{-30k} = \frac{67}{75} \dots\dots\dots (2)$$

Required: when $t = 1\frac{1}{2}$ hrs = 90 minutes, $m = ?$

$$m = 75 - M, M = \text{Amount converted}$$

$$\Rightarrow \ln|75 - M| = -90k + \ln 75$$

$$\Rightarrow e^{-90k} = \frac{75-M}{75}$$

$$\therefore 75e^{-90k} = 75 - M \Rightarrow M = 75 - 75e^{-90k}$$

$$M = 75 - 75(e^{-90k})^3$$

$$= 75 - 75\left(\frac{67}{75}\right)^3 = 21.55$$

- 2)** A research to investigate the effect of a certain chemical on a virus infection on crops revealed that the rate at which the virus population is destroyed is proportional to the population at that time. Initially the population was p_0 at t months later it was found to be p .

- Form a differential equation connecting p and t
- Given that the virus population reduced to $\frac{1}{3}$ of the initial population in 4 months solve the equation in (a) above
- Find:
 - how long it will take for only 5% of the original population
 - What percentage of the original virus population will be left after $2\frac{1}{2}$ months? (**Uneb 1999**)

Solution

a) $\frac{dp}{dt} \propto p$ or $\frac{dp}{dt} = -kp$

b) When $t = 4, p = \frac{1}{3}p_0$; $t = 0, p = p_0$

$$\Rightarrow \frac{dp}{p} = -k dt \Rightarrow \ln|p| = -kt + c$$

For $t = 4, p = \frac{1}{3}p_0, c = \ln|p_0|$

$$\therefore \ln\left|\frac{1}{3}p_0\right| = -4k + \ln|p_0|$$

$$\Rightarrow 4k = \ln|p_0| - \ln\left|\frac{1}{3}p_0\right| \therefore k = \frac{1}{4}\ln 3$$

$$\therefore \ln|p| = -\frac{t}{4}\ln 3 + \ln|p_0|$$

$$\Rightarrow \ln\left|\frac{p}{p_0}\right| = -\frac{t}{4}\ln 3 = \ln|3^{-t/4}|$$

$$\therefore p = p_0 \cdot 3^{-t/4}$$

c) (i) For $p = \frac{5p_0}{100} = \frac{p_0}{20}, t = ?$

$$\Rightarrow \frac{p_0}{20} = p_0 \cdot 3^{-t/4}$$

$$\Rightarrow 20 = 3^{\frac{t}{4}} \text{ or } \ln 20 = \frac{t}{4}\ln 3$$

$$\therefore t = \frac{4\ln 20}{\ln 3} = 10.907 \text{ hrs}$$

ii) Now for $t = 2\frac{1}{2} = \frac{5}{2}$ months, $p = ?$

$$\Rightarrow p = p_0 \cdot 3^{-\frac{5}{2} \times \frac{1}{4}} \therefore \frac{p}{p_0} = 3^{-5/8} = 0.503268$$

$$\therefore \frac{p}{p_0} = 50.3268\%$$

Wakissha 2009

A boy starts to sip a 900ml soda from a bottle at a rate of $10\text{cm}^3/\text{min}$. Given that the rate of consumption is inversely proportional to the square root of the volume of the soda remaining at any time, find the time it takes to empty the bottle.

Solution

Let volume of the soda remaining at any time t be v

$$\text{Given: } \frac{dv}{dt} \propto \frac{1}{\sqrt{v}} \Rightarrow \frac{dv}{dt} = -\frac{k}{\sqrt{v}}$$

$$\Rightarrow \sqrt{v} dv = -k dt, \text{ integrating}$$

$$\int \sqrt{v} dv = \int -k dt \quad \therefore \frac{2}{3} v^{3/2} = -k t + c$$

[$k = -ve$ because volume of the soda remaining keeps on decreasing]

Now at $t = 0, v = 900$

$$\Rightarrow \frac{2}{3} (900)^{3/2} = c \quad \therefore c = 18000$$

$$\therefore \frac{2}{3} \sqrt{v^3} = -k t + 18000$$

Now when $v = 0, t = T$

$$\frac{2}{3} (0) = -k T + 18000 \Rightarrow T = \frac{18000}{k} \dots\dots\dots (1)$$

$$\text{But } \frac{dv}{dt} = -10 = -\frac{k}{\sqrt{900}} \Rightarrow k = 300$$

$$\text{From eqn. (1), } T = \frac{18000}{300} = 60 \text{ minutes}$$

Thus the bottle will be finished in 60 minutes or 1 hour.

Alternatively

Let v = volume of soda sipped at any time t

$$\text{Given: } \frac{dv}{dt} \propto \frac{1}{\sqrt{900-v}} \Rightarrow \frac{dv}{dt} = +\frac{k}{\sqrt{900-v}}$$

$$\Rightarrow \sqrt{900-v} dv = k dt, \text{ integrating}$$

$$\int \sqrt{900-v} dv = \int k dt \quad \therefore \frac{(900-v)^{3/2}}{-3/2} = kt + c_1$$

$$\Rightarrow \frac{2}{3} (900-v)^{3/2} = -kt + c; c = -c_1$$

$$\text{At } t = 0, v = 0 \quad \therefore c = \frac{2}{3} (900)^{3/2} = 18000$$

$$\Rightarrow \frac{2}{3} (900-v)^{3/2} = -kt + 18000 \dots\dots\dots (1)$$

$$\text{Now } \frac{dv}{dt} = +10 = +\frac{k}{\sqrt{900-v}}, \text{ taking at } t = 0, v = 0$$

$$\Rightarrow k = 10\sqrt{900} = 300$$

Eqn. (1) becomes

$$\frac{2}{3} (900-v)^{3/2} = -300t + 18000$$

Now to empty the bottle, $v = 900, t = T$

$$\Rightarrow \frac{2}{3} (0) = -300T + 18000 \quad \therefore T = 60 \text{ minutes}$$

Thus the bottle will be finished within 60 minutes or 1 hour.

Uneb 2004 no. 16

An athlete runs at a speed proportional to the square root of the distance he has to cover. If the athlete starts running at 10 m/s and he has a distance of 1600m to cover, find how long he will take to cover the distance.

Solution

Let x = distance he still has to cover

$$\frac{dx}{dt} \propto \sqrt{x} \quad \text{or} \quad \frac{dx}{dt} = -k\sqrt{x} \quad \Rightarrow \quad \frac{dx}{\sqrt{x}} = -k \, dt, \text{ integrating}$$

$$\int \frac{dx}{\sqrt{x}} = \int -k \, dt \Rightarrow \int x^{-1/2} \, dx = \int -k \, dt$$

$$\therefore 2\sqrt{x} = -k \, t + c$$

$$\text{At } t = 0, x = 1600, c = 80$$

$$\Rightarrow 2\sqrt{x} = -k \, t + 80 \dots\dots\dots (1)$$

$$\text{Now } \frac{dx}{dt} = -k\sqrt{x} = -10 \Rightarrow k\sqrt{x} = 10$$

When he starts to cover the distance, $x = 1600$

$$\Rightarrow k\sqrt{1600} = 10 \quad \therefore k = \frac{1}{4}$$

Eqn. (1) becomes

$$2\sqrt{x} = -\frac{t}{4} + 80$$

To cover the whole distance 1600m, $x = 0, t = T$

$$0 = -\frac{T}{4} + 80 \quad \therefore T = 320$$

He takes 320 seconds to cover the whole distance.

Mixture Problems

- 1) A tank of volume 0.5m^3 is filled with brine containing 30kg of dissolved salt. Water runs in to the tank at a rate of $15 \times 10^{-5}\text{m}^3/\text{s}$ and the mixture, kept uniform by stirring, runs out at the same rate. How much salt is in the tank after 1 hour?

Solution

Let x = no. of kg of salt in the tank after t (seconds)

Concentration of salt/proportion of salt in brine at any time

$$\frac{x}{0.5} = 2x \text{ kg/m}^3$$

Considering a small change in t .i.e. dt for water to run in and out

$$\text{Quantity of salt out of tank in (m}^3\text{)} = (15 \times 10^{-5})dt$$

Now quantity of salt flowing out in dt

$$(15 \times 10^{-5})dt \times 2x = 3x \times 10^{-4} \, dt$$

Thus, the change dx of the amount of salt in the tank is

$$dx = 3x \times 10^{-4} \, dt$$

$$\Rightarrow \frac{1}{3 \times 10^{-4}} \cdot \frac{dx}{x} = -dt, \text{ integrating}$$

$$\frac{1}{3 \times 10^{-4}} \ln|x| = -t + c_1$$

$$\Rightarrow \ln|x| = -3 \times 10^{-4}t + c_1(3 \times 10^{-4})$$

$$\Rightarrow x = e^{-3 \times 10^{-4}t + c_1(3 \times 10^{-4})} = c e^{-3 \times 10^{-4}t}, \quad c = e^{c_1(-3 \times 10^{-4})}$$

Now when $t = 0, x = 30 \Rightarrow c = 30$

$$\therefore x = 30 e^{-3 \times 10^{-4}t}$$

Now when $t = 1 \text{ hr} = 3600 \text{ s}, x = ?$

$$\therefore x = 30 e^{-3 \times 10^{-4} \times 3600} = 10 \text{ kg}$$

- 2)** A tank being cleaned initially contains 3900 litres of water and 100 kg of dissolved dye. The mixture is kept uniform by stirring.

Pure water is run in at the rate of 130 litres per minute and the mixture is removed at the same rate. If the mass of dissolved dye remaining after t minutes is x kg, show that $\frac{dx}{dt} = -\frac{x}{30}$.

How long will it be before half the dye has been removed? Determine how much dye remains after 30 minutes.

Solution

Let x = mass of dissolved dye remaining after t (seconds)

Concentration of dye in solution = $\frac{x}{3900} = \text{kg/l}$

Let the mixture run in/out in dt minutes

Volume rate = 130 l/min

Quantity of dye running out = $(130 dt)$ litres

Mass of dye running out = $\frac{x}{3900} \times 130 dt = \frac{x}{30} dt$ kg

Thus the change dx of the dye in mixture

$$dx = -\frac{x}{30} dt \quad \therefore \frac{dx}{dt} = -\frac{x}{30}$$

Solving the differential equation

$$\frac{dx}{x} = -\frac{dt}{30} \Rightarrow \ln|x| = -\frac{t}{30} + c_1$$

$$x = e^{-\frac{t}{30} + c_1} \Rightarrow x = c e^{-\frac{t}{30}}, \quad c = e^{c_1}$$

$$\therefore x = c e^{-\frac{t}{30}}$$

Now when $x = 100 \text{ kg}, t = 0 \therefore c = 100$

$$\Rightarrow x = 100 e^{-\frac{t}{30}}$$

Now when $x = \frac{1}{2} \times 100 = 50 \text{ kg}, t = T$

$$\Rightarrow 50 = 100 e^{-\frac{T}{30}}$$

$$\Rightarrow \frac{1}{2} = e^{-\frac{T}{30}} \quad \text{or} \quad \ln\left|\frac{1}{2}\right| = -\frac{T}{30}$$

$$\therefore T = 30 \ln 2 \text{ minutes}$$

Also when $t = 30 \text{ min}, x = ?$

$$\Rightarrow x = 100 e^{-1} \text{ kg}$$

- 3)** A tank contains 50 litres of solution composed of 90% water and 10% alcohol

A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4 litres/min. as the second solution is being added, the tank is being drained at the rate of 5 litres/min.

Assuming the solution in the tank is stirred constantly, how much alcohol is in the tank after 10 minutes?

Solution

Let y = no. of litres of alcohol in the tank

At any time t

Amount of alcohol in tank initially i.e. at $t = 0$

$$= \frac{10}{100} \times 50 = 5$$

At $t = 0$, $y = 5$

No. of litres of solution in tank at any time t is $(50 - t)$

No. of litres lost/draind from tank = 5

∴ Litres of alcohol lost per minute = $\left(\frac{5}{50-t}\right) \cdot y$

No. of litres gained of alcohol per minute = $\frac{50}{100} \times 4 = 2$

Rate of change of alcohol in the tank

$$\frac{dy}{dt} = 2 - \left(\frac{5}{50-t}\right) \cdot y \quad \text{or} \quad \frac{dy}{dt} + \left(\frac{5}{50-t}\right) \cdot y = 2$$

$$\text{I.F.} = e^{\int \left(\frac{5}{50-t}\right) dt} = e^{-5 \ln|50-t|} = \frac{1}{(50-t)^5}$$

Using I.F. $y = \int \text{I.F.} \cdot Q \, dx$

$$\frac{y}{(50-t)^5} = \int \frac{2}{(50-t)^5} dt = \frac{1}{2(50-t)^4} + c$$

$$\therefore y = \frac{50-t}{2} - c(50-t)^5$$

$$\text{Now when } t = 0, y = 5 \quad \therefore c = -\frac{20}{50^5}$$

$$\therefore y = \frac{50-t}{2} - 20 \left(\frac{50-t}{50}\right)^5$$

Now when $t = 10$, $y = ?$

$$\therefore y = \frac{50-10}{2} - 20 \left(\frac{50-10}{50}\right)^5 = 13.45 \text{ l, which represents a solution containing } 33.6\% \text{ alcohol.}$$

- 4)** In a certain process the rate of production of yeast is $k \times$ grammes per minute, where x = amount produced and $k = 0.003$.

- a) Show that the amount of yeast is doubled in about 230 minutes
- b) If in addition, yeast is removed at a constant rate m grammes per minute, find the:

- i) Amount of yeast at time t , given that when $t = 0$, $x = p$ grammes
- ii) Value of m if $p = 20000$ grammes and the supply of yeast is exhausted in 100 minutes. **(Uneb 2009 no.16)**

Solution

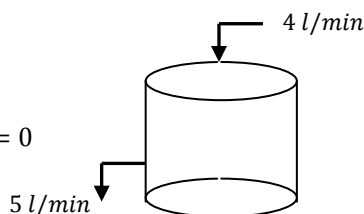
a) $\frac{dx}{dt} = kx \Rightarrow \frac{dx}{x} = k \, dt$, integrating

$$\ln|x| = k t + c_1 \quad \text{or} \quad x = e^{k t + c_1}$$

$$\therefore x = c e^{k t}; \quad c = e^{c_1}$$

Now when $t = 0$, $x = x_0$ and $t = T$, $x = 2x_0$

$$\therefore x_0 = c e^0 = c$$



$$\Rightarrow x = x_0 e^{k t}$$

$$\text{Now for } x = 2x_0, t = T$$

$$\Rightarrow 2x_0 = x_0 e^{0.003T} \quad \therefore 2 = e^{0.003T}$$

$$\Rightarrow \ln 2 = \ln e^{0.003T} = 0.003T$$

$$\therefore T = \frac{\ln 2}{0.003} \approx 231 \text{ min}$$

b) Rate of production of yeast = $0.003x$ grammes/min

Rate of removal of yeast = m grammes/min

Rate of change of production in yeast = $0.003x - m$

$$\therefore \frac{dx}{dt} = 0.003x - m \quad \text{or} \quad \frac{dx}{dt} - 0.003x = -m$$

$$\text{I.F} = e^{\int -0.003 dt} = e^{-0.003t}$$

$$\therefore x \cdot e^{-0.003t} = - \int m \cdot e^{-0.003t} dt = \frac{m}{0.003} e^{-0.003t} + c$$

Now at $t = 0, x = p$

$$\therefore p e^0 = \frac{m}{0.003} + c$$

$$\Rightarrow c = p - \frac{m}{0.003} = p - \frac{1000m}{3} = \frac{3p-1000m}{3}$$

$$\therefore x \cdot e^{-0.003t} = \frac{m}{0.003} e^{-0.003t} + \frac{3p-1000m}{3}$$

Now when $p = 20000, t = 100 \text{ min}, x = 0$

$$\Rightarrow 0 = \frac{m}{0.003} e^{-0.03} + \frac{60000-1000m}{3}$$

$$\Rightarrow 0 = \frac{1000m}{3} e^{-0.03} + \frac{60000-1000m}{3}$$

$$\Rightarrow 0 = \frac{1000m(e^{-0.03}-1)+60000}{3}$$

$$\therefore m = -\frac{60000}{1000(e^{-0.03}-1)} = \frac{60}{(1-e^{-0.03})} = 2030.15$$

Task

1. Glucose is added intravenously to the blood stream at a rate of q units per minute, and the body removes glucose from the blood stream at a rate proportional to the amount present.

Assume $Q(t)$ is the amount of glucose in the blood stream at time t ,

- a) Determine the differential equation describing the rate of change w.r.t time of glucose in the blood stream.
- b) Taking $Q = Q_0$ when $t = 0$, solve the differential equation

Answers

$$\text{a) } \frac{dQ}{dt} = q - kQ \quad \text{b) } Q = \frac{q}{k} + \left(Q_0 - \frac{q}{k}\right) e^{-kt}$$

2. A 200 m^3 tank is full of a solution containing 25 kg of concentrate.

Starting at $t = 0$, distilled water is admitted to the tank at a rate of $10 \text{ m}^3/\text{min}$, and the well stirred solution is withdrawn at the same rate.

- a) Find the amount of concentrate in the solution as a function of t
- b) Find the time at which the concentrate in the tank reaches 15 kg
- c) Assuming that the solution entering the tank contains 0.05 kg of concentrate per m^3 . Repeat (a) and (b) above.

Answer (ii)

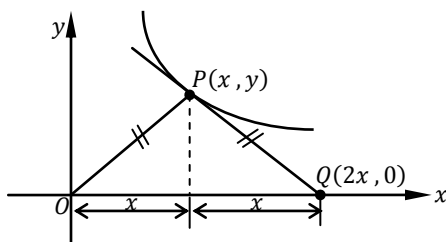
a) $q = 10 + 15e^{-0.005t}$

b) $t = -20 \ln \frac{1}{3} \approx 21.97 \text{ min}$

Tangent to the Curve

1. The tangent at any point P on a curve in the first quadrant cuts the x -axis at Q . Given that $OP = PQ$, where O is the origin and that the point $(1, 4)$ lies on the curve, find the equation of the curve.

Solution



Let $P(x, y)$ be the point
 ΔOPQ is isosceles
 $\Rightarrow Q(2x, 0)$

Now Gradient of a tangent at $P = \frac{dy}{dx}$. i.e. Grad $\overline{PQ}$

$$\Rightarrow \frac{dy}{dx} = \frac{0-y}{2x-x} = -\frac{y}{x}$$

Separating variables

$$\frac{dy}{y} = -\frac{1}{x} dx, \text{ integrating}$$

$$\int \frac{dy}{y} = \int -\frac{1}{x} dx \Rightarrow \ln|y| = -\ln|x| + c$$

$$\therefore \ln|x \cdot y| = c \Rightarrow e^c = x \cdot y$$

Now $(1, 4)$ is on this curve

$$\Rightarrow e^c = 1 \times 4 = 4$$

$$\therefore xy = 4$$

Task

1. The tangent at any point P on a curve cuts the x -axis at the point Q . given that $\angle OPQ = 90^\circ$ where O is the origin and that the point $(1, 2)$ lies on the curve, find the equation of the curve.

Ans: $x^2 + y^2 = 5$

2. The tangent at any point P on the curve cuts the x -axis at A and the y -axis at B . given that $2\overrightarrow{AP} = \overrightarrow{PB}$ and the curve passes through the point $(1, 1)$, find the equation of the curve. **Ans:** $y = 1/x^2$

3. The tangent of a curve at the point (x, y) cuts the x -axis at Q . if $OQ = kx$ (where O is the origin of the coordinate system) show that the points on the curve satisfy the differential equation. **Ans:** $\frac{dy}{dx} = \frac{y}{x-kx}$. By solving the differential equation show that the equation of the curve has the form $y = Ax^{1/(1-k)}$ where A is an arbitrary constant.

CIRCLES

Definition:

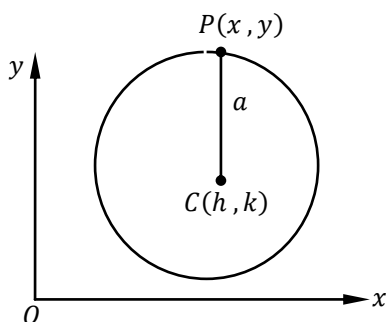
The locus of a point which moves in such a way that its distance from a fixed point is always a constant

The fixed point is called the *centre* and the fixed/constant distance is the *radius*.

Equation of a Circle (Cartesian Equation of Locus of a Point)

A. To find the Equation of the Circle given the Centre (h, k) and Radius, a

Consider the figure below



The point $P(x, y)$ whose locus is required lies on the circle.

From geometry,

$$\text{Distance } \overline{CP} = a \Rightarrow \sqrt{(x - h)^2 + (y - k)^2} = a$$

Squaring both sides gives the equation of the circle.

$$(x - h)^2 + (y - k)^2 = a^2$$

NOTE:

If the centre $C(h, k)$ is $(0, 0)$.i.e. at the origin the equation of the circle is

$x^2 + y^2 = a^2$, which is the simplest equation of a circle, called the *standard equation* of the circle.

Examples

1) Find the equation of a circle having

a) Radius 2 units and centre $(-1, 2)$

b) Radius $\sqrt{3}$ units and centre $(-2, -3)$

Solution

Using the result $(x - h)^2 + (y - k)^2 = a^2$

a) $(h, k) = (-1, 2) \therefore h = -1, k = 2$

Radius = $a = 2$

$$\Rightarrow (x - (-1))^2 + (y - 2)^2 = 2^2$$

$$\therefore (x + 1)^2 + (y - 2)^2 = 4$$

b) $(h, k) = (-2, -3) \therefore h = -2, k = -3$

$$\text{Radius} = a = \sqrt{3}$$

$$\Rightarrow (x - (-2))^2 + (y - (-3))^2 = (\sqrt{3})^2$$

$$\therefore (x + 2)^2 + (y + 3)^2 = 3$$

Task

Find the equation of a circle with

a) Radius 3 units and centre origin

Ans: $x^2 + y^2 = 9$

b) Radius $\frac{1}{2}\sqrt{5}$ units and centre $(\frac{1}{2}, \frac{1}{2})$

Ans: $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{5}{4}$

Stating the Centre and Radius of the circle in the form

$$(x - h)^2 + (y - k)^2 = a^2$$

Example

State the centre and radius of the circle;

i) $(x - 1)^2 + (y - 3)^2 = 4$

ii) $(x + 2)^2 + (y + 5)^2 = 5$

iii) $(x + 5)^2 + (y - 3)^2 - 8 = 0$

Solution

The above equations must be written in the form

$$(x - h)^2 + (y - k)^2 = a^2$$

i) $(x - 1)^2 + (y - 3)^2 = 4$

Re writing

$$\Rightarrow (x - 1)^2 + (y - 3)^2 = 2^2 \text{ [4 is written in terms of power 2]}$$

$$\therefore \text{Centre } (1, 3), \text{Radius} = 2 \text{ units}$$

ii) $(x + 2)^2 + (y + 5)^2 = 5$

Re writing

$$\Rightarrow [x - (-2)]^2 + [y - (-5)]^2 = (\sqrt{5})^2 \text{ [the -ve sign in the squared brackets has been introduced, 5 is written in terms of power 2]}$$

$$\therefore \text{Centre } (-2, -5), \text{Radius} = \sqrt{5} \text{ units}$$

Task

State the centre and radius of the circle;

i) $(x + 2)^2 + (y - \frac{1}{4})^2 = 10$ **Ans:** Centre $(-2, \frac{1}{4})$, Radius = $\sqrt{10}$ units

ii) $(x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 = 20$ **Ans:** Centre $(-\frac{1}{2}, \frac{1}{2})$, Radius = $\sqrt{20}$ units

iii) $(x + 5)^2 + (y + 4)^2 - 40 = 0$ **Ans:** Centre $(-5, -4)$, Radius = $\sqrt{40}$ units

General Equation of the Circle

From the equation of the circle $(x - h)^2 + (y - k)^2 = a^2$,

Expanding the brackets

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = a^2$$

$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0$

, which is the general equation

of a circle.

Rewriting this equation of the circle

Let $-h = g$; $-k = f$ and $h^2 + k^2 - a^2 = c$

Thus the general equation of a circle can be stated as;

$x^2 + y^2 + 2gx + 2fy + c = 0$

Deduction:

For the general equation to represent a circle,

- The coefficient of x^2 and y^2 must be equal
- No term involving xy
- It's a second degree equation

Examples

State with reasons whether the equations represent a circle

- a) $x^2 + y^2 - 2x + 4y + 2 = 0$
- b) $2x^2 + 2y^2 - 4x + 2y - 3 = 0$
- c) $4x^2 - 4y^2 + 4x + 2y - 3 = 0$
- d) $3x^2 + 2y^2 - 5x + y - 1 = 0$
- e) $3x^2 + 3y^2 - 5xy + x + y = 0$

Solution

- a) $x^2 + y^2 - 2x + 4y + 2 = 0$ is a circle because the coefficients of x^2 , y^2 are the same, no product of xy term
- b) $2x^2 + 2y^2 - 4x + 2y - 3 = 0$ is a circle because the coefficients of x^2 , y^2 are the same, no product of xy term
- c) $4x^2 - 4y^2 + 4x + 2y - 3 = 0$ is Not a circle because the coefficients of x^2 , y^2 are not the same.
- d) $3x^2 + 2y^2 - 5x + y - 1 = 0$ is Not a circle because the coefficients of x^2 , y^2 are not the same.
- e) $3x^2 + 3y^2 - 5xy + x + y = 0$ is Not a circle because of the product xy

STATING THE CENTRE AND RADIUS OF THE GENERAL EQUATION OF THE CIRCLE $x^2 + y^2 + 2gx + 2fy + c = 0$

Firs rewriting the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ in to the form

$$(x - h)^2 + (y - k)^2 = a^2$$

$$(x^2 + 2gx) + (y^2 + 2fy) = -c$$

Completing squares

$$\Rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = -c + g^2 + f^2$$

$$\Rightarrow (x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

$$\therefore [x - (-g)]^2 + [y - (-f)]^2 = [\sqrt{g^2 + f^2 - c}]^2$$

$\therefore$ Centre $(-g, -f)$, Radius is $\sqrt{g^2 + f^2 - c}$

$$\therefore x^2 + y^2 + 2gx + 2fy + c = 0 \text{ has centre } (-g, -f) \text{ and Radius } \sqrt{g^2 + f^2 - c}$$

NOTE:

In general, a circle expressed in the form $x^2 + y^2 + 2gx + 2fy + c = 0$ has:

Centre $(-g, -f) = \left[-\frac{1}{2} \times \text{coeff. of } x, -\frac{1}{2} \times \text{coeff. of } y\right]$ and

Radius

$$= \sqrt{g^2 + f^2 - c} = \sqrt{(\text{sum of squares of centre coodinates}) - \text{constant}}$$

Example

1) State the coordinates of centre and radius of the circle

i) $x^2 + y^2 + 2x - 4y - 4 = 0$

ii) $x^2 + y^2 - 3x + 4y = 0$

Solution

i) $x^2 + y^2 + 2x - 4y - 4 = 0$

Rewriting the equation

$$(x^2 + 2x) + (y^2 - 4y) = 4$$

Completing squares

$$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 4 + 1 + 4$$

$$(x + 1)^2 + (y - 2)^2 = 2^2$$

$\therefore$ Centre $(-1, 2)$ Radius = 2 units

ii) $x^2 + y^2 - 3x + 4y = 0$

Rewriting the equation

$$(x^2 - 3x) + (y^2 + 4y) = 0$$

Completing squares

$$\left(x^2 - 3x + \frac{9}{4}\right) + (y^2 + 4y + 4) = \frac{9}{4} + 4 = \frac{25}{4}$$

$$\left(x - \frac{3}{2}\right)^2 + (y + 2)^2 = \left(\frac{5}{2}\right)^2$$

$$\therefore \text{Centre } \left(\frac{3}{2}, 2\right) \text{ Radius} = \frac{5}{2} \text{ units}$$

2) State the coordinates of centre and radius of the circle

i) $2x^2 + 2y^2 - 3x + 4y + 2 = 0$

ii) $3x^2 + 3y^2 - 4x + 6y + 3 = 0$

Solution

i) $2x^2 + 2y^2 - 3x + 4y + 2 = 0$

Making coefficients of x^2, y^2 unity by dividing through by 2

$$x^2 + y^2 - \frac{3}{2}x + 2y + 1 = 0$$

Rewriting the equation

$$\left(x^2 - \frac{3}{2}x\right) + (y^2 + 2y) = -1$$

Completing squares

$$\left(x^2 - 3x + \frac{9}{16}\right) + (y^2 + 2y + 1) = -1 + \frac{9}{16} + 1$$

$$\left(x - \frac{3}{4}\right)^2 + (y + 1)^2 = \left(\frac{3}{4}\right)^2$$

$$\therefore \text{Centre } \left(-\frac{3}{4}, -1\right) \text{ Radius} = \frac{3}{4} \text{ units}$$

ii) $3x^2 + 3y^2 - 4x + 6y + 3 = 0$

Making coefficients of x^2, y^2 unity by dividing through by 3

$$x^2 + y^2 - \frac{4}{3}x + 2y + 1 = 0$$

Rewriting the equation

$$\left(x^2 - \frac{4}{3}x\right) + (y^2 + 2y) = -1$$

Completing squares

$$\left(x^2 - 3x + \frac{4}{9}\right) + (y^2 + 2y + 1) = -1 + \frac{4}{9} + 1$$

$$\left(x - \frac{2}{3}\right)^2 + (y + 1)^2 = \left(\frac{2}{3}\right)^2$$

$$\therefore \text{Centre } \left(\frac{2}{3}, -1\right) \text{ Radius} = \frac{2}{3} \text{ units}$$

Task

State the coordinates of centre and radius of the circle

i) $4x^2 + 4y^2 - 5x + 3y - 8 = 0$

ii) $x^2 + y^2 - 5x + 3y - 1 = 0$

iii) $Ax^2 + Ay^2 + Bx + Cy + D = 0$

Ans: (i) Centre: $\left(\frac{5}{8}, -\frac{3}{8}\right)$, Radius = $\frac{9}{4}\sqrt{2}$ units

Ans: (ii) Centre: $\left(\frac{5}{2}, -\frac{3}{2}\right)$, Radius = $\frac{\sqrt{19}}{2}$ units

Ans: (i) Centre: $\left(-\frac{D}{2A}, -\frac{F}{2A}\right)$, Radius = $\frac{1}{2A}\sqrt{D^2 + F^2 - 4AC}$ units

APPLICATION OF GENERAL EQUATION OF A CIRCLE $x^2 + y^2 + 2gx + 2fy + c = 0$

To find the equation of a circle passing through THREE points

Example

Find the Cartesian equation of a circle through points

- i) Origin, $A(1, 2)$ and $B(-2, -1)$
- ii) $A(1, 1)$, $B(2, -1)$ and $C(3, 2)$

Solution

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

- i) For $O(0, 0)$; $0 + 0 + 0 + 0 + c = 0 \quad \therefore c = 0$

For $A(1, 2)$; $1 + 4 + 2g + 4f + c = 0$

$$\therefore 2g + 4f = -5 \dots\dots\dots (1)$$

For $B(-2, -1)$; $4 + 1 - 4g - 2f + c = 0$

$$\therefore -4g - 2f = -5 \dots\dots\dots (2)$$

Solving eqn. (1) and eqn. (2) simultaneously

Eqn. (1) $\times 2$ + eqn. (2) yields

$$f = -\frac{5}{2} \dots\dots\dots (3)$$

Eqn. (3) in to eqn. (2)

$$\Rightarrow 2g + 4\left(-\frac{5}{2}\right) = -5, \text{ from which } g = \frac{5}{2}$$

$$\Rightarrow x^2 + y^2 + 2\left(\frac{5}{2}\right)x + 2\left(-\frac{5}{2}\right)y = 0$$

$$\therefore x^2 + y^2 + 5x - 5y = 0$$

- ii) For $A(1, 1)$; $1 + 1 + 2g + 2f + c = 0$

$$\therefore 2g + 2f + c = 0 \dots\dots\dots (1)$$

For $B(2, -1)$; $4 + 1 + 4g - 2f + c = 0$

$$\therefore 4g - 2f + c = -5 \dots\dots\dots (2)$$

For $C(3, 2)$; $9 + 4 + 6g + 4f + c = 0$

$$\therefore 6g + 4f + c = -13 \dots\dots\dots (3)$$

Eliminating c from eqn. (1) and eqn. (2)

Eqn. (1) - eqn. (2) yields

$$-2g + 4f = 3 \dots\dots\dots (4)$$

Eliminating c from eqn. (1) and eqn. (3)

Eqn. (3) - eqn. (2) yields

$$-4g - 2f = 11 \dots\dots\dots (5)$$

Now solving eqn. (4) and eqn. (5) simultaneously

Eqn. (4) $\times 2$ - eqn. (5) to eliminate gives

$$f = -\frac{1}{2} \dots\dots\dots (6)$$

Putting eqn. (6) in to eqn. (5) or eqn. (4) yields

$$g = -\frac{5}{2} \dots\dots\dots (7)$$

Putting eqn. (7) in to eqn. (1) or eqn. (2) or eqn. (3) yields

$$c = 4$$

Equation of the circle is

$$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$$

$$\therefore x^2 + y^2 - 5x - y + 4 = 0$$

Task

- a) Find the equation of a circle passing through points $A(-2, -1)$, $B(4, -2)$ and $C(2, 1)$

Ans: $8x^2 + 8y^2 - 15x + 50y - 20 = 0$

- b) Form an equation of a circle which passes through the points $A(-1, 4)$, $B(2, 5)$ and $C(0, 1)$ (Uneb 2006)

Ans: $x^2 + y^2 - 7x - 9y + 24 = 0$

- c) Find the equation of a circle which passes through the points $(5, 7)$, $(1, 3)$ and $(2, 2)$. (Uneb 2005)

Ans: $x^2 + y^2 - 7x - 9y + 24 = 0$

- d) Find the equation of a circle circumscribing the triangle whose vertices are $A(1, 3)$, $B(4, -5)$ and $C(9, -1)$. Find also the centre and radius.

Ans: $13x^2 + 13y^2 - 113x + 8y - 41 = 0$, centre: $\left(\frac{113}{26}, -\frac{8}{26}\right)$, Radius: 4.71

To find an Equation of the Circle passing through TWO given points and satisfying another condition

Example

Find an equation of a circle which passes through points

- a) $(4, 1)$ and $(6, 5)$ and having its centre on the line $4x + y - 16 = 0$
 b) $(0, 0)$ and $(1, 1)$ and having its radius as 4 units

Solution

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

- a) For $(4, 1)$; $16 + 1 + 8g + 2f + c = 0$

$$\therefore 8g + 2f + c = -17 \dots\dots\dots (1)$$

For $(6, 5)$; $36 + 25 + 12g + 10f + c = 0$

$$\therefore 12g + 10f + c = -61 \dots\dots\dots (2)$$

Centre of the circle $(-g, -f)$ lies on the line $4x + y - 16 = 0$

$$\therefore -4g - f = 16 \dots\dots\dots (3)$$

Solving eqn. (1) and eqn. (2) simultaneously, eliminating c

Eqn. (1) - eqn. (2) yields

$$-4g - 8f = 44 \dots\dots\dots (4)$$

Eliminating c from Eqn. (3) and eqn. (4)

Eqn. (3) - eqn. (4) yields

$$\Rightarrow 7f = -28, \text{ from which } f = -4 \dots\dots\dots (5)$$

Putting eqn. (5) in to eqn. (3) or eqn. (4)

$$g = -3 \dots\dots\dots (6)$$

Putting eqn. (6) in to eqn. (1) or eqn. (2)

$$c = 15$$

$$\Rightarrow x^2 + y^2 - 6x - 8y + 15 = 0$$

- b) For $(0, 0)$; $0 + 0 + 0 + 0 + c = 0 \Rightarrow c = 0$

For $(1, 1)$; $1 + 1 + 2g + 2f + c = 0$

$$\therefore g + f = -1 \dots\dots\dots (1)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = 4$$

$$\Rightarrow g^2 + f^2 = 16 \dots\dots\dots (2)$$

From eqn. (1), $g = -(f + 1)$

Now eqn. (2) becomes

$$[-(f + 1)]^2 + f^2 = 16$$

$$\Rightarrow f^2 + 2f + 1 + f^2 = 16 \text{ or } 2f^2 + 2f - 15 = 0$$

Solving the equation yields

$$\therefore f = \frac{-1 \pm \sqrt{31}}{2} \text{ or } f = \frac{-1 + \sqrt{31}}{2}, \frac{-1 - \sqrt{31}}{2}$$

From eqn. (1),

$$g = -(f + 1)$$

$$\text{For } f = \frac{-1 + \sqrt{31}}{2}, g = -\left(\frac{-1 + \sqrt{31}}{2} + 1\right) = \frac{\sqrt{31} - 1}{2}$$

$$\text{For } f = \frac{-1 - \sqrt{31}}{2}, g = -\left(\frac{-1 - \sqrt{31}}{2} + 1\right) = -\left(\frac{\sqrt{31} + 1}{2}\right)$$

Circle are:

$$x^2 + y^2 + 2\left(\frac{\sqrt{31} - 1}{2}\right)x + 2\left(\frac{-1 + \sqrt{31}}{2}\right)y = 0$$

$$\therefore x^2 + y^2 + (\sqrt{31} - 1)x - (\sqrt{31} + 1)y = 0$$

Also

$$x^2 + y^2 + 2\left(\frac{-\sqrt{31} - 1}{2}\right)x - 2\left(\frac{1 + \sqrt{31}}{2}\right)y = 0$$

$$\therefore x^2 + y^2 - (\sqrt{31} + 1)x - (\sqrt{31} - 1)y = 0$$

To find the equation of a circle passing through ONE given point and with a stated centre coordinates

Example

Find the equation of a circle with

- Centre $(-1, 2)$ and passing through point $(-3, 1)$
- Centre $(4, -1)$ and passing through point $(1, 3)$

Solution

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f)$

$$\text{a) } \Rightarrow (-g, -f) = (-1, 2) \therefore g = 1, f = -2 \dots\dots\dots (1)$$

$$\text{For } (-3, 1), 9 + 1 - 6g + 2f + c = 0$$

$$\therefore -6g + 2f + c = -10 \dots\dots\dots (2)$$

Putting eqn. (1) in to eqn. (2)

$$\Rightarrow -6 - 4 + c = -10 \therefore c = 0$$

The equation of the circle is

$$\therefore x^2 + y^2 + 2x - 4y = 0$$

$$\text{b) } \Rightarrow (-g, -f) = (4, -1) \therefore g = -4, f = 1 \dots\dots\dots (1)$$

$$\text{For } (1, 3), 1 + 9 + 2g + 6f + c = 0$$

$$\therefore 2g + 6f + c = -10 \dots\dots\dots (2)$$

Putting eqn. (1) in to eqn. (2)

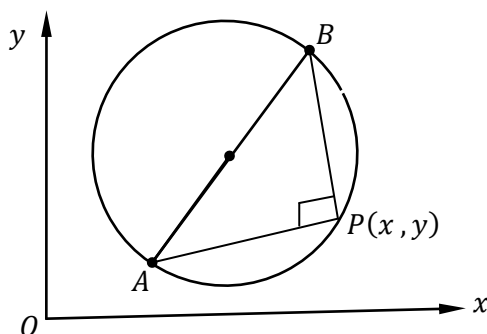
$$\Rightarrow -8 + 2 + c = -4 \therefore c = -4$$

The equation of the circle is

$$\therefore x^2 + y^2 - 8x + 2y - 4 = 0$$

DIAMETER FORM OF EQUATION OF A CIRCLE

To find the equation of a circle having $A(x_1, y_1)$ and $B(x_2, y_2)$ as ends of the diameter



Considering $P(x, y)$ to be the general point on the circle

From circle properties; $\angle APB = 90^\circ$

From geometry, $\text{slope } \overline{AP} \times \text{slope } \overline{PB} = -1$

$$\text{Slope } \overline{AP} = \frac{y-y_1}{x-x_1}$$

$$\text{Slope } \overline{PB} = \frac{y-y_2}{x-x_2}$$

$$\therefore \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

$$\therefore (y - y_1)(y - y_2) + (x - x_1)(x - x_2) = 0$$

, which is the

equation of the circle in diameter form.

Examples

Find the equation of a circle having extremities points given as diameter

a) $A(1, 2)$ and $B(-5, 6)$

b) $A(3, 0)$ and $B(7, -2)$

Solution

a) $A(1, 2)$ and $B(-5, 6)$

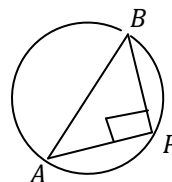
Taking a general point $P(x, y)$ on the circle,

$$\text{Grad } \overline{AP} = m_1 = \frac{y-2}{x-1}$$

$$\text{Grad } \overline{PB} = m_2 = \frac{y-6}{x+5}$$

By geometry $m_1 \times m_2 = -1$

$$\therefore \left(\frac{y-2}{x-1}\right) \times \left(\frac{y-6}{x+5}\right) = -1 \Rightarrow (x-1)(x+5) + (y-2)(y-6) = 0$$



$$\Rightarrow x^2 + 4x - 5 + y^2 - 8y + 12 = 0$$

$$\therefore x^2 + y^2 + 4x - 8y + 7 = 0$$

b)

Proceeding as in (a)

above, the equation of the circle is;

$$(x - 3)(x - 7) + y(y + 2) = 0 \text{ or } x^2 + y^2 - 10x + 2y + 21 = 0$$

NOTE:

The question on diameter equation of a circle can also be transformed in to “To find the equation of a circle passing through **one** point and having a stated centre *i.e.* midpoint of diameter extremity points.

Example

With reference to (a) above,

$A(1, 2)$ and $B(-5, 6)$

$$\text{Centre of } AB = \left(\frac{1+(-5)}{2}, \frac{2+6}{2} \right) = (-2, 6)$$

$$\text{Using } x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\text{For } A(1, 2); 1 + 4 + 2g + 4f + c = 0$$

$$\Rightarrow 2g + 4f + c = -5 \dots\dots\dots (1)$$

$$\text{Centre: } (-g, -f) = (-2, 6) \therefore g = 2, f = -6 \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1) to obtain c

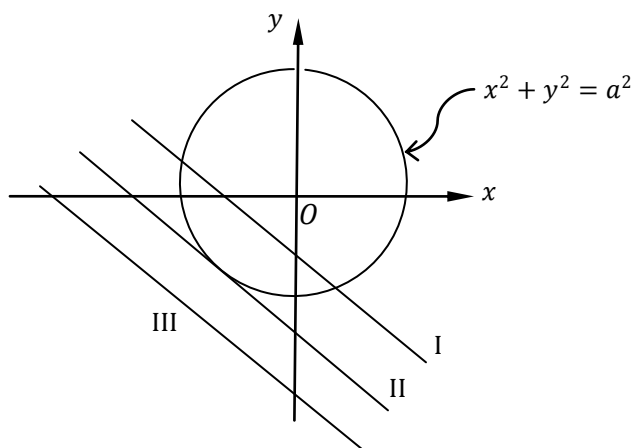
$$\Rightarrow 4 - 16 + c = -5 \therefore c = 7$$

$$\therefore x^2 + y^2 + 4x - 8y + 7 = 0$$

A LINE (TANGENT) AND A CIRCLE

A tangent is a line *touching* the circle at *only one point*. *i.e.* one point of contact.

Consider a circle $x^2 + y^2 = a^2$ and the line (tangent) $y = mx + c$



The *three* key questions asked about a line and tangents are:

To find the point(s) of intersection of a line and a circle

$$x^2 + y^2 = a^2 \dots\dots\dots (1)$$

$$y = mx + c \dots\dots\dots (2)$$

Solving simultaneously

$$x^2 + (mx + c)^2 = a^2$$

$$\Rightarrow x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$

$$x^2(1 + m^2) + (2mc)x + (c^2 - a^2) = 0$$

This equation is a quadratic in x which can be solved for values of x .

Three cases arise when solved.

Case I: *Two distinct points of contact*

For two distinct and real roots of a quadratic equation, $B^2 - 4AC > 0$

$$\Rightarrow (2mc)^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

$$\Rightarrow 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2) > 0$$

$$\Rightarrow -c^2 + a^2 + m^2a^2 > 0$$

$$a^2(1 + m^2) > c^2$$

Case II: *One coincident root .i.e. one point of contact*

This is a case of showing that a line is a tangent to the circle

For the quadratic equation $B^2 - 4AC = 0$

Condition for a line to be a tangent is

$$a^2(1 + m^2) = c^2 \text{ or } c = \pm a\sqrt{1 + m^2}$$

To find the point of contact of tangent and a circle

Putting $c = \pm a\sqrt{1 + m^2}$ in to $x^2(1 + m^2) + (2mc)x + (c^2 - a^2) = 0$

$$\Rightarrow x^2(1 + m^2) \pm (2ma\sqrt{1 + m^2})x + (a^2(1 + m^2) - a^2) = 0$$

$\Rightarrow x^2(1 + m^2) \pm (2ma\sqrt{1 + m^2})x + a^2m^2 = 0$ this should be factorizable to a perfect square

$$\Rightarrow (x\sqrt{1 + m^2} \pm am)^2 = 0 \text{ from which } x = \pm \frac{am}{\sqrt{1 + m^2}}$$

Obtaining the corresponding value of y ,

$$\text{From } y = mx + c, x = \pm \frac{am}{\sqrt{1 + m^2}} \text{ and } c = \pm a\sqrt{1 + m^2}$$

$$\Rightarrow y = \pm \frac{am^2}{\sqrt{1 + m^2}} \pm a\sqrt{1 + m^2} = \pm(am^2 + a(1 + m^2)) = \pm a(2m^2 + 1)$$

$\therefore$ Point of contact is

$$\left[\pm \frac{am}{\sqrt{1 + m^2}}, \pm a(2m^2 + 1) \right]$$

Case III: *Non real roots. i.e. No point of contact*

A line and a circle do not intersect if $B^2 - 4AC < 0$

$$a^2(1 + m^2) < c^2$$

Examples

Prove that the line

- a) $3x + 4y - 16 = 0$ touches the circle $x^2 + y^2 - 14x - 10y + 49 = 0$
 b) $3x - 4y + 1 = 0$ touches the circles $x^2 + y^2 - 4x - 16y + \frac{187}{9} = 0$ and $(x - 2)^2 + (y - 8)^2 = 25$
 c) $12x + 5y - 115 = 0$ touches the circle $x^2 + y^2 - 14x - 2y + 46 = 0$
 d) $3x - 4y - 7 = 0$ touches the circle $x^2 + y^2 - 14x - 2y + 46 = 0$. Hence find the point of contact in each of the above questions

Solution

a) $3x + 4y - 16 = 0$, $x^2 + y^2 - 14x - 10y + 49 = 0$

$$\Rightarrow x = \left(\frac{16-4y}{3}\right) \dots\dots\dots (1)$$

$$x^2 + y^2 - 14x - 10y + 49 = 0 \dots\dots\dots (2)$$

Eqn. (1) in to eqn. (2)

$$\Rightarrow \left(\frac{16-4y}{3}\right)^2 + y^2 - 14\left(\frac{16-4y}{3}\right) - 10y + 49 = 0$$

Simplifying

$$25y^2 - 50y + 25 = 0 \text{ or } y^2 - 2y + 1 = 0$$

$$\Rightarrow (y - 1)^2 = 0 \therefore y = 1, 1$$

Since there is only one value of $y = 1$ then there is only one point of contact. Hence the line is a tangent.

Point of contact

$$\text{For } y = 1, x = \frac{16-4 \times 1}{3} = 4$$

$\therefore (4, 1)$ is the point of contact.

b) Left as an exercise. Point of contact is $\left(\frac{7}{3}, 2\right)$

c) Left as an exercise

d) Left as an exercise

EQUATION OF A TANGENT TO A GIVEN CIRCLE

To find the equation of a tangent to a circle at a point $P(x_1, y_1)$ on the circle $x^2 + y^2 = a^2$

The general equation of a line passing through one point is:

$$y - y_1 = m(x - x_1) \dots\dots\dots (1)$$

Finding slope, m of the tangent

Since a tangent is passing through $P(x_1, y_1)$ which is on the circle, from differentiation, the derivative of a curve at a point is the slope of the tangent.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(a^2)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow m = \left. \frac{dy}{dx} \right|_{\text{at } (x_1, y_1)} = -\frac{x_1}{y_1} \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1) yields

$$y - y_1 = \left(-\frac{x_1}{y_1}\right)(x - x_1)$$

$$\Rightarrow yy_1 - y_1^2 = -xx_1 + x_1^2$$

$$xx_1 + yy_1 = x_1^2 + y_1^2 = a^2$$

, which is the equation of the tangent to the standard equation of a circle.

Example

Find the equation of a tangent to the circle at the given points;

- a) Circle: $x^2 + y^2 = 25$, point: $(-3, 4)$
 b) Circle: $x^2 + y^2 + 4x - 4y + 4 = 0$, point $(0, 2)$

Solution

- a) $x^2 + y^2 = 25$, point $(-3, 4)$ lies on the circle. i.e. $(-3)^2 + (4)^2 = 25$

Tangent equation

$$y - 4 = m(x + 3) \dots\dots\dots (1)$$

Finding the slope m

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow m = \left. \frac{dy}{dx} \right|_{\text{at } (-3, 4)} = -\frac{4}{-3} = \frac{4}{3}$$

$$\therefore y - 4 = \frac{4}{3}(x + 3), \text{ simplifying gives the tangent equation as}$$

$$\therefore 4x - 3y + 24 = 0$$

- b) $x^2 + y^2 + 4x - 4y + 4 = 0$, point $(0, 2)$ lies on the circle. i.e.

$$(0)^2 + (2)^2 + 4(0) - 4(2) + 4 = 0$$

Tangent equation

$$y - 2 = m(x - 0) \dots\dots\dots (1)$$

Finding the slope m

$$\frac{d}{dx}(x^2 + y^2 + 4x - 4y + 4) = \frac{d}{dx}(0)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} + 4 - 4 \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{4-2x}{2y-4}$$

$$\Rightarrow m = \left. \frac{dy}{dx} \right|_{\text{at } (0, 2)} = -\frac{4-2(0)}{2(2)-4} = -\frac{4}{0}$$

From eqn. (1),

$$\frac{y-2}{x} = m = -\frac{4}{0}$$

$$\therefore x = 0 \text{ is the tangent equation}$$

Task

Find the equation of the tangent to the circle

- i) $x^2 + y^2 + 4x - 4y + 4 = 0$ at the point $(-2, 0)$
 ii) $x^2 + y^2 - 4x + 2y + 1 = 0$ at the point $(2, -1)$
 iii) $x^2 + y^2 - 4x + 2y = 25$ at the point $(2, -3)$

Ans:

To find the equation of a circle touching a known line and satisfying atleast one other given condition.

Example

- 1) Find the equation of a circle whose centre is $\left(\frac{5}{2}, -3\right)$ and touches the line $3x + 2y + 5 = 0$
- 2) Find the equation(s) of the circle(s) which touch the y - axis and pass through points $(1, -2)$ and $(3, -4)$
- 3) Find the equation of the circle with radius 5 units and touching the line $3x + 4y - 16 = 0$ at the point $(4, 1)$
- 4) Find the equation of the circle which is inscribed in a triangle formed by the lines $3x + 4y - 15 = 0$, $3x - 4y - 7 = 0$ and $12x + 5y - 115 = 0$
- 5) Find the equation of the circle touching x - axis and y - axis at points $(0, 4)$ and $(6, 0)$
- 6) (i) If $x = 0$ and $y = 0$ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, show that $c = g^2 = f^2$
 ii) Given that the line $3x - 4y + 6 = 0$ is also a tangent to the circle in (i) above, determine the equation of the circle lying in the first quadrant.
(Uneb 2005)

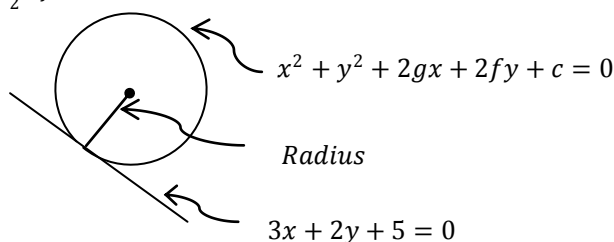
Solution

- 1) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Line $3x + 2y + 5 = 0$, centre $\left(\frac{5}{2}, -3\right)$

$$\left(\frac{5}{2}, -3\right) = (-g, -f)$$

$$\therefore g = -\frac{5}{2}, f = 3 \dots\dots\dots (1)$$



Radius = perpendicular distance of centre from tangent

$$= \frac{3(5/2) + 2(-3) + 5}{\sqrt{3^2 + 2^2}} = \frac{13}{2\sqrt{13}}$$

$$\text{But radius} = \sqrt{g^2 + f^2 - c}$$

$$\therefore \frac{13}{2\sqrt{13}} = \sqrt{g^2 + f^2 - c}, \text{ squaring both sides}$$

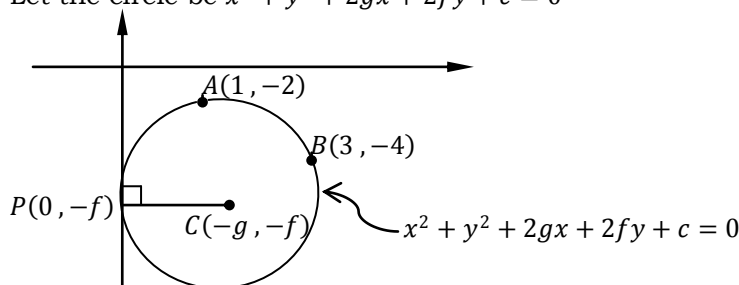
$$\Rightarrow \frac{13}{4} = \frac{25}{4} + 9 - c \therefore c = 12$$

Required equation of a circle is:

$$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2(3)y + 12 = 0$$

$$\therefore x^2 + y^2 - 5x + 6y + 12 = 0$$

2) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$



For $A(1, -2)$: $1 + 4 + 2g - 4f + c = 0$
 $\therefore 2g - 4f + c = -5$ (1)

For $B(3, -4)$: $9 + 16 + 6g - 8f + c = 0$
 $\therefore 6g - 8f + c = -25$ (2)

Let the circle touch y -axis at P

$\Rightarrow$ Coordinates of P are $(0, -f)$. This point lies on the circle

$\Rightarrow 0 + f^2 + 0 - 2f^2 + c = 0$
 $\therefore c = f^2$ (3)

Eliminating g from eqn. (1) and eqn. (2)

Eqn. (1) $\times 3$ - eqn. (2)
 $\Rightarrow 3(2g - 4f + c) - (6g - 8f + c) = -15 - -25$
 $\therefore -2f + c = 0$ (4)

Eqn. (3) in to eqn. (4)

$\Rightarrow -2f + f^2 = 0 \therefore f = 0, 2$

From eqn. (1): $2g - 4f + c = -5$

For $f = 0, c = 0, g = -\frac{5}{2}$

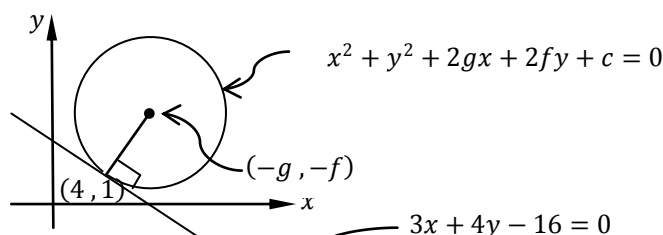
For $f = 2, c = 4, g = 2$

Thus the equation(s) of a circle are:

$\therefore x^2 + y^2 - 5x = 0$

$\therefore x^2 + y^2 + 4x + 4y + 4 = 0$

3)



Point $(4, 1)$ lies on the circle

$\Rightarrow 16 + 1 + 8g + 2f + c = 0$
 $\therefore 8g + 2f + c = -17$ (1)

Radius $= \sqrt{g^2 + f^2 - c} = 5$

$\Rightarrow 25 = g^2 + f^2 - c$ (2)

Also perpendicular distance of centre $(-g, -f)$, from the line $3x + 4y - 16 = 0$ is the radius

$$\therefore 5 = \left| \frac{3(-g) + 4(-f) - 16}{\sqrt{3^2 + 4^2}} \right| = \frac{-3g - 4f - 16}{5}$$

$$\Rightarrow -3g - 4f - 16 = 25 \text{ or } -3g - 4f = 41 \dots \dots \dots (3)$$

Eliminating c from eqn. (1) and eqn. (2)

Eqn. (1) + eqn. (2) yields

$$\therefore g^2 + f^2 + 8g + 2f = 8 \dots \dots \dots (4)$$

$$\text{From eqn. (3), } g = \left(\frac{41 + 4f}{-3} \right)$$

Substituting this in to eqn. (4)

$$\Rightarrow \left(\frac{41 + 4f}{-3} \right)^2 + f^2 + 8 \left(\frac{41 + 4f}{-3} \right) + 2f = 8$$

Simplifying the equation we get

$$f^2 + 10f + 25 = 0 \text{ or } (f + 5)^2 = 0 \therefore f = -5$$

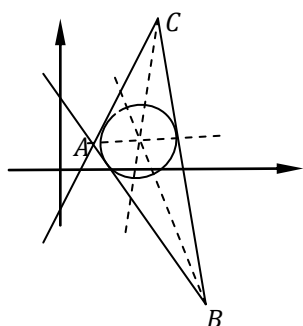
$$\text{Now from } g = \left(\frac{41 + 4f}{-3} \right), f = -5 \text{ gives}$$

$$\therefore g = -7$$

$$\text{From eqn. (1) } 8g + 2f + c = -17, g = -7, f = -5 \therefore c = 49$$

$$\therefore x^2 + y^2 - 14x - 10y + 49 = 0$$

4) Sketching the lines



$$\overline{AC} \text{ is } 3x - 4y - 7 = 0,$$

$$\overline{AB} \text{ is } 3x + 4y - 15 = 0$$

$$\overline{BC} \text{ is } 12x + 5y - 115 = 0$$

Dotted lines are Angle Bisectors

Finding the equations of Angle bisectors

Finding equations of angle bisectors between lines

$$3x + 4y - 15 = 0 \text{ and } 3x - 4y - 7 = 0$$

$$\frac{3x + 4y - 15}{\sqrt{3^2 + 4^2}} = \pm \frac{3x - 4y - 7}{\sqrt{3^2 + 4^2}}$$

Taking the +ve sign since the slope of the bisector is +ve

$$\frac{3x + 4y - 15}{5} = \frac{3x - 4y - 7}{5} \therefore y = 1 \dots \dots \dots (1)$$

Finding equations of angle bisectors between lines

$$3x + 4y - 15 = 0 \text{ and } 12x + 5y - 115 = 0$$

$$\frac{3x + 4y - 15}{\sqrt{3^2 + 4^2}} = \pm \frac{12x + 5y - 115}{\sqrt{12^2 + 5^2}}$$

Taking the -ve sign since the slope of the bisector is -ve

$$\frac{3x + 4y - 15}{5} = -\frac{12x + 5y - 115}{13} \therefore 9x + 7y - 70 = 0 \dots \dots \dots (2)$$

Solving eqn. (1) and eqn. (2) simultaneously, $x = 7, y = 1$

Thus in centre of the circle is $(7, 1)$

Radius = perpendicular distance of centre from any tangent

Using $3x + 4y - 15 = 0$

$$\text{Radius} = \frac{21+4-15}{5} = 2$$

Thus equation of the circle is:

$$\therefore (x - 7)^2 + (y - 1)^2 = 2^2 \text{ or } x^2 + y^2 - 14x - 2y + 46 = 0$$

Alternatively

Taking the centre of the circle to be $(-g, -f)$ for the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Perpendicular distance of the centre from each line is the radius.

This gives two equations which can be solved for values of the centre.

5) Left as an exercise

6) (i) Solving $x = 0$ and $x^2 + y^2 + 2gx + 2fy + c = 0$ simultaneously

$$\Rightarrow y^2 + 2fy + c = 0$$

For tangency, $B^2 - 4AC = 0$

$$\Rightarrow (2f)^2 = 4c \therefore f^2 = c$$

Solving $y = 0$ and $x^2 + y^2 + 2gx + 2fy + c = 0$ simultaneously

$$\Rightarrow x^2 + 2gx + c = 0$$

For tangency, $B^2 - 4AC = 0$

$$\Rightarrow (2g)^2 = 4c \therefore g^2 = c$$

$$\therefore c = g^2 = f^2$$

(ii) Since $\therefore c = g^2 = f^2$, the equation of the circle becomes

$$\Rightarrow x^2 + y^2 + 2fx + 2fy + f^2 = 0$$

Now $3x - 4y + 6 = 0 \Rightarrow x = \left(\frac{4y-6}{3}\right)$ is a tangent to this circle

Solving the equation of the circle and this line simultaneously

$$\Rightarrow \left(\frac{4y-6}{3}\right)^2 + y^2 + 2\left(\frac{4y-6}{3}\right) + 2fy + f^2 = 0$$

Simplifying the equation gives

$$\therefore 25y^2 + y(-48 + 42f) + (36 - 36f + 9f^2) = 0$$

Now for tangency $B^2 - 4AC = 0$

$$\Rightarrow (-48 + 42f)^2 = 4(25)(36 - 36f + 9f^2)$$

$$\Rightarrow 4(-24 + 21f)^2 = 4(25)(36 - 36f + 9f^2)$$

$$\Rightarrow 576 - 1008f + 441f^2 = 900 - 900f + 225f^2$$

Simplifying yields

$$\Rightarrow 2f^2 - f - 3 = 0 \text{ or } (f + 1)(2f - 3) = 0 \dots\dots\dots (*)$$

$$\therefore f = 1, -\frac{3}{2}$$

Since the centre of the circle is $(-g, -f)$

$$\Rightarrow -f = 1, -\frac{3}{2}$$

Taking $f = -1$ to bring the circle to the first quadrant, from $c = g^2 = f^2 = (-1)^2$

$\therefore (1, 1)$ is the centre of the circle.

Hence the equation of the circle in the first quadrant is

$$\Rightarrow x^2 + y^2 + 2(-1)x + 2(-1)y + 1 = 0$$

$$\therefore x^2 + y^2 - 2x - 2y + 1 = 0$$

Note

Equation (*) can also be obtained by considering the perpendicular distance of the centre $(-g, -f)$ from the tangent $3x - 4y + 6 = 0$

TANGENT IN SLOPE FORM

Referring to the standard equation of the circle $x^2 + y^2 = a^2$ and the tangent $y = mx + c$, it was shown that if this line is a tangent to the circle then $a^2(1 + m^2) = c^2$

$$\therefore c = \pm a\sqrt{(1 + m^2)}$$

Equation of tangent to the circle in slope form is

$$y = mx = \pm a\sqrt{(1 + m^2)}$$

Note

The lines $y = mx + a\sqrt{(1 + m^2)}$ and $y = mx - a\sqrt{(1 + m^2)}$ have the same gradient, hence two tangents can be drawn to the circle parallel to any given line.

Example

- 1) Find the equations of the tangents to the circle $x^2 + y^2 = 4$ which are parallel to the line $x + 2y + 3 = 0$. hence find their points of contact.

Solution

Let the required tangent be $y = mx + c$

Here we are to obtain the values of c and m

Obtaining slope m

If $y = mx + c$ and $x + 2y + 3 = 0$ are parallel, then their gradients are the same.

$$\Rightarrow m = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}x + c \text{ is the required tangent}$$

Now obtaining c , apply condition for tangency

Solving $y = -\frac{1}{2}x + c$ and $x^2 + y^2 = 4$ simultaneously

$$\Rightarrow x^2 + \left(-\frac{1}{2}x + c\right)^2 = 4$$

Simplifying the equation yields

$$5x^2 - 4cx + (4c^2 - 16) = 0$$

For tangency $B^2 - 4AC = 0$

$$\Rightarrow (-4c)^2 - 4(5)(4c^2 - 16) = 0$$

$$\Rightarrow 16c^2 = 80(c^2 - 4)$$

$$\Rightarrow c^2 = 5c^2 - 20 \text{ or } c^2 - 5 = 0$$

$$\therefore c = \pm\sqrt{5}$$

$$\text{Tangents are: } y = -\frac{1}{2}x \pm \sqrt{5}$$

Points of contact

From $5x^2 - 4cx + (4c^2 - 16) = 0$, $c = \pm\sqrt{5}$. Taking $c = +\sqrt{5}$

$$\Rightarrow 5x^2 - 4\sqrt{5}x + 4 = 0 \Rightarrow (\sqrt{5}x - 2)^2 = 0 \therefore x = \frac{2\sqrt{5}}{5}$$

$$\Rightarrow y = -\frac{1}{2}\left(\frac{2\sqrt{5}}{5}\right) + \sqrt{5} = \frac{4\sqrt{5}}{5}. \text{ Point of contact } \left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right)$$

Taking $c = -\sqrt{5}$,

$$\Rightarrow 5x^2 + 4\sqrt{5}x + 4 = 0 \Rightarrow (\sqrt{5}x + 2)^2 = 0 \therefore x = -\frac{2\sqrt{5}}{5}$$

$$\Rightarrow (\sqrt{5}x + 2)^2 = 0 \therefore x = -2/\sqrt{5}$$

$$\Rightarrow y = -\frac{1}{2}\left(-\frac{2\sqrt{5}}{5}\right) + \sqrt{5} = \frac{6\sqrt{5}}{5}. \text{ Point of contact } \left(-\frac{2\sqrt{5}}{5}, \frac{6\sqrt{5}}{5}\right)$$

2) Given that the line $y = mx + c$ is a tangent to the circle

$$x^2 + y^2 = a^2, \text{ show that } a^2(1 + m^2) = c^2$$

Hence find the equations of tangents to the circle $x^2 + y^2 = 4$ that are perpendicular to the line $3x + 4y + 5 = 0$

Solution

For the required proof see notes. i.e. left as an exercise

From $y = mx + c$ and $a^2(1 + m^2) = c^2$

Required tangent becomes

$$\therefore y = mx \pm a\sqrt{(1 + m^2)}$$

Obtaining m

If $y = mx + c$ is perpendicular to $3x + 4y + 5 = 0$ then from $m_1 \times m_2 = -1$,

slope of the tangent is $m = \frac{4}{3}$

Also comparing $x^2 + y^2 = a^2$ with $x^2 + y^2 = 4$, $\Rightarrow a = 2$

Thus equation of tangent becomes

$$\therefore y = mx \pm a\sqrt{(1 + m^2)} \text{ where } m = \frac{4}{3}, a = 2$$

$$\Rightarrow y = \frac{4}{3}x \pm 2\sqrt{\left(1 + \left(\frac{4}{3}\right)^2\right)}$$

$$\Rightarrow y = \frac{4}{3}x \pm 2 \times \frac{5}{3}$$

$$\therefore 3y = 4x \pm 10$$

3) Find the condition that the line $ax + by + c = 0$ is a tangent to the circle

$x^2 + y^2 + 2gx + 2fy + k = 0$. Hence or otherwise find the equations of tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ that are parallel to the line $x - y + 5 = 0$

Solution

For a line to be a tangent to the circle,

Radius = perpendicular distance of the centre from the tangent line

$$\text{Radius} = \sqrt{g^2 + f^2 - k} = \left| \frac{a(-g) + b(-f) + c}{\sqrt{a^2 + b^2}} \right|$$

Squaring both sides

$$\Rightarrow g^2 + f^2 - k = \frac{(-ag - bf + c)^2}{a^2 + b^2}$$

$$\therefore (g^2 + f^2 - k)(a^2 + b^2) = (-ag - bf + c)^2$$

Let the required tangent be $y = mx + \alpha$

This tangent is parallel to $x - y + 5 = 0$, hence $m = 1$

$\therefore y = x + \alpha$ is the tangent.

Rewriting this equation gives $x - y + \alpha = 0$

By comparison of;

$$x^2 + y^2 + 2gx + 2fy + k = 0 \text{ with } x^2 + y^2 - 2x - 4y - 4 = 0,$$

$$\Rightarrow g = -1, f = -2 \text{ and } k = -4$$

Also comparing $ax + by + c = 0$ with $x - y + \alpha = 0$

$$\Rightarrow a = 1, b = -1, \alpha = c$$

$$\text{Now from } (g^2 + f^2 - k)(a^2 + b^2) = (-ag - bf + c)^2$$

$$\Rightarrow ((-1)^2 + (-2)^2 - (-4))(1^2 + (-1)^2) = (-(-1)(-1) - (-1)(-2) + \alpha)^2$$

$$\Rightarrow 2 \times 9 = (\alpha + 1 - 2)^2$$

$$\Rightarrow (\alpha - 1)^2 = 18 \text{ or } \alpha - 1 = \pm\sqrt{18}$$

$$\therefore \alpha = 1 \pm 3\sqrt{2}$$

The required tangents are: $y = x + (1 \pm 3\sqrt{2})$

Note

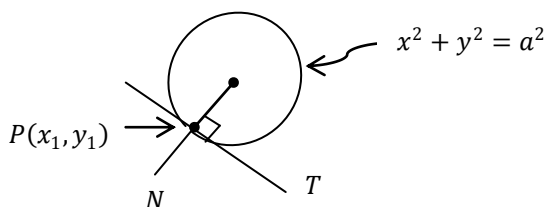
If $x^2 + y^2 - 2x - 4y - 4 = 0$ and $y = x + \alpha$ are solved simultaneously, the applying condition for tangency yields the values of

$\alpha = 1 \pm 3\sqrt{2}$, as before.

EQUATION OF THE NORMAL TO THE CIRCLE

To find the equation of the normal at any point $P(x_1, y_1)$ on the circle $x^2 + y^2 = a^2$

Consider the figure below



$\overline{PT}$ is a tangent,
 $\overline{NP}$ is a normal, both
 lines are at point P
 on the circle

Required equation of normal

$$y - y_1 = m'(x - x_1)$$

Finding m'

From $m \times m' = -1$, m = slope of the tangent at P

$$\Rightarrow m' = -\frac{1}{m}$$

First obtaining $m = \frac{dy}{dx}$ at point P

Differentiating the equation of the circle $x^2 + y^2 = a^2$ w.r. x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(a^2)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore m = -\frac{x_1}{y_1}$$

$$\text{Slope of the normal } m' = -\frac{1}{m} = \frac{1}{\left(-\frac{x_1}{y_1}\right)} = \frac{y_1}{x_1}$$

Equation of the normal is:

$$y - y_1 = m'(x - x_1) \Rightarrow y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$$\therefore x_1 y - x y_1 = 0$$

Examples

Find the equation of the normal at the given points

- a) $x^2 + y^2 = 25$ at $(-3, 4)$
- b) $x^2 + y^2 + 4x + 3y - 15 = 0$ at $(1, 2)$

Solution

- a) Required equation of the normal

$$y - 4 = m'(x + 3) \dots\dots\dots (1)$$

Finding m'

Let slope of the tangent be $m = \frac{dy}{dx}$ at $(-3, 4)$

Differentiating the equation of the circle $x^2 + y^2 = 25$ w.r. x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore m = -\frac{-3}{4} = \frac{3}{4}$$

$$\Rightarrow m' = -\frac{4}{3} \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow y - 4 = -\frac{4}{3}(x + 3) \text{ or } 4x + 3y = 0$$

- b) Required equation of the normal

$$y - 2 = m'(x - 1) \dots\dots\dots (1)$$

Finding m'

Let slope of the tangent be $m = \frac{dy}{dx}$ at $(1, 2)$

Differentiating the equation of the circle $x^2 + y^2 + 4x + 3y - 15 = 0$

w.r.t

$$\frac{d}{dx}(x^2 + y^2 + 4x + 3y - 15) = \frac{d}{dx}(0)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} + 4 + 3 \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{2x+4}{2y+3}$$

$$\therefore m = -\frac{2+4}{4+3} = -\frac{6}{7}$$

$$\Rightarrow m' = \frac{7}{6} \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow y - 2 = \frac{7}{6}(x - 1) \text{ or } 7x - 6y + 5 = 0$$

Task

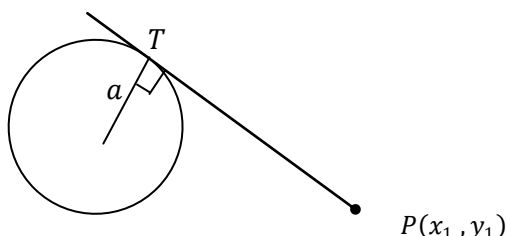
Find the equation of the normal at the given points

c) $x^2 + y^2 = 13$ at $(2, 3)$ **Ans:** $3x - 2y = 0$

d) $x^2 + y^2 - 2x + y = 20$ at $(-2, 3)$ **Ans:** $7x + 6y - 4 = 0$

TANGENTS TO THE CIRCLE FROM A POINT

To find the equation of a tangent to the circle $x^2 + y^2 = a^2$ from a point $P(x_1, y_1)$



Let the line $\overline{PT}$ be the tangent with the equation $y = mx + c$
 Obtaining c
 Since $P(x_1, y_1)$ lies on this line then;

$$y_1 = mx_1 + c \dots\dots\dots (1)$$

Obtaining slope, m

Since $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, solving the equations simultaneously and applying condition for tangency yields
 $a^2(1 + m^2) = c^2 \dots\dots\dots (2)$

Eliminating c from eqn. (1) and eqn. (2)

$$a^2(1 + m^2) = (y_1 - mx_1)^2$$

$$\Rightarrow m^2x_1^2 - 2mx_1y_1 + y_1^2 = a^2 + a^2m^2$$

$$\therefore m^2(x_1^2 - a^2) - (2mx_1y_1)m + (y_1^2 - a^2) = 0$$

, which is a quadratic in m . hence two values of m can be obtained.

NOTE:

Tangents corresponding to these values can be deduced from the quadratic equation.

CASE I: Real and distinct values of m

For real and distinct values of m , $B^2 - 4AC > 0$

$$\Rightarrow 4x_1^2y_1^2 > 4(x_1^2 - a^2)(y_1^2 - a^2)$$

$$\Rightarrow x_1^2y_1^2 > x_1^2y_1^2 - x_1^2a^2 - a^2y_1^2 + a^4$$

$$\therefore x_1^2 + y_1^2 > a^2 \text{ or } x_1^2 + y_1^2 - a^2 > 0$$

$\Rightarrow P(x_1, y_1)$ lies outside the circle. Thus if $P(x_1, y_1)$ lies outside the circle *two* real and distinct tangents can be drawn from $P(x_1, y_1)$

CASE II: *Coincident/Repeated roots of m*

$$\therefore x_1^2 + y_1^2 = a^2$$

This clearly shows that $P(x_1, y_1)$ lies on the circle. Thus *two* coincident tangents can be drawn from $P(x_1, y_1)$

CASE III: *Imaginary roots of m*

$$\therefore x_1^2 + y_1^2 < a^2 \text{ or } x_1^2 + y_1^2 - a^2 < 0$$

This shows that $P(x_1, y_1)$ lies inside the circle. Hence two imaginary tangents can be drawn from $P(x_1, y_1)$

Examples

- 1) Find the equations of the tangents to the circle from the given points
 - a) $x^2 + y^2 + 4x - 6y + 10 = 0$, $(0, 0)$
 - b) $x^2 + y^2 + 6x + 10y - 2 = 0$, $(2, 3)$

Solution

Before proceeding to the required solution, first test the point in the equation of the circle to judge whether the point is/out or on the circle.

- a) $x^2 + y^2 + 4x - 6y + 10 = 0$, $(0, 0)$

Checking the point

$\Rightarrow (0)^2 + (0)^2 + 4(0) - 6(0) + 10 \neq 0$. Hence the point is not on the circle.

Since $10 > 0$, point is outside the circle.

Let the required tangent be $y = mx + c$

Obtaining c

$(0, 0)$ lies on the line

$$\Rightarrow 0 = m(0) + c \therefore c = 0$$

$$\therefore y = mx \text{ or } mx - y = 0 \dots\dots\dots (1)$$

Now finding slope m

Two methods can be used

Method I

Solving $y = mx$ and $x^2 + y^2 + 4x - 6y + 10 = 0$ simultaneously

$$\Rightarrow x^2 + (mx)^2 + 4x - 6(mx) + 10 = 0$$

$$\Rightarrow x^2(1 + m^2) + x(4 - 6m) + 10 = 0$$

For tangency, $B^2 - 4AC = 0$

$$\Rightarrow (4 - 6m)^2 - 40(1 + m^2) = 0$$

$$\Rightarrow 4(2 - 3m)^2 - 40(1 + m^2) = 0$$

$$\Rightarrow 9m^2 - 12m + 4 = 10 + 10m^2$$

$$\therefore m^2 + 12m + 6 = 0 \dots\dots\dots (*)$$

Solving $m = -0.523, -11.477$

Thus tangents are:

$$y = -0.523x, y = -11.477x$$

Method II

Eqn. (*) can still be obtained by using the formula

$$\text{Radius} = \text{perpendicular distance of the centre from the tangent}$$

Centre of $x^2 + y^2 + 4x - 6y + 10 = 0$ is $(-2, 3)$,

$$\text{Radius} = \sqrt{(2)^2 + (-3)^2 - 10} = \sqrt{3}, \text{ tangent is } mx - y = 0$$

$$\Rightarrow \sqrt{3} = \left| \frac{-2m-3}{\sqrt{m^2+1}} \right|$$

Squaring both sides

$$\Rightarrow 3 = \frac{4m^2+12m+9}{m^2+1} \text{ or } m^2 + 12m + 6 = 0, \text{ which is as before.}$$

NOTE:

When finding the value of m , method I is hectic instead method II is quicker to the answer.

b) $x^2 + y^2 + 4x + 10y - 2 = 0, (2, 3)$

Let the tangent be $y = mx + c$

Obtaining c

$(2, 3)$ lies on the line

$$\Rightarrow 3 = m(2) + c \therefore c = 3 - 2m \dots\dots\dots (1)$$

Now finding slope m

Using

Radius = perpendicular distance of the centre from the tangent

Centre of $x^2 + y^2 + 6x + 10y - 2 = 0$ is $(-3, -5)$,

$$\text{Radius} = \sqrt{(3)^2 + (5)^2 - (-2)} = 6,$$

Tangent is $y = mx + c$ or $mx - y + c = 0$

$$\Rightarrow 6 = \left| \frac{-3m+5+c}{\sqrt{m^2+1}} \right|$$

Squaring both sides

$$\Rightarrow 36 = \frac{(-3m+5+c)^2}{m^2+1} \text{ or } 36m^2 + 36 = (-3m + 5 + c)^2 \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow 36m^2 + 36 = (-3m + 5 + 3 - 2m)^2$$

$$\Rightarrow 36m^2 + 36 = (-5m + 8)^2$$

$$\Rightarrow 36m^2 + 36 = 25m^2 - 80m + 64$$

$$\therefore 11m^2 + 80m - 28 = 0$$

Solving $m = 0.335, -7.607$

From eqn. (1), $c = 3 - 2m$

For $m = 0.335, c = 2.33$

For $m = -7.607, c = 18.214$

Thus tangents are:

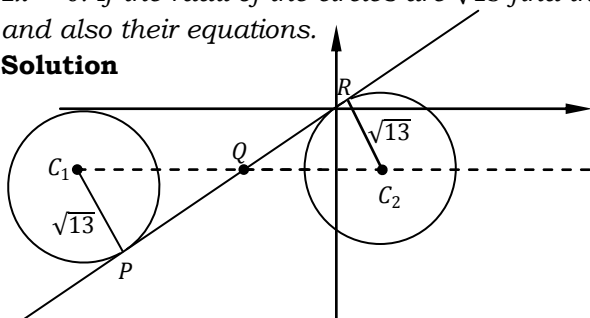
$$y = 0.335x + 2.33, y = -7.607x + 18.214$$

Task

- Find the equations of tangents to the circle $x^2 + y^2 = 25$ from point $(11, -2)$
Ans: $3x + 4y - 25 = 0$; $7x - 24y - 125 = 0$
- Find the equations of tangents to the circle $x^2 + y^2 - 10x + 8y + 5 = 0$ from point $(5, 4)$
Ans: $y = \sqrt{7}x + (4 - 5\sqrt{7})$
- Show that the line $y = x + 1$ touches the circle $x^2 + y^2 - 8x - 2y + 9 = 0$.
 Find the equation of the other tangent to the circle from the point $(0, 1)$
Ans: $x + y - 1 = 0$

- Two circles have their centres on the line $y + 3 = 0$ and touch the line $3y - 2x = 0$. If the radii of the circles are $\sqrt{13}$ find the coordinates of the centres and also their equations.

Solution



Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f)$ lies on $y + 3 = 0$

$$\Rightarrow -f + 3 = 0 \therefore f = 3$$

Using

Radius = perpendicular distance of the centre from the tangent $-2x + 3y = 0$

$$\Rightarrow \sqrt{13} = \frac{-2(-g) - 3f}{\sqrt{3^2 + 2^2}} = \frac{2g - 3f}{\sqrt{13}}$$

$$\Rightarrow 2g - 3f = 13. \text{ But } f = 3 \therefore g = 11$$

Now finding c

$$\text{Also Radius} = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \sqrt{13} = \sqrt{11^2 + 3^2 - c}$$

$$\Rightarrow 13 = 130 - c \therefore c = 117$$

$$\therefore 1^{\text{st}} \text{ equation of the circle is } x^2 + y^2 + 2(11)x + 2(3)y + 117 = 0$$

$$\therefore x^2 + y^2 + 22x + 6y + 117 = 0$$

To find the 2nd equation, first get the point of contact of the tangent to the first circle.

Point of contact

Solving $x^2 + y^2 + 22x + 6y + 117 = 0$ and $y = \frac{2x}{3}$ simultaneously

$$\Rightarrow x^2 + y^2 + 22x + 6\left(\frac{2x}{3}\right) + 117 = 0$$

Simplifying we get

$$x^2 + 18x + 81 = 0 \text{ or } (x + 9)^2 = 0$$

$$\therefore x = -9$$

$$\text{From } y = \frac{2x}{3}, x = -9 \Rightarrow y = -6$$

Point of contact is $(-9, -6)$

From the figure above ΔC_1PQ and ΔC_2PQ are similar

$$\Rightarrow \frac{\overline{C_1P}}{\overline{C_2P}} = \frac{\overline{PQ}}{\overline{QR}} = \frac{\overline{C_1Q}}{\overline{C_2Q}}$$

$$\Rightarrow \frac{\sqrt{13}}{\sqrt{13}} = \frac{\sqrt{(-9+9/2)^2 + (-6+3)^2}}{\overline{QR}} = \frac{\sqrt{(-11+9/2)^2 + (-3+3)^2}}{\overline{C_2Q}}$$

$$\Rightarrow 1 = \frac{\sqrt{117/2}}{\overline{QR}} = \frac{13/2}{\overline{C_2Q}}$$

$$\therefore \overline{C_2Q} = \frac{13}{2}, \overline{QR} = \frac{\sqrt{117}}{2}$$

Applying Pythagoras theorem on to ΔC_2RQ , let $C_2(-G, -3)$

$$\Rightarrow \frac{117}{4} + 13 = \left(-G + \frac{9}{2}\right)^2 + (-3 + 3)^2$$

$$\Rightarrow \left(-G + \frac{9}{2}\right)^2 = \frac{169}{4} \therefore -G + \frac{9}{2} = \pm \frac{13}{2}$$

$$\Rightarrow G = 11, -2 \text{ taking } G = -2$$

Required circle is: $\therefore x^2 + y^2 + 2(-2)x + 2(3)y + k = 0$

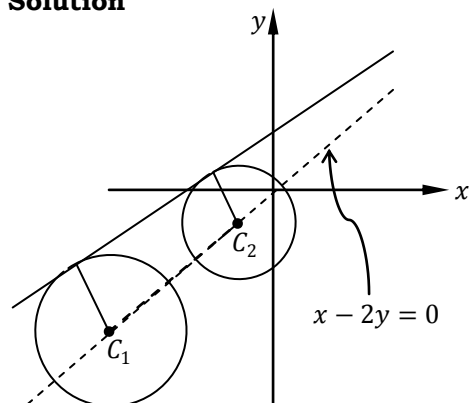
Using Radius = $\sqrt{G^2 + F^2 - k}$

$$\Rightarrow \sqrt{13} = \sqrt{(-2)^2 + (-3)^2 - k} \therefore k = 0$$

$$\therefore 2^{\text{nd}} \text{ equation of the circle is } x^2 + y^2 - 4x + 6y = 0$$

4. Two circles have their centres on the line $2y = x$ and touch the line $3x - 4y + 5 = 0$. If their radii are 5cm and 7cm respectively, find the coordinates of their centres and also their equations

Solution



Let $C_1(-g, -f)$ and $C_2(-G, -F)$

Centres lie on the line $2y = x$

$$\Rightarrow -g = -2f \Rightarrow g = 2f \dots\dots\dots (1)$$

$$\Rightarrow -G = -2F \Rightarrow G = 2F \dots\dots\dots (2)$$

Also from Radius = perpendicular distance of the centre from the tangent

$$\Rightarrow \frac{-3g+4f+5}{\sqrt{3^2+4^2}} = 7 \therefore -3g+4f = 30 \dots\dots\dots (5)$$

$$\text{Also } \frac{-3G+4F+5}{\sqrt{3^2+4^2}} = 5 \therefore -3G+4F = 20 \dots\dots\dots (6)$$

Eqn. (1) and eqn. (2) in to eqn. (5) and (6) respectively

$$\Rightarrow -3g+4f = 30, g = 2f \therefore f = -15 \text{ and } g = -30$$

$$\Rightarrow -3G+4F = 20, G = 2F \therefore F = -10 \text{ and } G = -20$$

Centres of the circle are: (30, 15) and (20, 10)

Thus equations of circle are:

$$\therefore (x - 30)^2 + (y - 15)^2 = 49$$

$$\therefore (x - 20)^2 + (y - 10)^2 = 25$$

Task

Two circles have radii 5 cm and 7 cm and touches the line $3x + 4y + 5 = 0$.

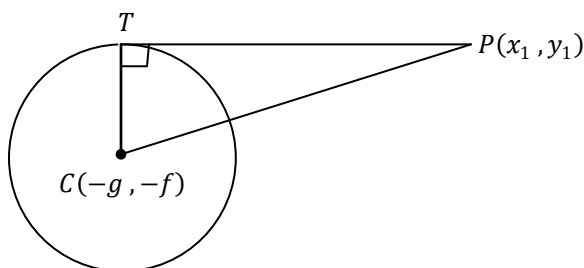
If the smaller and the larger circle touch respectively the y -axis and x -axis, find the equations of the circles.

Ans: $(x - 2/3)^2 + (y - 7)^2 = 49$, $(x - 5)^2 + (y - 5/4)^2 = 25$

LENGTH OF A TANGENT TO THE CIRCLE FROM A GIVEN POINT

$P(x_1, y_1)$

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point $P(x_1, y_1)$ lying outside the circle.



Let T = point of contact of tangent $\overline{PT}$ and the circle

By Pythagoras theorem

$$\overline{PT}^2 + \overline{CT}^2 = \overline{PC}^2 \quad \therefore \overline{PT} = \sqrt{\overline{PC}^2 - \overline{CT}^2}$$

$$\begin{aligned} \overline{PC} &= \sqrt{(x_1 + g)^2 + (y_1 + f)^2} \\ &= \sqrt{x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2} \end{aligned}$$

$$\overline{CT} = \text{Radius} = \sqrt{g^2 + f^2 - c}$$

$$\therefore \overline{PT} = \sqrt{x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - (g^2 + f^2 - c)}$$

$$\therefore \overline{PT} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Examples

Find the length of a tangent to the circle from the given point.

- $x^2 + y^2 + 4x - 6y + 10 = 0$, point (0, 0)
- $3x^2 + 3y^2 + 4x + 2y + 6 = 0$, point (2, 1)

Solution

- $x^2 + y^2 + 4x - 6y + 10 = 0$, point (0, 0)

Length of the tangent $= \sqrt{(0)^2 + (0)^2 + 4(0) - 6(0) + 10} = \sqrt{10}$ units

- $3x^2 + 3y^2 + 4x + 2y + 6 = 0$, point (2, 1)

Re writing the equation of the circle(note this step)

$$\Rightarrow x^2 + y^2 + \frac{4}{3}x + \frac{2}{3}y + 2 = 0$$

Length of the tangent

$$= \sqrt{(2)^2 + (1)^2 + 4(2) - 6(1) + 2} = 3 \text{ units}$$

Task

Find the length of a tangent to the circle from the given point.

a) $x^2 + y^2 = 16$, point $(3, -4)$ **Ans: 3**

b) $4x^2 + 4y^2 + 3x - 8y - 2 = 0$, point $(1, 1)$ **Ans: 1**

EQUATION OF A CIRCLE THROUGH THE POINTS OF INTERSECTION OF A GIVEN LINE AND CIRCLE

Consider a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and the line $ax + by + d = 0$

The required equation of a circle through points of intersection is:

$$\boxed{x^2 + y^2 + 2gx + 2fy + c + \lambda(ax + by + d) = 0}, \text{ where } \lambda \text{ is an arbitrary constant.}$$

The constant λ can be obtained from the given condition in the question.

NOTE:

The equations of a circle and the line must be equated to zero before using the general equation of a circle through the points of intersection.

Example

1. Find the equation of a circle through the points of intersection of the line $x - 3y = 0$ and the circle $x^2 + y^2 - 10x - 5y + 25 = 0$ with the centre lying on the line $x + y = 1$

Solution

Required equation of a circle:

$$x^2 + y^2 - 10x - 5y + 25 + \lambda(x - 3y) = 0$$

$$\Rightarrow x^2 + y^2 + x(\lambda - 10) + (-5 - 3\lambda)y + 25 = 0$$

Centre of the circle is:

$$\left(\frac{\lambda-10}{-2}, \frac{-5-3\lambda}{-2}\right) = \left(\frac{10-\lambda}{2}, \frac{5+3\lambda}{2}\right)$$

Since the centre lies on the line $x + y = 1$

$$\Rightarrow \frac{10-\lambda}{2} + \frac{5+3\lambda}{2} = 1 \quad \therefore \lambda = -\frac{13}{2}$$

Equation of the circle becomes

$$\Rightarrow x^2 + y^2 + x\left(-\frac{13}{2} - 10\right) + \left(-5 - 3\left(-\frac{13}{2}\right)\right)y + 25 = 0$$

$$\therefore x^2 + y^2 - \frac{33}{2}x - \frac{49}{2}y + 25 = 0$$

Alternatively

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f)$ lies on the line $x + y = 1$

$$\Rightarrow -g - f = 1 \text{ or } g + f = -1 \dots\dots\dots (1)$$

Solving the equation of given circle and line simultaneously to obtain points of intersection

$$x^2 + y^2 - 10x - 5y + 25 = 0, x = 3y$$

$$\Rightarrow (3y)^2 + y^2 - 10(3y) - 5y + 25 = 0$$

$$\Rightarrow 2y^2 - 7y + 5 = 0 \quad \therefore y = \frac{5}{2}, 1$$

Corresponding values of x to y are:

$$x = 3, y = 1, x = \frac{15}{2}, y = \frac{5}{2}$$

Points of intersection: $(3, 1), (\frac{15}{2}, \frac{5}{2})$

Since these points lie on the circle,

$$\text{For } (3, 1): 9 + 1 + 6g + 2f + c = 0$$

$$\Rightarrow 6g + 2f + c = -10 \dots\dots\dots (2)$$

$$\text{For } (\frac{15}{2}, \frac{5}{2}): \frac{225}{4} + \frac{25}{4} + \frac{30}{2}g + 5f + c = 0$$

$$\Rightarrow 15g + 5f + c = -\frac{125}{2} \dots\dots\dots (3)$$

Eliminating c from eqn. (1) and eqn. (2)

$$6g - f = -35 \dots\dots\dots (4)$$

Solving eqn. (1) and eq. (4) simultaneously

$$\Rightarrow g = -\frac{33}{4}, f = \frac{29}{4}, \text{ from eqn. (2) } c = 25$$

Required circle is

$$x^2 + y^2 + 2(-\frac{33}{4})x + 2(\frac{29}{4})y + 25 = 0$$

$$\therefore x^2 + y^2 - \frac{33}{2}x - \frac{49}{2}y + 25 = 0$$

- 2.** Find the equation of a circle passing through the points of intersection of the circle $x^2 + y^2 = 4$ and the line $x - y = 1$ as a diameter.

Solution

$$\text{Required circle: } x^2 + y^2 - 4 + \lambda(x - y - 1) = 0$$

$$\Rightarrow x^2 + y^2 + \lambda x - \lambda y - \lambda - 4 = 0 = 0$$

Centre $(-\frac{\lambda}{2}, \frac{\lambda}{2})$ lies on the diameter $x - y = 1$

$$\Rightarrow -\frac{\lambda}{2} - \frac{\lambda}{2} = 1 \quad \therefore \lambda = -1$$

$$\Rightarrow x^2 + y^2 - 4 - 1(x - y - 1) = 0$$

$$\therefore x^2 + y^2 - x + y - 3 = 0$$

Task

- 1.** Find the equation of a circle having the chord $x - y = 1$ as the diameter and passing through the points of intersection of this chord and the circle $x^2 + y^2 - x - 3y = 12.5$

Ans: $x^2 + y^2 - x - 3y = 12.5$

- 2.** Find the equation of a circle drawn on the line segment joining points of intersection of the circle $x^2 + y^2 = a^2$ and the line $x \cos \alpha + y \sin \alpha = p$ as the diameter.

Ans: $x^2 + y^2 - a^2 - 2p(x \cos \alpha + y \sin \alpha - p) = 0$

3. Show that the equation of the circle passing through the point of intersection of the circle $x^2 + y^2 = 2ax$ and the line $ax - by = 0$ having its centre on the line $\frac{x}{a} - \frac{y}{b} = 2$ is $x^2 + y^2 - 3ax + by = 0$
4. Find the equation of the circle containing the circle $(x - a)^2 + y^2 = a^2$ and the line $ax - by = 0$ as the diameter.
Ans: $(a^2 + b^2)(x^2 + y^2) = 2ab(bx + ay)$
5. Find the equation of a circle passing through the points of intersection of the circle $x^2 + y^2 + 6x - 10y + 7 = 0$ and the line $x - 6y = -6$ as its diameter.
6. Find the equation of a circle containing the chord $ax = by$ as its diameter and the circle $x^2 + y^2 + 2by + c^2 = 0$

EQUATION OF A CIRCLE THROUGH THE POINTS OF INTERSECTION OF TWO CIRCLES

Let the given circles be:

$$S_1 = x^2 + y^2 + 2gx + 2fy + c_1 = 0 \text{ and } S_2 = x^2 + y^2 + 2Gx + 2Fy + c_2 = 0$$

Required equation of the circle is:

$$x^2 + y^2 + 2gx + 2fy + c_1 + k(x^2 + y^2 + 2Gx + 2Fy + c_2) = 0$$

NOTE:

- 1) The two equations of the circle are in the form $S_1 = 0, S_2 = 0$. Hence the equation of a circle passing through the points of intersection of $S_1 = 0, S_2 = 0$ provided $k \neq -1$ is $S_1 + kS_2 = 0$
- 2) The arbitrary constant k can be obtained from the given condition
- 3) If $k = -1$, the 2^0 terms vanish and the equation in this case being 1^0 , represents a straight line passing through the points of intersection of two circles. The line is called the COMMON CHORD.

Examples

1. Find the equation of a circle passing through $(-1, 1)$ and the points of intersection of circles $x^2 + y^2 = 5$ and $x^2 + y^2 = 3x + y$

Solution

Required circle: $x^2 + y^2 - 5 + k(x^2 + y^2 - 3x - y) = 0$

This equation satisfies the point $(-1, 1)$

$$\Rightarrow 1 + 1 - 5 + k(1 + 1 + 3 - 1) = 0 \quad \therefore k = \frac{3}{4}$$

Required circle is:

$$x^2 + y^2 - 5 + \frac{3}{4}(x^2 + y^2 - 3x - y) = 0, \text{ simplifying gives}$$

$$\therefore 7x^2 + 7y^2 - 9x - 3y - 20 = 0$$

Note:

This question can be transformed in to “to find the equation of a circle through the points of intersection of a line and a circle. Here the line is the common chord.

Common chord of intersecting circles:

$$x^2 + y^2 = 5 \text{ and } x^2 + y^2 = 3x + y$$

Subtracting the equations to eliminate x^2, y^2

$$3x + y - 5 = 0$$

Required circle:

$$x^2 + y^2 - 3x - y + \lambda(3x + y - 5) = 0$$

This circle passes through $(-1, 1)$

$$\Rightarrow 1 + 1 - 5 + \lambda(-3 + 1 - 5) = 0 \therefore \lambda = -\frac{3}{7}$$

$$\Rightarrow x^2 + y^2 - 3x - y - \frac{3}{7}(3x + y - 5) = 0, \text{ simplifying gives}$$

$$\therefore 7x^2 + 7y^2 - 9x - 3y - 20 = 0 \text{ as before}$$

- 2.** Find the equation of a circle passing through the points of intersection of the circle $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2bx$ and having its centre on the line $\frac{x}{a} - \frac{y}{b} = 2$

Solution

$$\text{Required circle: } x^2 + y^2 - 2ax + k(x^2 + y^2 - 2bx) = 0$$

$$\Rightarrow x^2(1+k) + y^2(1+k) + x(-2a-2bk) = 0$$

$$\Rightarrow x^2 + y^2 + x\left(\frac{-2a-2bk}{1+k}\right) = 0$$

$$\therefore \text{centre is } \left(\frac{a+bk}{1+k}, 0\right) \text{ which lies on the line } \frac{x}{a} - \frac{y}{b} = 2$$

$$\Rightarrow \frac{\frac{a+bk}{1+k}}{a} = 2$$

$$\Rightarrow 2a(1+k) = a + bk \text{ or } k = \frac{a}{b-2a}$$

$$\Rightarrow x^2 + y^2 + x\left(\frac{-2a-2b\left(\frac{a}{b-2a}\right)}{1+\frac{a}{b-2a}}\right) = 0$$

$$\Rightarrow x^2 + y^2 + x\left(\frac{-2ab+4a^2-2ab}{b-2a+a}\right) = 0$$

$$\therefore (a+b)(x^2 + y^2) + 4ax(a-b) = 0$$

Task

- 1.** Find the equation of a circle passing through the points of intersection of the circles $x^2 + y^2 + 6x - 10y + 7 = 0$ and $x^2 + y^2 + 4x + 2y - 5 = 0$ and having the common chord as its diameter.

Ans: $37x^2 + 37y^2 + 168x - 46y - 65 = 0$

- 2.** Show that the equation of a circle passing through the points of intersection of the circles $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ and having its centre on the line $\frac{x}{a} - \frac{y}{b} = 2$ is $x^2 + y^2 - 3ax + by = 0$

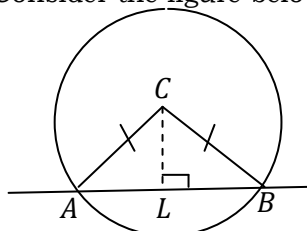
- 3.** Find the equation of a circle passing through the points of intersection of the circles $x^2 + y^2 - x - 3y - 12.5 = 0$ and $x^2 + y^2 - 2x - 2y - 13.5 = 0$ and having its centre on the line $x - y = 1$

Ans: $2x^2 + 2y^2 - 6x - 2y - 21 = 0$

LENGTH OF A CHORD

To find the length of a chord intercepted by a circle on a straight line

Consider the figure below



$\overline{CL}$ bisects AB

From $\triangle ACL$, Pythagoras theorem yields

$$\overline{AC}^2 = \overline{AL}^2 + \overline{CL}^2$$

$$\therefore \overline{AL} = \sqrt{\overline{AC}^2 - \overline{CL}^2}$$

$$\therefore \overline{AB} = \text{Length of chord} = 2\sqrt{\overline{AC}^2 - \overline{CL}^2}$$

$\text{Length of a chord} = 2\sqrt{\{\text{RADIUS}\}^2 - \{\perp \text{ DISTANCE OF CENTRE FROM CHORD}\}^2}$

Centre $(-g, -f)$ and Radius $= \sqrt{g^2 + f^2 - c}$

Let the chord be $ax + by + k = 0$

$\text{Length of the chord} = 2\sqrt{g^2 + f^2 - c - \left[\frac{ag + bf - k}{\sqrt{a^2 + b^2}}\right]^2}$

Note:

If the line is given in the form $y = mx + c$, first re write it in the form $mx - y + c = 0$, then get the perpendicular distance of the centre from this chord line.

Examples

1. Find the length of a chord intercepted by:

i) Circle $x^2 + y^2 - 6x + 4y - 12 = 0$ on the line $4x - 3y + 2 = 0$

ii) Circle $2x^2 + 2y^2 + 3x - 4y = 0$ on the line $x + y - 1 = 0$

Solution

i) $x^2 + y^2 - 6x + 4y - 12 = 0$, centre $(3, -2)$,

$$\text{Radius} = \sqrt{(-3)^2 + (2)^2 + 12} = 5$$

$\perp$ Distance of $(3, -2)$ from $4x - 3y + 2 = 0$ is

$$= \left| \frac{4(3) - 3(-2) + 2}{\sqrt{4^2 + 3^2}} \right| = 4$$

$$\therefore \text{Length of chord} = 2\sqrt{5^2 - 4^2} = 6 \text{ units}$$

ii) $2x^2 + 2y^2 + 3x - 4y = 0 \Rightarrow x^2 + y^2 + \frac{3}{2}x - 2y = 0$, centre $(-\frac{3}{4}, 1)$

$$\text{Radius} = \sqrt{\left(\frac{3}{4}\right)^2 + (1)^2} = \frac{5}{4}$$

$\perp$ Distance of $(-\frac{3}{4}, 1)$ from $x - y - 1 = 0$ is

$$= \left| \frac{-\frac{3}{4} + 1 - 1}{\sqrt{1^2 + 1^2}} \right| = \frac{3}{4\sqrt{2}}$$

$$\therefore \text{Length of chord} = 2\sqrt{\left(\frac{5}{4}\right)^2 - \left(\frac{3}{4\sqrt{2}}\right)^2} = 4.886 \text{ units}$$

Task

Find the length of a chord intercepted by:

Circle $x^2 + y^2 - 5x + 6y + 12 = 0$ on the line $3x + 2y + 5 = 0$

Ans: 0; $x + 2y + 5 = 0$ is a tangent

2. Find the length of a common chord between two circles

i) $x^2 + y^2 + 6x - 10y + 7 = 0$ and $x^2 + y^2 + 4x + 2y - 5 = 0$

ii) $x^2 + y^2 - 8x + 12 = 0$ and $x^2 + y^2 - 4x - 6y - 3 = 0$

Solution

i) Common chord: subtracting the equations to eliminate x^2, y^2

$$\Rightarrow x^2 + y^2 + 6x - 10y + 7 - (x^2 + y^2 + 4x + 2y - 5) = 0$$

$$\Rightarrow x - 6y + 6 = 0$$

Taking $x^2 + y^2 + 6x - 10y + 7 = 0$, centre $(-3, 5)$,

$$\text{Radius} = \sqrt{(3)^2 + (-5)^2 - 7} = \sqrt{27}$$

$\perp$ distance of $(-3, 5)$ from $x - 6y + 6 = 0$ is

$$= \left| \frac{(-3) - 6(5) + 6}{\sqrt{1^2 + (-6)^2}} \right| = \frac{27}{\sqrt{37}}$$

$$\therefore \text{Length of chord} = 2\sqrt{27 - \frac{27}{37}} = 10.25 \text{ units}$$

ii) Left as an exercise. **Ans: 2.49**

3. Prove that the length of a common chord of two circles whose equations are $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ is $\sqrt{4c^2 - 2(a - b)^2}$

Ans: Left as an exercise

3. Find the equation of a circle with centre $(\frac{5}{2}, -3)$ which

a) Cuts off a chord of length 6 units from the line $3x + 2y + 5 = 0$

b) Cuts off a chord of length 8 units from the line $3x - 4y - 2 = 0$

Solution

a) Let the required circle be: $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\text{Centre } (-g, -f) = \left(\frac{5}{2}, -3\right) \Rightarrow g = -\frac{5}{2}, f = 3$$

The equations becomes

$$x^2 + y^2 - 5x + 6y + c = 0$$

$$\text{Length of a chord} = 2\sqrt{\{\text{RADIUS}\}^2 - \{\perp \text{ DISTANCE OF CENTRE FROM CHORD}\}^2}$$

$$\text{Radius} = \sqrt{\left(-\frac{5}{2}\right)^2 + 3^2 - c} = \sqrt{\frac{25}{4} + 9 - c}$$

$\perp$ distance of $(\frac{5}{2}, -3)$ from $3x + 2y + 5 = 0$ is

$$= \left| \frac{3(5/2) + 2(-3) + 5}{\sqrt{3^2 + 2^2}} \right| = \frac{\sqrt{13}}{2}$$

$$\Rightarrow 6 = 2\sqrt{\frac{61}{4} - c - \frac{13}{4}}, \text{ squaring both sides}$$

$$\Rightarrow 9 = \frac{61}{4} - c - \frac{13}{4} \therefore c = 3$$

Thus the equation of the circle is:

$$x^2 + y^2 - 5x + 6y + 3 = 0$$

b) Left as an exercise. **Ans:** $x^2 + y^2 - 5x + 6y - \frac{61}{25} = 0$

ANGLE OF INTERSECTION OF TWO CIRCLES

When two circles intersect, the angle between them is the angle between the tangents to the circles at the common point of intersection.

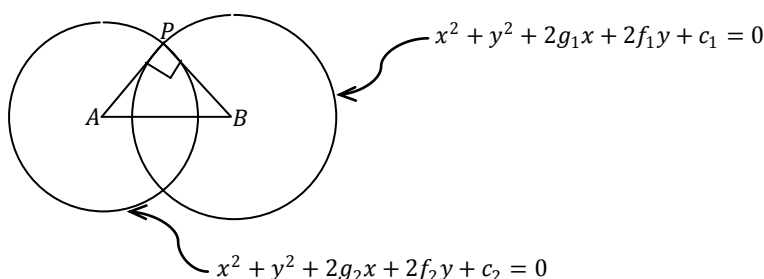
If the angle of intersection of two circles is a *right angle*. i.e. 90° , the circles are said to be *ORTHOGONALLY*, and for this case the tangents at the common point to either circle pass through the centre of the other.

The line joining their centres subtend a right angle at the point of intersection

ORTHOGONAL CIRCLES

To find the condition that the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ may cut each other orthogonally.

Consider the figure below



From $\triangle APB$, Pythagoras theorem yields

$$\overline{AP}^2 + \overline{PB}^2 = \overline{AB}^2$$

$$\overline{AP} = \text{Radius} = \sqrt{g_1^2 + f_1^2 - c_1}$$

$$\overline{PB} = \text{Radius} = \sqrt{g_2^2 + f_2^2 - c_2}$$

$$\text{Centres } A(-g_1, -f_1), B(-g_2, -f_2)$$

$$\overline{AB} = \sqrt{(-g_1 + g_2)^2 + (-f_1 + f_2)^2}$$

$$\therefore g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 = (-g_1 + g_2)^2 + (-f_1 + f_2)^2$$

Simplifying gives

$$-c_1 - c_2 = -2g_1g_2 - 2f_1f_2$$

$$\boxed{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

This condition can generally stated as

$$\boxed{\frac{1}{2} [\text{product of coeff. } x + \text{product of coeff. } y] = \text{sum of constants}}$$

Examples

1. Show that the circles are orthogonal

- a) $x^2 + y^2 - 6x - 8y + 9 = 0$, $x^2 + y^2 = 9$
- b) $x^2 + y^2 - 4x + 2 = 0$, $x^2 + y^2 + 6y - 2 = 0$
- c) $x^2 + y^2 - 6y + 8 = 0$, $x^2 + y^2 - 4x + 2y - 14 = 0$
- d) $x^2 + y^2 + 6x - 10y + 7 = 0$, $x^2 + y^2 + 4x + 2y - 5 = 0$

Solution

a) $x^2 + y^2 - 6x - 8y + 9 = 0$

Centre (3, 4), Radius = $\sqrt{(-3)^2 + (-4)^2 - 9} = 4$

$x^2 + y^2 = 9$

Centre (0, 0), Radius = $\sqrt{(0)^2 + (0)^2 + 9} = 3$

Distance between centres = $\sqrt{(3-0)^2 + (4-0)^2} = 5$

$\therefore (3)^2 + (4)^2 = 5^2$, hence circles are orthogonal

Alternatively

$\frac{1}{2}[-6 \times 0 + 0 \times -8] = -9 + 9 = 0$

b) , (c) and (d) have been left as an exercise to the students.

2. Find the equation of a circle which passes through the origin and cuts circles orthogonally the circles $x^2 + y^2 - 8y + 12 = 0$ and $x^2 + y^2 - 4x - 6y - 3 = 0$

Solution

$x^2 + y^2 - 8y + 12 = 0$ (1)

$x^2 + y^2 - 4x - 6y - 3 = 0$ (2)

Let the required circle be:

$x^2 + y^2 + 2gx + 2fy + c = 0$ (3)

Circles (1) and (3); (2) and (3) are orthogonal

$\Rightarrow \frac{1}{2}[2g \times 0 + 2f \times (-8)] = 12 + c$, simplifying gives

$\Rightarrow -8f - c = 12$ (4)

$\Rightarrow \frac{1}{2}[2g \times (-4) + 2f \times (-6)] = -3 + c$, simplifying gives

$\Rightarrow -4g - 6f = -3 + c$ (5)

Circle (3) passes through the origin

$\Rightarrow 0 + 0 + 0 + 0 + c = 0 \therefore c = 0$

Put $c = 0$ in to (4) and (5)

$\Rightarrow -8f = 12 \therefore f = -\frac{3}{2}$

Similarly

$\Rightarrow -4g - 6f = -3$, but $f = -\frac{3}{2}$

$\Rightarrow -4g - 6\left(-\frac{3}{2}\right) = -3 \therefore g = 3$

Eqn. (3) becomes

$x^2 + y^2 + 2(3)x + 2\left(-\frac{3}{2}\right)y + 0 = 0$

$$\therefore x^2 + y^2 + 6x - 3y = 0$$

3. Find the equation of a circle which cuts orthogonally the circle $x^2 + y^2 - 4x + 6y - 7 = 0$, passing through the point $(0, 3)$ and touches the x - axis

Solution

Let the required circle be:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots (1)$$

Case I

Eqn. (1) cuts $x^2 + y^2 - 4x + 6y - 7 = 0$ orthogonally

$$\Rightarrow \frac{1}{2}[-4 \times 2g + 6 \times 2f] = -7 + c$$

$$\therefore -4g + 6f = c - 7 \dots\dots\dots (2)$$

Case II

Eqn. (1) passes through $(0, 3)$

$$\Rightarrow 9 + 6f + c = 0 \dots\dots\dots (3)$$

Case III

$y = 0$ (x - axis) is a tangent to eqn. (1)

$$\Rightarrow x^2 + 2gx + c = 0$$

For tangency $B^2 - 4AC = 0$

$$\Rightarrow 4g^2 = 4(1)c \therefore g^2 = c \dots\dots\dots (4)$$

Eliminating f from eqn. (2) and eqn. (3)

Eqn. (2) - eqn. (3)

$$\Rightarrow 4g - 9 - c = c - 7 \text{ or } c = -2g - 1 \dots\dots\dots (5)$$

Eqn. (4) in to eqn. (5)

$$\Rightarrow g^2 + 2g + 1 = 0 \therefore g = -1$$

Putting $g = -1$ in to eqn. (5)

$$\Rightarrow c = 1$$

Putting $g = -1, c = 1$ in to eqn. (2)

$$\Rightarrow -4(-1) + 6f = 1 - 7 \therefore f = -\frac{5}{3}$$

Putting $g = -1, c = 1, f = -\frac{5}{3}$ into eqn. (1)

$$x^2 + y^2 + 2(-1)x + 2\left(-\frac{5}{3}\right)y + 1 = 0$$

$$\therefore 3x^2 + 3y^2 - 6x - 10y + 3 = 0$$

4. Find the equation of a circle with centre on the y - axis which cuts orthogonally each of the circles $x^2 + y^2 + 6x + 2y - 9 = 0$ and $x^2 + y^2 - 2x - 2y + 1 = 0$

Solution

Let the required circle be:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots (1)$$

Case I

Centre $(-g, -f)$ lies on y - axis . i. e. $x = 0$

$$\Rightarrow -g = 0 \therefore g = 0 \dots\dots\dots (2)$$

Circle in eqn. (1) cuts $x^2 + y^2 + 6x + 2y - 9 = 0$ orthogonally

$$\Rightarrow \frac{1}{2}[6 \times 2g + 2f \times 2] = c - 9 \quad \therefore 6g + 2f = c - 9. \text{ But } g = 0$$

$$\therefore 2f = c - 9 \dots\dots\dots (3)$$

Case II

Circle in eqn. (1) cuts $x^2 + y^2 - 2x - 2y + 1 = 0$ orthogonally

$$\Rightarrow \frac{1}{2}[-2 \times 2g + 2f \times -2] = c + 1 \quad \therefore -2g - 2f = c + 1. \text{ But } g = 0$$

$$\therefore -2f = c + 1 \dots\dots\dots (4)$$

Solving eqn. (3) and eqn. (4) simultaneously gives

$$\Rightarrow f = -\frac{5}{2}, \quad c = 4$$

Required equation of the circle is:

$$x^2 + y^2 - 5y + 4 = 0$$

Task

1. The circles $x^2 + y^2 - 14x + 6by + 42a = 0$ and $x^2 + y^2 - 1x + by - 23a = 0$ are orthogonal. If the first circle passes through the point (3, 9), find the values of a and b
2. Find the equation of a circle which passes through point (1, 2), cuts orthogonally the circle $x^2 + y^2 + 2x - 8 = 0$ and touches the line $3x - 4y + 4 = 0$
5. Prove that the circles which pass through points (0, a) and (0, $-a$) and touch the line $y = mx + c$ will cut orthogonally if $c^2 = a^2(2 + m^2)$

Solution

Let the equation of the circle be: $x^2 + y^2 + 2gx + 2fy + k = 0$

This circle passes through (0, a) and (0, $-a$)

$$\Rightarrow a^2 + 2fa + k = 0 \dots\dots\dots (1)$$

$$\Rightarrow a^2 - 2fa + k = 0 \dots\dots\dots (2)$$

If the line $y = mx + c$ or $mx - y + c = 0$ is a tangent to the circle $x^2 + y^2 + 2gx + 2fy + k = 0$ then

$$\Rightarrow \sqrt{g^2 + f^2 - k} = \left| \frac{m(-g) - (-f) + c}{\sqrt{m^2 + 1}} \right|, \text{ squaring both sides}$$

$$\Rightarrow g^2 + f^2 - k = \frac{(-mg + f + c)^2}{m^2 + 1} \dots\dots\dots (3)$$

Since k and f are not required in the proof, solving eqn. (1) and eqn. (2) simultaneously

Eliminating f , adding the equations

$$\Rightarrow 2a^2 + 2k = 0 \quad \therefore k = -a^2 \dots\dots\dots (4)$$

Putting $k = -a^2$ into eqn. (1)

$$\Rightarrow a^2 + 2fa - a^2 = 0 \quad \therefore f = 0 \dots\dots\dots (5)$$

Putting $k = -a^2, f = 0$ in to eqn. (3) to solve also for g since it is also unknown

$$\Rightarrow g^2 + 0 + a^2 = \frac{(-mg + 0 + c)^2}{m^2 + 1}$$

$$\Rightarrow g^2 m^2 + g^2 + a^2(m^2 + 1) = g^2 m^2 - 2mgc + c^2$$

$$\therefore g^2 + 2mcg + a^2(m^2 + 1) - c^2 = 0$$

From algebra, if g_1 and g_2 are the roots of the equation then
 $g_1 + g_2 = -2mc$, $g_1 \cdot g_2 = a^2(m^2 + 1) - c^2$ (6)

Now if circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ cut orthogonally then

$$\Rightarrow \frac{1}{2}[2g_1 \cdot 2g_2 + 2f_1 \cdot 2f_2] = c_1 + c_2, \text{ but } f = f_1 = 0, k = c_2 = c_1 = -a^2$$

$$\Rightarrow 2g_1 2g_2 = -a^2 - a^2$$

$$\Rightarrow g_1 g_2 = -a^2 \text{..... (7)}$$

Eqn. (7) in to eqn. (6) gives

$$a^2(m^2 + 1) - c^2 = -a^2 \text{ or } c^2 = a^2(m^2 + 1) + a^2$$

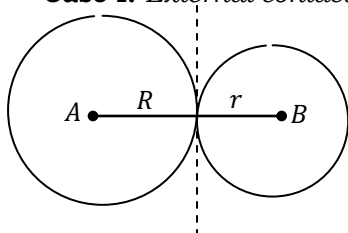
$$\therefore c^2 = a^2(2 + m^2)$$

TOUCHING CIRCLES

Two circles are said to *touch each other* at a point when they have the same tangent (common tangent) at that point called *point of contact* of the circles.

Two cases arise:

Case I: *External contact*

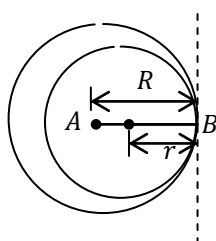


Circles touch each other *externally*.

Condition

$$R + r = \overline{AB} = \text{Distance between the centres}$$

Case II: *Internal contact*



Circles touch each other *internally*.

Condition

$$R - r = \overline{AB} = \text{Distance between the centres}$$

Examples

1. Prove that the circles

a) $x^2 + y^2 + 3x + y = 0$ and $x^2 + y^2 - 6x - 2y = 0$

b) $x^2 + y^2 + 5(2x + 1) = 0$ and $x^2 + y^2 + 5(y + 1) = 0$

c) $x^2 + y^2 + 2x - 2y = 7$ and $x^2 + y^2 - 4x - 10y + 25 = 0$

, touch each other. Find the point of contact and the equation of the tangent at that point of contact.

Solution

a) Circle can touch internally or externally

For $x^2 + y^2 + 3x + y = 0$, Centre $\left(-\frac{3}{2}, -\frac{1}{2}\right)$

$$\text{Radius} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 0} = \frac{\sqrt{10}}{2} = r$$

For $x^2 + y^2 - 6x - 2y = 0$, Centre $(3, 1)$

$$\text{Radius} = \sqrt{(-3)^2 + (-1)^2 - 0} = \sqrt{10} = R$$

$$\text{Distance between two centres} = \sqrt{\left(3 + \frac{3}{2}\right)^2 + \left(1 + \frac{1}{2}\right)^2} = \frac{3\sqrt{10}}{2} = d$$

Since $R + r = d$, hence circles touch each other externally

Note:

To find the point of contact, vectors, ratio theorem or solving the equations of the circles simultaneously can be used.

Point of contact

Using Ratio Theorem

Recall: If $A(x_1, y_1)$ and $B(x_2, y_2)$ are end points of the line segment AB and $P(x, y)$ divide AB internally in the ratio $m:n$ then

$$P(x, y) = P\left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right]$$



P divides $\overline{AB}$ in the ratio $\overline{AP}:\overline{PB} = R:r$

$$\Rightarrow \frac{\overline{AP}}{\overline{PB}} = \frac{R}{r} = \frac{\frac{\sqrt{10}}{2}}{\frac{\sqrt{10}}{2}} = \frac{2}{1}$$

Now coordinates of P are:

$$P\left[\frac{2 \times -\frac{3}{2} + 1 \times 3}{2+1}, \frac{2 \times -\frac{1}{2} + 1 \times 1}{2+1}\right] = P(0, 0) \text{ which}$$

is the point of contact

Tangent equation at the point of contact

This can be obtained either by differentiation at the point of contact for any circle or eliminating x^2, y^2 from the equations.

Eliminating x^2, y^2 from the equations,

By subtracting the equations gives

$$\Rightarrow x^2 + y^2 + 3x + y - (x^2 + y^2 - 6x - 2y) = 0$$

$$\Rightarrow 9x + 3y = 0 \quad \therefore 3x + y = 0$$

b) Left as an exercise. **Ans:** $(-1, -2)$, Tangent $2x - y = 0$

c) Left as an exercise. **Ans:** $(,)$, Tangent $3x + 4y = 16$

2. (i) Show that the circles $x^2 + y^2 + 2x - 2y = 7$ and $x^2 + y^2 - 4x - 10y + 25 = 0$ touch each other.

(ii) Find the coordinates of the point of contact of the circles in (i) above

(iii) Determine the coordinates of the point of intersection of the other common tangents to the circles which do not pass through the point of contact in (ii) above.

PARAMETRIC AND POLAR EQUATION OF A CIRCLE

Parametric Equations of a Circle

The parametric equations of a circle are:

$$x = a + c \cos \theta, y = b + c \sin \theta$$

To show that the equations represent a circle

$$x = a + c \cos \theta \Rightarrow x - a = c \cos \theta \dots\dots\dots (1)$$

$$y = b + c \sin \theta \Rightarrow y - b = c \sin \theta \dots\dots\dots (2)$$

Squaring and adding the equations

$$(x - a)^2 + (y - b)^2 = c^2 \cos^2 \theta + c^2 \sin^2 \theta = c^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\therefore (x - a)^2 + (y - b)^2 = c^2$$

Thus the equations $x = a + c \cos \theta, y = b + c \sin \theta$ represent an equation of a circle with *Centre* (a, b) and *Radius* c

To find the equation of a tangent and normal to the circle defined by $x = a + c \cos \theta, y = b + c \sin \theta$

Tangent equation at $(a + c \cos \theta, b + c \sin \theta)$

$$\frac{y - (b + c \sin \theta)}{x - (a + c \cos \theta)} = m_1 \dots\dots\dots (1)$$

$$m_1 = \frac{dy}{dx} \text{ at } (a + c \cos \theta, b + c \sin \theta)$$

$$\Rightarrow m_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$x = a + c \cos \theta \Rightarrow \frac{dx}{d\theta} = -c \sin \theta$$

$$y = b + c \sin \theta \Rightarrow \frac{dy}{d\theta} = c \cos \theta$$

$$\therefore m_1 = c \cos \theta \times \frac{1}{-c \sin \theta} = \frac{\cos \theta}{-\sin \theta} \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow \frac{y - (b + c \sin \theta)}{x - (a + c \cos \theta)} = \frac{\cos \theta}{-\sin \theta}$$

$$\Rightarrow -y \sin \theta + b \sin \theta + c \sin^2 \theta = x \cos \theta - a \cos \theta - c \cos^2 \theta$$

$$\Rightarrow -y \sin \theta + b \sin \theta + c(\sin^2 \theta + \cos^2 \theta) = x \cos \theta - a \cos \theta$$

$$\therefore x \cos \theta + y \sin \theta = a \cos \theta + b \sin \theta + c$$

Equation of the normal at $(a + c \cos \theta, b + c \sin \theta)$

$$\frac{y - (b + c \sin \theta)}{x - (a + c \cos \theta)} = m_2$$

$$m_2 = -\frac{1}{\text{Slope of tangent at } (a + c \cos \theta, b + c \sin \theta)}$$

$$= -1 / \cos \theta / -\sin \theta = \frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow \frac{y - (b + c \sin \theta)}{x - (a + c \cos \theta)} = \frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow y \cos \theta - b \cos \theta - c \sin \theta \cos \theta = x \sin \theta - a \sin \theta - c \cos \theta \sin \theta$$

$$\Rightarrow -b \cos \theta + a \sin \theta = x \sin \theta - y \cos \theta$$

$$\therefore x \sin \theta - y \cos \theta = a \sin \theta - b \cos \theta$$

Polar Equation of a Circle

The three basic equations used in expressing the polar equation of a circle in to Cartesian form and vis versa are:

$$x = r \cos \theta, y = r \sin \theta \text{ and } x^2 + y^2 = r^2, \text{ where } r = \text{modulus of point } P(x, y) \text{ from the origin and } \theta = \text{angle made by } \overline{OP} \text{ and } +ve \ x - \text{axis.}$$

Example

1. Show that the given equation represents a circle.

a) $r = 3 \cos \theta$

b) $r = 4 \sin \theta$

Solution

a) $r = 3 \cos \theta$ multiplying through by r

$$\Rightarrow r^2 = 3r \cos \theta, \text{ but } x^2 + y^2 = r^2 \text{ and } x = r \cos \theta$$

$$\Rightarrow x^2 + y^2 = 3x \therefore x^2 + y^2 - 3x = 0, \text{ which is an equation of a circle.}$$

b) $r = 4 \sin \theta$ multiplying through by r

$$\Rightarrow r^2 = 4r \sin \theta, \text{ but } x^2 + y^2 = r^2 \text{ and } y = r \sin \theta$$

$$\Rightarrow x^2 + y^2 = 4y \therefore x^2 + y^2 - 4y = 0, \text{ which is an equation of a circle.}$$

2. Express the given equation of a circle in to polar form

a) $x^2 + y^2 - 5x = 0$

b) $x^2 + y^2 + 2x - 4y = 0$

Solution

a) $x^2 + y^2 - 5x = 0$, using $x^2 + y^2 = r^2$ and $x = r \cos \theta$

$$\Rightarrow r^2 - 5r \cos \theta = 0 \therefore r = 5 \cos \theta$$

b) $x^2 + y^2 + 2x - 4y = 0$, using $x^2 + y^2 = r^2$, $x = r \cos \theta$ and $y = r \sin \theta$

$$\Rightarrow r^2 + 2r \cos \theta - 4r \sin \theta = 0 \therefore r = 4 \sin \theta - 2 \cos \theta$$

CONIC SECTION

Definition:

A conic section/a conic is the locus of a point that moves in such a way that its distance from a fixed point (FOCUS) bears a constant ratio to its distance from a fixed line (DIRECTRIX) not passing through the focus.

Terms used

A Focus:

Is a fixed point that lies on the axis of a conic.

A Directrix:

Is a line perpendicular to the axis of the conic not passing through the focus.

Eccentricity (e):

Is the constant ratio of distance of a point from a fixed point (focus) to the distance of a same point from the fixed line (Directrix).

Vertex of a conic:

A point (s) where the axis of the conic meets the conic.

Axis of the conic:

Is a line joining the focus and vertices of the conic and is perpendicular to the fixed line (Directrix).

Types of conic sections

- Circle
- Parabola
- Ellipse
- Hyperbola

The nature of locus of any point depends on the value of e , the eccentricity .i.e.

- If $e = 1$, the conic is called parabola
- If $0 < e < 1$, the conic is called ellipse
- If $e > 1$, the conic is called hyperbola

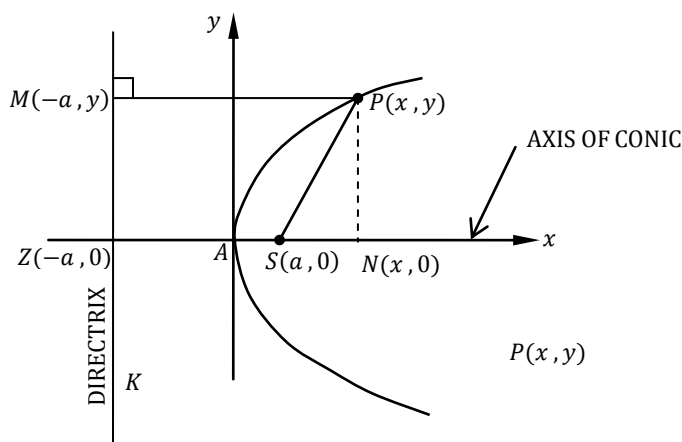
THE PARABOLA

Definition:

Is the locus of a point that moves in such away that its distance from a fixed point (Focus) is equal to its distance from a fixed line (Directrix) not passing through the focus.

Equation of a Parabola in Standard Form

Consider the figure below



Let $S(a, 0)$ be the focus and ZK be the Directrix.

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$$\Rightarrow \overline{PS} = \sqrt{(x-a)^2 + (y-0)^2}, \quad \overline{PM} = \sqrt{(x+a)^2 + (y-y)^2}$$

$$\Rightarrow \sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2}$$

Squaring both sides

$$(x-a)^2 + y^2 = (x+a)^2$$

$$\Rightarrow x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

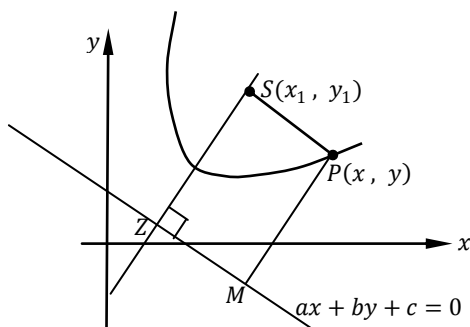
$y^2 = 4ax$ is the standard equation of a parabola

NOTE

- The parabola in standard form i.e. $y^2 = 4ax$ has:
 - Point $A(0, 0)$ as the vertex
 - Line ZAS as the axis of conic with the equation $y = 0$
 - Line MZK as the Directrix with the equation $x = -a$
- From $y^2 = 4ax$, if $a > 0$, the parabola opens to the right and if $a < 0$, the parabola opens to the left

General Equation of a Parabola

To find the Equation of a Parabola when the Coordinates of the Focus and Equation of the Directrix are given



$$\therefore \overline{PM} = \frac{ax+by+c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \sqrt{(x-x_1)^2 + (y-y_1)^2} = \frac{ax+by+c}{\sqrt{a^2+b^2}}$$

Squaring both sides gives

$$(a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = (ax + by + c)^2$$

, which on simplification can be written as:

$$(bx - ay)^2 + 2gx + 2fy + k = 0$$

NOTE

The 2^0 terms of the above equation form a perfect square.

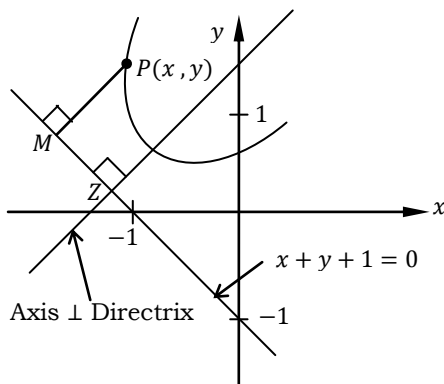
Examples

1) Find the equation of a parabola with;

- Focus $(-1, 1)$ and directrix $x + y + 1 = 0$
- Focus $(0, 2)$ and vertex as the origin
- Focus $(-7, 3)$ and vertex $(-2, 3)$

Solution

a) Focus: $S(-1, 1)$, Directrix: $x + y + 1 = 0$



Let the coordinates of focus be $S(x_1, y_1)$, and the equation of the directrix be $ax + by + c = 0$.

Let $P(x, y)$ be a point tracing the locus.

By definition

$$\overline{PS} = \overline{PM} \text{ or } \overline{AS} = \overline{AZ}$$

$$\overline{PS} = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

$$\overline{PM} = \perp \text{ Distance of P from directrix}$$

- Parabola opens to focus
- Directrix $\perp$ Axis of the conic
- A, the vertex is the midpoint of $\overline{SZ}$

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$$\overline{PS} = \sqrt{(x + 1)^2 + (y - 1)^2}$$

$$\overline{PM} = \perp \text{ Distance of P from directrix}$$

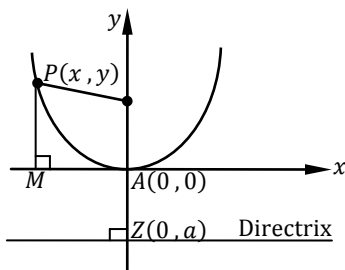
$$\overline{PM} = \frac{x+y+1}{\sqrt{1+1}} = \frac{x+y+1}{\sqrt{2}}$$

$$\Rightarrow \sqrt{(x + 1)^2 + (y - 1)^2} = \frac{x+y+1}{\sqrt{2}}, \text{ squaring}$$

$$\Rightarrow (x + 1)^2 + (y - 1)^2 = \frac{[(x+1)+y]^2}{2}$$

$$\begin{aligned}\therefore 2(x+1)^2 + 2(y-1)^2 &= (x+1)^2 + 2y(x+1) + y^2 \\ \Rightarrow (x+1)^2 + 2y^2 - 4y + 2 - y^2 - 2y(x+1) &= 0 \\ \Rightarrow x^2 + 2x + 1 + y^2 - 4y - 2xy - 2y + 2 &= 0 \\ \Rightarrow x^2 - 2xy + y^2 + 2x - 6y + 3 &= 0 \\ \therefore (x-y)^2 + 2x - 6y + 3 &= 0\end{aligned}$$

b) $S(0, 2)$, $A(0, 0)$



- Directrix $\perp$ Axis of the conic
- Parabola opens to focus
- A, the vertex is the midpoint of $\overline{SZ}$
First finding the equation of the directrix
 $A\left(\frac{0+0}{2}, \frac{2+a}{2}\right) \equiv A(0, 0)$
 $\Rightarrow \frac{2+a}{2} = 0, a = -2$
Equation of the directrix: $y + 2 = 0$

Now equation of parabola:

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$$\overline{PS} = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$$

$\overline{PM} = \perp$ Distance of P from directrix $y + 2 = 0 \Rightarrow 0x + y + 2 = 0$

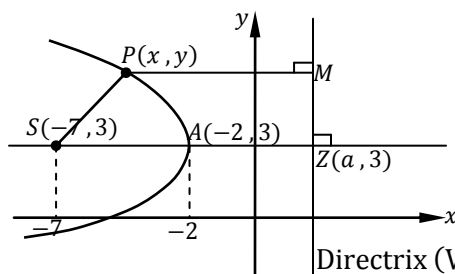
$$\overline{PM} = \frac{y+2}{\sqrt{0+1}} = y + 2$$

$$\therefore \sqrt{x^2 + (y-2)^2} = (y+2), \text{ squaring}$$

$$\Rightarrow x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$

$$\therefore x^2 = 8y$$

c) $S(-7, 3)$, $A(-2, 3)$



- Axis passes through SAZ
- Parabola opens to focus
- A, the vertex is the midpoint of $\overline{SZ}$

Directrix (Vertical line of the form $x = a$)

Finding the equation of the directrix

$$A(-2, 3) = A\left(\frac{-7+a}{2}, \frac{3+3}{2}\right) \Rightarrow -2 = \frac{-7+a}{2} \therefore a = 3$$

$$\therefore Z(a, 3) = Z(3, 3)$$

Equation of directrix is $x = 3$ or $x + 0y - 3 = 0$

By definition

$$\overline{PS} = \overline{PM} \text{ or } \overline{AS} = \overline{AZ}$$

$$\Rightarrow \sqrt{(x+7)^2 + (y-3)^2} = \frac{x-3}{\sqrt{0+1}} = x - 3, \text{ squaring both sides}$$

$$\Rightarrow (x+7)^2 + (y-3)^2 = (x-3)^2$$

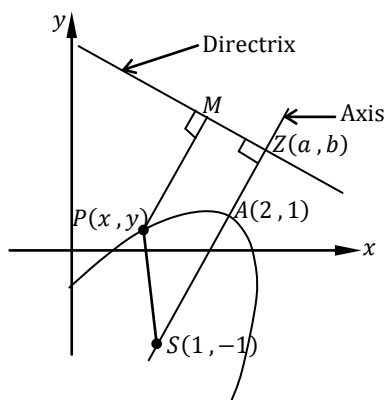
$$\Rightarrow x^2 + 14x + 49 + y^2 - 6y + 9 = x^2 - 6x + 9$$

$$\therefore y^2 - 6y + 20x + 49 = 0$$

- 2) Find the equation of a parabola with vertex $(2, 1)$ and focus $(1, -1)$.

Solution

Given: Vertex $A(2, 1)$ and Focus $S(1, -1)$



Finding the equation of the directrix:

$$A(2, 1) = \text{Midpoint of } \overline{SZ}$$

$$A(2, 1) = A\left(\frac{a+1}{2}, \frac{-1+b}{2}\right) \Rightarrow 2 = \frac{a+1}{2} \quad \therefore a = 3$$

$$\Rightarrow \frac{-1+b}{2} = 1 \quad \therefore b = 3$$

$$\therefore Z(3, 3)$$

But the directrix is $\perp$ to axis of conic

From coordinate geometry, $m_1 \times m_2 = -1$

$$m_1 = \text{Slope of axis of conic} = \text{Grad } \overline{AS}$$

$$= \frac{1+1}{2-1} = 2 \quad \therefore m_2 = -\frac{1}{2}$$

Now finding equation of a line (Directrix) passing through a known point $Z(3, 3)$ and perpendicular to a given line AS is:

$$\frac{y-3}{x-3} = -\frac{1}{2} \Rightarrow y-3 = -\frac{1}{2}(x-3) \text{ or } x+2y-9=0$$

Now equation of parabola:

$$\text{By definition, } \overline{PS} = \overline{PM} \text{ or } \overline{AS} = \overline{AZ}$$

$$\overline{PS} = \sqrt{(x-1)^2 + (y+1)^2}$$

$$\overline{PM} = \perp \text{ Distance of P from directrix } x+2y-9=0$$

$$\overline{PM} = \frac{x+2y-9}{\sqrt{1+4}} = \frac{x+2y-9}{\sqrt{5}}$$

$$\therefore \sqrt{(x-1)^2 + (y+1)^2} = \frac{x+2y-9}{\sqrt{5}}, \text{ squaring}$$

$$\Rightarrow x^2 - 2x + 1 + y^2 + 2y + 1 = \frac{[(x+2y)-9]^2}{5}$$

$$\Rightarrow 5x^2 - 10x + 5 + 5y^2 + 10y + 5 = (x+2y)^2 - 18(x+2y) + 81$$

$$\Rightarrow 5x^2 + 5y^2 - 10x + 10y + 10 = x^2 + 4xy + 4y^2 - 18x - 36y + 81$$

$$\Rightarrow 4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$$

$$\Rightarrow (2x-y)^2 + 8x + 46y - 71 = 0$$

Task

Find the equation of a parabola with:

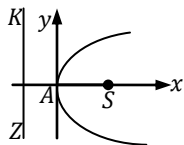
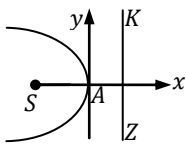
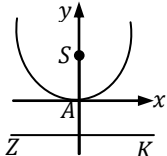
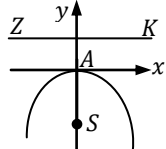
- a) Vertex $(2, 0)$, directrix $x = 0$

Ans: $y^2 = 8x - 16$

- b) Vertex $(3, 1)$, Focus at $(3, 3)$

Ans: $(x-3)^2 = 8(y-1)$

Summary of Standard Equation of a Parabola

Curve	Eqn. of curve	Focus	Directrix	Axis	Vertex	Tangent at vertex
	$y^2 = 4ax$	$(a, 0)$	$x = -a$	$y = 0$	$A(0, 0)$	$x = 0$
	$y^2 = -4ax$	$-(a, 0)$	$x = a$	$y = 0$	$A(0, 0)$	$x = 0$
	$x^2 = 4ay$	$(0, a)$	$y = -a$	$x = 0$	$A(0, 0)$	$y = 0$
	$x^2 = -4ay$	$(0, -a)$	$y = a$	$x = 0$	$A(0, 0)$	$y = 0$

The above table is used in sketching the conic, determining the vertex, equation of axis, focus and directrix of a parabola from a given equation.

Example

1. Find the vertex, axis, focus and directrix of a parabola

a) $y^2 = 5x - 4y - 9$

b) $x^2 = 9x - y - 14$

c) $y^2 - 6y + 20x + 49 = 0$

Solution

a) Expressing the equation in one of the above standard forms of a parabola i.e. $Y^2 = 4AX$. This form is taken because the perfect square variable is y .

$$y^2 = 5x - 4y - 9 \Rightarrow y^2 - 4y = 5x - 9$$

Completing squares

$$\Rightarrow y^2 - 4y + 4 = 5x - 9 + 4$$

$$\Rightarrow (y - 2)^2 = 5x - 5 = 5(x - 1), \text{ which is of the form } Y^2 = 4AX$$

By comparison

$$Y = y - 2, X = x - 1, 4A = 5 \therefore A = 5/4$$

But for a parabola $Y^2 = 4AX$,

Vertex $(0, 0)$, Axis $= 0$, Focus $S(A, 0)$, Directrix $X = -A$

For vertex, $(0, 0) = (X, Y)$

$$\Rightarrow y + 2 = 0 \therefore y = -2, \quad x - 1 = 0 \therefore x = 1$$

$\therefore$ Vertex is $(1, -2)$

$$\text{Axis, } Y = 0 = y + 2 \therefore y = -2$$

Focus, $S(A, 0) = S(X, Y)$

$$\Rightarrow Y = y + 2 = 0 \therefore y = -2, \quad x - 1 = A = 5/4 \therefore x = 9/4$$

Focus $S\left(\frac{9}{4}, -2\right)$

$$\text{Directrix, } X = -A \Rightarrow x - 1 = -\frac{5}{4} \therefore x = -1/4$$

b) $x^2 = 9x - y - 14 \Rightarrow x^2 - 9x = -y - 14$

Completing squares

$$\Rightarrow x^2 - 9x + (-9/2)^2 = -y - 14 + (-9/2)^2$$

$$\Rightarrow (x - 9/2)^2 = -y - 14 + \frac{81}{4} = -y + \frac{25}{4}$$

$$\therefore (x - 9/2)^2 = -y + 25/4, \text{ which is of the form } X^2 = 4AY$$

For this parabola,

Focus is $S(0, A) = S(X, Y)$, Axis is $X = 0$, Vertex $A(0, 0) = A(X, Y)$ and

Directrix $Y = -A$

Comparing $(x - 9/2)^2 = -y + 25/4$ with $X^2 = 4AY$

$X = x - 9/2, \quad Y = -y + 25/4, \quad 4A = 1 \therefore A = 1/4$

$$\text{Focus } X = 0 = x - \frac{9}{2} \therefore x = \frac{9}{2}; \quad Y = -y + \frac{25}{4} = A = \frac{1}{4} \therefore y = 6$$

Focus $S\left(\frac{9}{2}, 6\right)$

$$\text{Axis, } X = x - \frac{9}{2} = 0 \therefore x = \frac{9}{2}$$

$$\text{Vertex, } X = 0 = x - \frac{9}{2} = 0 \therefore x = \frac{9}{2}; \quad Y = 0 = -y + \frac{25}{4} \therefore y = \frac{25}{4}$$

Vertex $A\left(\frac{9}{2}, \frac{25}{4}\right)$

$$\text{Directrix, } Y = -A = -y + \frac{25}{4} = -\frac{1}{4} \therefore y = \frac{13}{2} \text{ is the directrix}$$

c) $y^2 - 6y + 20x + 49 = 0 \Rightarrow y^2 - 6y = -20x - 49$

Completing squares

$$\Rightarrow y^2 - 6y + 9 = -20x - 49 + 9$$

$$\Rightarrow (y - 3)^2 = -20(x + 2), \text{ which is of the form } Y^2 = 4AX$$

For this parabola

Focus $S(A, 0) = S(X, Y) \therefore X = A, Y = 0$

Vertex $A(0, 0) = A(X, Y) \therefore X = 0, Y = 0$

Axis $Y = 0$

Directrix $X = -A$

Comparing $(y - 3)^2 = -20(x + 2)$ with $Y^2 = 4AX$

$Y = y - 3, \quad X = x + 2, \quad 4A = -20 \therefore A = -5$

$$\text{For } X = A = -5 = x + 2 \therefore x = -7$$

For $Y = 0 = y - 3 \therefore y = 3$

Focus $S(-7, 3)$

Axis $Y = 0 = y - 3 \therefore y = 3$

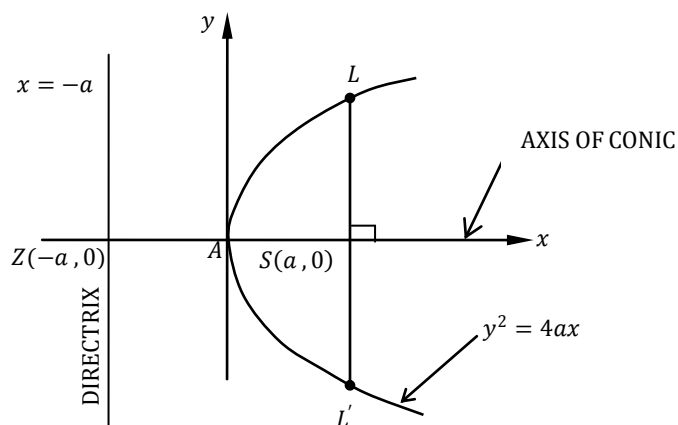
Vertex $X = 0 = x + 2 \therefore x = -2$

Vertex $A(-2, 3)$

Directrix $X = -A \Rightarrow x + 2 = 5 \therefore x = 3$

LATUS RECTUM

The chord of a parabola which passes through the focus and perpendicular to the axis of the parabola is called *Latus Rectum*.



From the figure, LSL' is the rectum.

$$LSL' = 2LS = 2L'S$$

Since the x - coordinate of $S(a, 0)$ is $x = a$, then the y - coordinate of ends of latus rectum are: $y^2 = 4a \cdot a \Rightarrow y = \pm 2a$

$\therefore L(a, 2a), L'(a, -2a)$, length $\overline{LL'} = 4a$

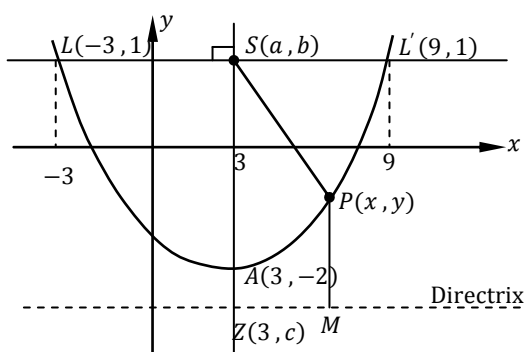
NOTE

Two parabolas are equal if latus rectum is the same.

Example

Find the equation of a parabola with vertex at $(3, -2)$, end points of latus rectum at $(-3, 1)$ and $(9, 1)$

Solution



Finding coordinates of focus $S(a, b)$

$S(a, b)$ is the midpoint of $\overline{LL'}$

$$\Rightarrow (a, b) = \left(\frac{-3+9}{2}, \frac{1+1}{2} \right) = (3, 1)$$

$$\therefore S(3, 1)$$

Also $\overline{AS} = \overline{AZ}$, $\overline{PS} = \overline{PM}$. From the

definition A is the midpoint of $\overline{SZ}$

$$\therefore (3, -2) = \left(\frac{a+3}{2}, \frac{b+c}{2} \right) = \left(\frac{3+3}{2}, \frac{1+c}{2} \right)$$

$$\Rightarrow \frac{1+c}{2} = -2 \quad \therefore c = -5$$

Directrix passes through $Z(3, -5)$

Since axis of conic passes through S,

V and Z and is vertical, the directrix

equation is: $y = -5$ or $y + 5 = 0$

$$\therefore \overline{PS}^2 = \overline{PM}^2 \text{ becomes;}$$

$$(x-3)^2 + (y-1)^2 = (y+5)^2$$

$$(x-3)^2 = (y+5)^2 - (y-1)^2 = (y+5+y-1)(y+5-y+1)$$

$$\therefore (x-3)^2 = 12(y+2)$$

A LINE AND A PARABOLA

The following questions must be understood in order to master this sub topic.

- a) **To Find the Point of Intersection of the Line $y = mx + c$ and the Parabola $y^2 = 4ax$**

Solving the line and the parabola simultaneously

$$y = mx + c \dots\dots\dots (1)$$

$$y^2 = 4ax \dots\dots\dots (2)$$

Eqn. (1) in to eqn. (2)

$$\Rightarrow (mx + c)^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mcx + c^2 - 4ax = 0$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0 \dots\dots\dots (3)$$

Solving this equation gives *two* values of x . Obtain the corresponding values from equation (1), and hence two points of intersection in the form (x, y)

- b) **To Show that a given Line $y = mx + c$ is a Tangent to the Parabola $y^2 = 4ax$.**

Here first solve the line and the parabola simultaneously to obtain eqn. (3) above

$$m^2x^2 + (2mc - 4a)x + c^2 = 0 \dots\dots\dots (3)$$

Now proceed in two ways

Approach I

Applying condition for tangency .i.e. condition for equal roots of a quadratic equation

For tangency, $B^2 = 4AC$

$$\Rightarrow (2mc - 4a)^2 = 4m^2c^2$$

$$\Rightarrow 4m^2c^2 - 16mca + 16a^2 = 4m^2c^2$$

$c = \frac{a}{m}$

, which is the condition for a line $y = mx + c$ to be a tangent to $y^2 = 4ax$

Approach II

Solving eqn. (3), two repeated/coincident values of x must be obtained.

Eqn. (3) can have coincident roots if the condition for tangency is used.

Put $c = a/m$ in to eqn. (3)

$$\Rightarrow m^2x^2 + \left(2m\frac{a}{m} - 4a\right)x + \left(\frac{a}{m}\right)^2 = 0$$

$$\Rightarrow m^2x^2 - 2ax + \frac{a^2}{m^2} = 0 \quad \therefore \left(mx - \frac{a}{m}\right)^2 = 0$$

$$\Rightarrow mx - \frac{a}{m} = 0 \quad \therefore x = \frac{a}{m^2}, \frac{a}{m^2}$$

Hence a line is a tangent to the parabola.

c) **To obtain the Point of Contact of a tangent line $y = mx + c$ to the parabola $y^2 = 4ax$**

Proceeding as in approach II above from eqn. (3), $x = \frac{a}{m^2}$, obtain the corresponding y - value from the line.

$$y = mx + c, \quad x = \frac{a}{m^2} \quad \therefore c = \frac{a}{m}$$

$$\Rightarrow y = m \times \frac{a}{m^2} = \frac{a}{m}$$

$$\therefore \text{Point of contact is } P\left(\frac{a}{m^2}, \frac{a}{m}\right)$$

d) **To state the General Point on the parabola $y^2 = 4ax$**

Using the line $y = mx + c$ as a tangent to the parabola $y^2 = 4ax$, obtain

the point of contact $P\left(\frac{a}{m^2}, \frac{a}{m}\right)$

$$\text{Let } t = \frac{1}{m} \Rightarrow P(at^2, 2at)$$

$$\therefore \text{General point on the parabola is } T(at^2, 2at)$$

NOTE

From the general point on the parabola, $y^2 = 4ax$, two parametric equations can be deduced.

The parametric equations of $y^2 = 4ax$ are: $x = at^2, y = 2at$

e) To state the Equation of the Tangent to the parabola in Standard Form

From the line $y = mx + c$ and condition for tangency $c = \frac{a}{m}$, the tangent in slope form is $y = mx + \frac{a}{m}$

NOTE

In answering questions (b), (c) and (d), you must first obtain eqn. (3) in question (a) above, the approach to finding point of intersection of a line and a parabola.

Example

1. Find the point of intersection of the line $x + y + 1 = 0$ and the parabola $y^2 = 4(x + 1)$

Solution

$$x + y + 1 = 0 \Rightarrow y = -x - 1 \dots\dots\dots (1)$$

$$y^2 = 4(x + 1) \dots\dots\dots (2)$$

Eqn. (1) in to eqn. (2)

$$(-x - 1)^2 = 4(x + 1) \Rightarrow (x + 1)^2 = 4(x + 1)$$

$$\Rightarrow (x + 1)(x + 1 - 4) = 0 \quad \therefore x = -1, 3 \dots\dots\dots (3)$$

Eqn. (3) in to eqn. (1)

$$\text{For } x = -1, y = 0; \quad x = 3, y = -4$$

Points of intersection are: $(-1, 0)$, $(3, -4)$

2. Prove that the line $3y - 6x - 2 = 0$ touches the parabola $3y^2 = 16x$. Hence find the coordinates of point of contact.

Solution

For a line to be a tangent, it should touch the parabola at only one point, the point of contact.

$$3y - 6x - 2 = 0 \Rightarrow y = \frac{6x+2}{3} \dots\dots\dots (1)$$

$$3y^2 = 16x \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$3\left(\frac{6x+2}{3}\right)^2 = 16x \Rightarrow 3\left(\frac{36x^2+24x+4}{9}\right) = 16x$$

$$\Rightarrow 36x^2 + 24x + 4 - 48x = 0 \text{ or } 36x^2 - 24x + 4 = 0$$

$$\Rightarrow 9x^2 - 6x + 1 = 0 \text{ or } (3x - 1)^2 = 0 \quad \therefore x = \frac{1}{3}, \frac{1}{3}$$

Hence line touches the parabola at one point since there is only one value of $x = \frac{1}{3}$

Point of contact:

$$\text{For } x = \frac{1}{3}, y = ?$$

$$\text{Using eqn. (1), } y = \frac{6(1/3)+2}{3} = \frac{4}{3} \quad \therefore \left(\frac{1}{3}, \frac{4}{3}\right) \text{ is the point of contact}$$

3. Find the condition that $lx + my + n = 0$ may touch the parabola $y^2 = 4ax$

Solution

$$lx + my + n = 0 \Rightarrow y = -\left(\frac{lx+n}{m}\right) \dots\dots\dots (1)$$

$$y^2 = 4ax \dots\dots\dots (2)$$

Solving the equations simultaneously

Eqn. (1) in to eqn. (2)

$$\left[-\left(\frac{lx+n}{m}\right)\right]^2 = 4ax \Rightarrow (lx + n)^2 = 4am^2x$$

$$\Rightarrow l^2x^2 + 2nlx + n^2 - 4am^2x = 0 \text{ or } l^2x^2 + (2nl - 4am^2)x + n^2 = 0$$

For tangency $B^2 = 4AC$

$$\Rightarrow (2nl - 4am^2)^2 = 4l^2n^2 \text{ or } 4l^2n^2 - 16nlam^2 + 16a^2m^4 = 4l^2n^2$$

$$\Rightarrow 16nlam^2 = 16a^2m^4, \text{ dividing through by } 16am^2$$

$$\therefore ln = am^2$$

Other questions involving Tangents and Parabolas

4. A line $x + 2y + k = 0$ is a tangent to the parabola $y^2 = 8x$. Find the possible value(s) of k , hence deduce the point(s) of contact.

Solution

$$x + 2y + k = 0 \Rightarrow x = -(2y + k) \dots\dots\dots (1)$$

$$y^2 = 8x \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$y^2 = 8[-(2y + k)] \Rightarrow y^2 + 16y + 8k = 0 \dots\dots\dots (3)$$

For tangency $B^2 = 4AC$

$$\Rightarrow (16)^2 = 4(8k) \therefore k = 8$$

$\therefore x + 2y + 8 = 0$ is a tangent.

Point of contact:

From eqn. (3), substituting the value of k ,

$$\Rightarrow y^2 + 16y + 64 = 0 \text{ or } (y + 8)^2 = 0 \therefore y = -8 \dots\dots\dots (4)$$

From eqn. (1), putting $y = -8, k = 8$

$$\Rightarrow x = -(2 \times 8 + 8) = -8$$

$\therefore$ Point of contact is: $(-8, -8)$

5. Find the equation of the tangent to the given parabola;

a) $y^2 = 12x$ which is parallel to the line $x - y + 5 = 0$

b) $y^2 = 16x$ which is inclined to x -axis at an angle of 135°

c) $y^2 = -24x$ which is perpendicular to the line $x + 2y + 10 = 0$

d) $y^2 = 8x$ which is inclined to x -axis at an angle of $\sin^{-1} \frac{5}{13}$

Solution

- a) Let the required tangent be $y = mx + c \dots\dots\dots (1)$

In this equation we have a task of finding values of c and m

Obtaining m

Given: $x - y + 5 = 0$ or $y = x + 5$ is $\parallel$ to $y = mx + c$

For parallel lines, $m = 1$

Eqn. (1) becomes

$$y = x + c \dots\dots\dots (2)$$

Obtaining c :

Eqn. (2) in to equation of the parabola $y^2 = 12x$

$$\Rightarrow (x + c)^2 = 12x \text{ or } x^2 + (2c - 12)x + c^2 = 0$$

For tangency, $B^2 = 4AC$

$$\Rightarrow (2c - 12)^2 = 4(1)c^2 \text{ or } 4c^2 - 48c + 144 = 4c^2$$

$$\Rightarrow c = 3$$

$\therefore$ Tangent equation is: $y = x + 3$

- b) Let the required tangent be $y = mx + c \dots\dots\dots (1)$

Finding the values of c and m

Obtaining m

Given: Angle of inclination to x -axis $135^\circ \Rightarrow m = \tan \theta = \tan 135^\circ = -1$

Eqn. (1) becomes

$$y = -x + c \dots\dots\dots (2)$$

Obtaining c :

Eqn. (2) in to equation of the parabola $y^2 = 16x$

$$\Rightarrow (-x + c)^2 = 16x \text{ or } x^2 + (-2c - 16)x + c^2 = 0$$

For tangency, $B^2 = 4AC$

$$\Rightarrow (-2c - 16)^2 = 4(1)c^2 \text{ or } 4c^2 + 64c + 256 = 4c^2$$

$$\Rightarrow c = -4$$

$\therefore$ Tangent equation is: $y = -x - 4$

- c) Let the required tangent be $y = mx + c \dots\dots\dots (1)$

Finding the values of c and m

Obtaining m

Given: tangent is $\perp$ $x + 2y + 10 = 0 \Rightarrow m = -1/(-1/2) = 2$

Eqn. (1) becomes

$$y = 2x + c \dots\dots\dots (2)$$

Obtaining c :

Eqn. (2) in to equation of the parabola $y^2 = -24x$

$$\Rightarrow (2x + c)^2 = -24x \text{ or } 4x^2 + (4c + 24)x + c^2 = 0$$

For tangency, $B^2 = 4AC$

$$\Rightarrow (4c + 24)^2 = 4(4)c^2 \text{ or } 16c^2 + 192c + 576 = 16c^2$$

$$\Rightarrow 192c + 576 = 0 \text{ or } c^2 + 3 = 0$$

$$\therefore c = -3$$

$\therefore$ Tangent equation is: $y = 2x - 3$

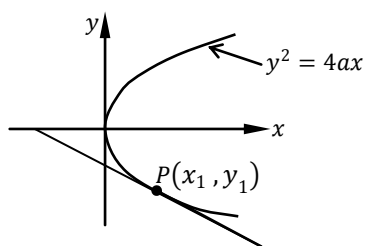
- d) Left as an exercise. **Ans:** $65y = 25x + 338$

Task

1. Prove that the line $9x - 3y + 1 = 0$ touches $y^2 = 4x$
2. Find the equation of a common tangent to $y^2 = 4x$ and $x^2 = 4y$.
Ans: $x + y + 1 = 0$
3. Prove that the line $6y = 18x + 1$ touches $y^2 = 2x$
4. Find the equation of a straight line which touches both $y^2 = 8ax$ and $x^2 + y^2 = 2a^2$
Ans: $y = x + 2a$
5. Find the equation of the parabola whose focus is $(0, 0)$ and whose directrix is $2x = y + 1$. Show also that the line $2y = 4x - 1$ touches the parabola.
Ans: $(x + 2y)^2 = 2y - 4x + 1$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON A PARABOLA $y^2 = 4ax$

To Find an Equation of a Tangent at any point (x_1, y_1) on the Parabola $y^2 = 4ax$



Required line:

$$y - y_1 = m(x - x_1) \dots\dots\dots (1)$$

Now finding slope m of the line

From $y^2 = 4ax$, differentiating w.r.t x

$$\Rightarrow 2y \frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\left. \frac{dy}{dx} \right|_{\text{at } P(x_1, y_1)} = \frac{2a}{y_1}$$

$$\therefore y - y_1 = \frac{2a}{y_1}(x - x_1) \text{ or } yy_1 - y_1^2 = 2a(x - x_1)$$

$$\therefore yy_1 = 2ax + y_1^2 - 2ax_1 \dots\dots\dots (2)$$

But (x_1, y_1) lies on the parabola

$$\Rightarrow y_1^2 = 4ax_1$$

Eqn. (2) becomes

$$\therefore yy_1 = 2ax + 4ax_1 - 2ax_1 = 2a(x + x_1)$$

$yy_1 = 2a(x + x_1)$, which is the required equation of the tangent at $P(x_1, y_1)$ on to the parabola $y^2 = 4ax$.

To Find an Equation of a Tangent at any point $(at^2, 2at)$ on the Parabola $y^2 = 4ax$

Following the above presentation, and using the equation of a tangent at any point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ which is $yy_1 = 2a(x + x_1)$, the required tangent at $(at^2, 2at)$ is:

$$\text{Comparing } P(x_1, y_1) = (at^2, 2at) \Rightarrow x_1 = at^2, y_1 = 2at$$

$$\Rightarrow y(2at) = 2a(x + at^2) \text{ or } ty = x + at^2$$

$ty = x + at^2$

To Find the Equation of a Tangent to the Parabola $y^2 = 4ax$ from a point $P(x_1, y_1)$ not on the parabola

Let the required tangent be: $y = mx + c$ (1)

The task is to find the values of m and c

Obtaining the value(s) of m

Solving the equations of tangent and the parabola simultaneously gives:

$$m^2x^2 + (2mc - 4a)x + c^2 = 0$$

Now applying condition for tangency $B^2 = 4AC$ gives $c = \frac{a}{m}$ (2)

Thus equation of tangent becomes:

$$y = mx + \frac{a}{m} \text{..... (3)}$$

To find c , point $P(x_1, y_1)$ lies on the tangent

$$\Rightarrow c = y_1 - mx_1 \text{..... (4)}$$

Solving equation (2) and (4) simultaneously gives:

$$\frac{a}{m} = y_1 - mx_1 \text{ or } m^2x_1 - my_1 + a = 0. \text{ Hence two values of } m \text{ are obtained.}$$

From eqn. (3), two tangents can be obtained as:

$$y = m_1x + \frac{a}{m_1}, \quad y = m_2x + \frac{a}{m_2}$$

Alternatively

Considering the given point $P(x_1, y_1)$ to be the general point on the parabola $y^2 = 4ax$ as $T(at^2, 2at)$, the tangent equation at $T(at^2, 2at)$ is $ty = x + at^2$ (1)

Now point $P(x_1, y_1)$ is on the tangent

$$\Rightarrow ty_1 = x_1 + at^2 \text{ or } at^2 - ty_1 + x_1 = 0, \text{ which is a quadratic in } t.$$

Two values of t can be obtained. Using these values in eqn. (1), two tangents can be deduced.

$$t_1y = x + at_1^2 \text{ and } t_2y = x + at_2^2$$

NORMAL AT ANY POINT $P(x_1, y_1)$ TO THE PARABOLA $y^2 = 4ax$

To Find the Equation of the Normal at any point $P(x_1, y_1)$ on the Parabola $y^2 = 4ax$

Using $m_1 \times m_2 = -1$, m_1 = Gradient of tangent

, and m_2 = Gradient of the normal

Grad $PT = m_1 = ?$

From $y^2 = 4ax$, differentiating w.r.t x

$$2y \frac{dy}{dx} = 4a, \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \left. \frac{dy}{dx} \right|_{\text{at } P(x_1, y_1)} = \frac{2a}{y_1} = m_1$$

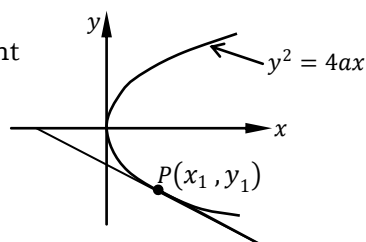
$$\therefore m_2 = \text{Grad NP} = -\frac{1}{m_1} = -\frac{y_1}{2a}$$

Equation of the normal is:

$$y - y_1 = m_2(x - x_1)$$

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

, which is an equation of the normal at



$P(x_1, y_1)$ onto the parabola $y^2 = 4ax$

To Find an Equation of a Normal at any point $T(at^2, 2at)$ on the Parabola $y^2 = 4ax$

Comparing $P(x_1, y_1)$ with $T(at^2, 2at)$, and using the equation of the normal above at $P(x_1, y_1)$ which is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

The required equation of the normal at $T(at^2, 2at)$ is:

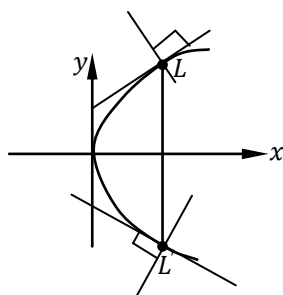
$$\Rightarrow y - 2at = -\frac{2at}{2a}(x - at^2) = -tx + at^3$$

$$tx + y = at^3 + 2at$$

General Examples

1. Find the equation of the tangents and normals to the parabola $y^2 = 4ax$ at the ends of latus rectum.

Solution



$L(a, 2a)$, $L'(a, -2a)$ are end points of the latus rectum

Equation of tangent at $L(a, 2a)$

$$y - 2a = m(x - a)$$

Obtaining slope m

Differentiating $y^2 = 4ax$ w.r.t x

$$\Rightarrow 2y \frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{dy}{dx} \text{ at } P(a, 2a) = \frac{2a}{2a} = 1$$

$$\Rightarrow y - 2a = x - a \quad \therefore y = x + a$$

Equation of normal at $L(a, 2a)$

$$y - 2a = m'(x - a)$$

Finding slope m'

From $m \times m' = -1$, $m' = -\frac{1}{m} = -\frac{1}{1} = -1$ m = slope of tangent

$$\Rightarrow y - 2a = -1(x - a)$$

$$\therefore y = -x + 3a$$

Equation of tangent at $L'(a, -2a)$

$$y + 2a = m_1(x - a)$$

Obtaining slope m_1

Differentiating $y^2 = 4ax$ w.r.t x

$$\Rightarrow 2y \frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{dy}{dx} \text{ at } P(a, -2a) = \frac{2a}{-2a} = -1$$

$$\Rightarrow y + 2a = -1(x - a)$$

$$\therefore y = -x - a$$

Equation of normal at $L'(a, -2a)$

$$y + 2a = m_2(x - a)$$

Finding slope m_2

$$\text{But } m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$$

$$\Rightarrow y + 2a = (x - a)$$

$$\therefore y = x - 3a$$

2. Show that the equation of the tangent to the curve $y^2 = 4ax$ at the point $(at^2, 2at)$ is $ty = x + at^2$ and the equation of the tangent to the curve $x^2 = 4by$ at the point $(2bp, bp^2)$ is $y = px - bp^2$.

The curves $y^2 = 32x$ and $x^2 = 4y$ intersect at the origin and at A. find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate area of triangle ABC.

Solution

$$y^2 = 4ax, \text{ required tangent: } ty = x + at^2$$

Tangent equation at $(at^2, 2at)$

$$y - 2at = m(x - at^2) \dots\dots\dots (1)$$

Obtaining slope, m

For $y^2 = 4ax$, differentiating w.r.t x

$$2y \frac{dy}{dx} = 4a, \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\text{Now } \left. \frac{dy}{dx} \right|_{\text{at } (at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t} = m \dots\dots\dots (2)$$

Eqn. (2) in to eqn. (1)

$$\Rightarrow y - 2at = \frac{1}{t} \cdot (x - at^2)$$

$$\Rightarrow ty - 2at^2 = x - at^2 \quad \therefore ty = x + at^2$$

For $x^2 = 4by$, differentiating w.r.t x

$$2x \frac{dy}{dx} = 4b, \quad \frac{dy}{dx} = \frac{x}{2b}$$

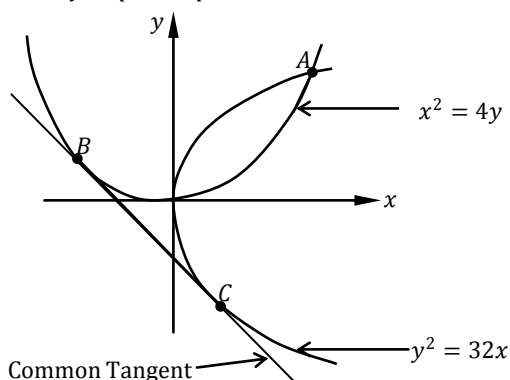
$$\text{Now } \left. \frac{dy}{dx} \right|_{\text{at } (2bp, bp^2)} = \frac{2bp}{2b} = p = m'$$

Tangent equation at $(2bp, bp^2)$

$$y - bp^2 = m'(x - 2bp) \Rightarrow y - bp^2 = p(x - 2bp)$$

$$\Rightarrow y = px - 2bp^2 + bp^2$$

$$\therefore y = px - bp^2$$



Comparing curves $y^2 = 32x$ and $y^2 = 4ax \Rightarrow 32 = 4a \quad \therefore a = 8$

Point on $y^2 = 4ax$ is $C(at^2, 2at)$

$$\therefore C(8t^2, 16t)$$

Now equation of tangent to $y^2 = 4ax$

$$\text{at } (at^2, 2at) \text{ is } ty = x + at^2$$

$\Rightarrow$ Equation of tangent at $C(8t^2, 16t)$

to the curve $y^2 = 32x$ is

$$ty = x + 8t^2 \dots\dots\dots (1)$$

Eqn. (1) meets $x^2 = 4y$ at B

From curve $x^2 = 4by$ and $x^2 = 4y$, by comparison, $4b = 4 \therefore b = 1$

Equation of the tangent to the curve $x^2 = 4y$ at $B(2bp, bp^2)$ is $y = px - bp^2$

Now equation of the tangent to the curve $x^2 = 4y$ at $B(2p, p^2)$ is:

$$y = px - p^2 \dots\dots\dots (2)$$

Since the tangent at B and the tangent at C are the same, comparing the two equations,

$$ty = x + 8t^2 \Rightarrow y = \frac{1}{t}x + 8t ; y = px - p^2$$

$$\Rightarrow \frac{1}{t} = p , \text{ and } 8t = -p^2, \text{ solving the two equations}$$

$$\Rightarrow 8t = -\frac{1}{t^2} \therefore 8t^3 + 1 = 0$$

$$\Rightarrow (2t + 1)(4t^2 - 2t + 1) = 0 \therefore t = -\frac{1}{2}$$

$$\Rightarrow t = -\frac{1}{2}, p = -2$$

Using eqn. (2) and $p = -2$

Equation of the common tangent is:

$$y = -2x - 4$$

Alternatively

Tangent at B: $y = px - p^2$ is also a tangent to the curve $y^2 = 32x$

Solving the two equations simultaneously and applying the condition for tangency $B^2 = 4AC$

$$(px - p^2)^2 = 32x$$

$$p^2x^2 - 2p^3x + p^4 - 32x = 0 \Rightarrow p^2x^2 - (2p^3 + 32)x + p^4 = 0$$

For tangency $B^2 = 4AC$

$$\Rightarrow [-(2p^3 + 32)]^2 = 4p^2 \cdot p^4$$

$$\Rightarrow 4p^6 + 128p^3 + 1024 = 4p^6 \text{ or } p^3 + 8 = 0$$

$$\Rightarrow (p + 2)(p^2 - 2p + 4) = 0 \therefore p = -2$$

From eqn. (2) $y = px - p^2$, $p = -2$

Equation of the common tangent is:

$$y = -2x - 4$$

Now obtaining the coordinates of B and C

$$B(2p, p^2) \equiv B(-4, 4)$$

To find $C(8t^2, 16t)$, lies on the tangent $y = -2x - 4$

$$\Rightarrow 16t = -16t^2 - 4 \text{ or } 4t^2 + 4t + 1 = 0, t = -\frac{1}{2}$$

$$\therefore C(8t^2, 16t) \equiv C(2, -8)$$

3. Find the equation of a tangent to the parabola from the given points:

- a) Parabola: $y^2 = 8x$, point $(-10, -8)$
- b) Parabola: $y^2 = -32x$, point $(1, -12)$
- c) Parabola: $y^2 = 4ax$, point $(-4a, -3a)$
- d) Parabola: $y^2 = -16ax$, point $(4a, 2a)$

Solution

a) **Approach I:**

Let the required tangent be: $y = mx + c$

Obtaining value of c

$(-10, -8)$ lies on this tangent

$$\Rightarrow c = -8 + 10m \dots\dots\dots (1)$$

Obtaining the slope, m

Solving the equation of the tangent and parabola simultaneously

$$(mx + c)^2 = 8x \Rightarrow m^2x^2 + (2mc - 8)x + c^2 = 0$$

For tangency, $B^2 = 4AC$

$$\Rightarrow (2mc - 8)^2 = 4m^2c^2$$

$$\Rightarrow 4m^2c^2 - 32mc + 64 = 4m^2c^2$$

$$\therefore mc = 2 \dots\dots\dots (2)$$

Eqn. (1) into eqn. (2)

$$m(-8 + 10m) = 2 \text{ or } 5m^2 - 4m - 1 = 0 \therefore m = 1, -\frac{1}{5}$$

$$\text{For } m = 1, c = -8 + 10(1) = 2$$

$$\text{For } m = -\frac{1}{5}, c = -8 + 10\left(-\frac{1}{5}\right) = -10$$

$$\therefore y = x + 2, y = -\frac{1}{5}x - 10 \Rightarrow -5y = x + 50$$

Approach II:

The equation of the tangent to the parabola $y^2 = 4ax$ at the point

$T(at^2, 2at)$ is $ty = x + at^2$

Now comparing $y^2 = 4ax$ with $y^2 = 8x$, $a = 2$

Thus the general point on the given parabola is: $P(2t^2, 4t)$

The tangent at $P(2t^2, 4t)$ passes through $(-10, -8)$

$$\text{Tangent is } ty = x + 2t^2 \dots\dots\dots (1)$$

$$\Rightarrow -8t = -10 + 2t^2 \text{ or } t^2 + 4t - 5 = 0 \therefore t = 1, -5$$

From equation (1), the tangents are:

$$\text{For } t = 1, y = x + 2$$

$$\text{For } t = -5, -5y = x + 50$$

Approach III:

The general tangent to the parabola $y^2 = 4ax$ with slope m is: $y = mx + \frac{a}{m}$

This tangent passes through $(-10, -8)$.

Comparing $y^2 = 4ax$ with $y^2 = 8x$, $a = 2$

Required tangent is:

$$y = mx + \frac{2}{m} \dots\dots\dots (1)$$

Eqn. (1) is satisfied by the point $(-10, -8)$

$$\Rightarrow -8 = -10m + \frac{2}{m} \text{ or } 5m^2 - 4m - 1 = 0 \therefore m = 1, -\frac{1}{5}$$

Tangent equations are:

$$y = x + 2, y = -\frac{1}{5}x - 10 \Rightarrow -5y = x + 50$$

b) Left as an exercise. **Ans:** $y = 2x - 4, 3y = 18x - 4$

c) Using approach III, let the tangent be:

$$y = mx + \frac{a}{m} \dots\dots\dots (1)$$

This tangent is the general tangent on $y^2 = 4ax$

Point $(-4a, -3a)$ lies on this tangent

$$\Rightarrow -3a = -4am + \frac{a}{m} \text{ or } 4m^2 - 3m - 1 = 0 \therefore m = 1, -1/4$$

Tangent equations are:

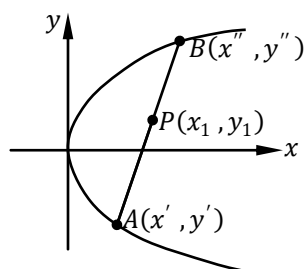
$$y = x + a, y = -\frac{1}{4}x - 4a \Rightarrow -4y = x + 4a$$

d) Left as an exercise. **Ans:** $y = 2x - 2a, 2y = -x + 8a$

CHORD WITH GIVEN MIDPOINT

To find the equation of a chord of the parabola $y^2 = 4ax$ having $M(x_1, y_1)$ as its midpoint

Consider the figure below.



Equation of chord is

$$y - y_1 = m(x - x_1) \dots\dots\dots (1)$$

Finding slope of the chord

$$P(x_1, y_1) = P\left[\frac{x' + x''}{2}, \frac{y' + y''}{2}\right]$$

$$\therefore 2x_1 = x' + x'' \dots\dots\dots (2)$$

$$\therefore 2y_1 = y' + y'' \dots\dots\dots (3)$$

Also A and B lie on the parabola

$$\Rightarrow y'^2 = 4ax' \dots\dots\dots (3)$$

$$\Rightarrow y''^2 = 4ax'' \dots\dots\dots (4)$$

Eqn. (3) - eqn. (4)

$$y'^2 - y''^2 = 4a(x' - x'')$$

$$(y' + y'')(y' - y'') = 4a(x' - x'')$$

$$\Rightarrow \text{Grad APB} = \frac{y' - y''}{x' - x''} = \frac{4a}{y' + y''} \dots\dots\dots (5)$$

Eqn. (3) into eqn. (5)

$$\text{Grad APB} = \frac{4a}{2y_1} = \frac{2a}{y_1}$$

From eqn. (1), $y - y_1 = m(x - x_1)$ becomes

$$yy_1 - y_1^2 = 2a(x - x_1)$$

$yy_1 - y_1^2 = 2a(x - x_1)$, which is the required equation

Rewriting this equation

$$yy_1 = y_1^2 - 4ax_1 - 2ax + 2ax_1$$

$$\boxed{yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1}$$

To Find the Equation of a Chord joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lying of the Parabola $y^2 = 4ax$

This is generally given as

$$(y - y_1)(y - y_2) = y^2 - 4ax$$

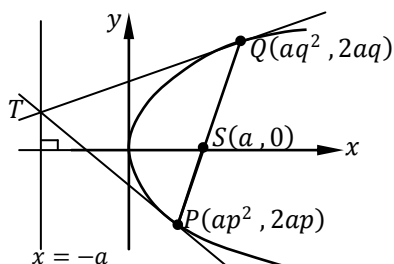
NOTE

A line joining two points on parabola and passing through the focus is called a *Focal Chord*.

Example

1. Prove that the tangent at the extremities of the focal chord of a parabola $y^2 = 4ax$ intersects at right angles on the directrix. Take coordinates of extremities to be $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$.

Solution



Slope of the tangent to the curve $y^2 = 4ax$

$$2y \frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{dy}{dx} \text{ at } P = \frac{2a}{2ap} = \frac{1}{p}, \quad \frac{dy}{dx} \text{ at } Q = \frac{2a}{2aq} = \frac{1}{q}$$

Equation of tangent at P

$$y - 2ap = \frac{1}{p}(x - ap^2) \dots\dots\dots (1)$$

Equation of tangent at Q

$$y - 2aq = \frac{1}{q}(x - aq^2) \dots\dots\dots (2)$$

Solving for point of intersection

Eqn. (1) - eqn. (2)

$$2aq - 2ap = \frac{x - ap^2}{p} - \frac{x - aq^2}{q} \dots\dots\dots \times pq$$

$$pq(2aq - 2ap) = qx - ap^2q - px + apq^2$$

$$2apq(q - p) = (q - p)x - apq(q - p) \text{ or } 2apq = x - apq \quad \therefore x = apq$$

For the focal chord, $\text{Grad } \overline{PS} = \text{Grad } \overline{SQ}$

$$\Rightarrow \frac{2ap - 0}{ap^2 - a} = \frac{2aq - 0}{aq^2 - a} \text{ or } \frac{2p}{p^2 - 1} = \frac{2q}{q^2 - 1}$$

$$\Rightarrow pq^2 - p = p^2q - q \text{ or } pq(q - p) = p - q \quad \therefore pq = -1$$

eqn. (3) - eqn. (4)

$x = -a$, which is the x - coordinate of point of intersection, and is the equation of directrix.

Now for intersection at right angles,

$$\text{Grad } \overline{PT} \times \text{Grad } \overline{QT} = -1$$

$$\text{Grad } \overline{PT} = \frac{1}{p}, \quad \text{Grad } \overline{QT} = \frac{1}{q}$$

$$\Rightarrow \frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1 = \text{RHS}$$

2. Prove that the tangent to the parabola $y^2 = 4ax$ at points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ meet at $T(apq, a(p+q))$.

If M is the midpoint of PQ , prove that TM is bisected by the parabola.

Solution

First case, see solution above in example (1)

For TM to be bisected by the parabola, midpoint of TM must lie on the parabola.

$$\Rightarrow M\left(\frac{ap^2+aq^2}{2}, \frac{2ap+2aq}{2}\right) = M\left(\frac{a(p^2+q^2)}{2}, a(p+q)\right)$$

$$T(apq, a(p+q)), M\left(\frac{a(p^2+q^2)}{2}, a(p+q)\right)$$

Let N = midpoint of TM

$$\Rightarrow N\left(\frac{apq + \frac{a(p^2+q^2)}{2}}{2}, \frac{a(p+q) + a(p+q)}{2}\right) = N\left(\frac{a(p^2+2pq+q^2)}{4}, a(p+q)\right)$$

$$\Rightarrow N\left(\frac{a(p+q)^2}{4}, a(p+q)\right)$$

$$\text{Let } t = p + q \Rightarrow N\left(\frac{at^2}{4}, at\right), y^2 = 4ax$$

$$\Rightarrow (at)^2 = 4a\left(\frac{at^2}{4}\right) = a^2t^2$$

Hence point N lies on parabola, which is the midpoint of TM . i.e. parabola bisects TM .

Uneb 2011

- a) Find the equation of a tangent to the parabola $y^2 = \frac{x}{16}$ at the point $\left(t^2, \frac{t}{4}\right)$
 b) If the tangents to the parabola in (a) above at points $P\left(p^2, \frac{p}{4}\right)$ and $Q\left(q^2, \frac{q}{4}\right)$ meet on the line $y = 2$
 i) Show that $p + q = 16$
 ii) Deduce the midpoint of PQ lies on the line $y = 2$

Solution

- a) Required tangent: $y - \frac{t}{4} = m(x - t^2)$ (1)

Finding the slope of the tangent at $\left(t^2, \frac{t}{4}\right)$

Differentiating $y^2 = \frac{x}{16}$ w.r.t x

$$2y \frac{dy}{dx} = \frac{1}{16} \quad \therefore \frac{dy}{dx} = \frac{1}{32y}$$

$$M = \frac{dy}{dx} \text{ at } \left(t^2, \frac{t}{4}\right) = \frac{1}{32 \times \frac{t}{4}} = \frac{1}{8t}$$

$$\Rightarrow y - \frac{t}{4} = \frac{1}{8t}(x - t^2) \text{ or } 8ty - 2t^2 = x - t^2$$

$$\therefore x - 8ty + t^2 = 0$$

- b) (i) Tangent at P is $x - 8py + p^2 = 0$ (2)

Tangent at Q is $x - 8qy + q^2 = 0$ (3)

Solving for point of intersection

Eqn. (2) – eqn. (3)

$$(8q - 8p)y + (p^2 - q^2) = 0$$

$$y = \frac{q^2 - p^2}{8(q - p)} = \frac{(q + p)(q - p)}{8(q - p)} = \frac{p + q}{8}$$

But y – coordinate of point of intersection is $y = 2$

$$\Rightarrow 2 = \frac{p + q}{8} \quad \therefore p + q = 16$$

ii) Midpoint of PQ is $\left(\frac{q^2 + p^2}{2}, \frac{p + q}{8}\right)$

For this point to lie on $y = 2$, y – coordinate must be equal to 2.

$$\Rightarrow \frac{p + q}{8} = \frac{16}{8} = 2, \text{ as required}$$

3. a) Show that the equation of the normal to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ is $y + tx = 2at + at^3$

b) the normal at P meets x – axis

at G and the midpoint of PG is N. Find the coordinates of G and N

Solution

a) Required normal:

$$y - 2at = m_1(x - at^2) \dots\dots\dots (1)$$

Finding slope of the normal m_1

$m_1 \times m_2 = -1$, m_2 = Slope of the tangent at P

For $y^2 = 4ax$, differentiating w.r.t x

$$2y \frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$m_2 = \left. \frac{dy}{dx} \right|_{P(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

$$\therefore m_1 = -t \quad [\because m_1 \times m_2 = -1]$$

$$\Rightarrow y - 2at = -t(x - at^2) \quad \therefore y + tx = 2at + at^3, \text{ as required.}$$

b) For x – intercept, $y = 0$

$$\Rightarrow 0 + tx = 2at + at^3 \quad \therefore x = 2a + at^2$$

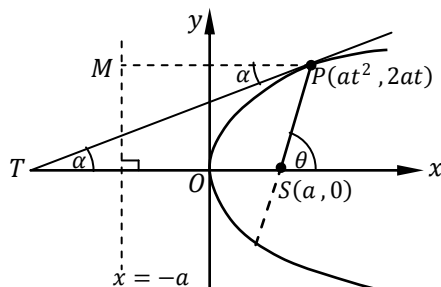
$$\therefore G(2at + at^3, 0)$$

Midpoint of PN: $P(at^2, 2at)$, $G(2a + at^2, 0)$

$$\Rightarrow N\left(\frac{at^2 + at^2 + 2a + 2at + 0}{2}, \frac{2at + 0}{2}\right) = N[a(t^2 + 1), at]$$

4. Show that the tangent at any point P on the parabola $y^2 = 4ax$ bisects the angle between the focal chord through P and perpendicular from P onto the directrix.

Solution



From the figure coordinates of $M(-a, 2at)$

Let m_1 = Slope of $\overline{PS}$, m_2 = Slope of $\overline{PT}$

$$m_1 = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} = \tan \theta$$

$$m_2 = \left. \frac{dy}{dx} \right|_{at^2, 2at} = \frac{2a}{2at} = \frac{1}{t} = \tan \alpha$$

$$\angle T\hat{P}S = \theta - \alpha$$

Introducing tangents

$$\begin{aligned} \tan \angle T\hat{P}S &= \tan(\theta - \alpha) = \frac{\tan \theta - \tan \alpha}{1 + \tan \theta \tan \alpha} \\ &= \frac{\frac{2t}{t^2 - 1} - \frac{1}{t}}{1 + \frac{2t}{t^2 - 1} \cdot \frac{1}{t}} = \frac{2t^2 - t^2 + 1}{(t^2 - 1 + 2)t} \\ &= \frac{t^2 + 1}{t(t^2 + 1)} = \frac{1}{t} \end{aligned}$$

$$\Rightarrow \tan \angle T\hat{P}S = \frac{1}{t} = \tan \alpha \quad \therefore \angle T\hat{P}S = \alpha$$

Hence tangent bisects $\angle MPS$

LOCUS PROBLEMS IN PARABOLA

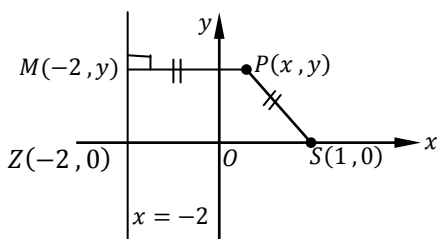
1. Find the locus of a point $P(x, y)$ which moves in a such a way that:

- It is equidistant from the point $S(1, 0)$ and the line $x = -2$
- It is equidistant from point $S(-1, 1)$ and the line $y = -3$
- It is equidistant from $S(2, -1)$ and the line $3x - 4y = 0$

Hence describe and sketch the locus in each case above.

Solution

a) Using a very clear diagram,



$$\text{Given: } \overline{PS} = \overline{PM}$$

$$\Rightarrow \sqrt{(x - -2)^2 + (y - y)^2} = \sqrt{(x - 1)^2 + (y - 0)^2}$$

Squaring both sides

$$\Rightarrow x^2 + 4x + 4 = x^2 - 2x + 1 + y^2$$

$$y^2 = 6x + 3 = 3(2x + 1)$$

$$\therefore y^2 = 3(2x + 1), \text{ is the locus of P.}$$

The locus is a parabola with focus $S(1, 0)$ and equation of directrix $x = -2$

Sketching $y^2 = 3(2x + 1)$

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix, hence axis is x -axis. The parabola is symmetrical about the axis

Vertex is the midpoint of $\overline{SZ}$

$$= A\left(\frac{-2+1}{2}, \frac{0+0}{2}\right) = A\left(-\frac{1}{2}, 0\right)$$

Obtaining intercepts:

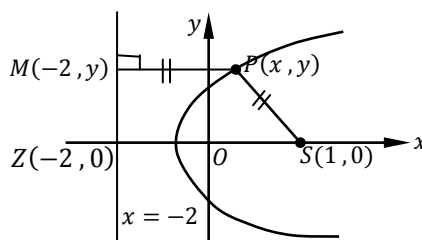
For $x = 0 \Rightarrow y^2 = 3(2(0) + 1) \therefore y = \pm\sqrt{3}$

$\therefore B(0, +\sqrt{3}), B'(0, -\sqrt{3})$, are y - intercepts.

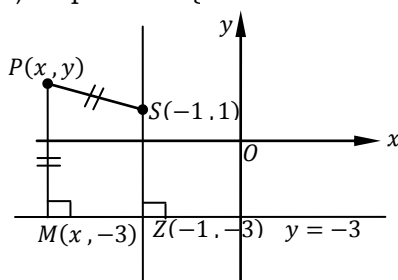
For $y = 0 \Rightarrow 0 = 3(2x + 1) \therefore x = -\frac{1}{2}$

$A\left(-\frac{1}{2}, 0\right)$, is an x - intercept.

Note: the x - intercept is the vertex



b) Representing the information on a clearer diagram.



Given: $\overline{PS} = \overline{PM}$

$$\Rightarrow \sqrt{(x - -1)^2 + (y - 1)^2} = \sqrt{(x - x)^2 + (y + 3)^2}$$

Squaring both sides

$$\Rightarrow x^2 + 2x + 1 + y^2 - 2y + 1 = y^2 + 6y + 9$$

$$x^2 + 2x - 6y - 7 = 0$$

$\therefore x^2 + 2x - 6y - 7 = 0$, is the locus of P.

The locus is a parabola with focus $S(-1, 1)$ and directrix $y = -3$

Sketching $x^2 + 2x - 6y - 7 = 0$

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix, hence axis is $x = -1$. The parabola is symmetrical about the axis

Vertex is the midpoint of $\overline{SZ}$

$$= A\left(\frac{-1-1}{2}, \frac{1+(-3)}{2}\right) = A(-1, -1)$$

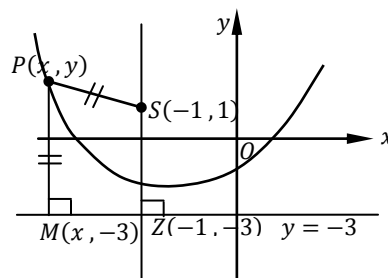
Obtaining intercepts:

For $x = 0$, $\frac{7}{6}$

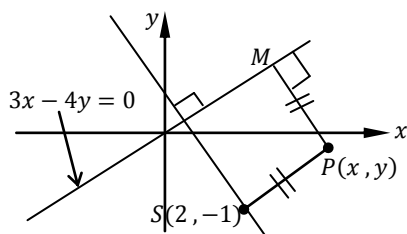
$\therefore A(0, +7/6)$, is the y - intercepts.

For $y = 0 \Rightarrow x^2 + 2x - 7 = 0$

$\therefore B(1.828, 0), B'(-1.828, 0)$ are x - intercepts



c) Using a clearer diagram,



$$\text{Given: } \overline{PS} = \overline{PM}$$

$$\Rightarrow \overline{PS} = \sqrt{(x-2)^2 + (y+1)^2}$$

$\overline{PM}$ = Perpendicular distance of $P(x, y)$ from the line $3x - 4y = 0$

$$\Rightarrow \overline{PM} = \left| \frac{3x-4y}{\sqrt{3^2+4^2}} \right| = \frac{1}{5} |3x-4y|$$

Now $\sqrt{(x-2)^2 + (y+1)^2} = \frac{1}{5} |3x-4y|$, squaring both sides

$$\Rightarrow x^2 - 4x + 4 + y^2 + 2y + 1 = \frac{9x^2 - 24xy + 16y^2}{25}$$

$$\Rightarrow 16x^2 + 9y^2 + 24xy - 100x + 125 + 50y = 0$$

$$\Rightarrow (4x + 3y)^2 - 100x + 50y + 125 = 0, \text{ which is the required locus of } P.$$

The locus is a parabola with focus $S(2, -1)$ and directrix $3x - 4y = 0$

Sketching $(4x + 3y)^2 - 100x + 50y + 125 = 0$

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix.

Since the directrix is perpendicular to the axis, slope of the axis is:

$$= -\frac{1}{3/4} = -\frac{4}{3}$$

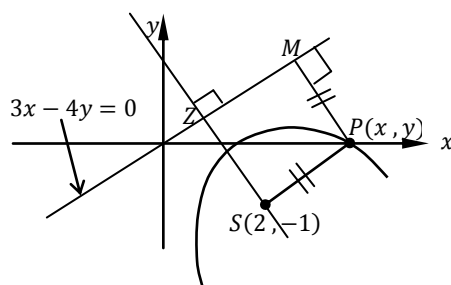
Equation of the axis must pass through the focus

$$\Rightarrow y + 1 = -\frac{4}{3}(x - 2) \quad \therefore 4x + 3y - 5 = 0$$

Obtaining the point of intersection of the axis and the directrix in order to find the vertex

Solving $4x + 3y - 5 = 0$ and $3x - 4y = 0$, point of intersection is $Z\left(\frac{4}{5}, \frac{3}{5}\right)$

Vertex is the midpoint of $\overline{SZ}$



$$= A\left(\frac{2+4/5}{2}, \frac{-1+3/5}{2}\right) = A\left(\frac{7}{5}, -\frac{1}{5}\right)$$

Obtaining intercepts:

For $x = 0 \Rightarrow 9y^2 + 50y + 125 = 0$, hence no real values of y .

No y -intercept

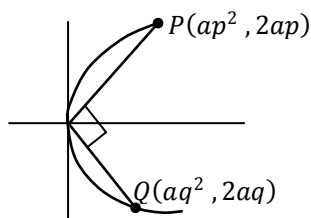
For $y = 0 \Rightarrow 16x^2 - 100x + 125 = 0$

$$\therefore x = 4.52, 1.73$$

$\therefore B(4.52, 0), B'(1.73, 0)$, are intercepts

- 2.** P and Q are two points on the parabola $y^2 = 4ax$ whose coordinates are $(ap^2, 2ap)$ and $(aq^2, 2aq)$. O is the origin and $\overline{OP} \perp \overline{OQ}$, show that $pq + 4 = 0$.

Solution



Let $m_1 = \text{Grad OP}$, $m_2 = \text{Grad OQ}$

$$m_1 = \frac{2ap}{ap^2} = \frac{2}{p} , m_2 = \frac{2aq}{aq^2} = \frac{2}{q}$$

But $m_1 \times m_2 = -1$

$$\Rightarrow \frac{2}{p} \times \frac{2}{q} = -1$$

$$\Rightarrow 4 = -pq$$

$$\therefore pq + 4 = 0$$

3. a) Show that the equation of a chord joining points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on a parabola $y^2 = 4ax$ is given as $y(p + q) = 2(x + apq)$
 b) A variable focal chord PQ of a parabola $y^2 = 4ax$ is moving with its midpoint R . prove that the locus of point R defines a parabola of length of latus rectum $2a$. Find the focus and the equation of the directrix of locus of R .

Solution

- a) General equation of the chord joining any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the parabola is $(y - y_1)(y - y_2) = y^2 - 4ax$
 Equation of the required chord is:
 $(y - 2ap)(y - 2aq) = y^2 - 4ax$
 $\Rightarrow y^2 - 2aqty - 2apy + 4a^2pq = y^2 - 4ax$
 $\Rightarrow y(-2ap - 2aq) = -4ax - 4a^2pq$
 $\Rightarrow y(p + q) = 2(x + apq)$
 b) Since PQ is a variable focal chord, then PQ passes through the focus $S(a, 0)$
 $y(p + q) = 2(x + apq) \Rightarrow 0 = 2(a + apq)$
 $\Rightarrow 1 + pq = 0 \therefore pq = -1$
 $\therefore$ Equation of the focal chord is $y(p + q) = 2(x - a)$ (1)
 But R is the midpoint of PQ
 $\Rightarrow R\left[\frac{ap^2 + aq^2}{2}, \frac{2ap + 2aq}{2}\right] = R\left[\frac{a(p^2 + q^2)}{2}, a(p + q)\right]$
 Let the coordinates of R be $R(X, Y)$
 $R(X, Y) = R\left[\frac{a(p^2 + q^2)}{2}, a(p + q)\right]$
 $\Rightarrow X = \frac{a(p^2 + q^2)}{2}$ (2)
 $\Rightarrow Y = a(p + q)$ (3)
 Solving eqn. (2) and eqn. (3) simultaneously, eliminating the variables p and q in the coordinates,
 From eqn. (2)
 $\frac{2X}{a} = p^2 + q^2 = (p + q)^2 - 2pq$
 $\Rightarrow \frac{2X}{a} = (p + q)^2 + 2 \quad [\because pq = -1]$ (4)
 Eqn. (3) into eqn. (4) to eliminate p and q
 $\Rightarrow \frac{2X}{a} = \left(\frac{Y}{a}\right)^2 + 2$ or $2aX = Y^2 + 2a^2$

$$\Rightarrow Y^2 = 2a(X - a)$$

$\therefore$ the locus of R is a parabola $y^2 = 2a(x - a)$

NOTE

When handling numbers of locus, the following points are worth noting.

- (a) When mention of any point, line or any angle is made, you must first look for what has been mentioned in case it is not given in the question.
- (b) The locus of a point needed is obtained by first letting the point to be (X, Y)
- (c) Formulate two equations from step (b) above.
- (d) Solve simultaneously the two formulated equations eliminating the variables/parameter involved in the coordinates given.
- (e) Write the locus in terms of the coordinate variables on the curve.
Usually write it in terms of variables x and y .

- 4.** a) Find the equation of a line parallel to the normal to the parabola $y^2 = 4ax$ at the point $(p^2, 2p)$ passing through $(1, 0)$.
- b) The tangent to $y^2 = 4ax$ at the point $P(at^2, 2at)$ meets the y -axis at G and the normal at P meets the x -axis at H
- i) Show that the midpoint M of HG has coordinates $\left(a + \frac{at^2}{2}, \frac{at}{2}\right)$
 - ii) Find the locus of M as P moves on the parabola.

Solution

For parallel lines, gradients are the same .i.e. $m_1 = m_2$

Let the required parallel line with slope m_1 be:

$$y - 0 = m_1(x - 1) \text{ or } y = m_1(x - 1) \dots \dots \dots (1)$$

Obtaining the slope, m_2 of the normal at $(p^2, 2p)$

Given: $y^2 = 4ax$, differentiating w.r.t x to obtain the slope of the tangent

$$\Rightarrow 2y \frac{dy}{dx} = 4a \text{ or } \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore m' = \left. \frac{dy}{dx} \right|_{\text{at } (p^2, 2p)} = \frac{2a}{2p} = \frac{a}{p} = \text{Slope of the tangent at } (p^2, 2p)$$

$$\Rightarrow m_2 = -\frac{1}{m'} = -\frac{p}{a}$$

$$\therefore \text{Slope of the required line is } m_1 = m_2 = -\frac{p}{a}$$

$$\therefore \text{Required equation of the line is: } y = -\frac{p}{a}(x - 1) \text{ or } px + ay = p$$

- b) Obtaining the tangent equation at $P(at^2, 2at)$

$$y - 2at = m(x - at^2) \dots \dots \dots (1)$$

Obtaining slope m

$$m = \left. \frac{dy}{dx} \right|_{\text{at } P(at^2, 2at)}, \text{ differentiating } y^2 = 4ax \text{ w.r.t } x$$

$$\Rightarrow 2y \frac{dy}{dx} = 4a \text{ or } \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore m = \left. \frac{dy}{dx} \right|_{\text{at } P(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

From eqn. (1), the tangent equation is:

$$y - 2at = \frac{1}{t}(x - at^2) \Rightarrow ty - 2at^2 = x - at^2$$

$\therefore ty = x + at^2$, now we have the tangent.

This line meets y - axis at G. This is the y - intercept which occurs when $x = 0$

$$\Rightarrow ty = at^2 \text{ or } y = at$$

$\therefore G(0, at)$, now we have G

Obtaining the equation of the Normal at $P(at^2, 2at)$

$$y - 2at = m'(x - at^2) \dots\dots\dots (2)$$

Obtaining slope m

$$m' = -\frac{1}{\text{Slope of the tangent at P}} = -\frac{1}{1/t} = -t$$

From eqn. (2)

$$y - 2at = -t(x - at^2) \text{ or } y = -tx + 2at + at^3$$

$\therefore y = -tx + 2at + at^3$, now we have the normal

This line meets the x -axis at H. this is the x -intercept which occurs when $y = 0$

$$\Rightarrow 0 = -tx + 2at + at^3 \quad \therefore x = 2a + at^2$$

$\therefore H(2a + at^2, 0)$, now we have also H

$$\text{Midpoint of HG} = \left(\frac{0+2a+at^2}{2}, \frac{at+0}{2} \right) = \left(a + \frac{at^2}{2}, \frac{at}{2} \right)$$

Obtaining the locus of M

$$\text{Let } M(X, Y) = \left(a + \frac{at^2}{2}, \frac{at}{2} \right)$$

$$\Rightarrow X = a + \frac{at^2}{2} \text{ or } 2X = 2a + at^2 \dots\dots\dots (1)$$

$$\Rightarrow Y = \frac{at}{2} \text{ or } t = \frac{2Y}{a} \dots\dots\dots (2)$$

Eliminating t from eqn. (1) and eqn. (2), eqn. (2) into eqn. (1)

$$2X = 2a + a \left(\frac{2Y}{a} \right)^2 \Rightarrow 2aX = 2a^2 + 4Y^2$$

$$\therefore Y^2 = \frac{a}{2}(X - a)$$

$$\therefore \text{The locus of M is } \therefore y^2 = \frac{a}{2}(x - a)$$

5. The normal to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ meets the x - axis at G and the midpoint of PG is N.

- i) Find the locus of N as P moves on the parabola
- ii) If the focus of the parabola is S, prove that $SN \perp PG$
- iii) If ΔSRG is equilateral, find the coordinates of R
- iv) If $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ and $S(a, 0)$ respectively are points in $x - y$ plane, show that $\frac{1}{SP} + \frac{1}{SQ}$ is independent of the parameter t

Solution

- i) Obtaining the equation of the normal to the curve $y^2 = 4ax$ at $P(at^2, 2at)$
See solution in 4(b) above.

$\therefore y = -tx + 2at + at^3$, is now the equation of the normal.

This normal meets the x -axis at G (the x -intercept) when $y = 0$

$$\Rightarrow 0 = -tx + 2at + at^3 \quad \therefore x = 2a + at^2$$

$\therefore G(2a + at^2, 0)$, is now known

Obtaining the midpoint of PG .i.e. N

$$N\left(\frac{2a+at^2+at^2}{2}, \frac{0+2at}{2}\right) = N(at^2 + a, at), \text{ is also now known}$$

To obtain the locus of N, let $N(X, Y)$

$$\Rightarrow N(X, Y) = N(at^2 + a, at)$$

$$X = at^2 + a \dots\dots\dots (1)$$

$$Y = at \Rightarrow t = \frac{Y}{a} \dots\dots\dots (2)$$

Eqn. (2) into eqn. (1)

$$X = a\left(\frac{Y}{a}\right)^2 + a \Rightarrow aX = Y^2 + a^2 \text{ or } Y^2 = a(X - a)$$

$\therefore$ Locus of N is: $y^2 = a(x - a)$, which is a parabola.

ii) To prove that $SN \perp PG$, $S(a, 0)$, $N(at^2 + a, at)$

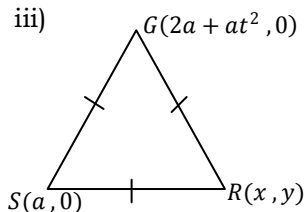
For $SN \perp PG$, $\text{Grad } \overline{SN} \times \text{Grad } \overline{PG} = -1$

$$\text{Grad } \overline{SN} = \frac{at-0}{at^2+a-a} = \frac{1}{t}$$

$$\text{Grad } \overline{PG} = \frac{2at-0}{at^2-(2a+at^2)} = \frac{2at}{-2a} = -t$$

$$\therefore SN \perp PG \text{ if } \frac{1}{t} \times -t = -1 \quad [m_1 \times m_2 = -1]$$

iii)



By coordinate geometry, $\overline{SR}^2 = \overline{SG}^2 = \overline{GR}^2$

$$\text{For } \overline{SR}^2 = \overline{GR}^2$$

$$(x - a)^2 + y^2 = [x - (2a + at^2)]^2 + y^2$$

$$x^2 - 2ax + a^2 + y^2 = x^2 - 2x(2a + at^2) + (2a + at^2)^2 + y^2$$

$$-2ax + a^2 = -2x(2a + at^2) + (2a + at^2)^2$$

$$\Rightarrow 2x(2a + at^2 - a) = (2a + at^2)^2 - a^2 = (3a + at^2)(a + at^2)$$

$$\Rightarrow x = \frac{(3a+at^2)(a+at^2)}{2(a+at^2)} = \frac{a(3+t^2)}{2}$$

$$\text{For } \overline{SR}^2 = \overline{SG}^2$$

$$\Rightarrow (x - a)^2 + y^2 = [(2a + at^2) - a]^2 + 0$$

$$\Rightarrow (x - a)^2 + y^2 = (a + at^2)^2 = a^2(1 + t^2)^2$$

$$\text{But } x = \frac{a(3+t^2)}{2}$$

$$\Rightarrow \left(\frac{a(3+t^2)}{2} - a\right)^2 + y^2 = (a + at^2)^2 = a^2(1 + t^2)^2$$

$$\Rightarrow y^2 = a^2(1 + t^2)^2 - \frac{a^2(1+t^2)^2}{4}$$

$$\Rightarrow y^2 = \frac{3}{4}a^2(1 + t^2)^2 \quad \therefore y = \frac{a(1+t^2)\sqrt{3}}{2}$$

$$\therefore R\left(\frac{a(3+t^2)}{2}, \frac{a(1+t^2)\sqrt{3}}{2}\right)$$

NOTE

The x – coordinate of R can still be obtained from the midpoint of SG since these two lie on the same horizontal line.

$$x = \frac{a+2a+at^2}{2} = \frac{a(3+t^2)}{2}$$

To find the y – coordinate, use the formula for angle between two lines, say $\overline{SG}$ and $\overline{SR}$

$$\tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$\text{Let } m_2 = \text{Grad } \overline{SR} = \frac{y-0}{x-a} = \frac{y}{x-a}$$

$$m_1 = \text{Grad } \overline{SG} = \frac{0-0}{2a+at^2-a} = 0$$

$$\therefore \tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1} = m_2, \text{ since } m_1 = 0$$

$$\therefore \tan 60^\circ = \frac{y}{x-a} \Rightarrow y = \sqrt{3} \cdot (x - a)$$

$$y = \sqrt{3} \cdot \left[\frac{a(3+t^2)}{2} - a \right] = \frac{a(1+t^2)\sqrt{3}}{2}$$

$$\therefore R \left(\frac{a(3+t^2)}{2}, \frac{a(1+t^2)\sqrt{3}}{2} \right)$$

$$\text{iv) } P(at^2, 2at), Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right), S(a, 0)$$

$$\frac{1}{SP} + \frac{1}{SQ} = ?$$

$$\overline{SP} = \sqrt{(a - at^2)^2 + (2at - 0)^2} = \sqrt{a^2(1 - 2t^2 + t^4 + 4t^2)}$$

$$= a\sqrt{(t^4 + 2t^2 + 1)}$$

$$= a\sqrt{(t^2 + 1)^2} = a(t^2 + 1)$$

$$\overline{SQ} = \sqrt{\left(a - \frac{a}{t^2}\right)^2 + \left(0 - -\frac{2a}{t}\right)^2} = \sqrt{a^2 \left(1 - \frac{2}{t^2} + \frac{1}{t^4} + \frac{4}{t^2}\right)}$$

$$= \frac{a}{t^2} \sqrt{(t^4 + 2t^2 + 1)}$$

$$= \frac{a}{t^2} \sqrt{(t^2 + 1)^2} = \frac{a}{t^2} (t^2 + 1)$$

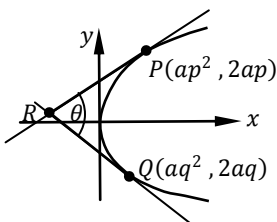
$$\therefore \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t^2+1)} + \frac{1}{\frac{a}{t^2}(t^2+1)}$$

$$= \frac{(t^2+1)}{a(t^2+1)} = \frac{1}{a}, \text{ which is independent of } t$$

6. Find the coordinates of the point of intersection R of the tangents to the parabola $y^2 = 4ax$ at the points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$.

If the tangents at P, Q are inclined to each other at an angle of 45° , show that the locus of R is the curve $y^2 = x^2 + 6ax + a^2$.

Solution



Tangents at P and Q can be obtained as

$py = x + ap^2$ and $qy = x + aq^2$ respectively.

Solving for point of intersection we obtain:

$R(apq, a(p+q))$ (Left for student as an exercise)

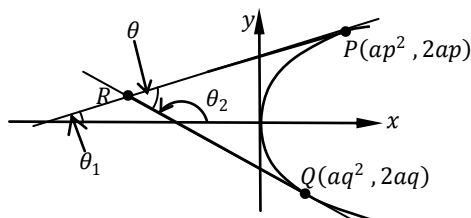
Given: $\theta = 45^\circ$

Let $m_1 = \text{Grad } \overline{PR}$ and $m_2 = \text{Grad } \overline{QR}$ where $m_1 > m_2$

$$\Rightarrow m_1 = \frac{1}{p}, m_2 = \frac{1}{q}$$

Let $\overline{PR}$ be inclined to the x -axis at an angle θ_1 and ,

$\overline{QR}$ be inclined to the x -axis at an angle θ_2



From the figure

$$\theta_1 + 180^\circ - \theta_2 = \theta \Rightarrow \theta - 180^\circ = (\theta_1 - \theta_2)$$

Introduce tangent on both sides

$$\frac{\tan \theta - \tan 180^\circ}{1 + \tan \theta \cdot \tan 180^\circ} = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \cdot \tan \theta_2}$$

$$\tan \theta = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \cdot \tan \theta_2} = \frac{m_1 - m_2}{1 + m_1 \cdot m_2}$$

$$\tan \theta = \frac{\frac{1}{p} - \frac{1}{q}}{1 + \frac{1}{p} \cdot \frac{1}{q}} = -\frac{(p-q)}{pq+1}$$

$$\Rightarrow \tan 45^\circ = -\frac{(p-q)}{pq+1} \text{ or } pq + 1 = -(p - q) \dots \dots \dots$$

(1)

$$\text{Let } R(X, Y) = R(apq, a(p + q))$$

$$\Rightarrow X = apq \text{ or } pq = \frac{X}{a} \dots \dots \dots (2)$$

$$\Rightarrow Y = a(p + q) \dots \dots \dots (3)$$

Expressing eqn. (1) in terms of $p + q$ and pq which are in equations (2) and (3)

$pq + 1 = -(p - q)$, squaring both sides

$$\Rightarrow (pq + 1)^2 = p^2 - 2pq + q^2$$

$$\Rightarrow (pq + 1)^2 = (p + q)^2 - 2pq - 2pq$$

$$\therefore (pq + 1)^2 = (p + q)^2 - 4pq \dots \dots \dots (4)$$

Eqn. (2), eqn. (3) into eqn. (4)

$$(p + q)^2 = (pq + 1)^2 + 4pq$$

$$\Rightarrow \left(\frac{Y}{a}\right)^2 = \left(\frac{X}{a} + 1\right)^2 + \frac{4X}{a} \dots \dots \dots (\times a^2)$$

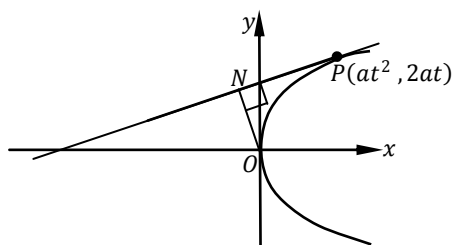
$$\Rightarrow Y^2 = (X + a)^2 + 4aX$$

$$\Rightarrow Y^2 = X^2 + 2aX + a^2 + 4aX \text{ or } Y^2 = X^2 + 6aX + a^2$$

$$\therefore \text{Locus of R is } y^2 = x^2 + 6ax + a^2$$

- 7.** $P(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$. N is the foot of the perpendicular drawn from the origin to the tangent at P . show that as P varies the locus of N is the curve $x(x^2 + y^2) + ay^2 = 0$

Solution



Equation of the tangent at P can be obtained as:

$$ty = x + at^2 \dots\dots\dots (1)$$

Obtaining the equation of the normal through the origin and perpendicular to the tangent

$$y - 0 = m(x - 0) \text{ or } y = m_1x$$

But for perpendicular lines, $m_1 \times m_2 = -1$

$$\therefore m_1 = -\frac{1}{1/t} = -t$$

$$\text{Equation of normal: } y = -tx \dots\dots\dots (2)$$

Solving for the coordinates of N

Eqn. (1) into eqn. (2)

$$-t^2x = x + at^2 \Rightarrow x = -\frac{at^2}{t^2+1} \dots\dots\dots (3)$$

Eqn. (3) into eqn. (2)

$$\Rightarrow y = \frac{at^3}{t^2+1}$$

$$\therefore \text{Coordinates of } N \left(-\frac{at^2}{t^2+1}, \frac{at^3}{t^2+1} \right)$$

$$\text{Let } N(X, Y) = N \left(-\frac{at^2}{t^2+1}, \frac{at^3}{t^2+1} \right)$$

$$\Rightarrow X = -\frac{at^2}{t^2+1} \dots\dots\dots (1)$$

$$\Rightarrow Y = \frac{at^3}{t^2+1} \dots\dots\dots (2)$$

Dividing the equations to simplify in eliminating parameter t

$$\Rightarrow \frac{X}{Y} = -\frac{at^2}{t^2+1} \times \frac{t^2+1}{at^3} = -\frac{1}{t}$$

$$\Rightarrow t = -\frac{Y}{X} \dots\dots\dots (3)$$

Eqn. (3) into eqn. (1)

$$\Rightarrow X = -\frac{a\left(-\frac{Y}{X}\right)^2}{\left(-\frac{Y}{X}\right)^2+1} = -\frac{aY^2}{Y^2+X^2} \text{ or } X(X^2 + Y^2) = -aY^2$$

$$\Rightarrow X(X^2 + Y^2) + aY^2 = 0$$

$$\therefore \text{Required locus is } x(x^2 + y^2) + ay^2 = 0$$

Task

1. A variable chord through the focus of the parabola $y^2 = 4ax$ cuts the curve at P and Q. the straight line joining P to the point (0,0) cuts the line joining Q to the point $(-a, 0)$ at R. show that the equation of the locus of point R is $y^2 + 8x^2 + 4ax = 0$

2. (i) Show that the locus of the midpoint of the line joining the parabola $y^2 = 8x$ and the point (8, 0) is a parabola.

Hint: $y^2 = 8x$ can be compared with $y^2 = 4ax \therefore a = 2$. The general point on the parabola is $(at^2, 2at) = (2t^2, 4t)$

- ii) Determine the points at which lines from the new focus are perpendicular to the parabola $y^2 = 8x$

- iii) Find the y – coordinate of the point at which the tangent at one of the points meet the y – axis.
4. A tangent from the point $T(t^2, 2t)$ touches $y^2 = 4x$. Find:
- The equation of the tangent
 - The equation of the line L parallel to the normal at $T(t^2, 2t)$ and passing through $(1, 0)$
 - The point of intersection X of line L and tangent.
(**Note:** point T lies on parabola)
5. Find the locus of the point of intersection of tangents to $y^2 = 4ax$ which are perpendicular to each other. **Ans:** $x + a = 0$
6. The normal at P to $y^2 = 4ax$ cuts the x – axis at G . A line through the midpoint of PG perpendicular to Ox cuts the curve at Q and Ox at N . prove that $QN = PG$
7. The normal to $y^2 = 4ax$ at P meets the x – axis in G . GP is produced so that $PQ = PG$. Find the locus of Q . **Ans:** $y^2 = 16a(x + 2a)$
8. P is any point on $y^2 = 4ax$. O is the origin and M is a point on Ox so that $MPO = 90^\circ$. If PN is the ordinate of P prove that $MN = 4a$
9. The focus of the parabola $y^2 = 4ax$ is S . Any tangent to the curve meets the directrix in M and the latus rectum in L . prove that $SM = SL$
10. The normal at the point $P(4a, 4a)$ to $y^2 = 4ax$ cuts the curve again at Q . if S is the focus prove that angle $PSQ = 90^\circ$
11. P is any point on the parabola $y^2 = 4ax$. The normal at P cuts the x – axis in Q . find the locus of the midpoint of PQ . **Ans:** $y^2 = a(x - a)$
12. The normal at P to the parabola $y^2 = 4ax$ cuts the curve again at Q . O is the origin and S is the focus. A line through P parallel to OQ meets the x – axis in R . prove that $OR = 2PS$
13. The vertex of the parabola $y^2 = 4ax$ is O . OP and OQ are two chords of the parabola and angle $POQ = 90^\circ$. Show that PQ cuts the x – axis in a fixed point.
14. Chords of the parabola $y^2 = 4ax$ are drawn so that the chords all subtend 90° at $O(0, 0)$. Prove that the midpoint of the chords all lie on the parabola $y^2 = 2a(x - 4a)$
15. The normal at P to the parabola $y^2 = 4ax$ cuts the x – axis in G . the normals at Q and R pass through the midpoint of PG . Prove that QR cuts the x – axis at $(-a, 0)$
16. PQ is a focal chord of $y^2 = 4ax$. O is the origin and PO cuts the directrix at M . prove that MQ is parallel to $y = 0$
17. P is the point $(at^2, 2at)$ on $y^2 = 4ax$. P is joined to point $S(a, 0)$ and the line PS cuts the curve again at Q . the tangents at P and Q cut at R . Prove that $RP.RQ = a^2 \left(\frac{t^2+1}{t} \right)^3$

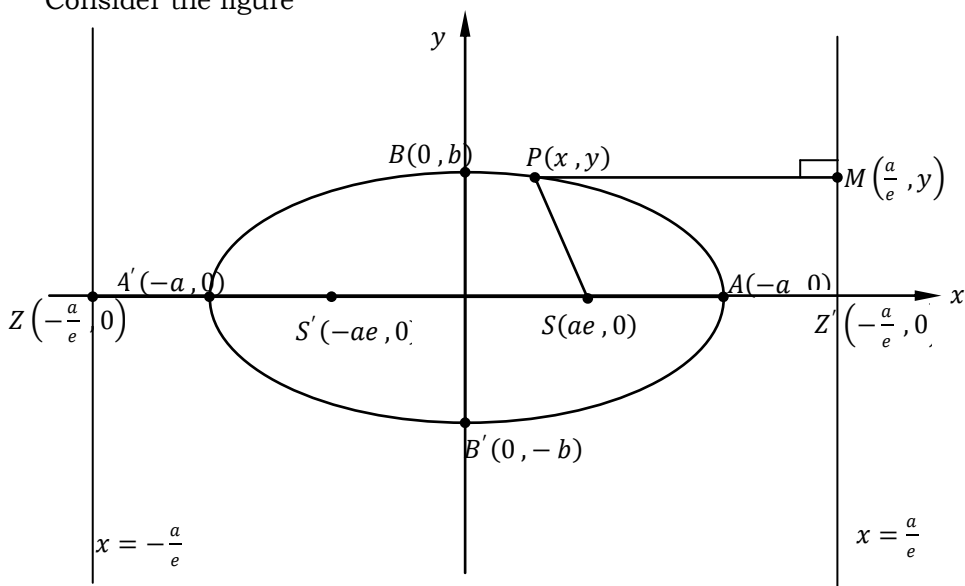
ELLIPSE

Definition:

An ellipse is the locus of a point which moves in such a way that its distance from a fixed point (Focus) bears a constant ratio e , ($0 < e < 1$) to its distance from a fixed line (Directrix) not passing through the focus.

Equation of an Ellipse in Standard Form

Consider the figure



Point A divides ZS internally and externally in the ratio 1: e

From the definition

$$\overline{PS} = e\overline{PM} \text{ or } \overline{AS} = e\overline{AZ}$$

$$\overline{PS} = \sqrt{(x - ae)^2 + (y - 0)^2}, \quad \overline{PM} = \sqrt{\left(x - \frac{a}{e}\right)^2 + (y - y)^2}$$

$$\Rightarrow \overline{PS}^2 = e^2 \overline{PM}^2$$

$$\Rightarrow (x - ae)^2 + (y - 0)^2 = e^2 \left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow x^2 - 2aex + a^2e^2 + y^2 = (xe - a)^2$$

$$\Rightarrow x^2 - 2aex + a^2e^2 + y^2 = x^2e^2 - 2aex + a^2$$

$$\Rightarrow x^2(1 - e^2) + y^2 = a^2(1 - e^2) \dots\dots\dots \div a^2(1 - e^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1, \text{ let } b^2 = a^2(1 - e^2)$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2(1 - e^2)$

, which is the standard equation of an ellipse

Definitions

- 1) AA' is called the *Major Axis* of the ellipse and its length is $2a$
- 2) BB' is called the *Minor Axis* of the ellipse and its length is $2b$
- 3) The points A and A' are *Vertices* of the ellipse
- 4) *Eccentricity* of the ellipse is given by the equation $b^2 = a^2(1 - e^2)$
- 5) *Intercepts* of the ellipse are: $A(a, 0), A'(-a, 0), B(0, b), B'(0, -b)$

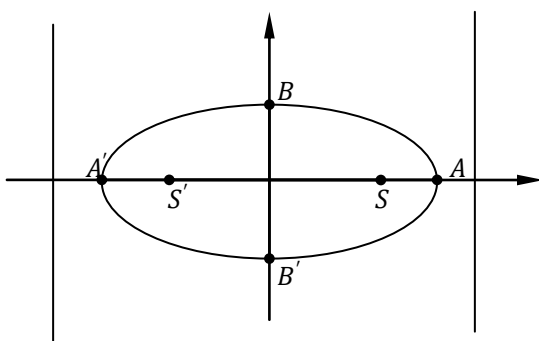
Note:

The points A and A' are x – intercepts and the points B and B' are y – intercepts.

The major axis is also called Transverse axis.

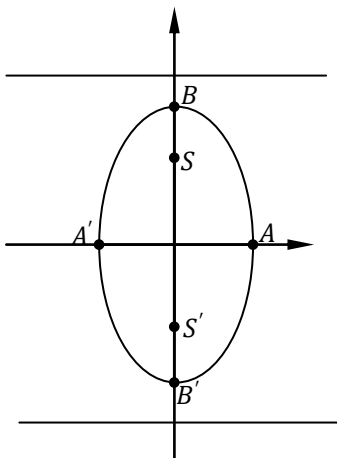
Summary of Standard Equation of an Ellipse

Case I: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ where $b^2 = a^2(1 - e^2)$



Foci	$(\pm ae, 0)$
Directrices	$x = \pm \frac{a}{e}$
Vertices	$(\pm a, 0), (0, \pm b)$
Major axis	x –axis or $y = 0$
Minor axis	y –axis or $x = 0$

Case II: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a < b$ where $a^2 = b^2(1 - e^2)$



Foci	$(0, \pm be)$
Directrices	$y = \pm \frac{b}{e}$
Vertices	$(\pm a, 0), (0, \pm b)$
Major axis	y –axis or $y = 0$
Minor axis	x –axis or $x = 0$

NOTE:

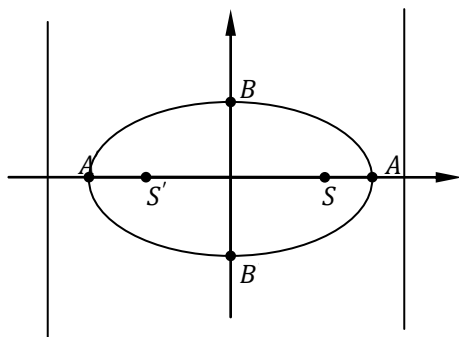
1. If the transverse axis (Major axis) is the y -axis rather than the x -axis, the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a < b$ where $a^2 = b^2(1 - e^2)$
2. The table above is used in stating the focus, equation of Directrices, vertices, eccentricity and equation of the ellipse.

Example

1. Find the equation of the ellipse with:
 - i) Foci at $(\pm 3, 0)$, vertices $(\pm 5, 0)$
 - ii) Foci at $(0, \pm 4)$, vertices $(0, \pm 7)$
 - iii) Foci at $(\pm 2, 0)$, minor axis of length 3 units
 - iv) Foci at $(0, \pm 5)$, major axis of length 24 units
 - v) Vertices $(\pm 8, 0)$, contains $(4, \sqrt{3})$

Solution

- i) Foci at $(\pm 3, 0)$, vertices $(\pm 5, 0)$
 $\Rightarrow S(3, 0), S'(-3, 0), A(5, 0), A'(-5, 0)$



Since the major axis is the x -axis,
minor axis is the y -axis

The equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

$$\text{Vertices: } (\pm a, 0) = (\pm 5, 0) \therefore a = 5$$

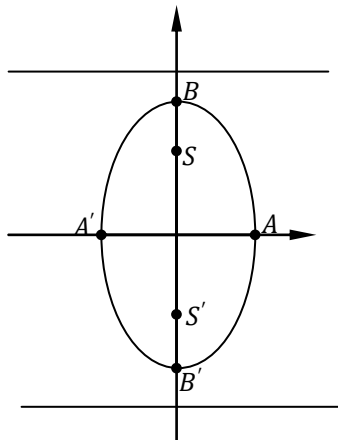
$$\text{Foci: } (\pm ae, 0) = (\pm 3, 0) \therefore ae = 3$$

$$\Rightarrow e = \frac{3}{5}$$

$$\text{Also } b^2 = a^2(1 - e^2) = 25\left(1 - \frac{9}{25}\right) = 16$$

$$\therefore \frac{x^2}{25} + \frac{y^2}{16} = 1, \text{ is the required ellipse.}$$

- ii) Foci at $(0, \pm 4)$, vertices $(0, \pm 7)$
 $\Rightarrow S(0, 4), S'(0, -4), B(0, 7), B'(0, -7)$



Since the major axis is the y -axis,
minor axis is the x -axis

The equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$$

$$\text{Vertices: } (0, \pm b) = (0, \pm 7) \therefore b = 7$$

$$\text{Foci: } (0, \pm be) = (0, \pm 4) \therefore be = 4$$

$$\Rightarrow e = \frac{4}{7}$$

$$\text{Also } a^2 = b^2(1 - e^2) = 49\left(1 - \frac{16}{49}\right) = 33$$

$$\therefore \frac{x^2}{33} + \frac{y^2}{49} = 1$$

iii) Foci at $(\pm 2, 0)$, minor axis of length 3 units

Since the major axis is the x -axis,
minor axis is the y -axis

The equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$$

$$\text{Foci: } (\pm ae, 0) = (\pm 2, 0) \quad \therefore ae = 2$$

$$\text{Length of minor axis} = 2b = 3 \quad \therefore b = \frac{3}{2}$$

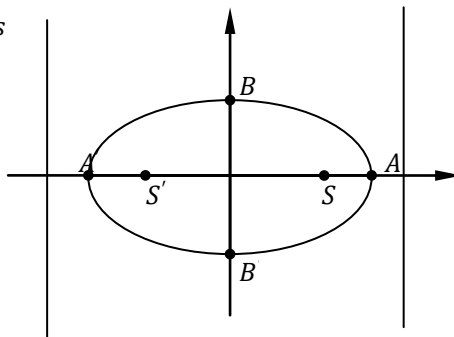
$$\text{Since } a > b, \text{ using } b^2 = a^2(1 - e^2)$$

$$\Rightarrow \frac{9}{4} = a^2(1 - e^2) = a^2 - a^2e^2$$

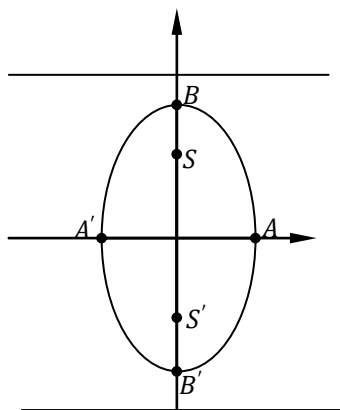
$$\Rightarrow \frac{9}{4} = a^2 - 4 \quad [\because ae = 2 \Rightarrow a^2e^2 = 4]$$

$$\therefore a^2 = \frac{25}{4}$$

$$\text{Required ellipse: } \frac{x^2}{25/4} + \frac{y^2}{9/4} = 1 \quad \text{or} \quad \frac{4x^2}{25} + \frac{4y^2}{9} = 1$$



iv) Foci at $(0, \pm 5)$, major axis of length 24 units



Since the major axis is the y -axis,
minor axis is the x -axis

The equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a < b$$

$$\text{Foci: } (0, \pm be) = (0, \pm 5) \quad \therefore be = 5$$

$$\text{Length of Major axis} = 24 = 2b \quad \therefore b = 12$$

$$\Rightarrow e = 5/12$$

$$\text{Also } a^2 = b^2(1 - e^2) = 144 \left(1 - \frac{25}{144}\right) = 119$$

$$\therefore \frac{x^2}{119} + \frac{y^2}{144} = 1$$

v) Vertices $(\pm 8, 0)$ contains $(4, \sqrt{3})$

Vertices $(\pm 8, 0) = (\pm a, 0) \Rightarrow a = 8$, since the vertices are on the x -axis,
it is the major axis.

$$\text{Using } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$$

$$\Rightarrow \frac{x^2}{64} + \frac{y^2}{b^2} = 1, \text{ point } (4, \sqrt{3}) \text{ is on the curve}$$

$$\Rightarrow \frac{4^2}{64} + \frac{(\sqrt{3})^2}{b^2} = 1 \quad \text{or} \quad 1 - \frac{1}{4} = \frac{3}{b^2} \quad \therefore b^2 = 4$$

$$\therefore \frac{x^2}{64} + \frac{y^2}{4} = 1, \text{ is the ellipse}$$

2. Find the major axis, length of the major axis and semi minor axis and coordinates of the foci

i) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

ii) $9x^2 + 25y^2 = 225$

iii) $32x^2 + 16y^2 = 64$

iv) $162x^2 + 128y^2 = 10368$

Solution

- i) $\frac{x^2}{9} + \frac{y^2}{25} = 1$, which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a < b$, major axis is y - axis

Foci: $(0, \pm be)$; $a^2 = b^2(1 - e^2)$

Now $a^2 = 9 \Rightarrow a = \pm 3$, $b^2 = 25 \Rightarrow b = \pm 5$

$\Rightarrow 9 = 25(1 - e^2) \therefore e = +4/5$

Major axis: $x = 0$ (y - axis)

Length of major axis: $2b = 10$ units

Length of semi minor axis: $a = 3$ units

Foci: $(0, \pm 5 \times \frac{4}{5}) = (0, \pm 4)$

- ii) $9x^2 + 25y^2 = 225$($\div 225$)

$\frac{x^2}{25} + \frac{y^2}{9} = 1$, which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$

$\Rightarrow a^2 = 25 \therefore a = \pm 5$; $b^2 = 9 \therefore b = \pm 3$

Using $b^2 = a^2(1 - e^2)$

$\Rightarrow 9 = 25(1 - e^2) \therefore e = +4/5$

Transverse axis: $y = 0$ (x - axis)

Length of major axis: $2a = 10$ units

Semi minor axis: $b = 3$ units

Foci: $(\pm 5 \times \frac{4}{5}, 0) = (\pm 4, 0)$

- iii) Left as an exercise

- iv) Left as an exercise

3. Find the eccentricity, foci and Directrices of the ellipse;

a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

c) $4x^2 + y^2 = 1$

b) $16x^2 + 25y^2 = 100$

d) $\frac{x^2}{4} + y^2 = 1$

e) $\frac{x^2}{2} + \frac{2y^2}{9} = 2$

Solution

- a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$

$a^2 = 25$, $b^2 = 9$, Foci: $(\pm ae, 0)$, Directrices: $x = \pm \frac{a}{e}$, $b^2 = a^2(1 - e^2)$

$\Rightarrow 25 = 9(1 - e^2) \therefore e = +4/5$

Foci: $(\pm 5 \times \frac{4}{5}, 0) = (\pm 4, 0)$

Directrices: $x = \pm \frac{5}{4/5} = \pm \frac{25}{4}$

- b) $16x^2 + 25y^2 = 100 \Rightarrow \frac{x^2}{25/4} + \frac{y^2}{4} = 1$, which of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$

$a^2 = 25/4$, $b^2 = 4$, Foci: $(\pm ae, 0)$, Directrices: $x = \pm \frac{a}{e}$, $b^2 =$

$a^2(1 - e^2)$

$\Rightarrow \frac{25}{4} = 4(1 - e^2) \therefore e = +3/5$

Foci: $(\pm \frac{5}{2} \times \frac{3}{5}, 0) = (\pm \frac{3}{2}, 0)$

Directrices: $x = \pm \frac{5/2}{3/5} = \pm \frac{25}{6}$

c) $4x^2 + y^2 = 1 \Rightarrow \frac{x^2}{1/4} + \frac{y^2}{1} = 1$ which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a < b$

$a^2 = 1/4$, $b^2 = 1$, Foci: $(0, \pm be)$, Directrices: $y = \pm \frac{b}{e}$, $a^2 = b^2(1 - e^2)$

$\Rightarrow \frac{1}{4} = 1(1 - e^2) \therefore e = +\sqrt{3}/2$

Foci: $(0, \pm 1 \times \frac{\sqrt{3}}{2}) = (0, \pm \frac{\sqrt{3}}{2})$

Directrices: $y = \pm \frac{1}{+\sqrt{3}/2} = \pm \frac{2}{\sqrt{3}}$

d) Left as an exercise. **Ans:** $e = \frac{\sqrt{3}}{2}$, $(\pm\sqrt{3}, 0)$, $x = \pm \frac{4\sqrt{3}}{3}$

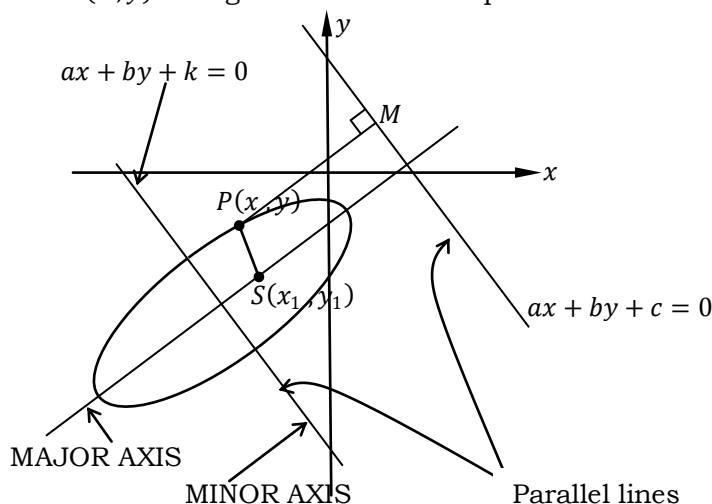
e) Left as an exercise. **Ans:** $e = \frac{\sqrt{5}}{3}$, $(0, \pm\sqrt{5})$, $y = \pm \frac{9}{5}\sqrt{5}$

GENERAL EQUATION OF AN ELLIPSE

To Find the Equation of an Ellipse given coordinates of Focus and Equation of Directrix

Let the focus have coordinates $S(x_1, y_1)$ and the equation of the directrix $ax + by + c = 0$

Let $P(x, y)$ be a general point on ellipse.



By definition, $PS = ePM$ or $PS^2 = e^2 PM^2$

$PM \perp$ Distance of P from the directrix, $ax + by + c = 0$

$$PM = \pm \frac{ax + by + c}{\sqrt{a^2 + b^2}}$$

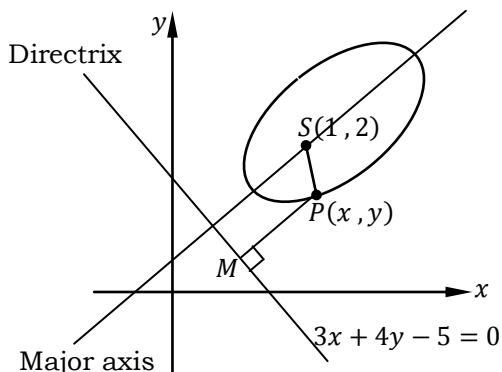
$$PS = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$

$$(x - x_1)^2 + (y - y_1)^2 = \frac{(ax + by + c)^2}{a^2 + b^2}$$

Example

Find the equation of the ellipse with focus $(1, 2)$, equation of directrix $3x + 4y = 5$ and eccentricity $e = 1/2$.

Solution



$$\text{Using } \overline{PS} = e\overline{PM} \text{ or } \overline{PS}^2 = e^2\overline{PM}^2$$

$\overline{PM} \perp$ Distance of P from the directrix,

$$3x + 4y - 5 = 0$$

$$\overline{PM} = \pm \frac{3x+4y-5}{\sqrt{3^2+4^2}} = \pm \frac{3x+4y-5}{5}$$

$$\overline{PS} = \sqrt{(x-1)^2 + (y-2)^2}$$

$$\therefore (x-1)^2 + (y-2)^2 = \frac{1}{4} \cdot \frac{(3x+4y-5)^2}{25}$$

$$100x^2 + 100y^2 - 200x - 400y + 500 =$$

$$9x^2 + 16y^2 + 24xy - 30x - 40y$$

$$\therefore 91x^2 + 84y^2 - 24xy - 170x - 360y + 475 = 0$$

USING STANDARD EQUATION OF AN ELLIPSE TO DEDUCE THE CENTRE, LENGTH OF AXES, ECCENTRICITY, FOCI AND VERTICES OF A GIVEN ELLIPSE

Example

Find the centre, length of the axes, eccentricity, foci and vertices of the ellipse

i) $x^2 + 4y^2 - 4x + 24y + 31 = 0$

ii) $12x^2 + 4y^2 + 24x - 16y + 25 = 0$

iii) $4x^2 + 9y^2 - 48x + 72y + 144 = 0$

iv) $x^2 + 4y^2 - 10x - 8y + 13 = 0$

Solution

- i) Rewriting the equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Completing squares

$$x^2 - 4x + 4 - 4 + 4(y^2 + 6y + 9 - 9) + 31 = 0$$

$$(x-2)^2 + 4(y+3)^2 - 4 - 36 + 31 = 0$$

$$(x-2)^2 + 4(y+3)^2 = 9$$

$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{9/4} = 9, \text{ which is of the form } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

By comparison

$$X = x - 2, Y = y + 3, a^2 = 9, b^2 = 9/4$$

$$\text{Centre: } (0, 0) = (X, Y)$$

$$\Rightarrow x - 2 = 0 \therefore x = 2$$

$$\Rightarrow y + 3 = 0 \therefore y = -3$$

$$\therefore \text{Centre: } (2, -3)$$

$$\text{Using } b^2 = a^2(1 - e^2) \Rightarrow \frac{9}{4} = 9(1 - e^2) \therefore e = +\frac{\sqrt{3}}{2}$$

$$\therefore \text{Eccentricity: } e = +\frac{\sqrt{3}}{2}$$

$$\text{Vertices: } (\pm a, 0) = (X, Y) \text{ and } (0, \pm b) = (X, Y)$$

$$\Rightarrow x - 2 = \pm 3 \therefore x = 5, -1$$

$$\Rightarrow y + 3 = 0 \therefore y = -3$$

$$\therefore \text{Vertices: } (5, -3), (-1, -3)$$

$$\Rightarrow x - 2 = 0 \therefore x = 2$$

$$\Rightarrow y + 3 = \pm \frac{3}{2} \therefore y = -\frac{3}{2}, -\frac{9}{2}$$

$$\therefore \text{Vertices: } \left(2, -\frac{3}{2}\right), \left(2, -\frac{9}{2}\right)$$

$$\text{Foci: } (\pm ae, 0) = (X, Y)$$

$$\Rightarrow x - 2 = \pm 3 \times \frac{\sqrt{3}}{2} \therefore x = 2 \pm \frac{3\sqrt{3}}{2}$$

$$\Rightarrow y + 3 = 0 \therefore y = -3$$

$$\therefore \text{Foci: } \left(2 \pm \frac{3\sqrt{3}}{2}, -3\right)$$

$$\text{Length of major axis: } 2a = 2 \times 3 = 6 \text{ units}$$

$$\text{Length of minor axis: } 2b = 2 \times \frac{3}{2} = 3 \text{ units}$$

ii) Rewriting the equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Completing squares

$$12(x^2 + 2x + 1 - 1) + 4(y^2 - 4y + 4 - 4) + 25 = 0$$

$$12(x + 1)^2 + 4(y - 2)^2 - 12 - 16 + 25 = 0$$

$$12(x + 1)^2 + 4(y - 2)^2 = 3$$

$$\frac{(x+1)^2}{1/4} + \frac{(y-2)^2}{3/4} = 1, \text{ which is of the form } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$$

By comparison

$$X = x + 1, Y = y - 2, a^2 = 1/4, b^2 = 3/4$$

$$\text{Centre: } (0, 0) = (X, Y)$$

$$\Rightarrow x + 1 = 0 \therefore x = -1$$

$$\Rightarrow y - 2 = 0 \therefore y = 2$$

$$\therefore \text{Centre: } (-1, 2)$$

$$\text{Using } a^2 = b^2(1 - e^2) \Rightarrow \frac{1}{4} = \frac{3}{4}(1 - e^2) \therefore e = +\frac{\sqrt{6}}{3}$$

$$\therefore \text{Eccentricity: } e = +\frac{\sqrt{6}}{3}$$

$$\text{Vertices: } (\pm a, 0) = (X, Y) \text{ and } (0, \pm b) = (X, Y)$$

$$\Rightarrow x + 1 = \pm \frac{1}{2} \therefore x = -\frac{1}{2}, -\frac{3}{2}$$

$$\Rightarrow y - 2 = 0 \therefore y = 2$$

$$\therefore \text{Vertices: } \left(-\frac{1}{2}, 2\right), \left(-\frac{3}{2}, 2\right)$$

$$\Rightarrow x + 1 = 0 \therefore x = -1$$

$$\Rightarrow y - 2 = \pm \frac{\sqrt{3}}{2} \therefore y = 2 \pm \frac{\sqrt{3}}{2}$$

$$\therefore \text{Vertices: } \left(-1, 2 \pm \frac{\sqrt{3}}{2}\right)$$

$$\text{Foci: } (0, \pm be) = (X, Y)$$

$$\Rightarrow x + 1 = 0 \quad \therefore x = -1$$

$$\Rightarrow y - 2 = \pm \frac{\sqrt{3}}{2} \times \frac{\sqrt{6}}{3} = \pm \frac{\sqrt{2}}{2} \quad \therefore y = 2 \pm \frac{\sqrt{2}}{2}$$

$$\therefore \text{Foci: } \left(-1, 2 \pm \frac{\sqrt{2}}{2}\right)$$

$$\text{Length of major axis: } 2b = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \text{ units}$$

$$\text{Length of minor axis: } 2a = 2 \times \frac{1}{2} = 1 \text{ unit}$$

iii) Left as an exercise. **Ans:** centre $(2, 0)$, Foci $(2, \pm 2\sqrt{3})$, $e = \sqrt{3}/2$

iv) Left as an exercise. **Ans:** centre $(5, 1)$, Foci $(5 \pm 2\sqrt{3}, 1)$, $e = \sqrt{3}/2$

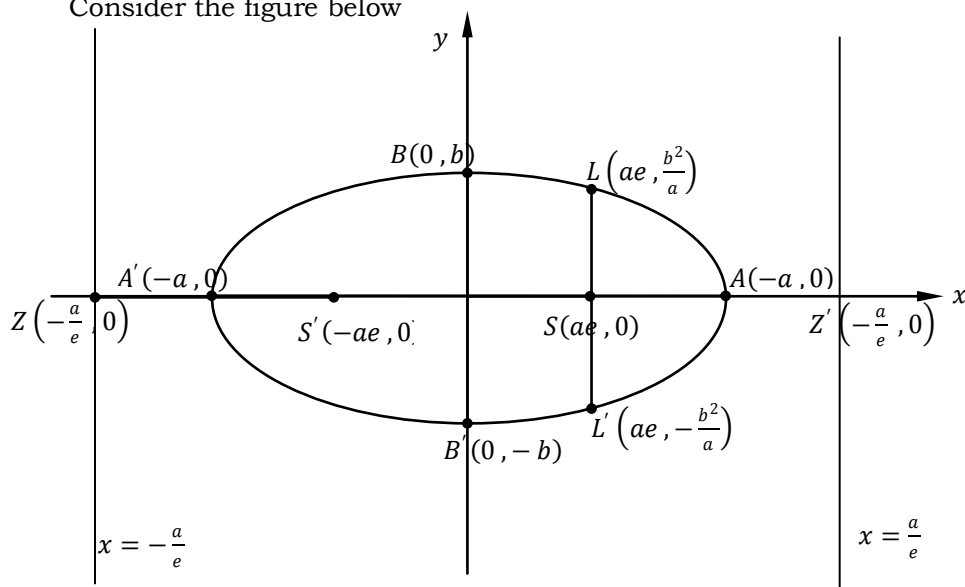
A CHORD, FOCAL CHORD AND LATUS RECTUM OF $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Chord: is any segment whose endpoints are on the ellipse

Focal Chord: is a chord containing the focus

Latus Rectum: the focal chord which is perpendicular to the transverse axis (Major axis)

Consider the figure below



The line LSL' is latus rectum (latus recta in plural).

Finding the coordinates of L' and L

The x -coordinate of L' and L is $x = +ae$

$$\frac{(ae)^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{a^2e^2}{a^2} = 1 - e^2$$

$$\text{But from } b^2 = a^2(1 - e^2) \Rightarrow 1 - e^2 = \frac{b^2}{a^2}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{b^2}{a^2} \quad \therefore y = \pm \frac{b^2}{a}$$

$$\therefore L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right)$$

Length of latus rectum is $\frac{2b^2}{a}$

Example

Find the equation of the ellipse in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ given that:

- i) Distance between the foci is 6 and distance between the Directrices is 24
- ii) Latus rectum is $10\frac{2}{3}$ and distance between the Directrices is 36

Solution

- i) Foci: $(\pm ae, 0)$ $\therefore$ Distance = $2ae = 6$
 $\therefore ae = 3$ (1)
 Distance between Directrices = $\frac{2a}{e} = 24$ $\therefore a = 12e$ (2)
 Eqn. (1) into eqn. (2)
 $\Rightarrow 12e^2 = 3$ $\therefore e = +\frac{1}{2}$
 From eqn. (2), $a = 12 \times \frac{1}{2} = 6 \Rightarrow a^2 = 36$
 Using $b^2 = a^2(1 - e^2)$
 $\Rightarrow b^2 = 36(1 - (1/2)^2)$ $\therefore b^2 = 27$
 $\therefore$ Required ellipse is $\frac{x^2}{36} + \frac{y^2}{27} = 1$
- ii) Length of latus rectum is $\frac{2b^2}{a} = 10\frac{2}{3}$ $\therefore b^2 = \frac{16}{3}a$ (1)
 Distance between Directrices = $\frac{2a}{e} = 36$ $\therefore a = 18e$ (2)
 Using $b^2 = a^2(1 - e^2)$
 $\Rightarrow \frac{16}{3}a = a^2 \left(1 - \frac{a^2}{324}\right)$ or $1728 = 324a - a^3$
 $\Rightarrow a^3 - 324a + 1728 = 0$ $\therefore a = 14.2337, -20.2337, 6$
 Taking the most accurate value, the whole number, $a = 6$
 $\Rightarrow a^2 = 36$, $b^2 = \frac{16}{3} \times 36 = 32$
 $\therefore$ Required ellipse: $\frac{x^2}{36} + \frac{y^2}{32} = 1$

A LINE AND AN ELLIPSE

Just like in the parabola, the following questions must be well understood in mastering this sub topic.

f) To Find the Point of Intersection of the Line $y = mx + c$ and the

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solving the equations simultaneously

$$y = mx + c \dots\dots\dots (1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots\dots\dots (2)$$

Eqn. (1) into eqn. (2)

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

$$\Rightarrow (b^2 + a^2m^2)x^2 + (2mca^2)x + a^2c^2 - a^2b^2 = 0 \dots\dots\dots (3)$$

Solving this equation gives *two* values of x . Obtain the corresponding values from equation (1), and hence two points of intersection in the form (x, y)

NOTE

Points of intersection can be deduced from the roots of the equation above, which either can be

- Real and distinct
- Coincident/Repeated or
- Imaginary

, depending on the nature of the discriminant $B^2 - 4AC$

For real and distinct roots, $B^2 - 4AC > 0$

For Repeated roots, $B^2 - 4AC = 0$

For Imaginary roots, $B^2 - 4AC < 0$

g) To Show that a given Line $y = mx + c$ is a Tangent to the Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solving the equations of the line and the ellipse simultaneously, obtain eqn. (3) above as

$$(b^2 + a^2m^2)x^2 + (2mca^2)x + (a^2c^2 - a^2b^2) = 0 \dots\dots\dots (3)$$

Proceed in two ways

Approach I:

Applying condition for tangency .i.e. condition for equal roots of a quadratic equation

For tangency, $B^2 = 4AC$

$$\Rightarrow (2mca^2)^2 = 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2)$$

$$\Rightarrow 4m^2c^2a^4 = 4a^2(b^2 + a^2m^2)(c^2 - b^2) \dots\dots\dots (\div 4a^2)$$

$$\Rightarrow m^2c^2a^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$$

$$\Rightarrow b^2c^2 - b^4 - a^2m^2b^2 = 0 \dots\dots\dots (\div b^2)$$

$$\Rightarrow c^2 - b^2 - a^2m^2 = 0 \quad \therefore c^2 = b^2 + a^2m^2$$

$$c^2 = a^2m^2 + b^2 \text{ or } c = \pm\sqrt{a^2m^2 + b^2}$$

, which is the condition for a line $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Approach II

Solving eqn. (3), two repeated/coincident values of x must be obtained. Eqn. (3) can have coincident roots if the condition for tangency is used.

Put $c = \pm\sqrt{a^2m^2 + b^2}$ in to eqn. (3)

$$(b^2 + a^2m^2)x^2 \pm (2ma^2\sqrt{a^2m^2 + b^2})x + [a^2(a^2m^2 + b^2) - a^2b^2] = 0$$

$$(b^2 + a^2m^2)x^2 \pm (2ma^2\sqrt{a^2m^2 + b^2})x + (a^4m^2 + a^2b^2 - a^2b^2) = 0$$

$$(b^2 + a^2m^2)x^2 \pm (2ma^2\sqrt{a^2m^2 + b^2})x + a^4m^2 = 0 \dots\dots\dots (4)$$

$$\Rightarrow (x\sqrt{a^2m^2 + b^2} \pm ma^2)^2 = 0 \quad \therefore x = \pm \frac{ma^2}{\sqrt{a^2m^2 + b^2}}, \text{ which will be a repeated}$$

value. Hence the line is a tangent to the ellipse.

- h) **To obtain the Point of Contact of a tangent line $y = mx + c$ to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Proceeding as in approach II above, from eqn. (4), $x = \pm \frac{ma^2}{\sqrt{a^2m^2 + b^2}}$, obtain the corresponding y - value from the line.

$$\text{Taking } x = + \frac{ma^2}{\sqrt{a^2m^2 + b^2}}, \quad c = -\sqrt{a^2m^2 + b^2}$$

$$\Rightarrow y = m \times \frac{ma^2}{\sqrt{a^2m^2 + b^2}} - \sqrt{a^2m^2 + b^2} = \frac{a^2m^2 - a^2m^2 - b^2}{\sqrt{a^2m^2 + b^2}} = -\frac{b^2}{\sqrt{a^2m^2 + b^2}}$$

$$\therefore \text{Point of contact is } P \left(\frac{ma^2}{\sqrt{a^2m^2 + b^2}}, -\frac{b^2}{\sqrt{a^2m^2 + b^2}} \right)$$

$$\text{Similarly, taking } x = -\frac{ma^2}{\sqrt{a^2m^2 + b^2}}, \quad c = +\sqrt{a^2m^2 + b^2}$$

$$\Rightarrow y = -m \times \frac{ma^2}{\sqrt{a^2m^2 + b^2}} + \sqrt{a^2m^2 + b^2} = \frac{-a^2m^2 + a^2m^2 + b^2}{\sqrt{a^2m^2 + b^2}} = \frac{b^2}{\sqrt{a^2m^2 + b^2}}$$

$$\therefore \text{Point of contact is } P' \left(-\frac{ma^2}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}} \right)$$

- i) **To state the Equation of the Tangent to the parabola in Standard Form**

From the line $y = mx + c$ and condition for tangency $c = \pm\sqrt{a^2m^2 + b^2}$, the tangent in slope form is $y = mx \pm \sqrt{a^2m^2 + b^2}$

NOTE

In answering questions (b) and (c), you must first obtain eqn. (3) in question (a) above, the approach to finding point of intersection of a line and a parabola.

Example

1. Find the point of intersection of the line and the ellipse

Line: $y = x$ and the ellipse $4x^2 + 5y^2 = 49$

Solution

Solving the equations simultaneously

$$4x^2 + 5y^2 = 49, y = x$$

$$\Rightarrow 4x^2 + 5x^2 = 49 \quad \therefore x = \pm 7$$

From the equation of the line, $y = x = \pm 7$

$\therefore (7, 7), (-7, -7)$ are the points of intersection

2. Prove that $y = 2x + 3$ touches the ellipse $\frac{x^2}{2} + y^2 = 1$ and that the point of contact is $\left(-\frac{4}{3}, \frac{1}{3}\right)$

Solution

$$y = 2x + 3 \dots\dots\dots (1)$$

$$\frac{x^2}{2} + y^2 = 1 \Rightarrow x^2 + 2y^2 = 2 \dots\dots\dots (2)$$

Solving the equations simultaneously

Eqn. (1) into eqn. (2)

$$\Rightarrow x^2 + 2(2x + 3)^2 = 2$$

$$\Rightarrow x^2 + 2(4x^2 + 12x + 9) = 2$$

$$\Rightarrow 9x^2 + 24x + 16 = 0 \text{ or } (3x + 4)^2 = 0$$

$\therefore x = -\frac{4}{3}, -\frac{4}{3}$, which is a repeated value. Since there is only one value of

$x = -\frac{4}{3}$, then there is only one point of contact. Hence the line is a tangent to the ellipse.

Point of contact

For $x = -\frac{4}{3}, y = ?$

$$\text{From eqn. (1), } y = 2 \times -\frac{4}{3} + 3 = \frac{1}{3}$$

$\therefore$ Point of contact: $\left(-\frac{4}{3}, \frac{1}{3}\right)$

Task

- 1) Prove that $y = 2x + 5$ touches $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ and find the point of contact.

Ans: $\left(-\frac{8}{5}, \frac{9}{5}\right)$

- 2) Prove that $y = x - 5$ touches $\frac{1}{16}x^2 + \frac{1}{9}y^2 = 1$ and find the point of contact.

Ans: $\left(\frac{16}{5}, -\frac{9}{5}\right)$

- 3) Show that the line $x - 2y + 10 = 0$ is a tangent to the ellipse $\frac{x^2}{64} + \frac{y^2}{9} = 1$.

Hence deduce the point of contact. (Uneb 2000) **Ans:** $\left(-\frac{32}{4}, \frac{9}{5}\right)$

Other questions involving Tangents and Ellipse

3. Find the equations of tangents to an ellipse;

- a) $3x^2 + 4y^2 = 12$, which are parallel to the line $x - 2y + 1 = 0$.
Hence find the points of contact of the tangents to the ellipse.
- b) $\frac{x^2}{16} + y^2 = 1$, which make an angle of 60° with x - axis

Solution

- a) The equation of tangent line parallel to $x - 2y + 1 = 0$ is given as:
 $x - 2y + k = 0$ or $x = 2y - k$ (1)
 For point of intersection of the tangent line and ellipse, solve the two equations simultaneously
 Putting eqn. (1) into $3x^2 + 4y^2 = 12$
 $\Rightarrow 3(2y - k)^2 + 4y^2 = 12$
 $\Rightarrow 3(4y^2 - 4ky + k^2) + 4y^2 = 12$ or $16y^2 - 12ky + (3k^2 - 12) = 0$ (2)
 For tangency, $B^2 = 4AC$
 $\Rightarrow (-12k)^2 = 4(16)(3k^2 - 12)$
 $\Rightarrow 144k^2 = 192k^2 - 768 \quad \therefore k = \pm 4$
 $\therefore$ Equations of tangents are: $x - 2y \pm 4 = 0$
Point of contact
 Putting $k = \pm 4$ into eqn. (2)
 $16y^2 - 12k y + (3k^2 - 12) = 0$, $k = +4$
 $\Rightarrow 16y^2 - 48y + (48 - 12) = 0$ or $\Rightarrow 16y^2 - 48y + 36 = 0 \quad \therefore y = \frac{3}{2}$
 From eqn. (1), for $y = \frac{3}{2}$, $x = 2 \times \frac{3}{2} - 4 = -1$
 $\therefore$ Point of contact: $\left(-1, \frac{3}{2}\right)$
 $16y^2 - 12k y + (3k^2 - 12) = 0$, $k = -4$
 $\Rightarrow 16y^2 + 48y + (48 - 12) = 0$ or $\Rightarrow 16y^2 + 48y + 36 = 0 \quad \therefore y = -\frac{3}{2}$
 From eqn. (1), for $y = -\frac{3}{2}$, $x = 2 \times -\frac{3}{2} + 4 = 1$
 $\therefore$ Point of contact: $\left(1, -\frac{3}{2}\right)$
- b) Let the equation of the tangent be: $y = mx + c$
 Given: $m = \tan 60^\circ = \sqrt{3}$
 $y = \sqrt{3}x + c$ (1)
 Solving the equation of tangent and ellipse simultaneously
 $\frac{x^2}{16} + y^2 = 1$ (2)
 Eqn. (1) into eqn. (2)
 $\frac{x^2}{16} + (\sqrt{3}x + c)^2 = 1$ or $x^2 + 16(3x^2 + 2\sqrt{3}x + c^2) - 16 = 0$
 $49x^2 + (32\sqrt{3}c)x + 16(c^2 - 1)$
 For tangency, $B^2 = 4AC$
 $\Rightarrow (32\sqrt{3}c)^2 = 4(49)(16)(c^2 - 1)$
 $\Rightarrow 3072c^2 = 3136c^2 - 3136$ or $64c^2 = 3136$

$$\Rightarrow c^2 = 49 \quad \therefore c = \pm 7$$

$$\therefore \text{Equations of tangents are: } y = \sqrt{3}x \pm 7$$

Task

Find the equations of tangents to the ellipse $4x^2 + 9y^2 = 8$ which are parallel to $2x - 3y = 0$ **Ans:** $3y = 2x \pm 4$

4. Show that if the line $y = mx + c$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the $c^2 = a^2m^2 + b^2$. Hence find the equation of tangents to the ellipse;
- $4x^2 + 9y^2 = 1$ which are perpendicular to $y = 2x + 3$
 - $\frac{x^2}{64} + \frac{y^2}{9} = 1$, that pass through $(-2, 4)$
 - $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which are inclined to x - axis at an angle $\sin^{-1}\left(\frac{2}{5}\sqrt{5}\right)$
 - $\frac{x^2}{3} + \frac{y^2}{4} = 1$, that pass through $(0, \sqrt{31})$
 - $x^2 + 2y^2 = 8$ which are parallel to $y = 2x$

Solution

The proof of a line being a tangent to the ellipse is left as an exercise.

- a) Required tangent: $y = mx \pm \sqrt{a^2m^2 + b^2}$
 Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $4x^2 + 9y^2 = 1$ or $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$
 $a^2 = \frac{1}{4}$, $b^2 = \frac{1}{9}$
 $\Rightarrow y = mx \pm \sqrt{\frac{m^2}{4} + \frac{1}{9}}$, m = slope of the tangent which must be $-\frac{1}{\text{Slope of the given line}}$ $\therefore m = -\frac{1}{2}$
 $\therefore y = -\frac{1}{2}x \pm \sqrt{\frac{1}{16} + \frac{1}{9}}$ or $y = -\frac{1}{2}x \pm \frac{5}{12}$
- b) Required tangent: $y = mx \pm \sqrt{a^2m^2 + b^2}$
 Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $\frac{x^2}{64} + \frac{y^2}{9} = 1$
 $a^2 = 64$, $b^2 = 9$
 $\Rightarrow y = mx \pm \sqrt{64m^2 + 9}$, m = slope of the tangent which must be obtained
 Point $(-2, 4)$ lies on this line
 $\Rightarrow 4 = -2m \pm \sqrt{64m^2 + 9}$
 $\Rightarrow (4 + 2m) = \pm \sqrt{64m^2 + 9}$, squaring both sides
 $\Rightarrow 16 + 16m + 4m^2 = 64m^2 + 9$ or $60m^2 - 16m - 7 = 0 \quad \therefore m = \frac{1}{2}, -\frac{7}{30}$
 $\therefore$ Tangents are obtained from: $y = mx \pm \sqrt{64m^2 + 9}$
 For $m = \frac{1}{2}$, $y = \frac{1}{2}x \pm 5$
 For $m = -\frac{7}{30}$, $y = -\frac{7}{30}x \pm \frac{53}{15}$
- c) Left as an exercise. **Ans:** $y = 2x \pm 5$
- d) Left as an exercise. **Ans:** $y = 3x - \sqrt{31}$, $y = -3x + \sqrt{31}$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON AN ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

To Find an Equation of a Tangent at any point $P(x_1, y_1)$ on the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Required tangent: $y - y_1 = m(x - x_1)$ (1)

Finding slope of the Tangent, m

From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, differentiating w.r.t x

$$\left. \frac{dy}{dx} \right|_{\text{at } P(x_1, y_1)} = m$$

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0, \text{ multiplying through by } \frac{a^2 b^2}{2}$$

$$\Rightarrow b^2 x + a^2 y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

$$\therefore \left. \frac{dy}{dx} \right|_{\text{at } P(x_1, y_1)} = m = -\frac{b^2 x_1}{a^2 y_1} \dots\dots\dots (2)$$

Eqn. (2) into eqn. (1)

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1} (x - x_1)$$

$$a^2 y_1 y - a^2 y_1^2 = -b^2 x_1 x + b^2 x_1^2$$

$$\Rightarrow a^2 y_1 y + b^2 x_1 x = a^2 y_1^2 + b^2 x_1^2 \dots\dots\dots (\div a^2 b^2)$$

$$\Rightarrow \frac{x x_1}{a^2} + \frac{y y_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

Since point $P(x_1, y_1)$ lies on the ellipse, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$

, which is the equation of the tangent at $P(x_1, y_1)$

To Find an Equation of the Normal at any point $P(x_1, y_1)$ on the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Required tangent: $y - y_1 = m'(x - x_1)$ (1)

Finding slope of the Normal, m'

$$\text{Slope of tangent, } m = -\frac{b^2 x_1}{a^2 y_1}$$

$$\therefore \text{Slope of Normal, } m' = \frac{a^2 y_1}{b^2 x_1} \quad [\because m \times m' = -1]$$

$$\Rightarrow y - y_1 = \frac{a^2 y_1}{b^2 x_1} (x - x_1)$$

$\frac{y - y_1}{\frac{y_1}{b^2}} = \frac{y - x_1}{\frac{x_1}{b^2}} \quad \text{or} \quad \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

Example

Find the equation of tangent and normal to an ellipse at the given point;

i) $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at $(0, 4)$

ii) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at $(3, 0)$

iii) $4x^2 + 5y^2 = 120$ at $(-5, -2)$

iv) $9x^2 + 16y^2 = 40$ at $(2, \frac{1}{2})$

Solution

- i) Point $(0, 4)$ lies on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Required Tangent

$$y - 4 = m(x - 0) = mx \text{ or } y = mx + 4$$

Finding the slope, m

$$m = \left. \frac{dy}{dx} \right|_{\text{at } (0, 4)}$$

Differentiating the curve w.r.t x

$$\frac{2x}{9} + \frac{2y}{16} \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{16x}{9y}$$

$$\Rightarrow m = \left. \frac{dy}{dx} \right|_{\text{at } (0, 4)} = -\frac{0}{72} = 0$$

$$\therefore y = 4$$

Required equation of the normal

$$y - 4 = m'(x - 0) = m'x \text{ or } y = m'x + 4$$

Finding the slope, m'

$$m' = -\frac{1}{\text{Slope of the tangent at } (0, 4)} = -\frac{1}{m} = -\frac{1}{0}$$

Rewriting the equation of the normal as:

$$\frac{y-4}{(x-0)} = m' = -\frac{1}{0}$$

$$\Rightarrow x - 0 = 0 \quad \therefore x = 0$$

- ii) Left as an exercise. **Ans:** Tangent: $x = 3$, Normal: $y = 0$

- iii) $x^2 + 5y^2 = 120$, $(-5, -2)$

Required Tangent

$$y + 2 = m(x + 5)$$

Finding the slope, m

$$m = \left. \frac{dy}{dx} \right|_{\text{at } (-5, -2)}$$

Differentiating the curve w.r.t x

$$2x + 10y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{4x}{5y}$$

$$\Rightarrow m = \left. \frac{dy}{dx} \right|_{\text{at } (-5, -2)} = \frac{-4 \times -5}{5 \times -2} = -2$$

$$\therefore y + 2 = -2(x + 5) \text{ or } 2x + y + 12 = 0$$

Required equation of the normal

$$y + 2 = m'(x + 5)$$

Finding the slope, m'

$$m' = -\frac{1}{\text{Slope of the tangent at } (0, 4)} = -\frac{1}{m} = -\frac{1}{-2} = \frac{1}{2}$$

Rewriting the equation of the normal as:

$$y + 2 = \frac{1}{2}(x + 5)$$

$$\Rightarrow 2y + 4 = x + 5 \quad \therefore x - 2y + 1 = 0$$

iv) Left as an exercise. **Ans:** Tangent: $9x + 4y = 20$, Normal: $8x - 18y - 7 = 0$

Task

a) Show that the line $5y = 4x + 25$ is a tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

b) Find the equation of the normal to the ellipse at the point of contact

Determine the eccentricity of the ellipse. (**Unneb 2004**) **Ans:** $\left(-4, \frac{9}{5}\right)$, $e = \frac{4}{5}$

To Find the Equation of a Tangent to the Ellipse from a Point

$P(x_1, y_1)$ not on the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

1. Find the equation of tangent(s) to the ellipse:

a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ passing through the point $\left(-8, -\frac{7}{5}\right)$

b) $\frac{x^2}{64} + \frac{y^2}{9} = 1$ from the point $(10, 15)$

Solution

To attempt this question, two approaches can be used.

Approach I:

Let the equation of the tangent be:

$$y = mx + c \dots\dots\dots (1)$$

The task is to find the values of m and c

Obtaining c

The point $\left(-8, -\frac{7}{5}\right)$ lies on the tangent

$$\Rightarrow -\frac{7}{5} = -8m + c \quad \therefore c = 8m - \frac{7}{5} \dots\dots\dots (1)$$

Obtaining the slope, m

Solve the equation of the tangent and the ellipse simultaneously

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \dots\dots\dots (2)$$

Eqn. (1) into eqn. (2)

$$\frac{x^2}{25} + \frac{(mx+c)^2}{9} = 1 \dots\dots\dots (\times 225)$$

$$9x^2 + 25(m^2x^2 + 2mcx + c^2) = 225$$

$$(9 + 25m^2)x^2 + 50mcx + (25c^2 - 225) = 0$$

Applying the condition for repeated roots $B^2 = 4AC$

$$(50mc)^2 = 4(9 + 25m^2)(25c^2 - 225) = (36 + 100m^2)(25c^2 - 225)$$

$$2500m^2c^2 = 900c^2 - 8100 + 2500m^2c^2 - 22500m^2$$

$$0 = 900c^2 - 8100 - 22500m^2 \dots\dots\dots (\div 900)$$

$$c^2 - 9 - 25m^2 = 0 \dots\dots\dots (3)$$

Eqn. (1) into eqn. (3)

$$\left(8m - \frac{7}{5}\right)^2 - 9 - 25m^2 = 0 \text{ or } (40m - 7)^2 - 225 - 625m^2 = 0$$

$$1600m^2 - 560m + 49 - 225 - 625m^2 = 0$$

$$975m^2 - 560m - 176 = 0 \quad \therefore m = \frac{4}{5}, -\frac{44}{195}$$

From eqn. (1),

For $m = \frac{4}{5}$, $c = 8 \times \frac{4}{5} - \frac{7}{5} = 5 \quad \therefore y = \frac{4}{5}x + 5$

For $m = -\frac{44}{195}$, $c = 8 \times -\frac{44}{195} - \frac{7}{5} = -\frac{125}{39} \quad \therefore y = -\frac{44}{195}x - \frac{125}{39}$

Approach II:

Using the general tangent in slope form to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let the required tangent be:

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $\frac{x^2}{25} + \frac{y^2}{9} = 1 \Rightarrow a^2 = 25, b^2 = 9$

Tangent equation becomes:

$$y = mx \pm \sqrt{25m^2 + 9} \dots\dots\dots (1)$$

Point $(-8, -\frac{7}{5})$ lies on this tangent

$$\Rightarrow -\frac{7}{5} = -8m \pm \sqrt{25m^2 + 9}$$

$$\Rightarrow 40m - 7 = \pm 5\sqrt{25m^2 + 9}, \text{ squaring both sides}$$

$$\Rightarrow 1600m^2 - 560m + 49 = 25(25m^2 + 9)$$

$$\Rightarrow 1600m^2 - 560m + 49 = 625m^2 + 225$$

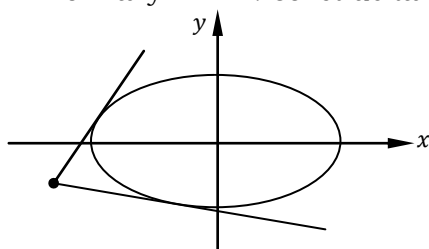
$$\Rightarrow 975m^2 - 560m - 176 = 0 \quad \therefore m = \frac{4}{5}, -\frac{44}{195} \dots\dots\dots (2)$$

Eqn. (2) into eqn. (1)

$$\text{For } m = \frac{4}{5}, y = \frac{4}{5}x \pm \sqrt{25\left(\frac{4}{5}\right)^2 + 9} \Rightarrow y = \frac{4}{5}x \pm 5$$

$$\text{For } m = -\frac{44}{195}, y = -\frac{44}{195}x \pm \sqrt{25\left(-\frac{44}{195}\right)^2 + 9} \Rightarrow y = -\frac{44}{195}x \pm \frac{125}{39}$$

It appears that we have four solutions. From a point, two tangents are normally drawn. So let us take the only two tangents required.



The ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ has its major axis as the x -axis. Plot of the point suggests that two tangents can be drawn to the ellipse.

A tangent with $m = \frac{4}{5}$ has a $+ve$ y -intercept and a tangent with $m = -\frac{44}{195}$ has a $-ve$ y -intercept.

Thus the required tangents are: $\therefore y = \frac{4}{5}x + 5, y = -\frac{44}{195}x - \frac{125}{39}$

The other tangents do not pass through the point $(-8, -\frac{7}{5})$

2. The line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- a) Obtain an expression for c in terms of a, b and m
- b) Calculate the gradients of the tangents to the ellipse through the point $(\sqrt{a^2 + b^2}, 0)$ (**Uneb 2018**)

Solution

- a) Solving the equation of the line and ellipse simultaneously, and then applying the condition for repeated roots of a quadratic equation

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1 \dots\dots\dots (\times a^2 b^2)$$

$$a^2(m^2 x^2 + 2mcx + c^2) + b^2 x^2 - a^2 b^2 = 0$$

$$(a^2 m^2 + b^2)x^2 + 2mca^2 x + (a^2 c^2 - a^2 b^2) = 0$$

For tangency, $B^2 = 4AC$

$$(2mca^2)^2 = 4(a^2 m^2 + b^2)(a^2 c^2 - a^2 b^2)$$

$$4m^2 c^2 a^4 = 4a^2 (b^2 + a^2 m^2)(c^2 - b^2) \dots\dots\dots (\div 4a^2)$$

$$m^2 c^2 a^2 = b^2 c^2 - b^4 + a^2 m^2 c^2 - a^2 m^2 b^2$$

$$b^2 c^2 - b^4 - a^2 m^2 b^2 = 0 \dots\dots\dots (\div b^2)$$

$$\Rightarrow c^2 - b^2 - a^2 m^2 = 0$$

$$\therefore c^2 = b^2 + a^2 m^2$$

- b) The point $(\sqrt{a^2 + b^2}, 0)$ does not lie on the ellipse. It is regarded to be a point not on the ellipse.

Tangent in slope form is:

$$y = mx \pm \sqrt{b^2 + a^2 m^2}$$

The point $(\sqrt{a^2 + b^2}, 0)$ lies on this tangent

$$\Rightarrow 0 = m\sqrt{a^2 + b^2} \pm \sqrt{b^2 + a^2 m^2}$$

$$\Rightarrow -m\sqrt{a^2 + b^2} = \pm \sqrt{b^2 + a^2 m^2}, \text{ squaring both sides}$$

$$\Rightarrow m^2(a^2 + b^2) = b^2 + a^2 m^2$$

$$\Rightarrow m^2(a^2 + b^2 - a^2) = b^2 \text{ or } m^2 = 1 \therefore m = \pm 1$$

$\therefore$ Gradients of the tangents are: ± 1

Task

1. Show that the x – coordinates of any point of intersection of the line $y = mx + c$ and the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ are given by the solutions of the quadratic equation $(4 + 9m^2)x^2 + 18mcx + (9c^2 - 36) = 0$
If the line $y = mx + c$ is a tangent of the ellipse, prove that $c^2 = 4 + 9m^2$.
The line $y = mx + c$ passes through the point $(2, 3)$ write down a second equation connecting m and c , hence prove that m must satisfy the equation $5m^2 + 12m - 5 = 0$.
Prove that the two tangents drawn from the point $(2, 3)$ to the ellipse are perpendicular to each other.
2. Find the equation of the tangent and normal to $ax^2 + by^2 = c$ at the point (x_1, y_1) . **Ans:** $axx_1 + byy_1 = c$; $bxy_1 - ax_1y = (b - a)x_1y_1$

PARAMETRIC COORDINATES

The general coordinates on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $(a \cos \theta, b \sin \theta)$ or generally denoted as θ .

The parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are:

$$x = a \cos \theta, y = b \sin \theta$$

To Find the Equation of a Tangent and Normal to an Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a \cos \theta, b \sin \theta)$$

Tangent Equation

$$y - b \sin \theta = m_1(x - a \cos \theta) \dots\dots\dots (1)$$

Obtaining slope, m_1

Method I:

$$m_1 = \left. \frac{dy}{dx} \right|_{\text{at } (a \cos \theta, b \sin \theta)}$$

Differentiating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ w.r.t x

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

$$\Rightarrow m_1 = \left. \frac{dy}{dx} \right|_{\text{at } (a \cos \theta, b \sin \theta)} = -\frac{b^2 a \cos \theta}{a^2 b \sin \theta} = -\frac{b \cos \theta}{a \sin \theta}$$

$$\therefore m_1 = -\frac{b \cos \theta}{a \sin \theta}$$

Method II:

Formulating parametric equations

$$x = a \cos \theta, y = b \sin \theta$$

$$m_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dx}{d\theta} = -a \sin \theta, \frac{dy}{d\theta} = b \cos \theta$$

$$\therefore m_1 = b \cos \theta \times \frac{1}{-a \sin \theta} = -\frac{b \cos \theta}{a \sin \theta}$$

Tangent equation becomes:

$$y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta} (x - a \cos \theta)$$

$$\Rightarrow ya \sin \theta - ab \sin^2 \theta = -b \cos \theta x + ab \cos^2 \theta$$

$$\Rightarrow ya \sin \theta + b \cos \theta x = ab(\cos^2 \theta + \sin^2 \theta)$$

$$\therefore ay \sin \theta + bx \cos \theta = ab \quad [\because \cos^2 \theta + \sin^2 \theta = 1] \dots\dots\dots (\div ab)$$

$$\therefore \frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

Equation of the Normal

$$y - b \sin \theta = m_2(x - a \cos \theta) \dots\dots\dots (2)$$

Obtaining slope, m_2

$$m_2 = -\frac{1}{\text{Slope of the tangent at } (a \cos \theta, b \sin \theta)} = -\frac{1}{m_1} = \frac{a \sin \theta}{b \cos \theta}$$

$$y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta)$$

$$\begin{aligned}
 yb \cos \theta - b^2 \cos \theta \sin \theta &= a \sin \theta x - a^2 \cos \theta \sin \theta \\
 \therefore a \sin \theta x - by \cos \theta &= (a^2 - b^2) \cos \theta \sin \theta \quad (\div \cos \theta \sin \theta) \\
 \Rightarrow a \frac{\sin \theta}{\cos \theta \sin \theta} x - by \frac{\cos \theta}{\cos \theta \sin \theta} &= a^2 - b^2 \\
 \therefore ax \sec \theta - by \operatorname{cosec} \theta &= a^2 - b^2
 \end{aligned}$$

To show that the parametric equations $x = a \cos \theta$, $y = b \sin \theta$ represent an equation of an ellipse

$$x = a \cos \theta \Rightarrow \cos \theta = \frac{x}{a}$$

$$y = b \sin \theta \Rightarrow \sin \theta = \frac{y}{b}$$

Using the trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$ to eliminate parameter θ

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ which is an ellipse.}$$

General Examples

1. a) A conic section is given by $x = 4 \cos \theta$ and $y = 3 \sin \theta$. Show that the conic is an ellipse. Determine its eccentricity
- b) Given that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $c^2 = a^2 m^2 + b^2$. Hence determine the equations of the tangents to at the point $(-3, 3)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Solution

$$\text{a) } x = 4 \cos \theta \Rightarrow \cos \theta = \frac{x}{4}$$

$$y = 3 \sin \theta \Rightarrow \sin \theta = \frac{y}{3}$$

From $\cos^2 \theta + \sin^2 \theta = 1$

$$\Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \quad \therefore \frac{x^2}{16} + \frac{y^2}{9} = 1, \text{ which is an ellipse.}$$

Eccentricity

$$\text{Using } b^2 = a^2(1 - e^2)$$

$$\Rightarrow 9 = 16(1 - e^2)$$

$$\Rightarrow e^2 = +\frac{\sqrt{7}}{4}$$

- b) Left as an exercise. See notes and the equation of tangents to the ellipse.

Ans: $y = -3, 7y = 18x - 25$

2. a) Find the equation of the tangent and normal to the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ at the point $P(2 \cos \theta, \sin \theta)$
- b) if the tangent in (a) cuts the y - axis at point A and the x - axis at point B, and the normal cuts the x - axis at point C, find the coordinates of A, B and C.

Solution

- a) **Tangent Equation**

$$y - \sin \theta = m_1(x - 2 \cos \theta) \dots\dots\dots (1)$$

Obtaining slope, m_1

$$m_1 = \left. \frac{dy}{dx} \right|_{\text{at } (2 \cos \theta, \sin \theta)}$$

Differentiating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ w.r.t x

$$\frac{2x}{a^2} + 2y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{x}{4y}$$

$$\Rightarrow m_1 = \left. \frac{dy}{dx} \right|_{\text{at } (2 \cos \theta, \sin \theta)} = -\frac{2 \cos \theta}{4 \sin \theta} = -\frac{\cos \theta}{2 \sin \theta}$$

$$\therefore m_1 = -\frac{\cos \theta}{2 \sin \theta}$$

Tangent equation becomes:

$$y - \sin \theta = -\frac{\cos \theta}{2 \sin \theta} (x - 2 \cos \theta)$$

$$\Rightarrow 2y \sin \theta - \sin^2 \theta = -b \cos \theta x + 2 \cos^2 \theta$$

$$\Rightarrow 2y \sin \theta + x \cos \theta = 2(\cos^2 \theta + \sin^2 \theta)$$

$$\therefore 2y \sin \theta + x \cos \theta = 2 \quad [\because \cos^2 \theta + \sin^2 \theta = 1]$$

Equation of the Normal

$$y - \sin \theta = m_2(x - 2 \cos \theta) \dots\dots\dots (2)$$

Obtaining slope, m_2

$$m_2 = -\frac{1}{\text{Slope of the tangent at } (2 \cos \theta, \sin \theta)} = -\frac{1}{m_1} = \frac{2 \sin \theta}{\cos \theta}$$

$$y - \sin \theta = \frac{2 \sin \theta}{\cos \theta} (x - 2 \cos \theta)$$

$$y \cos \theta - \cos \theta \sin \theta = 2 \sin \theta x - 4 \cos \theta \sin \theta$$

$$\therefore 2 \sin \theta x - y \cos \theta = 3 \cos \theta \sin \theta \dots\dots\dots (\div \cos \theta \sin \theta)$$

$$\Rightarrow a \frac{\sin \theta}{\cos \theta \sin \theta} x - b y \frac{\cos \theta}{\cos \theta \sin \theta} = a^2 - b^2$$

$$\therefore 2x \sec \theta - y \operatorname{cosec} \theta = 3$$

b) $2y \sin \theta + x \cos \theta = 2$, y - intercept $x = 0$

$$\Rightarrow 2y \sin \theta = 1 \quad \therefore y = \frac{1}{2 \sin \theta} = \frac{1}{2} \operatorname{cosec} \theta$$

$$\therefore A \left(0, \frac{1}{2} \operatorname{cosec} \theta \right)$$

$$x$$
 - Intercept, $y = 0$

$$\Rightarrow x \cos \theta = 1 \quad \therefore x = \sec \theta$$

$$\therefore B(\sec \theta, 0)$$

$$\text{Normal: } 2x \sec \theta - y \operatorname{cosec} \theta = 3$$

$$x$$
 - Intercept, $y = 0$

$$\Rightarrow 2x \sec \theta = 3 \quad \therefore x = \frac{3}{2 \sec \theta} = \frac{3 \cos \theta}{2}$$

$$\therefore C \left(\frac{3 \cos \theta}{2}, 0 \right)$$

3. The coordinates of a point $P(x, y)$ on the curve is given parametrically by the equations $x = a \cos \theta$, $y = b \sin \theta$ where a and b are constants and θ is a parameter. Find:

- The Cartesian equation of the curve and identify the curve
- The equation of the tangent to the curve at the point at which the parameter $\theta = \phi$

- iii) The relation between φ_1 and φ_2 if the tangents at the points $(a \cos \varphi_1, b \sin \varphi_1)$ and $(a \cos \varphi_2, b \sin \varphi_2)$ are at right angles to one another.

Solution

$$\text{i) } x = a \cos \theta \Rightarrow \cos \theta = \frac{x}{a}$$

$$y = b \sin \theta \Rightarrow \sin \theta = \frac{y}{b}$$

Using the trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$ to eliminate parameter θ

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ which is an ellipse.}$$

- ii) Obtaining the tangent at $(a \cos \varphi, b \sin \varphi)$ on to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, see notes. Try to obtain the equation as:

$$ay \sin \varphi + bx \cos \varphi = ab$$

- iii) For tangents to intersect at right angles, then $m_1 \times m_2 = -1$

$$m_1 = \left. \frac{dy}{dx} \right|_{\text{at } (a \cos \varphi_1, b \sin \varphi_1)} = -\frac{b \cos \varphi_1}{a \sin \varphi_1}$$

$$m_2 = \left. \frac{dy}{dx} \right|_{\text{at } (a \cos \varphi_2, b \sin \varphi_2)} = -\frac{b \cos \varphi_2}{a \sin \varphi_2}$$

$$\Rightarrow m_1 \times m_2 = -\frac{b \cos \varphi_1}{a \sin \varphi_1} \times -\frac{b \cos \varphi_2}{a \sin \varphi_2} = -1$$

$$\Rightarrow \frac{b^2}{a^2} \cdot \frac{\cos \varphi_1 \cos \varphi_2}{\sin \varphi_1 \sin \varphi_2} = -1 \text{ or } a^2 \sin \varphi_1 \sin \varphi_2 + b^2 \cos \varphi_1 \cos \varphi_2 = 0$$

4. Prove that the equation of the tangent at the point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

The tangent at the point $(2 \cos \theta, \sqrt{3} \sin \theta)$ on the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ passes through the point $P(2, 1)$. Show that $\sqrt{3} \cos \theta + \sin \theta = \sqrt{3}$

Without using tables/calculator, find all the solutions to this equation which are in the range $0^\circ \leq \theta \leq 360^\circ$. Hence obtain the coordinates of the point of contact Q and R , of the tangents to the ellipse from P .

Verify that the line through the origin and the point P passes through the midpoint of the line QR .

Solution

The student should derive the tangent equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ as in notes.

$$\text{Let } (x_1, y_1) = (2 \cos \theta, \sqrt{3} \sin \theta)$$

$$\Rightarrow \frac{x(2 \cos \theta)}{a^2} + \frac{y(\sqrt{3} \sin \theta)}{b^2} = 1, \text{ but } a^2 = 4, b^2 = 3$$

$$\therefore \frac{2x \cos \theta}{4} + \frac{y\sqrt{3} \sin \theta}{3} = 1 \text{ or } \therefore \frac{x}{2} \cos \theta + \frac{y\sqrt{3}}{3} \sin \theta = 1 \text{ is the tangent at}$$

$$(2 \cos \theta, \sqrt{3} \sin \theta)$$

Point $P(2, 1)$ lies on the tangent

$$\Rightarrow \frac{2}{2} \cos \theta + \frac{\sqrt{3}}{3} \sin \theta = 1 \text{ or } \cos \theta + \frac{\sqrt{3}}{3} \sin \theta = 1 \dots\dots\dots(\times \sqrt{3})$$

$$\therefore \sqrt{3} \cos \theta + \sin \theta = \sqrt{3}, \text{ as required.}$$

$$\begin{aligned} \text{Let } \sqrt{3} \cos \theta + \sin \theta &= R \sin(\theta + \alpha) \\ &= R \sin \theta \cos \alpha + R \cos \theta \sin \alpha \end{aligned}$$

Comparing

$$\Rightarrow \sqrt{3} = R \sin \alpha, 1 = R \cos \alpha$$

Dividing the equations

$$\sqrt{3} = \tan \alpha \quad \therefore \alpha = 60^\circ$$

Squaring and adding the equations

$$3 + 1 = R^2(\sin^2 \alpha + \cos^2 \alpha) \quad \therefore R = 2$$

$$\Rightarrow \sqrt{3} = 2 \sin(\theta + 60^\circ) \text{ or } \frac{\sqrt{3}}{2} = \sin(\theta + 60^\circ)$$

$$\Rightarrow \theta + 60^\circ = 60^\circ, 120^\circ, 420^\circ$$

$$\therefore \theta = 0^\circ, 60^\circ, 360^\circ$$

Taking $\theta = 0^\circ, 60^\circ$ and from the tangent equation $\frac{x}{2} \cos \theta + \frac{y\sqrt{3}}{3} \sin \theta = 1$

$$\text{For } \theta = 0^\circ, \quad \frac{x}{2} = 1 \quad \therefore x = 2$$

$$\text{For } \theta = 60^\circ, \quad \frac{x}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{3} y = 1 \quad \therefore x + 2y = 4$$

$$\text{From the ellipse } \frac{x^2}{4} + \frac{y^2}{3} = 1, \text{ for } x = 2, \frac{y^2}{3} = 0 \quad \therefore y = 0$$

$\therefore$ Point of contact is $(2, 0)$

From $x + 2y = 4$, $\frac{x^2}{4} + \frac{y^2}{3} = 1$, solving the equations simultaneously

$$x = 4 - 2y \text{ into eqn. } \frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\Rightarrow \frac{(4-2y)^2}{4} + \frac{y^2}{3} = 1, \text{ simplifying we get}$$

$$4y^2 - 12y + 9 = 0 \quad \therefore y = 3/2$$

$$\text{From } x + 2y = 4, y = \frac{3}{2}, x = 4 - 3 = 1$$

$\therefore$ Point of contact is $\left(1, \frac{3}{2}\right)$

To prove that $\overline{OP}$ passes through midpoint of $\overline{QR}$

$$O(0, 0), P(2, 1), Q\left(1, \frac{3}{2}\right), R(2, 0)$$

Equation of the line through $\overline{OP}$

$$\frac{y-0}{x-0} = \frac{1-0}{2-0} \quad \therefore y = \frac{1}{2}x$$

Midpoint of $\overline{QR} = \left(\frac{1+2}{2}, \frac{\frac{3}{2}+0}{2}\right) = \left(\frac{3}{2}, \frac{3}{4}\right)$, this point must satisfy the line

$$y = \frac{1}{2}x$$

For $x = \frac{3}{2}, y = \frac{3}{4} = y$ - coordinate. Hence proved as required.

Task

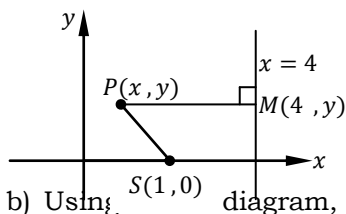
- Find the parametric coordinates of any point on $4x^2 + 9y^2 = 16$. Find also the equation of the tangent at this point.
Ans: $(2 \cos \theta, \frac{4}{3} \sin \theta)$, $2x \cos \theta + 3y \sin \theta = 4$
- Show that the equation of the tangent to $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ at $(2 \cos \theta, 3 \sin \theta)$ is $3x \cos \theta + 2y \sin \theta = 6$
- The normal at any point P to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes in A and B . prove AB is divided externally at P in a constant ratio.
- S and S_1 are foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. P is a point on the ellipse so that PS is parallel to the y -axis. The tangent at P meets the y -axis in M . prove MS_1 is parallel to the normal at P
- The tangent at any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the minor axis at Q and the tangent at one end of the major axis in R . if S is the focus prove $QR = QS$

LOCUS PROBLEMS IN ELLIPSE

- Find the locus of a point $P(x, y)$ such that twice its distance from $S(1, 0)$ is equal to its distance from the line $x = 4$. Identify the locus.
 - Find the locus of a point $P(x, y)$ such that $3PS = PM$ where $S(-3, 0)$ and M is a point on the line $x = -27$ such that PM is parallel to x -axis. Identify the locus.

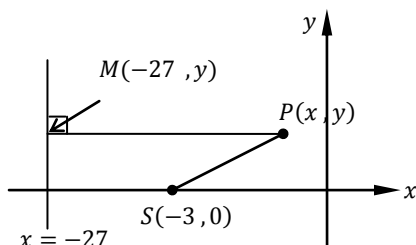
Solution

- a) Using a clear diagram



$$\begin{aligned}
 \text{Given: } 2\overline{PS} &= \overline{PM} \text{ or } 4\overline{PS}^2 = \overline{PM}^2 \\
 \Rightarrow 4[(x-1)^2 + (y-0)^2] &= (x-4)^2 + (y-y)^2 \\
 \Rightarrow 4x^2 - 8x + 4 + 4y^2 &= x^2 - 8x + 16 \\
 \Rightarrow 3x^2 + 4y^2 &= 12 \text{ or } \frac{x^2}{4} + \frac{y^2}{3} = 1, \text{ which is an ellipse}
 \end{aligned}$$

- b) Using diagram,



$$\begin{aligned}
 \text{Given: } 3\overline{PS} &= \overline{PM} \text{ or } 9\overline{PS}^2 = \overline{PM}^2 \\
 \Rightarrow 9[(x+3)^2 + (y-0)^2] &= (x+27)^2 + (y-y)^2 \\
 \Rightarrow 9x^2 + 54x + 81 + 9y^2 &= x^2 + 54x + 729 \\
 \Rightarrow 8x^2 + 9y^2 &= 648 \text{ or } \frac{x^2}{81} + \frac{y^2}{72} = 1, \text{ which is an ellipse}
 \end{aligned}$$

- The normal at $P(2 \cos \theta, \sin \theta)$ of an ellipse meets x -axis in Q and y -axis in R . A point S is taken on QR produced such that $QS = RQ$, prove that the locus is a circle whose equation is $x^2 + y^2 = 9$

Solution

$P(2 \cos \theta, \sin \theta)$ lies on ellipse

$$x = 2 \cos \theta \Rightarrow \cos \theta = \frac{x}{2}, y = \sin \theta$$

$$\text{Using } \cos^2 \theta + \sin^2 \theta = 1$$

The Cartesian equation of the curve is: $\frac{x^2}{4} + \frac{y^2}{1} = 1$, which is an ellipse

Obtaining the equation of the normal.

$$y - \sin \theta = m_1(x - 2 \cos \theta) \dots\dots\dots (1)$$

$$m_1 = - \frac{1}{\left. \frac{dy}{dx} \right|_{\text{at } P(2 \cos \theta, \sin \theta)}}$$

$$\text{By chain rule, } \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\text{For } x = 2 \cos \theta \Rightarrow \frac{dx}{d\theta} = -2 \sin \theta$$

$$\text{For } y = \sin \theta \Rightarrow \frac{dy}{d\theta} = \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \cos \theta \times \frac{1}{-2 \sin \theta}$$

$$\Rightarrow m_1 = - \frac{1}{\left. \frac{dy}{dx} \right|_{\text{at } P(2 \cos \theta, \sin \theta)}} = \frac{2 \sin \theta}{\cos \theta}$$

$$\Rightarrow y - \sin \theta = \frac{2 \sin \theta}{\cos \theta}(x - 2 \cos \theta)$$

$$\Rightarrow y \cos \theta - \sin \theta \cos \theta = 2x \sin \theta - 2 \sin \theta \cos \theta$$

$$\therefore 2x \sin \theta - y \cos \theta = 3 \sin \theta \cos \theta, \text{ which is the normal at P}$$

For x -intercept, $y = 0$

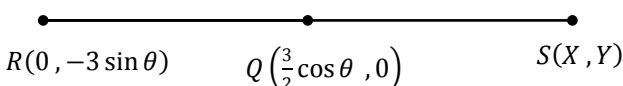
$$2x \sin \theta = 3 \sin \theta \cos \theta \Rightarrow x = \frac{3}{2} \cos \theta$$

$$\therefore Q\left(\frac{3}{2} \cos \theta, 0\right)$$

For y -intercept, $x = 0$

$$-y \cos \theta = 3 \sin \theta \cos \theta \Rightarrow y = -3 \sin \theta$$

$$\therefore R(0, -3 \sin \theta)$$



Q is the midpoint of SR

$$\Rightarrow \left(\frac{3}{2} \cos \theta, 0\right) = \left(\frac{X+0}{2}, \frac{-3 \sin \theta + Y}{2}\right)$$

$$\Rightarrow \frac{X}{2} = \frac{3}{2} \cos \theta \quad \therefore \cos \theta = \frac{X}{3} \dots\dots\dots (1)$$

$$\Rightarrow \frac{-3 \sin \theta + Y}{2} = 0 \quad \therefore \sin \theta = \frac{Y}{3} \dots\dots\dots (2)$$

Using $\cos^2 \theta + \sin^2 \theta = 1$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{9} = 1 \quad \therefore x^2 + y^2 = 9$$

3. The normal at the point $P(5 \cos \theta, 4 \sin \theta)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ meets the major and minor axes at L and M respectively. Show that the locus of R , the midpoint of LM is an ellipse having the same eccentricity as the given ellipse.

Solution

The equation of the normal *at the point* $P(5 \cos \theta, 4 \sin \theta)$ is:

$$5x \sec \theta - 4y \operatorname{cosec} \theta = 9 \dots\dots\dots (1)$$

Let the student look for this equation.

The major axis is the x -axis. So L is the x -intercept

$$\text{For } y = 0, 5x \sec \theta = 9 \quad \therefore x = \frac{9 \cos \theta}{5}$$

$$\therefore L \left(\frac{9 \cos \theta}{5}, 0 \right)$$

The minor axis is the y -axis. So M is the y -intercept

$$\text{For } x = 0, -4y \operatorname{cosec} \theta = 9 \quad \therefore y = -\frac{9 \sin \theta}{4}$$

$$\therefore M \left(0, -\frac{9 \sin \theta}{4} \right)$$

Let $R(X, Y)$ be the midpoint of LM

$$R(X, Y) = R \left(\frac{\frac{9 \cos \theta}{5} + 0}{2}, \frac{0 - \frac{9 \sin \theta}{4}}{2} \right)$$

$$\therefore X = \frac{9 \cos \theta}{10} \Rightarrow \cos \theta = \frac{10X}{9} \dots\dots\dots (2)$$

$$\therefore Y = -\frac{9 \sin \theta}{8} \Rightarrow \sin \theta = -\frac{8Y}{9} \dots\dots\dots (3)$$

Using $\cos^2 \theta + \sin^2 \theta = 1$

$$\left(\frac{10X}{9} \right)^2 + \left(-\frac{8Y}{9} \right)^2 = 1 \text{ or } \frac{X^2}{81/100} + \frac{Y^2}{81/64} = 1$$

$$\text{Required locus is: } \frac{x^2}{81/100} + \frac{y^2}{81/64} = 1$$

The locus is an ellipse with its major axis as the y -axis

$$\text{Calculating Eccentricity for } \frac{x^2}{81/100} + \frac{y^2}{81/64} = 1$$

$$\text{Using } a^2 = b^2(1 - e^2)$$

$$\Rightarrow \frac{81}{100} = \frac{81}{64}(1 - e^2)$$

$$\Rightarrow e^2 = \frac{36}{100} \quad \therefore e = +\frac{3}{5}$$

$$\text{Calculating Eccentricity for } \frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\text{Using } b^2 = a^2(1 - e^2)$$

$$\Rightarrow 16 = 25(1 - e^2)$$

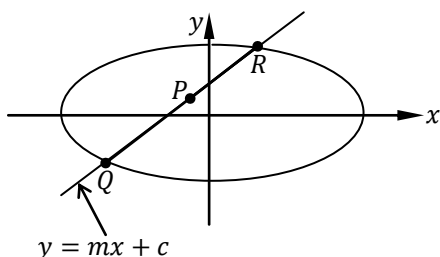
$$\Rightarrow e^2 = \frac{9}{16} \quad \therefore e = +\frac{3}{5}$$

The eccentricity for the required locus is the same as that of the given locus. i.e. $e = +\frac{3}{5}$

3. A variable straight line with constant gradient m meets the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } Q, R. \text{ find the locus of } P \text{ the midpoint of } QR.$$

Solution



Let $Q(x_1, y_1)$ and $R(x_2, y_2)$ be the points of intersection of the line and the ellipse.

Let $y = mx + c$ be the line.

(c is unknown, m is known)

Solving the equations simultaneously,

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1, \text{ simplifying}$$

$$x^2(a^2m^2 + b^2) + (2mca^2)x + a^2(c^2 - b^2) = 0 \dots\dots\dots (1)$$

This equation is quadratic in x , which gives the x - coordinates of points of intersection for Q and R

Using sum of roots

$$x_1 + x_2 = -\frac{2mca^2}{a^2m^2 + b^2} \dots\dots\dots (2)$$

Let the midpoint of QR be $P(X, Y)$

$$P(X, Y) = P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\therefore X = -\frac{2mca^2}{a^2m^2 + b^2} \times \frac{1}{2} = -\frac{mca^2}{a^2m^2 + b^2} \dots\dots\dots (3)$$

In a similar way, solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $y = mx + c$ simultaneously we obtain

$$\frac{\left(\frac{y-c}{m}\right)^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ simplifying}$$

$$y^2(a^2m^2 + b^2) - (2cb^2)y + b^2(c^2 - a^2m^2) = 0 \dots\dots\dots (4)$$

Using sum of roots

$$y_1 + y_2 = \frac{cb^2}{a^2m^2 + b^2} \dots\dots\dots (5)$$

$$\therefore Y = \frac{2cb^2}{a^2m^2 + b^2} \times \frac{1}{2} = \frac{cb^2}{a^2m^2 + b^2} \dots\dots\dots (6)$$

Since c is unknown eliminating it from equations (3) and (6) to obtain the locus of P

$$\Rightarrow \frac{X}{Y} = -\frac{mca^2}{a^2m^2 + b^2} \div \frac{cb^2}{a^2m^2 + b^2} = \frac{-ma^2}{b^2}$$

$$\Rightarrow b^2X = -ma^2Y \text{ or } b^2X + ma^2Y = 0$$

$$\therefore \text{Required locus is: } b^2x + ma^2y = 0$$

4. A variable tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at

$T(4 \cos \theta, 2 \sin \theta)$ meets the parabola $y^2 = 4ax$ at L and M . Find the locus of the midpoint of LM .

Solution

$$\text{Tangent at } T(4 \cos \theta, 2 \sin \theta) \text{ is } x \cos \theta + 2y \sin \theta = 4 \dots\dots\dots (1)$$

(Student, look for this equation)

Solving this equation and parabola $y^2 = 4ax$ simultaneously to obtain points of intersection L and M

$$x = \frac{2y \sin \theta - 4}{\cos \theta} \text{ into } y^2 = 4ax$$

$$\Rightarrow y^2 = 4a \left(\frac{2y \sin \theta - 4}{\cos \theta} \right)$$

$$y^2 \cos \theta + 8a \sin \theta y - 16a = 0 \dots \dots \dots (2)$$

This equation is quadratic in y , which gives the y – coordinates of points of intersection for L and M

Using sum of roots

$$y_1 + y_2 = -\frac{8a \sin \theta}{\cos \theta} \dots \dots \dots (3)$$

Let the midpoint of QR be $P(X, Y)$

In a similar way, solving $x \cos \theta + 2y \sin \theta = 4$ and $y^2 = 4ax$ simultaneously we obtain:

$$y = \frac{4 - x \cos \theta}{2 \sin \theta}$$

$$\Rightarrow \left(\frac{4 - x \cos \theta}{2 \sin \theta} \right)^2 = 4ax \text{ or } 16 - 8x \cos \theta + x^2 \cos^2 \theta = 16a \sin^2 \theta x$$

$$x^2 \cos^2 \theta - x(8 \cos \theta + 16a \sin^2 \theta) + 16 = 0$$

$$x_1 + x_2 = \frac{8 \cos \theta + 16a \sin^2 \theta}{\cos^2 \theta} \dots \dots \dots (4)$$

Let the points of intersection be:

$L(x_1, y_1)$ and $M(x_2, y_2)$

$$\Rightarrow P(X, Y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$\Rightarrow 2X = \frac{8 \cos \theta + 16a \sin^2 \theta}{\cos^2 \theta} = 8 \sec \theta + 16a \tan^2 \theta$$

$$\therefore 2X = 8 \sec \theta + 16a \tan^2 \theta \dots \dots \dots (5)$$

$$\therefore 2Y = -8a \tan \theta \dots \dots \dots (6)$$

From eqn. (5) $2X = 8 \sec \theta + 16a \tan^2 \theta$

$\Rightarrow (X - 8a \tan^2 \theta) = 4 \sec \theta$, squaring both sides

$$\Rightarrow (X - 8a \tan^2 \theta)^2 = 16 \sec^2 \theta = 16(1 + \tan^2 \theta) \dots \dots \dots (7)$$

$$\text{From eqn. (6) } Y = -4a \tan \theta \Rightarrow \tan \theta = \frac{Y}{-4a} \dots \dots \dots (8)$$

Eqn. (8) into eqn. (7)

$$\Rightarrow \left(X - 8a \left(\frac{Y}{-4a} \right)^2 \right)^2 = 16 \left(1 + \left(\frac{Y}{-4a} \right)^2 \right)$$

$$\Rightarrow (16a^2 X - 8a Y^2)^2 = 16(16a^2 + Y^2)$$

$$\therefore 4a^2(2aX - Y^2)^2 = 16a^2 + Y^2$$

$$\text{Required locus: } 4a^2(2ax - y^2)^2 = 16a^2 + y^2$$

Task

- 1) The tangent at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the minor axis at Q and the tangent at one end of the major axis in R . S is a focus prove that $QR = QS$

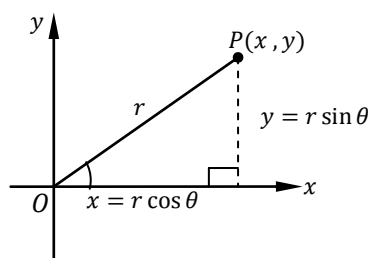
- 2) A perpendicular OY is drawn from the origin to any tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Prove the locus of Y is $x^2 + y^2 = a^2x^2 + b^2y^2$
- 3) The normal at point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the major axis at G .
show that the locus of the midpoint of PG is an ellipse.
Ans: $\frac{4a^2x^2}{(2a^2-b^2)^2} + \frac{4y^2}{b^2} = 1$
- 4) The tangent at any point P on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts Ox in T . if PN is the ordinate of P prove that $ON \cdot OT = a^2$
- 5) The normal at the point $P(a \cos \theta, b \sin \theta)$ on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the y -axis at Q . if S is either focus of the ellipse, prove $\frac{SQ}{PQ}$ is independent of θ

POLAR COORDINATES

These specify the position of a point in a plane.

Consider a point $P(x, y)$ having coordinates in vector form. i.e.

Rectangular Coordinates



$OP = r$

The ordered pair of numbers (r, θ) is called the *Polar Coordinate of P*

The equations $x = r \cos \theta$ and $y = r \sin \theta$ help in conversion of polar coordinates to rectangular coordinates.

Given r, θ the x, y can be found using

$$x = r \cos \theta, y = r \sin \theta$$

Conversely if x, y are given, then

$$r^2 = x^2 + y^2, \tan \theta = \frac{y}{x}$$

Angle θ is determined for $r = +\sqrt{x^2 + y^2}$ and should lie in the interval $-\pi < \theta < \pi$

NOTE

If $r = +\sqrt{x^2 + y^2}$, then

$$\theta = \tan^{-1} \left(\frac{y}{x} \right); x > 0$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right) + \pi; x < 0$$

Examples

- 1) a) Find the rectangular coordinates of $(3, 150^\circ)$
 b) Find the polar coordinates of $(2, -2)$ for $r > 0$

Solution

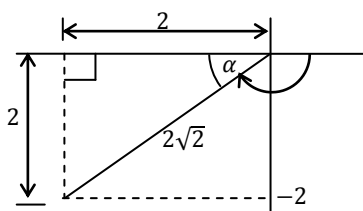
- a) Polar coordinates: $(r, \theta) = (3, 150^\circ)$

$$\Rightarrow r = 3, \theta = 150^\circ$$

$$\text{Using } x = r \cos \theta, y = r \sin \theta$$

$$x = 3 \cos 150^\circ = \left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$$

- b) Polar coordinates: $(r, \theta) = ?$



$$\tan \alpha = \frac{2}{2} = 1 \quad \therefore \alpha = 45^\circ$$

$$\therefore \theta = -135^\circ$$

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 2^2} \\ = 2\sqrt{2}$$

$$\therefore (r, \theta) = (2\sqrt{2}, -135^\circ)$$

2. Find

- i) The Cartesian coordinates of the point whose polar coordinates are $\left(5, \frac{\pi}{4}\right)$
 ii) The polar coordinates whose Cartesian coordinates are $(-1, 1)$

Solution

- a) Cartesian coordinates: $x = r \cos \theta, y = r \sin \theta$

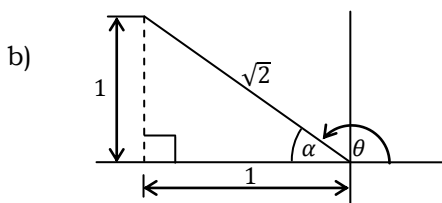
$$\text{Polar coordinates } \left(5, \frac{\pi}{4}\right) = (r, \theta)$$

$$r = 5, \theta = \frac{\pi}{4}$$

$\therefore$ Cartesian coordinates:

$$x = 5 \cos \frac{\pi}{4} = \frac{5}{\sqrt{2}}, y = 5 \sin \frac{\pi}{4} = \frac{5}{\sqrt{2}}$$

$$\therefore \left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$$



$$\tan \alpha = \frac{1}{1} = 1 \quad \therefore \alpha = 45^\circ = \frac{\pi}{4}$$

$$\therefore \theta = 180^\circ - 45^\circ = 135^\circ = \frac{3\pi}{4}$$

Polar coordinates:

$$(r, \theta) = \left(\sqrt{2}, \frac{3\pi}{4}\right)$$

θ

Expressing Equations in terms of Polar Coordinates

Expressing the equations in polar coordinates

- a) $x = 3$ c) $x^2 + y^2 = 16$ e) $y^2 = 4(x + 3)$
 b) $2x - y = 3$ d) $\frac{x^2}{4} + y^2 = 1$

Solution

Using $x = r \cos \theta$, $y = r \sin \theta$

- a) $x = 3 \Rightarrow r \cos \theta = 3$
 b) $2x - y = 3 \Rightarrow 2r \cos \theta - r \sin \theta = 3$
 $\therefore r(2 \cos \theta - \sin \theta) = 3$
 c) $x^2 + y^2 = 16 \Rightarrow r^2 \cos^2 \theta + r^2 \sin^2 \theta = 16$
 $\Rightarrow r^2(\cos^2 \theta + \sin^2 \theta) = 16 \therefore r = 4$
 d) $\frac{x^2}{4} + y^2 = 1 \Rightarrow \frac{r^2 \cos^2 \theta}{4} + r^2 \sin^2 \theta = 1$
 $\therefore r^2(\cos^2 \theta + 4 \sin^2 \theta) = 4$
 e) $y^2 = 4(x + 3) \Rightarrow r^2 \sin^2 \theta = 4(r \cos \theta + 3)$
 $r^2 \sin^2 \theta = 4(r \cos \theta + 3)$

Task

1. (I) Find the coordinates of the points where the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ cuts the axes
 i) Express the equation in polar form
 ii) If the line $y = mx + c$ is a tangent to the ellipse, show that $c^2 = 4m^2 + 9$
2. Find the Cartesian equations corresponding to
 (i) $r = a \sec \theta$ (ii) $r^2 \sin 2\theta = a^2$ **Ans:** (i) $x = a$ (ii) $2xy = a^2$
3. Find the polar equations corresponding to:
 (i) $(x^2 + y^2)^2 = 2a^2 xy$ (ii) $(x^2 + y^2)^2 = a^2(x^2 + y^2)$
 (iii) $x(x^2 + y^2)ay^2$ (iv) $y^2(2a - x) = x^3$
Ans: (i) $r^2 = a^2 \cos 2\theta$ (ii) $r = a \sin \theta \tan \theta$ (iii) $r^2 = a^2 \sin 2\theta$
 (iv) $r = 2a \sin^2 \theta \sec \theta$

THE HYPERBOLA

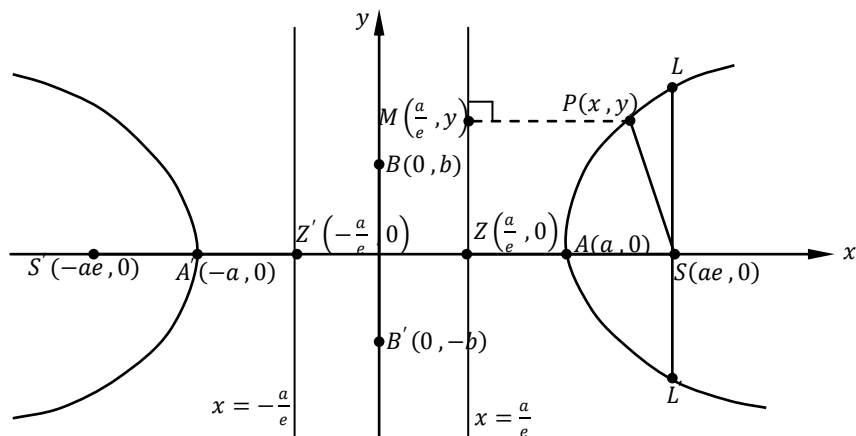
Definition:

Is the locus of a point which moves in such a way that its distance from a fixed point (Focus) bears a constant ratio, $e > 1$ to its distance from a fixed line (Directrix)

Standard Equation of a Hyperbola

To Find the equation of a Hyperbola in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Consider the figure below.



A and A' divide $\overline{SZ}$ internally and externally in the ratio $\overline{SA}:\overline{AZ} = e:1$

From the definition,

$$\overline{PS}:\overline{PM} = e:1 \quad \therefore \overline{PS} = e \overline{PM} \text{ or } \overline{PS}^2 = e^2 \overline{PM}^2 \text{ or } \overline{AS}:\overline{AZ} = e:1$$

$$\overline{PS} = \sqrt{(x - ae)^2 + (y - 0)^2} = \sqrt{x^2 - 2aex + a^2e^2}$$

$$\overline{PM} = \sqrt{\left(x - \frac{a}{e}\right)^2 + (y - y)^2} = \sqrt{x^2 - \frac{2a}{e}x + \frac{a^2}{e^2}}$$

$$\Rightarrow x^2 - 2aex + a^2e^2 = e^2 \left(x^2 - \frac{2a}{e}x + \frac{a^2}{e^2} + y^2 \right)$$

$$\Rightarrow x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$$

$$\Rightarrow x^2(e^2 - 1) - y^2 = a^2(e^2 - 1), \text{ dividing through by } a^2(e^2 - 1)$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

$$\text{Let } b^2 = a^2(e^2 - 1)$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad b^2 = a^2(e^2 - 1)}$$

Important Observations about the Standard Equation of a Hyperbola

1. $A(a, 0)$ and $A'(-a, 0)$ are its vertices and line $\overline{AA'}$ is called the *Transverse Axis* of a hyperbola.
2. Line $\overline{BB'}$.i.e. $B(0, b)$ and $B'(0, -b)$ is called the *Conjugate Axis*, thus $\overline{AA'}$ and $\overline{BB'}$ are axes of a hyperbola.
3. Foci are: $S(ae, 0)$ and $S'(-ae, 0)$
4. Equation of Directrices are: $x = \pm \frac{a}{e}$
5. If (x_1, y_1) lies on the curve, so is the point $(-x_1, -y_1)$. But (x_1, y_1) and $(-x_1, -y_1)$ are points on the line through the origin. Thus every chord through the origin is bisected at O. thus O is the *Centre* of the curve.
6. Since the powers of x and y are even, the curve is symmetrical about the coordinate axes.
7. When $x \rightarrow \infty$ also $y \rightarrow \infty$
8. The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be written as $y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$. When $x < a$, y is imaginary hence no part of the curve lies between the lines $x = \pm a$
9. The chord through any focus and perpendicular to the transverse axis is called *Latus Rectum*.

From the figure, line LSL' is latus rectum.

Obtaining the coordinates at the end points of latus rectum

From $x = ae$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes

$$\frac{a^2 e^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2(e^2 - 1)$$

$$\therefore y = \pm \sqrt{b^2(e^2 - 1)}$$

$$\text{But } b^2 = a^2(e^2 - 1) \text{ or } e^2 - 1 = \frac{b^2}{a^2}$$

$$\Rightarrow y = \pm \sqrt{\frac{b^2}{a^2} \cdot b^2} = \pm \frac{b^2}{a}$$

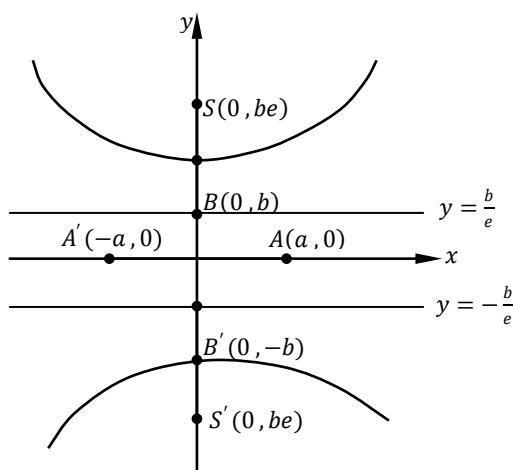
$$\therefore L\left(ae, \frac{b^2}{a}\right), S(ae, 0), L'\left(ae, -\frac{b^2}{a}\right)$$

$$\text{Length of Latus Rectum } \overline{LL'} = \frac{2b^2}{a}$$

10. Equations of latus recta are $x = \pm ae$

CONJUGATE HYPERBOLA

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is another standard equation of a hyperbola



Observations

1. $\overline{AA'}$ is the conjugate axis, $\overline{BB'}$ is the transverse axis
2. Vertices are: $(0, \pm b)$
3. Directrices are: $y = \pm \frac{b}{e}$
4. Foci are: $(0, \pm be)$
5. Length of Latus rectum = $\frac{2a^2}{b}$
6. Equations of latus recta: $y = \pm be$
7. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$,
 $a^2 = b^2(e^2 - 1)$

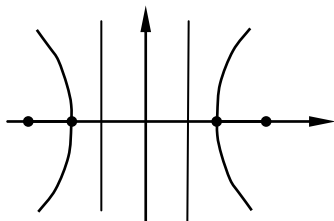
Illustrative Examples

1. Find the equation of the hyperbola with:

- i) Foci $(\pm 5, 0)$ and equation of Directrices $x = \pm \frac{9}{5}$
- ii) Foci $(0, \pm 5)$ and vertices are $(0, \pm 4)$
- iii) Foci $(\pm 6, 0)$ and vertices are $(\pm 3, 0)$

Solution

i)



Since the foci are on the x -axis, then $y = 0$ is a transverse axis of hyperbola. Hence

using $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Foci: $(\pm ae, 0) = (\pm 5, 0) \therefore ae = 5 \dots \dots \dots (1)$

Directrices: $x = \pm \frac{a}{e} = \pm \frac{9}{5} \therefore \frac{a}{e} = \frac{9}{5} \dots \dots \dots (2)$

Solving simultaneously eqn. (1) $\times$ eqn. (2)

$$ae \times \frac{a}{e} = \frac{9}{5} \times 5 \therefore a^2 = 9$$

Also using $b^2 = a^2(e^2 - 1)$

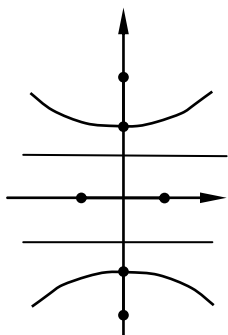
$$\Rightarrow b^2 = 9(e^2 - 1)$$

From eqn. (1), $e = \frac{5}{a} = \frac{5}{3}$

$$\therefore b^2 = 9\left(\frac{25}{9} - 1\right) = 16$$

$\therefore$ Required equation of hyperbola is: $\frac{x^2}{9} - \frac{y^2}{16} = 1$

ii)



Since the transverse axis is through the foci points, the equation of the required hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Given: Foci: $(0, \pm 5) = (0, \pm be) \therefore be = 5 \dots \dots \dots (1)$

Vertices: $(0, \pm 4) = (0, \pm b) \therefore b = 4 \dots \dots \dots (2)$

Solving eqn. (1) and eqn. (2) simultaneously $e = \frac{5}{4}$

Using $a^2 = b^2(e^2 - 1)$

$$a^2 = b^2(e^2 - 1) = 16\left(\frac{25}{16} - 1\right) = 9$$

$\therefore$ Required equation of hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = -1$

iii) Left as an exercise. **Ans:** $\frac{x^2}{9} - \frac{y^2}{27} = -1$

2. Find the eccentricity, foci, equation of Directrices, centre of the hyperbola

i) $16x^2 - 9y^2 = 144$

iii) $9x^2 - 16y^2 + 18x + 32y - 151 = 0$

ii) $x^2 - y^2 = 32$

iv) $9x^2 - 16y^2 + 144 = 0$

Solution

i) $16x^2 - 9y^2 = 144$, rewriting the equation in standard form ($\div 144$)

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \therefore a^2 = 9, b^2 = 16, b^2 = a^2(e^2 - 1)$$

$$\Rightarrow 16 = 9(e^2 - 1) \quad \therefore e = 5/3$$

$$\text{Foci: } (\pm ae, 0) = \left(\pm 3 \times \frac{5}{3}, 0\right) = (\pm 5, 0)$$

$$\text{Directrices: } x = \pm \frac{a}{e} = \pm \frac{3}{5/3} = \pm \frac{9}{5} \quad \therefore x = \pm \frac{9}{5}$$

Centre: $(0, 0)$

ii) Left as an exercise.

Ans: Foci: $(\pm 8, 0)$, Directrices: $x = \pm 4$, centre: $(0, 0)$

iii) $9x^2 - 16y^2 + 18x + 32y - 151 = 0$

$(9x^2 + 18x) + (-16y^2 + 32y) = 151$, completing squares,

$$9(x^2 + 2x + 1) - 16(y^2 - 2y + 1) = 151 + 9 - 16$$

$$9(x + 1)^2 - 16(y - 1)^2 = 144, \text{ dividing through by } 144$$

$$\frac{(x+1)^2}{16} - \frac{(y-1)^2}{9} = 1, \text{ which is of the form } \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

$$\therefore X = x + 1, Y = y - 1, a^2 = 16, b^2 = 9$$

$$\text{But for hyperbola } \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

$$\text{Foci: } (\pm Ae, 0) = (X, Y)$$

$$\Rightarrow Ae = X = x + 1, Y = 0 = y - 1 \quad \therefore y = 1$$

$$\text{Also } b^2 = a^2(e^2 - 1)$$

$$9 = 16(e^2 - 1) \quad \therefore e = \frac{5}{4}$$

$$\therefore Ae = 4 \times \frac{5}{4} = x + 1 \Rightarrow x = 4$$

∴ Foci: $(\pm 4, 1)$

Directrices: $X = \pm \frac{A}{e} = \pm \frac{4}{5/4} = \pm \frac{16}{5}$

$\Rightarrow x + 1 = \pm \frac{16}{5}$ or $x = -\frac{21}{5}, \frac{11}{5}$

∴ Directrices are $5x - 11 = 0, 5x + 21 = 0$

Centre: $(0, 0) = (X, Y)$

$X = x + 1 = 0 \quad \therefore x = -1$

$Y = 0 = y - 1 \quad \therefore y = 1$

∴ Centre: $(-1, 1)$

iv) Left as an exercise. **Ans:** Foci: $(0, \pm 5)$, Directrices: $y = \pm \frac{9}{5}$, centre: $(0, 0)$

NOTE

Most results of ellipse obtained earlier for the standard equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ also hold pretty well for the hyperbola in standard form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

When b^2 is changed into $-b^2$, the method of proof being exactly the same, the results are as follows:

A LINE AND A HYPERBOLA

j) **To Find the Point of Intersection of the Line $y = mx + c$ and the**

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solving the two equations simultaneously, obtain

$$(b^2 - a^2m^2)x^2 - (2mca^2)x - (a^2c^2 - a^2b^2) = 0 \dots\dots\dots (1)$$

This equation gives two values of x , hence two points of intersection.

k) **To Show that a given Line $y = mx + c$ is a Tangent to the Hyperbola**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Approach I:

First solve the two equations simultaneously to obtain

$$(b^2 - a^2m^2)x^2 - (2mca^2)x - (a^2c^2 - a^2b^2) = 0 \dots\dots\dots (1)$$

Applying the condition for tangency $B^2 = 4AC$ to obtain

$$c^2 = a^2m^2 - b^2 \quad \text{or} \quad c =$$

$$\pm \sqrt{a^2m^2 - b^2} \dots\dots\dots (2)$$

Approach II:

Solving eqn. (1) for point of contact

Eqn. (2) into eqn. (1) to obtain

$$(a^2m^2 - b^2)x^2 \pm (2ma^2\sqrt{a^2m^2 - b^2})x + a^4m^2 = 0$$

$$(x\sqrt{a^2m^2 - b^2} \pm ma^2)^2 = 0 \quad \therefore x = \pm \frac{ma^2}{\sqrt{a^2m^2 - b^2}}, \text{ which will be a repeated}$$

value. Hence the line is a tangent to the ellipse

- l) **To obtain the Point of Contact of a tangent line $y = mx + c$ to the**

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Using $x = \pm \frac{ma^2}{\sqrt{a^2m^2 - b^2}}$ into $y = mx + c$, obtain point of contact as:

$\therefore$ Point of contact is $P\left(\frac{ma^2}{\sqrt{a^2m^2 - b^2}}, -\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$ for $x = +\frac{ma^2}{\sqrt{a^2m^2 - b^2}}, c = -\sqrt{a^2m^2 - b^2}$

$\therefore$ Point of contact is $P'\left(-\frac{ma^2}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$ for $x = -\frac{ma^2}{\sqrt{a^2m^2 - b^2}}, c = +\sqrt{a^2m^2 - b^2}$

- m) **To state the Equation of the Tangent to the parabola in Standard Form**

From the line $y = mx + c$ and condition for tangency $c = \pm\sqrt{a^2m^2 - b^2}$, the tangent in slope form is $y = mx \pm \sqrt{a^2m^2 - b^2}$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON

A HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

To Find an Equation of a Tangent at any point $P(x_1, y_1)$ on the

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proceeding as in ellipse, the tangent equation is:

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

To Find an Equation of the Normal at any point $P(x_1, y_1)$ on the

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proceeding as in ellipse obtain

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

To Find an Equation of a Chord with the Midpoint $P(x_1, y_1)$ on the

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Follow the steps in ellipse for this question to obtain

$$\frac{x^2x_1}{a^2} - \frac{y^2y_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

General Examples

Find the equation of the tangent and the normal to the

a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $(0, 3)$

b) $4x^2 - 9y^2 = 36$ at $\left(2, \frac{4\sqrt{5}}{3}\right)$

Solution

a) Required tangent: $y - 3 = m(x - 0) = mx \dots\dots\dots (1)$

$$m = \left. \frac{dy}{dx} \right|_{at (0,3)}$$

$$\frac{2x}{16} - \frac{2y}{9} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{9x}{16y}$$

$$\therefore m = \frac{0}{48} = 0$$

$\therefore$ Required tangent is $y - 3 = 0$

Required normal: $y - 3 = m'(x - 0) = m'x \dots\dots\dots (2)$

$$m' = -\frac{1}{m} = -\frac{1}{0}$$

Rewriting the required normal

$$\frac{y-3}{x} = m' = -\frac{1}{0}$$

$\therefore$ Required tangent is $x = 0$

b) Left as an exercise. **Ans:** $2x - 3\sqrt{5}y + 16 = 0$; $9\sqrt{5}x + 6y + 10\sqrt{5} = 0$

ASYMPTOTES

Definition:

Are lines which meet the curve in two points at infinity but does not wholly lie at infinity.

To Find the Asymptotes of Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Two methods can be used.

Method I:

Let $y = mx + c$ be the asymptote.

Solving $y = mx + c$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ obtain:

$$(b^2 - a^2m^2)x^2 - (2mca^2)x - (a^2c^2 - a^2b^2) = 0$$

Equating coefficients of higher powers of x to zero

$$\text{For } x^2: b^2 - a^2m^2 = 0 \Rightarrow m = \pm \frac{b}{a}$$

$$\text{For } x: 2mca^2 = 0 \Rightarrow c = 0$$

Equation of asymptotes: $y = \pm \frac{b}{a}x$

Method II:

From $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \dots\dots\dots (\times b^2x^2)$

$$\frac{b^2}{a^2} - \frac{y^2}{x^2} = \frac{b^2}{x^2} \text{ or } \frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2}$$

$$\text{As } x \rightarrow \pm\infty, \frac{y^2}{x^2} \rightarrow \frac{b^2}{a^2} \text{ or } \frac{y}{x} \rightarrow \pm \frac{b}{a}$$

$$\therefore y = \pm \frac{b}{a}x$$

General Equation of Asymptotes of Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

From $y = \pm \frac{b}{a}x$ or $\frac{y}{b} = \pm \frac{x}{a}$

$$\Rightarrow \frac{x}{a} - \frac{y}{b} = 0, \quad \frac{x}{a} + \frac{y}{b} = 0$$

Multiplying the equations

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}$$

Relationship between Equation of Hyperbola and its Asymptotes

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ differs from its asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ by a constant, i.e. 1, and also the equation of a conjugate hyperbola differs from its asymptotes by the same constant.

Generally if $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the equation of asymptotes combined is:

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$$

Then the equation of the hyperbola which differs from that of asymptotes by a constant k is:

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = k$$

Example

1. Find the equation of the hyperbola passing through the point (4, 6) and whose asymptotes are $y = \pm\sqrt{3}x$

Solution

$$y = \pm\sqrt{3}x \text{ or } \sqrt{3}x - y = 0 \text{ and } \sqrt{3}x + y = 0$$

Combined equation of asymptotes:

$$(\sqrt{3}x + y)(\sqrt{3}x - y) = 0 \Rightarrow 3x^2 - y^2 = 0$$

Equation of hyperbola differs from that of asymptotes by a constant

$$\Rightarrow 3x^2 - y^2 = k$$

Now (4, 6) lies on the hyperbola

$$\Rightarrow 48 - 36 = k \therefore k = 12$$

$$\therefore 3x^2 - y^2 = 12$$

2. Find the points of intersection of the tangent to hyperbola $\frac{x^2}{16} - \frac{y^2}{25} = 1$ at (0, 5) and its asymptotes.

Solution

$$\text{Equation of asymptotes to } \frac{x^2}{16} - \frac{y^2}{25} = 1 \text{ are: } \frac{x^2}{16} - \frac{y^2}{25} = 0 \text{ or } \left(\frac{x}{4} + \frac{y}{5}\right)\left(\frac{x}{4} - \frac{y}{5}\right) = 0$$

$$\text{Considering } \frac{x}{4} + \frac{y}{5} = 0 \text{ or } x = -\frac{4}{5}y \dots\dots\dots (1)$$

Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Thus the tangent at $(0, 5)$ to the hyperbola $\frac{x^2}{16} - \frac{y^2}{25} = 1$ is:

$$\frac{x(0)}{16} - \frac{y(5)}{25} = 1 \quad \therefore y = 5 \quad \dots\dots\dots (2)$$

Considering $y = 5$ and $\frac{x}{4} + \frac{y}{5} = 0$

$$\Rightarrow \frac{x}{4} + 1 = 0 \quad \therefore x = -4$$

$\therefore (-4, 5)$

Considering $y = 5$ and $\frac{x}{4} - \frac{y}{5} = 0$

$$\Rightarrow \frac{x}{4} - 1 = 0 \quad \therefore x = 4$$

$\therefore (4, 5)$

- 3.** Show that the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ have the same foci.

Find the point of intersection of the curves.

Solution

For the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1, a > b$

Using $b^2 = a^2(1 - e^2)$

$$7 = 16(1 - e^2) \quad \therefore e = \frac{3}{4}$$

$$\text{Foci: } (\pm ae, 0) = \left(\pm 4 \times \frac{3}{4}, 0\right) = (\pm 3, 0)$$

For the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ or $\frac{x^2}{144/25} - \frac{y^2}{81/25} = 1$

Using $b^2 = a^2(1 - e^2)$

$$\frac{81}{25} = \frac{144}{25}(1 - e^2) \quad \therefore e = \frac{5}{4}$$

$$\text{Foci: } (\pm ae, 0) = \left(\pm \frac{12}{5} \times \frac{5}{4}, 0\right) = (\pm 3, 0)$$

Hence foci are the same.

Point of intersection of the curves

$$\frac{x^2}{16} + \frac{y^2}{7} = 1 \Rightarrow 7x^2 + 16y^2 = 112 \quad \dots\dots\dots (1)$$

$$\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25} \Rightarrow 2025x^2 - 3600y^2 = 11664 \quad \dots\dots\dots (2)$$

Solving the equations, $x^2 = 10.24, y^2 = 2.52$

$$\Rightarrow x = \pm \frac{16}{5}, y = \pm \frac{3\sqrt{7}}{5}$$

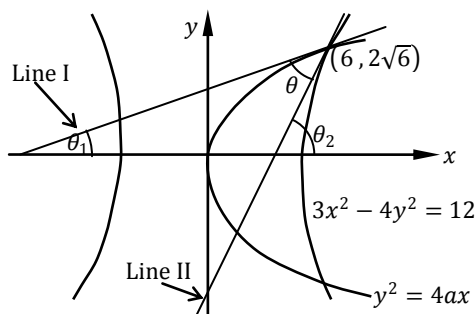
$\therefore$ Points of intersection are: $\left(\pm \frac{16}{5}, \pm \frac{3\sqrt{7}}{5}\right)$

- 4.** Find the point of intersection in the first quadrant of the hyperbola $3x^2 - 4y^2 = 12$ and the parabola $y^2 = 4x$, and also find the angle at which the curves intersect.

Solution

Point of intersection is: $(6, 2\sqrt{6})$

The student should find this point. Left as an exercise



From the figure $\theta = \theta_2 - \theta_1$
 Finding slope of Line I $= m_1 = \tan \theta_1$
 For $y^2 = 4x$, $2y \frac{dy}{dx} = 4 \therefore \frac{dy}{dx} = \frac{2}{y}$
 $\therefore m_1 = \frac{dy}{dx} \Big|_{\text{at } (6, 2\sqrt{6})} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$
 Finding slope of Line II $= m_2 = \tan \theta_2$
 For $3x^2 - 4y^2 = 12$, $6x - 8y \frac{dy}{dx} = 0$
 $\therefore \frac{dy}{dx} = \frac{3x}{4y}$

$$\therefore m_2 = \frac{dy}{dx} \Big|_{\text{at } (6, 2\sqrt{6})} = \frac{3 \times 6}{4 \times 2\sqrt{6}} = \frac{9}{4\sqrt{6}}$$

From $\theta = \theta_2 - \theta_1$

Introducing tangents on both sides

$$\begin{aligned} \tan \theta &= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1} = \frac{\frac{9}{4\sqrt{6}} - \frac{1}{\sqrt{6}}}{1 + \frac{9}{4\sqrt{6}} \times \frac{1}{\sqrt{6}}} \\ &= \frac{\frac{1}{\sqrt{6}}(\frac{9}{4} - 1)}{1 + \frac{9}{24}} = \frac{10}{11\sqrt{6}} \\ \therefore \theta &= \tan^{-1} \left(\frac{10}{11\sqrt{6}} \right) \end{aligned}$$

PARAMETRIC EQUATIONS OF THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The general point $(a \sec \theta, b \tan \theta)$ lies on the hyperbola. Thus parametric equations are:

$$x = a \sec \theta, \quad y = b \tan \theta$$

To Show that the Parametric Equations represent a Hyperbola

For $x = a \sec \theta \Rightarrow \sec \theta = \frac{x}{a}$

$y = b \tan \theta \Rightarrow \tan \theta = \frac{y}{b}$

Using $\sec^2 \theta = 1 + \tan^2 \theta \Rightarrow \sec^2 \theta - \tan^2 \theta = 1$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

To Find the Equation of a Tangent and Normal to a Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$

Tangent Equation

$y - b \tan \theta = m_1(x - a \sec \theta) \dots\dots\dots (1)$

Obtaining slope, m_1

Method I:

$$m_1 = \left. \frac{dy}{dx} \right|_{(a \sec \theta, b \tan \theta)}$$

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t x

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$$\Rightarrow m_1 = \left. \frac{dy}{dx} \right|_{(a \sec \theta, b \tan \theta)} = -\frac{b^2 a \sec \theta}{a^2 b \tan \theta} = -\frac{b \sec \theta}{a \tan \theta}$$

$$\therefore m_1 = \frac{b \sec \theta}{a \tan \theta}$$

Method II:

Formulating parametric equations

$$x = a \sec \theta, \quad y = b \tan \theta$$

$$m_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta, \quad \frac{dy}{d\theta} = b \sec^2 \theta$$

$$\therefore m_1 = b \sec^2 \theta \times \frac{1}{a \sec \theta \tan \theta} = \frac{b \sec \theta}{a \tan \theta}$$

Tangent equation becomes:

$$y - b \tan \theta = \frac{b \sec \theta}{a \tan \theta} (x - a \sec \theta)$$

$$\Rightarrow ya \tan \theta - ab \tan^2 \theta = b \sec \theta x - ab \sec^2 \theta$$

$$\Rightarrow b \sec \theta x - ya \tan \theta = ab(\sec^2 \theta - \tan^2 \theta)$$

$$\therefore ay \tan \theta - bx \sec \theta = ab \quad [\because \sec^2 \theta - \tan^2 \theta = 1] \dots\dots\dots (\div ab)$$

$$\therefore \frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

Equation of the Normal

$$y - b \tan \theta = m_2 (x - a \sec \theta) \dots\dots\dots (2)$$

Obtaining slope, m_2

$$m_2 = -\frac{1}{\text{Slope of the tangent at } (a \cos \theta, b \sin \theta)} = -\frac{1}{m_1} = -\frac{a \tan \theta}{b \sec \theta}$$

$$y - b \tan \theta = -\frac{a \tan \theta}{b \sec \theta} (x - a \sec \theta)$$

$$yb \sec \theta - b^2 \tan \theta \sec \theta = -a \tan \theta x + a^2 \tan \theta \sec \theta$$

$$\therefore a \tan \theta x + by \sec \theta = (a^2 + b^2) \tan \theta \sec \theta \dots\dots\dots (\div \cos \theta \sin \theta)$$

$$\Rightarrow a \frac{\tan \theta}{\tan \theta \sec \theta} x + by \frac{\sec \theta}{\tan \theta \sec \theta} = a^2 + b^2$$

$$\therefore \frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2$$

Examples

1. (i) Show that the equation of the tangent to the hyperbola $(a \sec \theta, b \tan \theta)$ is $bx - ay \sin \theta - ab \cos \theta = 0$
- ii) Find the equations of tangents to $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at the points where $\theta = 45^\circ$ and where $\theta = -135^\circ$

Solution

- i) Required equation:

$$y - b \tan \theta = m(x - a \sec \theta) \dots\dots\dots (1)$$

Obtaining slope, m

Method:

$$m = \left. \frac{dy}{dx} \right|_{at (a \sec \theta, b \tan \theta)}$$

$$\text{Differentiating } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ w.r.t } x$$

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$$\Rightarrow m = \left. \frac{dy}{dx} \right|_{at (a \sec \theta, b \tan \theta)} = -\frac{b^2 a \sec \theta}{a^2 b \tan \theta} = -\frac{b \sec \theta}{a \tan \theta}$$

$$\therefore m = \frac{b \sec \theta}{a \tan \theta}$$

$$\Rightarrow y - b \tan \theta = \frac{b \sec \theta}{a \tan \theta} (x - a \sec \theta)$$

$$\Rightarrow ya \tan \theta - ab \tan^2 \theta = b \sec \theta x - ab \sec^2 \theta$$

$$\Rightarrow b \sec \theta x - ya \tan \theta = ab(\sec^2 \theta - \tan^2 \theta)$$

$$\therefore bx \sec \theta - ay \tan \theta = ab \quad [\because \sec^2 \theta - \tan^2 \theta = 1] \dots\dots\dots (\div ab)$$

Expressing the equation in terms of $\sin \theta$ and $\cos \theta$

$$\Rightarrow \frac{bx}{\cos \theta} - \frac{ay \sin \theta}{\cos \theta} = ab \dots\dots\dots (\times \cos \theta)$$

$$\therefore bx - ay \sin \theta - ab \cos \theta = 0$$

ii) Equation of tangent at $(a \sec \theta, b \tan \theta)$ is $bx - ay \sin \theta - ab \cos \theta = 0$

Now at $\theta = 45^\circ$

$$bx - ay \sin 45^\circ - ab \cos 45^\circ = 0 \Rightarrow bx - \frac{ay}{\sqrt{2}} - \frac{ab}{\sqrt{2}} = 0$$

$$\therefore 3\sqrt{2}x - 2y - 6 = 0$$

Now at $\theta = -135^\circ$

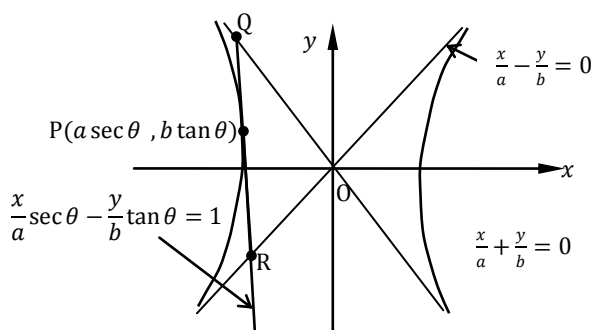
$$bx - ay \sin(-135^\circ) - ab \cos(-135^\circ) = 0 \Rightarrow bx + \frac{ay}{\sqrt{2}} + \frac{ab}{\sqrt{2}} = 0$$

$$\therefore 3\sqrt{2}x + 2y + 6 = 0$$

2. Prove that the portion of a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepted between asymptotes is bisected at the point of contact and the area of the triangle formed by the asymptotes and the tangent is a constant.

Solution

Equation of asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ or $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$



Let the point of contact on the hyperbola be $P(a \sec \theta, b \tan \theta)$

Obtaining the tangent at P gives

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1 \dots\dots\dots (1)$$

Let the student look for this tangent

Solving for the point of intersection Q and R

$$\frac{x}{a} + \frac{y}{b} = 0 \text{ or } x = -\frac{ay}{b} \dots\dots\dots (2)$$

eqn. (2) Into eqn. (1)

$$\Rightarrow -\frac{ay}{b} \cdot \frac{1}{a} \sec \theta - \frac{y}{b} \tan \theta = 1 \text{ or } -\frac{y}{b} \sec \theta - \frac{y}{b} \tan \theta = 1$$

$$\Rightarrow y = -\frac{b}{\sec \theta + \tan \theta} \dots\dots\dots (3)$$

Eqn. (3) into eqn. (2)

$$x = -\frac{a}{b} \times -\frac{b}{\sec \theta + \tan \theta} = \frac{a}{\sec \theta + \tan \theta}$$

$$\therefore Q \left(\frac{a}{\sec \theta + \tan \theta}, -\frac{b}{\sec \theta + \tan \theta} \right) \text{ or } Q[a(\sec \theta - \tan \theta), -b(\sec \theta - \tan \theta)]$$

Now finding R

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ or } x = \frac{ay}{b} \dots\dots\dots (4)$$

eqn. (4) into eqn. (1)

$$\Rightarrow \frac{ay}{b} \cdot \frac{1}{a} \sec \theta - \frac{y}{b} \tan \theta = 1 \text{ or } \frac{y}{b} \sec \theta - \frac{y}{b} \tan \theta = 1$$

$$\Rightarrow y = \frac{b}{\sec \theta - \tan \theta} = b(\sec \theta + \tan \theta) \dots\dots\dots (5)$$

Eqn. (5) into eqn. (4)

$$x = \frac{a}{b} \times b(\sec \theta + \tan \theta) = a(\sec \theta + \tan \theta)$$

$$\therefore R[a(\sec \theta + \tan \theta), b(\sec \theta + \tan \theta)]$$

Midpoint of QR

$$\left(\frac{a(\sec \theta + \tan \theta) + a(\sec \theta - \tan \theta)}{2}, \frac{-b(\sec \theta - \tan \theta) + b(\sec \theta + \tan \theta)}{2} \right) = (a \sec \theta, b \tan \theta),$$

which lies on a hyperbola

$$\text{Area of the triangle } QOR = \frac{1}{2} \times \overline{QR} \times \text{Height}$$

$H = \perp$ Distance of $(0, 0)$ from QR

$$H = \frac{1}{\sqrt{\left(\frac{\sec \theta}{a}\right)^2 + \left(\frac{\tan \theta}{b}\right)^2}} = \frac{ab}{\sqrt{b^2 \sec^2 \theta + a^2 \tan^2 \theta}}$$

$$\therefore \overline{QR} = \sqrt{(2a \tan \theta)^2 + (2b \sec \theta)^2} = 2\sqrt{b^2 \sec^2 \theta + a^2 \tan^2 \theta}$$

$$\therefore \text{Area of } \Delta QOR = \frac{1}{2} \times 2\sqrt{b^2 \sec^2 \theta + a^2 \tan^2 \theta} \times \frac{ab}{\sqrt{b^2 \sec^2 \theta + a^2 \tan^2 \theta}} = ab \text{ (Constant)}$$

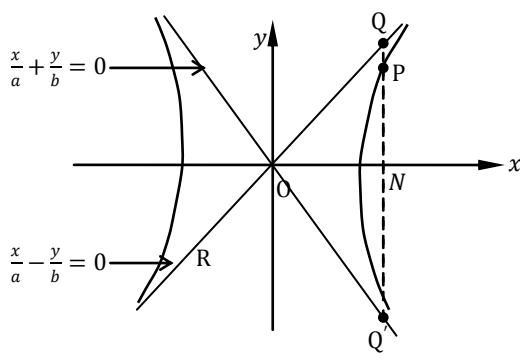
3. If the ordinate of the point $P(a \sec \theta, b \tan \theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be produced to meet the asymptotes in Q and Q' , prove that $PQ \cdot PQ' = b^2$

Solution

In the figure below, the x - coordinate (abscissa) of P which is $x = a \sec \theta$ is the same as that of Q and Q' .

The y - coordinate of Q is:

$$\frac{a \sec \theta}{a} - \frac{y}{b} = 0 \therefore y = b \sec \theta$$



$$\therefore Q(a \sec \theta, b \sec \theta)$$

The y -coordinate of Q' is:

$$\frac{a \sec \theta}{a} + \frac{y}{b} = 0 \therefore y = -b \sec \theta$$

$$\therefore Q'(a \sec \theta, -b \sec \theta)$$

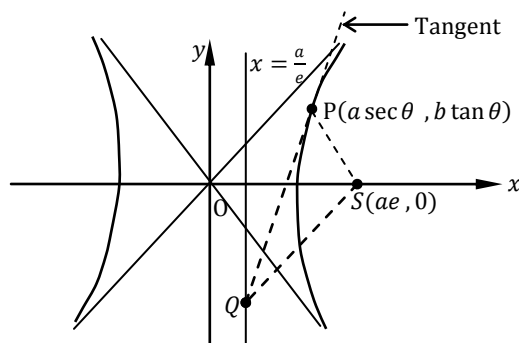
$$\overline{PQ} = b \sec \theta - b \tan \theta = b(\sec \theta - \tan \theta)$$

$$\overline{PQ'} = b \tan \theta + b \sec \theta = b(\tan \theta + \sec \theta)$$

$$\therefore \overline{PQ} \cdot \overline{PQ'} = b^2(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) \\ = b^2(\sec^2 \theta - \tan^2 \theta) = b^2$$

4. A tangent to a hyperbola at $P(a \sec \theta, b \tan \theta)$ meets the directrix at Q . if S is the corresponding focus, prove that PQ subtends a right angle at S

Solution



Tangent at P is:

$$bx - ay \sin \theta - ab \cos \theta = 0$$

Finding the coordinates of Q

$$\text{When } x = \frac{a}{e}, y = ?$$

$$\frac{ba}{e} - ay \sin \theta - ab \cos \theta = 0$$

$$\therefore y = \frac{b(1 - e \cos \theta)}{e \sin \theta}$$

$$\therefore Q\left(\frac{a}{e}, \frac{b(1 - e \cos \theta)}{e \sin \theta}\right)$$

$$\text{Slope of } QS = m_1 = \frac{\frac{b(1 - e \cos \theta)}{e \sin \theta}}{\frac{a}{e} - ae} = \frac{b(1 - e \cos \theta)}{a(1 - e^2) \cos \theta}$$

$$\text{Slope of } PS = m_2 = \frac{b \tan \theta}{a \sec \theta - ae} = \frac{b \sin \theta}{a(1 - e \cos \theta)}$$

$$\text{Now } m_1 \times m_2 = \frac{b(1 - e \cos \theta)}{a(1 - e^2) \cos \theta} \times \frac{b \sin \theta}{a(1 - e \cos \theta)} = \frac{b^2}{a^2(1 - e^2)}$$

$$\text{But } b^2 = a^2(e^2 - 1) \Rightarrow -b^2 = a^2(1 - e^2)$$

$$\therefore m_1 \times m_2 = \frac{b^2}{-b^2} = -1$$

Thus $SQ \perp SP$, so $\overline{PQ}$ subtends $< 90^\circ$ at the focus.

LOCUS PROBLEMS IN HYPERBOLA

1. P is a variable point given by the parametric equations

$x = \frac{a}{2} \left(t + \frac{1}{t} \right)$, $y = \frac{b}{2} \left(t - \frac{1}{t} \right)$. Show that the locus of P is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. State the asymptotes of the curve.

Solution

$$x = \frac{a}{2} \left(t + \frac{1}{t} \right) \Rightarrow t + \frac{1}{t} = \frac{2x}{a} \dots\dots\dots (1)$$

$$y = \frac{b}{2} \left(t - \frac{1}{t} \right) \Rightarrow t - \frac{1}{t} = \frac{2y}{b} \dots\dots\dots (2)$$

Squaring eqn. (1) and eqn. (2) to create the squares on x and y

$$\Rightarrow \left(t + \frac{1}{t} \right)^2 = \frac{4x^2}{a^2} \quad \text{or} \quad \frac{4x^2}{a^2} = t^2 + \frac{1}{t^2} + 2 \dots\dots\dots (3)$$

$$\Rightarrow \left(t - \frac{1}{t} \right)^2 = \frac{4y^2}{b^2} \quad \text{or} \quad \frac{4y^2}{b^2} = t^2 + \frac{1}{t^2} - 2 \dots\dots\dots (4)$$

Eqn. (3) – eqn. (4)

$$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4 \quad \therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ is the required locus.}$$

$$\text{Asymptotes: } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

2. P is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and Q is the point (a, b) . Find the locus of the point dividing PQ in the ratio 2:1.

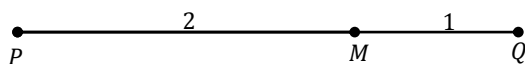
Solution

Let $M(X, Y)$ be the point dividing PQ in the ratio 2:1

Ratio of division; $PM:MQ = 2:1$

Let $P(a \sec \theta, b \tan \theta)$, obtaining the coordinates of M in terms of a and b

From coordinate theorem, applying the ratio theorem



$$\Rightarrow M(X, Y) = \left(\frac{1 \times a \sec \theta + 2 \times a}{2+1}, \frac{1 \times b \tan \theta + 2 \times b}{2+1} \right)$$

$$\Rightarrow X = \frac{a}{3} (\sec \theta + 2) \quad \therefore \sec \theta = \frac{3X}{a} - 2$$

$$\Rightarrow Y = \frac{b}{3} (\tan \theta + 2) \quad \therefore \tan \theta = \frac{3Y}{b} - 2$$

Using $1 + \tan^2 \theta = \sec^2 \theta$

$$\Rightarrow 1 + \left(\frac{3Y}{b} - 2 \right)^2 = \left(\frac{3X}{a} - 2 \right)^2$$

$$\Rightarrow 1 + \frac{9Y^2}{b^2} - \frac{12Y}{b} + 4 = \frac{9X^2}{a^2} - \frac{12Y}{a} + 4 \dots\dots\dots (\times a^2 b^2)$$

$$\therefore 9b^2 X^2 - 9a^2 Y^2 = 12ab^2 X + 12a^2 bY - a^2 b^2 = 0$$

Thus the locus is:

$$9b^2 x^2 - 9a^2 y^2 = 12ab^2 x + 12a^2 by - a^2 b^2 = 0$$

3. The normal at $P(a \sec \theta, b \tan \theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes at E and F . find the locus of the midpoint of EF .

Solution

Equation of the normal: $ax \sin \theta + by - (a^2 + b^2) \tan \theta = 0$

The student should derive the equation

Finding the x and y – intercepts

When $x = 0$, $by = (a^2 + b^2) \tan \theta \quad \therefore E \left(0, \frac{(a^2 + b^2) \tan \theta}{b} \right)$

When $y = 0$, $ax \sin \theta - (a^2 + b^2) \tan \theta = 0 \quad \therefore F \left(\frac{(a^2 + b^2) \tan \theta}{a \sin \theta}, 0 \right)$

Let $M(X, Y)$ be the midpoint

$$\Rightarrow X = \frac{(a^2 + b^2) \tan \theta}{2a \sin \theta} = \frac{(a^2 + b^2)}{2 \cos \theta} \quad \therefore X = \frac{(a^2 + b^2) \sec \theta}{2a} \dots\dots\dots (1)$$

$$\Rightarrow Y = \frac{(a^2 + b^2) \tan \theta}{2b} \dots\dots\dots (2)$$

From eqn. (1)

$$\sec \theta = \frac{2aX}{(a^2 + b^2)} \Rightarrow \sec^2 \theta = \frac{2a^2 X^2}{(a^2 + b^2)^2}$$

From eqn. (2)

$$\tan \theta = \frac{2bY}{(a^2 + b^2)} \Rightarrow \tan^2 \theta = \frac{4b^2 Y^2}{(a^2 + b^2)^2}$$

From $1 + \tan^2 \theta = \sec^2 \theta$

$$\Rightarrow 1 + \frac{4b^2 Y^2}{(a^2 + b^2)^2} = \frac{2a^2 X^2}{(a^2 + b^2)^2} \text{ or } (a^2 + b^2)^2 = 4(a^2 X^2 - b^2 Y^2)$$

$$\therefore (a^2 + b^2)^2 = 4(a^2 x^2 - b^2 y^2)$$

Task

- The normal at any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x – axis at G. Q is a point on either asymptote such that PQ is parallel to the y – axis. Show that GQ is perpendicular to the asymptote on which Q lies.
- Find the eccentricity, the foci and the directrices of:
 - $\frac{1}{9}x^2 - \frac{1}{16}y^2 = 1$ **Ans:** $\frac{5}{3}, (\pm 5, 0), 5x \pm 9 = 0$
 - $5x^2 - 4y^2 = 20$ **Ans:** $\frac{3}{2}, (\pm 3, 0), 3x \pm 4 = 0$
 - $\frac{1}{4}x^2 - \frac{1}{9}y^2 = 1$ **Ans:** $\frac{1}{2}\sqrt{13}, (\pm\sqrt{13}, 0), \sqrt{13}x \pm 4 = 0$
- Prove that the locus of a point which moves so that the difference of its distances from the points $(\pm ae, 0)$ is $2a$ is $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$
- Find the equation of the tangent and normal to $\frac{1}{25}x^2 - \frac{1}{16}y^2 = 1$ at the point $\left(6\frac{1}{4}, 3\right)$. **Ans:** $4x - 3y = 16, 16y + 12x - 123 = 0$
- Find the equation of the tangent to $x = a \sec \varphi, y = b \tan \varphi$ at the point $\varphi = 1/4\pi$ **Ans:** $\frac{x\sqrt{2}}{a} - \frac{y}{b} = 1$
- Find the equation of the hyperbola whose focus is $-1, 1$ and corresponding directrix $x = y$, and whose eccentricity is 2. **Ans:** $x^2 + y^2 - 4xy + 2y - 2x - 2 = 0$

- 7.** Find the equation of a hyperbola whose focus is $(1, 0)$, directrix $x = 3$, and eccentricity $\frac{4}{3}$. **Ans:** $7x^2 - 9y^2 - 78x + 135 = 0$
- 8.** Prove that the line is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2l^2 - b^2m^2 = n^2$
- 9.** P is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P cuts Ox in T and PN is the ordinate of P. prove that $ON \cdot OT = a^2$
- 10.** ON is drawn perpendicular to the tangent at any point P on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from the origin O. prove that the locus of N is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

THE RECTANGULAR HYPERBOLA

Definition: If the asymptotes of the hyperbola are perpendicular to each other, it is called a *rectangular hyperbola*.

i) Condition for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to be a Rectangular Hyperbola

Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 0 \text{ or } y = -\frac{b}{a}x \quad \therefore m_1 = -\frac{b}{a}$$

$$\Rightarrow \frac{x}{a} - \frac{y}{b} = 0 \text{ or } y = \frac{b}{a}x \quad \therefore m_2 = \frac{b}{a}$$

For lines to be perpendicular to each other, $m_1 \times m_2 = -1$

$$\Rightarrow -\frac{b}{a} \times \frac{b}{a} = -1 \text{ or } a^2 = b^2$$

$$\therefore a = b$$

Thus the equation of a rectangular hyperbola is:

$$x^2 - y^2 = a^2$$

ii) Eccentricity of a Rectangular Hyperbola $x^2 - y^2 = 0$

From $b^2 = a^2(e^2 - 1)$, $a = b$

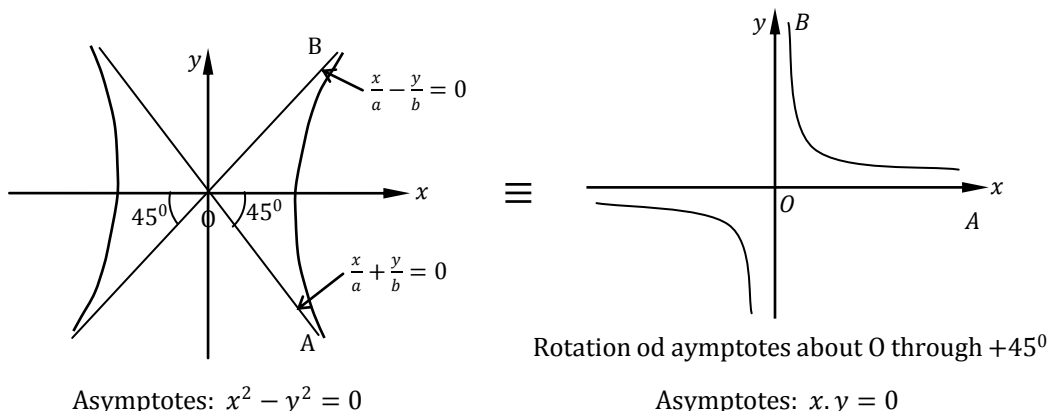
$$\Rightarrow a^2 = a^2(e^2 - 1) \quad \therefore e = \sqrt{2}$$

Eccentricity of a Rectangular Hyperbola $x^2 - y^2 = a^2$ is $e = \sqrt{2}$

iii) Equation of a Rectangular Hyperbola referred to its Asymptotes as Coordinate Axes is $xy = c^2$

Equation of the rectangular hyperbola is $x^2 - y^2 = a^2$

Asymptotes: $x^2 - y^2 = 0$ or $(x + y)(x - y) = 0$



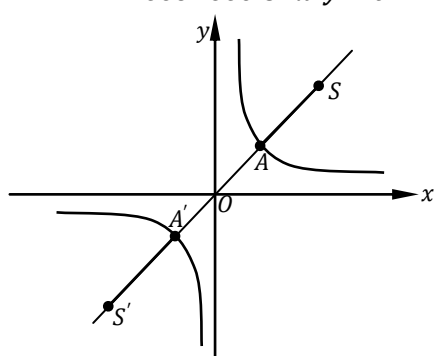
Joint equation of asymptotes after rotation through $+45^\circ$ about the origin is

$$x.y = 0$$

Since a hyperbola and its asymptotes differ by a constant, say c^2 , the required equation of a rectangular hyperbola referred to its asymptotes as axes is:

$$x.y = c^2$$

iv) Stating the Vertices, coordinates of the Foci and Equations of Directrices of $x.y = c^2$



Comparing $x.y = c^2$ with $xy = \frac{1}{2}a^2$ (Standard equation of rectangular hyperbola)

$$\Rightarrow c^2 = \frac{1}{2}a^2 \quad \therefore a = \pm c\sqrt{2}$$

From the figure, $\overline{OA} = \overline{OA'} = c\sqrt{2}$; $\overline{OS} = \overline{OS'} = ae$

Since the hyperbola is rectangular $e = \sqrt{2}$

$$\therefore ae = 2c = \overline{OS} = \overline{OS'}$$

Now the axis $SAOA'S'$ is inclined to the axes at 45°

$$x - \text{coordinate of point A} = \overline{OA} \cos 45^\circ = c\sqrt{2} \times \frac{1}{\sqrt{2}} = c = \overline{OA'}$$

$$y - \text{coordinate of point A} = \overline{OA} \sin 45^\circ = c\sqrt{2} \times \frac{1}{\sqrt{2}} = c$$

$\therefore$ Vertices are: $(\pm c, \pm c)$

$$x - \text{coordinate of point S} = \overline{OS} \cos 45^\circ = 2c \times \frac{\sqrt{2}}{2} = c\sqrt{2}$$

$$y - \text{coordinate of point S} = \overline{OS} \sin 45^\circ = 2c \times \frac{\sqrt{2}}{2} = c\sqrt{2}$$

$\therefore$ Coordinates of foci are: $(\pm c\sqrt{2}, \pm c\sqrt{2})$

Since directrices are perpendicular to the axis, let their distance from the origin be $\overline{OZ} = \frac{a}{e} = \frac{c\sqrt{2}}{\sqrt{2}} = c$

$$x - \text{coordinate of point Z} = \overline{OZ} \cos 45^\circ = c \times \frac{\sqrt{2}}{2} = \frac{c\sqrt{2}}{2}$$

$$y - \text{coordinate of point } Z = \overline{OZ} \sin 45^\circ = c \times \frac{\sqrt{2}}{2} = \frac{c\sqrt{2}}{2}$$

$\therefore$ Directrices pass through points $\left(\pm \frac{c\sqrt{2}}{2}, \pm \frac{c\sqrt{2}}{2}\right)$, and have a slope of

$$m = -1$$

Taking $\left(\frac{c\sqrt{2}}{2}, \frac{c\sqrt{2}}{2}\right)$

$$y - \frac{c\sqrt{2}}{2} = -\left(x - \frac{c\sqrt{2}}{2}\right) \therefore x + y = c\sqrt{2}$$

$\therefore$ Equation of directrices are: $x + y = \pm c\sqrt{2}$

Task

Find the coordinates of the vertices, foci and equations of directrices of the hyperbola $xy = 18$

Ans: $(\pm 3\sqrt{2}, \pm 3\sqrt{2}), (\pm 6, \pm 6), x + y = \pm 6$

v) Parametric Equations of Rectangular Hyperbola $x.y = c^2$

The general point on the rectangular hyperbola $x.y = c^2$

is $P\left(ct, \frac{c}{t}\right)$. Thus the parametric equations are:

$$x = ct, \quad y = \frac{c}{t}$$

vi) Equation of the Tangent to Hyperbola $x.y = c^2$ at $P\left(ct, \frac{c}{t}\right)$

Required Tangent:

$$y - \frac{c}{t} = m(x - ct) \dots \dots \dots (1)$$

Obtaining slope, m

Approach I:

$$m = \frac{dy}{dx} \bigg|_{at P\left(ct, \frac{c}{t}\right)}$$

Differentiating the hyperbola $x.y = c^2$

$$\Rightarrow x \frac{dy}{dx} + y = 0 \therefore \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow m = \frac{dy}{dx} \bigg|_{at P\left(ct, \frac{c}{t}\right)} = -\frac{c/t}{ct} = -\frac{1}{t^2}$$

Approach II:

$$m = \frac{dy}{dt} \times \frac{dt}{dx}$$

From $x = ct, \quad y = \frac{c}{t}$

$$\frac{dx}{dt} = c, \quad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\Rightarrow m = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{c}{t^2} \times \frac{1}{c} = -\frac{1}{t^2}$$

Eqn. (1) becomes

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) \dots\dots\dots (\times t^2)$$

$$t^2 y - ct = -x + ct$$

$$\boxed{x + t^2 y = 2 ct}$$

vii) Equation of the Normal to Hyperbola $x.y = c^2$ at $P\left(ct, \frac{c}{t}\right)$

Required Tangent:

$$y - \frac{c}{t} = m'(x - ct) \dots\dots\dots (1)$$

Obtaining slope, m'

$$m' = -\frac{1}{\text{Slope of the Tangent at P}} = -\frac{1}{-1/t^2} = t^2$$

$$\therefore y - \frac{c}{t} = t^2(x - ct) \text{ or } t^3 x - ty = c(t^4 - 1)$$

$$\boxed{t^3 x - ty = c(1 + t^4)}$$

viii) Equation of a Tangent and Normal to $x.y = c^2$ at $P(x_1, y_1)$

This point is on the curve $x.y = c^2$

Tangent equation:

From equation of tangent at $P\left(ct, \frac{c}{t}\right)$

$$x + t^2 y = 2 ct \dots\dots\dots (\div ct)$$

$$\frac{x}{ct} + \frac{t^2 y}{ct} = 2, P\left(ct, \frac{c}{t}\right) = P(x_1, y_1)$$

$$\Rightarrow \frac{x}{x_1} + \frac{y}{y_1} = 2$$

$\therefore$ Equation of the tangent to $x.y = c^2$ at $P(x_1, y_1)$ is:

$$\boxed{\frac{x}{x_1} + \frac{y}{y_1} = 2}$$

From equation of the Normal at $P\left(ct, \frac{c}{t}\right)$

$$y - \frac{c}{t} = t^2(x - ct), \text{ but } P\left(ct, \frac{c}{t}\right) = P(x_1, y_1) \text{ and } t^2 = \frac{ct}{\frac{c}{t}} = \frac{x_1}{y_1}$$

$$\Rightarrow y - y_1 = \frac{x_1}{y_1}(x - x_1)$$

$$\boxed{x.x_1 - yy_1 = x_1^2 - y_1^2}$$

ix) To Find the Equation of a Tangent to the curve $x.y = c^2$ from a point $Q(x_1, y_1)$

This point is not on the curve.

First obtaining the equation of the tangent to the curve at $P\left(ct, \frac{c}{t}\right)$.

This gives:

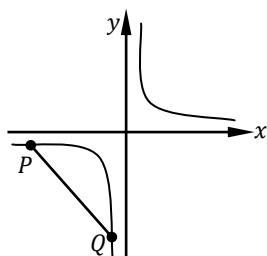
$$x + t^2 y = 2 ct \dots\dots\dots (1)$$

The point $Q(x_1, y_1)$ lies on this tangent

$$\Rightarrow x_1 + t^2 y_1 = 2 ct \text{ or } t^2 y_1 - 2 ct + x_1 = 0 \dots\dots\dots (2)$$

Eqn. (2) gives two values of t hence two tangents from point Q .

x) To Find the Equation of a Chord joining $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ on the rectangular hyperbola $x.y = c^2$



Required equation of the chord is:

$$y - \frac{c}{p} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} (x - cp)$$

$$\text{i.e. } y - \frac{c}{p} = -\frac{1}{pq} (x - cp)$$

$$\text{i.e. } y + \frac{x}{pq} = c \left(\frac{1}{p} + \frac{1}{q} \right)$$

$$\text{i.e. } pqy + x = c(p + q)$$

To Find the Equation of a Chord of the Rectangular Hyperbola $x.y = c^2$ having $M(x_1, y_1)$ as its midpoint

Let the chord have end points as $P(x', y')$ and $Q(x'', y'')$

Required chord:

$$y - y_1 = m(x - x_1) \dots\dots\dots (1)$$

Obtaining the slope of the chord

$$M(x_1, y_1) = P \left[\frac{x' + x''}{2}, \frac{y' + y''}{2} \right]$$

$$2x_1 = x' + x'' \dots\dots\dots (2)$$

$$\therefore 2y_1 = y' + y'' \dots\dots\dots (3)$$

Also P and Q lie on the hyperbola

$$\Rightarrow x' \cdot y' = c^2 \dots\dots\dots (4)$$

$$\Rightarrow x'' \cdot y'' = c^2 \dots\dots\dots (5)$$

Eqn. (3) + eqn. (4)

$$\Rightarrow x' + x'' = \frac{c^2}{y'} + \frac{c^2}{y''}$$

$$\Rightarrow x' + x'' = c^2 \left(\frac{y'' + y'}{y' y''} \right) \therefore y' y'' = \frac{c^2 \cdot 2y_1}{2x_1} = \frac{c^2 \cdot y_1}{x_1} \dots\dots\dots (6)$$

$$m = \frac{y'' - y'}{x'' - x'} = \frac{\frac{y'' - y'}{\frac{c^2}{y''} - \frac{c^2}{y'}}}{\frac{y'' - y'}{y' y''}}$$

$$= \frac{y'' - y'}{-c^2 \left(\frac{y'' - y'}{y' y''} \right)} = -\frac{y' y''}{c^2}$$

$$= -\frac{y_1}{x_1}$$

Eqn. (1) becomes

$$y - y_1 = -\frac{y_1}{x_1} (x - x_1) \Rightarrow x_1 y - x_1 y_1 = -x y_1 + x_1 y_1$$

$$\Rightarrow x_1 y + x y_1 = x_1 y_1 + x_1 y_1 = 2x_1 y_1 \dots\dots\dots (\div x_1 y_1)$$

$$\therefore \frac{x}{x_1} + \frac{y}{y_1} = 2$$

Examples

1. Find the equation of a chord joining points $(ct_1, \frac{c}{t_1})$ to $(ct_2, \frac{c}{t_2})$ on a hyperbola. Hence deduce the equation of the tangent at $(ct, \frac{c}{t})$. Find the equations of the tangents to the hyperbola $x = 4t$ $y = \frac{4}{t}$ which pass through the point $(4, 3)$.

Solution

Let the required chord be:

$$y - \frac{c}{t_1} = m(x - ct_1) \dots\dots\dots (1)$$

$$m = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} = \frac{t_1 - t_2}{(t_2 - t_1)t_1t_2} = -\frac{1}{t_1t_2}$$

$$\therefore y - \frac{c}{t_1} = -\frac{1}{t_1t_2}(x - ct_1) \dots\dots\dots (\times t_1t_2)$$

$$\therefore yt_1t_2 - ct_2 = -x + ct_1$$

$$\therefore x + yt_1t_2 = ct_2 + ct_1 = c(t_1 + t_2)$$

$$\therefore x + yt_1t_2 = c(t_1 + t_2)$$

For a line (chord) to be a tangent to the hyperbola, $t_1 \rightarrow t_2$ i.e. at $(ct_2, \frac{c}{t_2})$

$$\therefore x + yt_1t_2 = c(t_1 + t_2), \text{ becomes}$$

$$x + yt_2t_2 = c(t_2 + t_2)$$

$$x + yt_2^2 = 2ct_2, \text{ but at } (ct, \frac{c}{t}), t = t_2$$

$$\therefore x + yt^2 = 2ct, \text{ which is the tangent at } (ct, \frac{c}{t})$$

Tangents to the hyperbola $x = 4t$ $y = \frac{4}{t}$ through $(4, 3)$

From $x + yt^2 = 2ct$ and $(ct, \frac{c}{t})$, comparing $(ct, \frac{c}{t})$ with $(4t, \frac{4}{t})$

$$c = 4$$

$$\therefore \text{Tangent at } (4t, \frac{4}{t}) \text{ is: } x + yt^2 = 8t$$

This tangent pass through $(4, 3)$

$$\Rightarrow 4 + 3t^2 = 8t \text{ or } 3t^2 - 8t + 4 = 0 \text{ from which } t = 2, \frac{2}{3}$$

$\therefore$ Tangents are:

$$x + 4y = 16, \quad x + \frac{4y}{9} = \frac{16}{3} \text{ or } x + 4y = 16, \quad 9x + 4y = 48$$

2. The normal to the rectangular hyperbola $xy = 8$ at the point $(4, 2)$ meets the asymptotes at M and N . find the length MN .

Solution

Asymptotes are: $xy = 0 \Rightarrow x = 0, y = 0$

Finding the equation of the normal at $(4, 2)$

$$y - 2 = m(x - 4) \dots\dots\dots (1)$$

$$m = -\frac{1}{\left.\frac{dy}{dx}\right|_{\text{at } (4, 2)}}$$

$x.y = 8$, differentiating w.r.t x

$$y + x \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore m = -\frac{1}{-2/4} = 2$$

$$\therefore y - 2 = 2(x - 4) \text{ or } y = 2x - 6$$

$$\text{For } x = 0, y = -6 \quad \therefore M(0, -6)$$

$$\text{For } y = 0, x = 3 \quad \therefore N(3, 0)$$

$$\overline{MN} = \sqrt{(3-0)^2 + (0+6)^2} = \sqrt{45} = 3\sqrt{5} \text{ units}$$

3. Find the equation of the normal at (3, 4) to the rectangular hyperbola $x.y = 12$ and the coordinates of its second point of intersection with the curve.

Solution

$$\text{Required normal: } y - 4 = m(x - 3) \dots\dots\dots (1)$$

$$m = -\frac{1}{\left.\frac{dy}{dx}\right|_{\text{at } (4,2)}}$$

$$x.y = 12, \text{ differentiating w.r.t } x$$

$$y + x \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore m = -\frac{1}{-4/3} = \frac{3}{4}$$

$$\therefore y - 4 = \frac{3}{4}(x - 3) \text{ or } 3x - 4y + 7 = 0 \dots\dots\dots (2)$$

$$\text{From } x.y = 12 \Rightarrow x = \frac{12}{y}, \text{ putting into eqn. (2) for points of intersection}$$

$$\frac{36}{y} - 4y + 7 = 0 \text{ or } 4y^2 - 7y - 36 = 0 \quad \therefore y = 4, -\frac{9}{4}$$

$$\text{From } x = \frac{12}{y}, \text{ for } y = -\frac{9}{4}, x = -\frac{16}{3}$$

$$\therefore \left(-\frac{16}{3}, -\frac{9}{4}\right)$$

Alternatively

$$\text{Comparing } xy = 12 \text{ with } xy = c^2, c = \sqrt{12} = 2\sqrt{3}$$

$$\text{General point on rectangular hyperbola } xy = c^2 \text{ is } \left(ct, \frac{c}{t}\right)$$

$$\text{Thus the general point on } xy = 12 \text{ is } \left(2\sqrt{3}t, \frac{2\sqrt{3}}{t}\right)$$

$$\text{Obtaining the equation of the normal at } \left(2\sqrt{3}t, \frac{2\sqrt{3}}{t}\right)$$

$$y - \frac{2\sqrt{3}}{t} = m(x - 2\sqrt{3}t) \dots\dots\dots (1)$$

$$m = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\text{For } x = 2\sqrt{3}t \Rightarrow \frac{dx}{dt} = 2\sqrt{3}$$

$$\text{For } y = \frac{2\sqrt{3}}{t} \Rightarrow \frac{dy}{dt} = -\frac{2\sqrt{3}}{t^2}$$

$$\therefore m = -\frac{2\sqrt{3}}{t^2} \times \frac{1}{2\sqrt{3}} = -\frac{1}{t^2}$$

$$\therefore y - \frac{2\sqrt{3}}{t} = -\frac{1}{t^2}(x - 2\sqrt{3}t)$$

Comparing $(2\sqrt{3}t, \frac{2\sqrt{3}}{t})$ with $(3, 4)$ since they represent the same point on the curve

$$\Rightarrow 2\sqrt{3}t = 3 \quad \therefore t = \frac{1}{2}\sqrt{3}$$

$$\therefore y - \frac{2\sqrt{3}}{t} = -\frac{1}{t^2}(x - 2\sqrt{3}t), \text{ becomes}$$

$$y - \frac{2\sqrt{3}}{\sqrt{3}/2} = -\frac{1}{(\sqrt{3}/2)^2}(x - 2\sqrt{3}(\frac{1}{2}\sqrt{3}))$$

$$\therefore y - 4 = -\frac{4}{3}(x - 3), \text{ which is eqn. (2) in approach (1)}$$

$$\therefore 3x - 4y + 7 = 0 \text{ is the normal.}$$

But $(2\sqrt{3}t, \frac{2\sqrt{3}}{t})$ lies on this normal

$$3(2\sqrt{3}t) - 4\left(\frac{2\sqrt{3}}{t}\right) + 7 = 0 \text{ or } 6\sqrt{3}t^2 + 7t - 8\sqrt{3} = 0$$

$$\therefore t = \frac{-7 \pm \sqrt{7^2 - 4(6\sqrt{3})(-8\sqrt{3})}}{12\sqrt{3}} = \frac{-7 \pm 25}{12\sqrt{3}}$$

$$\therefore t = t = \frac{1}{2}\sqrt{3}, \quad t = -\frac{8}{9}\sqrt{3}$$

Putting $t = -\frac{8}{9}\sqrt{3}$ into $(2\sqrt{3}t, \frac{2\sqrt{3}}{t})$

$$\left(2\sqrt{3} \times \left(-\frac{8}{9}\sqrt{3}\right), \frac{2\sqrt{3}}{-\frac{8}{9}\sqrt{3}}\right) = \left(-\frac{16}{3}, -\frac{9}{4}\right)$$

Task

- Find the equation of the tangent and normal to $xy = 4$ at the point $(1, 4)$.

Ans: $4x + y = 8, 4y = x + 15$

- Find c if the line $y = mx + c$ is a tangent to $xy = a^2$ for all values of m .

Ans: $c = \pm 2a\sqrt{-m}$

- Find the coordinates of the vertices, coordinates of the foci and directrices of $xy = c^2$. **Ans:** $(\pm c, \pm c), (\pm c\sqrt{2}, \pm c\sqrt{2}), x + y = \pm c\sqrt{2}$

- The normal at $P(ct, \frac{c}{t})$ cuts the curve again at Q. prove that $PQ = \frac{c(1+t^4)^2}{t^3}$

- Prove that the equation of a chord of $xy = c^2$ whose midpoint is (h, k) is $\frac{x}{h} + \frac{y}{k} = 2$

- Show that the equation of the chord joining the points $(a \sec \theta, b \tan \theta)$

$$\text{and } (a \sec \phi, b \tan \phi) \text{ on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\text{Is } \frac{x}{a} \cos \frac{1}{2}(\theta - \phi) - \frac{y}{b} \sin \frac{1}{2}(\theta + \phi) = \cos \frac{1}{2}(\theta + \phi)$$

Deduce the equation of the tangent at the point $(a \sec \theta, b \tan \theta)$.

- Find the condition for the line $lx + my = n$ to touch the

$$\text{hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

By writing $+b^2$ in place of $-b^2$ deduce the condition for the same line to

$$\text{meet the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Ans: $a^2l^2 - b^2m^2 = n^2; a^2l^2 + b^2m^2 = n^2$

8. Show that the point $\left[\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{b}{2}\left(t - \frac{1}{t}\right)\right]$ always lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of t
9. Find the equations and the points of contact of the tangents to the hyperbola $2x^2 - 3y^2 = 5$ which are parallel to $8x = 9y$.
Ans: $8x - 9y = \pm 5$; $(\pm 4, \pm 3)$
10. Prove that the straight line $lx + my = n$ touches the rectangular hyperbola $xy = c^2$ if $n^2 = 4lmc^2$. Find the co-ordinates of the point of contact.
Ans: $\left(\frac{n}{2l}, \frac{2c^2l}{n}\right)$
11. Show that if the line $y = mx + c$ touches the hyperbola $x^2 - 3y^2 = 1$, then $3m^2 = 3c^2 + 1$.
 Obtain an equation for the gradients of the two tangents to the hyperbola from the point $P(x_0, y_0)$. Show that if these tangents are perpendicular then P lies on the circle $x^2 + y^2 = \frac{2}{3}$.
Ans: $m^2(x_0^2 - 1) - 2x_0y_0m + y_0^2 + \frac{1}{3} = 0$
12. Show that if the line $y = mx + c$ is a tangent to the rectangular hyperbola $x^2 - y^2 = a^2$ then $c^2 = a^2(m^2 - 1)$ and the co-ordinates of the point of contact T are $\left(-\frac{ma^2}{c}, -\frac{a^2}{c}\right)$.
 If the line meets the asymptotes at P, Q, show that T is the midpoint of PQ.
11. Find the equation of the tangents to the hyperbola $x^2 - y^2 = 7$ which are parallel to $3y = 4x$ and find their points of contact. Find the area of the triangle which one of these tangents makes with the asymptotes.
Ans: $4x - 3y = \pm 7$, $(4, 3)$ and $(-4, -3)$. 7 square units

LOCUS PROBLEMS ON RECTANGULAR HYPERBOLA $x \cdot y = c^2$

1. The tangent at any point $P(ct, c/t)$ on the hyperbola $x \cdot y = c^2$ meets x and y - axis at A and B respectively. O is the origin. Prove that:
- i) $\overline{AP} = \overline{PB}$ ii) The area of $\triangle AOB$ is a constant

Solution

Find the tangent equation as:

$$x + t^2y = 2ct \dots\dots\dots (1)$$

Obtaining the x, y - intercept

For y - intercept, $x = 0$

$$\Rightarrow t^2y = 2ct \text{ or } y = \frac{2c}{t} \therefore B(0, 2c/t)$$

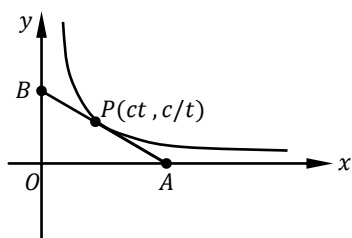
For x - intercept, $y = 0$

$$\Rightarrow x = 2ct \text{ or } \therefore A(2ct, 0)$$

$$\text{Obtaining } \overline{AP} = \sqrt{(2ct - ct)^2 + (0 - c/t)^2} = c\sqrt{t^2 + 1/t^2}$$

$$\text{Obtaining } \overline{PB} = \sqrt{(0 - ct)^2 + \left(\frac{c}{t} - \frac{2c}{t}\right)^2} = c\sqrt{t^2 + 1/t^2}$$

$$\therefore \overline{AP} = \overline{PB} = c\sqrt{t^2 + 1/t^2}$$

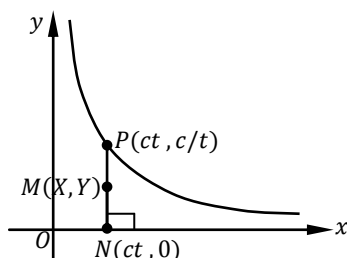


$$\begin{aligned}\text{Area of } \triangle AOB &= \frac{1}{2} \times \overline{OA} \times \overline{OB} \\ &= \frac{1}{2} \times 2ct \times \frac{2c}{t} = 2c^2\end{aligned}$$

Hence there is no variable t in the expression of area.

2. PN is the perpendicular to an asymptote from a point on a rectangular hyperbola $x.y = c^2$. Prove that the locus of the midpoint of PN is a rectangular hyperbola with the same axes.

Solution



$$\text{Midpoint of } PN = M(X, Y) = M\left(\frac{ct+ct}{2}, \frac{\frac{c}{t}+0}{2}\right)$$

$$\therefore X = ct, Y = \frac{c}{2t}$$

Multiplying the equations

$$X.Y = ct \times \frac{c}{2t} = \frac{c^2}{2}$$

$\therefore xy = \frac{c^2}{2}$, which is a rectangular hyperbola with x, y - axes

3. PQ is a chord of the rectangular hyperbola $x.y = c^2$ and R is the midpoint of PQ . If PQ has a constant length k , find the locus of R .

Solution

Let $P(cp, c/p)$, $Q(cq, c/q)$ and $R(X, Y)$ be the given points

$$\Rightarrow X = \frac{1}{2}c(p+q) \dots \dots \dots (1)$$

$$\Rightarrow Y = \frac{1}{2}c\left(\frac{p+q}{pq}\right) \dots \dots \dots (2)$$

$$\text{But } \overline{PQ} = k \text{ or } \overline{PQ}^2 = k^2$$

$$\Rightarrow \overline{PQ}^2 = k^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$$

$$\Rightarrow k^2 = c^2(p - q)^2 + c^2\left(\frac{1}{p} - \frac{1}{q}\right)^2$$

$$\Rightarrow k^2 = c^2(p - q)^2 + c^2\left(\frac{q-p}{pq}\right)^2$$

$$\Rightarrow k^2 = c^2(p - q)^2 \left[1 + \frac{1}{p^2q^2}\right]$$

$$\text{But } (p - q)^2 = p^2 + q^2 - 2pq = (p + q)^2 - 4pq$$

$$\therefore k^2 = c^2[(p + q)^2 - 4pq] \left[1 + \frac{1}{p^2q^2}\right] \dots \dots \dots (3)$$

The task is to eliminate variables p, q from the formulated equations

From eqn. (1)

$$p + q = \frac{2X}{c} \dots \dots \dots (4)$$

$$\text{Eqn. (1)} \div \text{eqn. (2)}$$

$$\frac{x}{y} = pq \dots \dots \dots (5)$$

Eqn. (4), eqn. (5) into eqn. (3)

$$\Rightarrow k^2 = c^2 \left[\left(\frac{2X}{c} \right)^2 - 4 \left(\frac{X}{Y} \right) \right] \left[1 + \frac{1}{(X/Y)^2} \right]$$

$$\Rightarrow k^2 = c^2 \left[\frac{4X^2}{c^2} - \frac{4X}{Y} \right] \left[\frac{Y^2 + X^2}{X^2} \right]$$

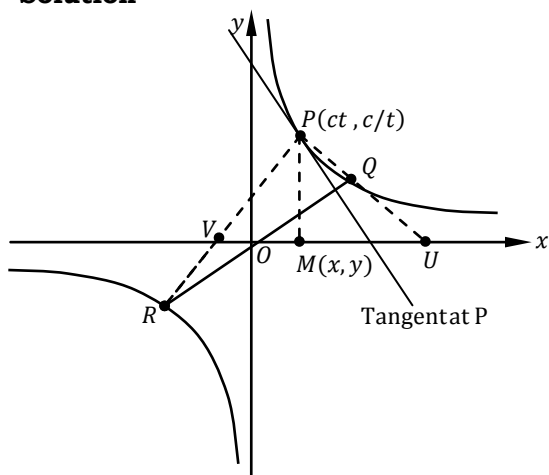
$$\Rightarrow X^2 Y k^2 = (4X^2 Y - 4X c^2)(X^2 + Y^2), \text{ reducing both sides by } X$$

$$\Rightarrow XY k^2 = (4XY - 4c^2)(X^2 + Y^2)$$

$$\therefore 4(X^2 + Y^2)(xy - c^2) = xy k^2, \text{ is the required locus}$$

4. The perpendicular from the origin to the tangent at a point P on the rectangular hyperbola $x.y = c^2$ meets the curve at Q and R. the chords PQ and PR meet the x -axis at u and v . prove that the midpoint of uv is the foot of the perpendicular from P to the x -axis

Solution



To prove: $M(x, y)$ is the midpoint of uv .

Let $P(ct, c/t)$ be any point on $x.y = c^2$

$$x = ct, y = c/t$$

Obtaining the tangent at P

$$x + t^2 y = 2ct \dots \dots \dots (1)$$

Obtaining the equation of a line through origin and perpendicular to the tangent
 $y - 0 = m'(x - ct) \Rightarrow y = m'x$

$$\text{But } m' = -\frac{1}{\frac{dy}{dx} \big|_{\text{at } P}} = -\frac{1}{-1/t^2} = t^2$$

$$\therefore y = t^2 x$$

Obtaining the points of intersection of $xy = c^2$ and $y = t^2 x$

$$\Rightarrow t^2 x^2 = c^2 \quad \therefore x = \pm \frac{c}{t}$$

$$\text{For } x = \frac{c}{t}, y = t^2 \times \frac{c}{t} = ct \quad \therefore Q \left(\frac{c}{t}, ct \right)$$

$$\text{For } x = -\frac{c}{t}, y = t^2 \times -\frac{c}{t} = -ct \quad \therefore R \left(-\frac{c}{t}, -ct \right)$$

Obtaining the equation of the chord PQ

$$y - \frac{c}{t} = m_1(x - ct), \text{ but } m_1 = \frac{\frac{c}{t} - ct}{ct - c/t} = -1$$

$$\Rightarrow y - \frac{c}{t} = -1(x - ct) \text{ or } y = -x + ct + c/t$$

Finding coordinates of $u, y = 0$

$$0 = -x + ct + \frac{c}{t} \quad \therefore x = ct + c/t$$

$$\therefore u \left(ct + \frac{c}{t}, 0 \right)$$

Obtaining the equation of the chord PR

$$y - \frac{c}{t} = m_2(x - ct), \text{ but } m_1 = \frac{\frac{c}{t} + ct}{ct + c/t} = 1$$

$$\Rightarrow y - \frac{c}{t} = (x - ct) \text{ or } y = x - ct + c/t$$

Finding coordinates of v, y = 0

$$0 = x - ct + \frac{c}{t} \quad \therefore x = ct - c/t$$

$$\therefore u \left(ct - \frac{c}{t}, 0 \right)$$

$$\text{Now midpoint of } uv = M(x, y) = \left(\frac{ct + \frac{c}{t} + ct - \frac{c}{t}}{2}, \frac{0+0}{2} \right) = (ct, 0)$$

But abscissa of P is $x = ct$

$\therefore$ Foot of perpendicular from P to x - axis is $(ct, 0)$

Task

1. A chord PQ of the rectangular hyperbola $x.y = c^2$ subtends a right angle at another point R on the curve. Prove PQ parallel to the normal at R.
2. The normal at a point P on $x.y = c^2$ meets the asymptotes in Q and R and the transverse axis in G. prove that $PG^2 = PQ.PR$
3. N is the foot of the perpendicular from the point $(0, 0)$ on to the tangent at point P on $x.y = c^2$. Prove that the locus of N is $(x^2 + y^2)^2 = 4c^2xy$
4. If chords of a constant length $2d$ are drawn in the curve $x.y = c^2$ prove the locus of their midpoint is $(x^2 + y^2)(xy - c^2) = d^2xy$
5. Show that the equation of the line joining points $(cp, c/p), (cq, c/q)$ on the rectangular hyperbola $xy = c^2$ is $x + pqy = c(p + q)$. Deduce the equation of the tangent at the point P. **Ans:** $x + p^2y = 2cp$
6. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$ meets the curve again at $Q(ct_1, c/t_1)$ such that $t^3t_1 = -1$
7. Show that the locus of the midpoint of the normal chords of a rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$

Conclusion

When handling numbers involving conic section,

- Knowledge of Coordinate Geometry I is relevant
- Try to interpret the question with an aid of a clear and illustrative diagram
- Knowledge of Algebra when handling equations involving parameters is also necessary
- Your ability to handle and interpret the question carefully determines how quick can you reach to the required answer
- DO NOT ASSUME THAT CONIC SECTION IS HARD.

MACLAURIN'S THEOREM

This is used in polynomial approximations for a function, estimating of values of constants etc.

Under certain conditions, a function $f(x)$ is expressed as an infinite series of ascending powers of x .

Derivation of Maclaurin's Theorem

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_rx^r + \dots$

Differentiating with respect to x

$$f'(x) = a_1 + a_2x + 2a_3x^2 + \dots + ra_rx^{r-1} + \dots$$

$$f''(x) = 2a_2 + 6a_3x + \dots + r(r-1)a_rx^{r-2} + \dots$$

$$f'''(x) = 6a_3 + \dots + r(r-1)(r-2)a_rx^{r-3} + \dots$$

$$f^r(x) = r(r-1)(r-2) \dots 3 \times 2 \times a_r + \dots$$

Substituting $x = 0$

$$f(0) = a_0 \qquad \qquad \qquad \therefore a_0 = f(0)$$

$$f'(0) = a_1 \qquad \qquad \qquad \therefore a_1 = f'(0)$$

$$f''(0) = 2a_2 \qquad \qquad \qquad \therefore a_2 = \frac{f''(0)}{2}$$

$$f'''(0) = 3 \cdot 2 a_3 = 6a_3 \qquad \qquad \qquad \therefore a_3 = \frac{f'''(0)}{3!}$$

$$\text{And } f^r(0) = r \cdot (r-1)(r-2) \dots 3 \times 2 a_r \qquad \qquad \qquad \therefore a_r = \frac{f^r(0)}{r!}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^r(0)}{r!}x^r$$

NOTE

- The Maclaurin's series for a function $f(x)$ can be obtained if the derivatives of $f'(0), f''(0), \dots$ can be determined.
- For some functions, Maclaurin's theorem holds for only within restricted range of values of x .

Derivation of Various Important Expressions by Maclaurin's Theorem

a) Binomial Theorem

Let $f(x) = (1+x)^n$, n is a real number

$$f'(x) = n(1+x)^{n-1}$$

$$f''(x) = n(n-1)(1+x)^{n-2}$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$$

$$\dots \dots \dots f^r(x) = n(n-1)(n-2) \dots (n-r+1)(1+x)^{n-r}$$

$$\therefore f(0) = 1, f'(0) = n, f''(0) = n(n-1), f'''(0) = n(n-1)(n-2), \text{ and}$$

$$f^r(0) = n(n-1)(n-2) \dots (n-r+1)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

, which is valid for $|x| < 1$ or $-1 < x < 1$

b) Exponential Series, e^x

Let $f(x) = e^x$, then $f(0) = e^0 = 1$

$f'(x) = e^x$, then $f'(0) = e^0 = 1$

$f''(x) = e^x$, then $f''(0) = e^0 = 1$

... ..

$f^r(x) = e^x$, then $f^r(0) = e^0 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

, which valid for all values of x

c) Logarithm Series, $\ln(1+x)$

Let $f(x) = \ln(1+x)$, then $f(0) = \ln(1) = 0$

$f'(x) = (1+x)^{-1}$, then $f'(0) = 1$

$f''(x) = -(1+x)^{-2}$, then $f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$, then $f'''(0) = 2$

$f^4(x) = -6(1+x)^{-4}$, then $f^4(0) = -6$

... ..

$f^r(x) = (-1)^{r+1}(r-1)!(1+x)^{-r}$, then $f^r(0) = (-1)^{r+1}(r-1)!$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{r+1} \frac{x^r}{r} + \cdots$$

NOTE

- The function $\ln x$ is not defined at $x = 0$ and thus no Maclaurin's series
- The expansion of $\ln(1+x)$ is valid for $-1 < x \leq 1$

d) Expansion of $\sin x$

Let $f(x) = \sin x$, then $f(0) = \sin 0 = 0$

$f'(x) = \cos x$, then $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$, then $f''(0) = \sin 0 = 0$

$f'''(x) = -\cos x$, then $f'''(0) = -\cos 0 = -1$

$f^4(x) = \sin x$, then $f^4(0) = \sin 0 = 0$

$f^5(x) = \cos x$, then $f^5(0) = \cos 0 = 1$

... ..

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

e) **Expansion of $\cos x$**

In a similar way

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

The expansion of $\sin x$ and $\cos x$ are valid for all values of x (in radians)

General considerations for Validity of an expansion

Expansion	Condition
$(1+x)^n$	$ x < 1$ or $-1 < x < 1$
$\ln(1+x)$	$-1 < x \leq 1$
$\tan^{-1} x$	$-1 \leq x \leq 1$
$e^x, \sin x, \cos x$	valid for all values of x

Examples

1. Assuming that x is so small that terms in x^3 and higher powers may be neglected, find a quadratic approximation to $\sqrt{\frac{1-x}{1+2x}}$

Solution

$$\begin{aligned}\sqrt{\frac{1-x}{1+2x}} &= (1-x)^{1/2} (1+2x)^{-1/2} \\ (1-x)^{1/2} &= 1 + \frac{1}{2}(-x) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-x)^2 + \dots \\ &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \dots \\ (1+2x)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)(2x) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(2x)^2 + \dots \\ &= 1 - x + \frac{3}{2}x^2 + \dots\end{aligned}$$

Neglecting terms in x^3 and higher power terms

$$\begin{aligned}\sqrt{\frac{1-x}{1+2x}} &\approx \left(1 - \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 - x + \frac{3}{2}x^2\right) \\ &\approx 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2 \\ \therefore \sqrt{\frac{1-x}{1+2x}} &\approx 1 - \frac{3}{2}x + \frac{15}{8}x^2\end{aligned}$$

2. Expand $\frac{1}{1+x+2x^2}$ in ascending powers of x up to and including the term in x^3

Solution

Let $y = x + 2x^2$

$$\begin{aligned}\frac{1}{1+y} &= (1+y)^{-1} = 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \frac{(-1)(-2)(-3)}{3!}y^3 + \dots \\ &= 1 - y + y^2 - y^3 + \dots\end{aligned}$$

Substituting $y = x + 2x^2$

$$\begin{aligned}\frac{1}{1+x+2x^2} &= 1 - (x + 2x^2) + (x + 2x^2)^2 - (x + 2x^2)^3 + \dots \\ &= 1 - x - 2x^2 + x^2 + 4x^3 - x^3 + \dots \\ &= 1 - x - x^2 + 3x^3 + \dots\end{aligned}$$

3. Expand $\frac{1}{(1-x)(1+2x)}$ in ascending powers of x

Solution

$$\frac{1}{(1-x)(1+2x)} = (1+x-2x^2)^{-1} = 1 - (x - 2x^2) + (x - 2x^2)^2 + \dots$$

Alternatively

$$\begin{aligned}\frac{1}{(1-x)(1+2x)} &= (1-x)^{-1}(1+2x)^{-1} \\ &= (1+x+x^2+\dots)(1-2x+4x^2+\dots)\end{aligned}$$

NOTE

If the general formula for the term in x^r is required, a method of partial fractions can be used.

4. Expand $\frac{1}{(1-x)(1+2x)}$ in ascending powers of x giving the first three terms and the term in x^r . State the values of x for which the expansion is valid.

Solution

$$\begin{aligned}\frac{1}{1-x} &= (1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \dots + \frac{(-1)(-2)\dots(-r)}{r!}(-x)^r \\ \therefore \frac{1}{1-x} &= 1 + x + x^2 + \dots + x^r + \dots\end{aligned}$$

Similarly

$$\frac{1}{1+2x} = (1+2x)^{-1} = 1 - 2x + 4x^2 + \dots + (-2)^r \cdot x^r + \dots$$

$$\text{Let } \frac{1}{(1-x)(1+2x)} = \frac{A}{1-x} + \frac{B}{1+2x}$$

$$\Rightarrow 1 \equiv A(1+2x) + B(1-x)$$

$$\text{Let } x = 1, 1 = 3A \quad \therefore A = \frac{1}{3}$$

$$x = -\frac{1}{2}, 1 = \frac{3}{2}B \quad \therefore B = \frac{2}{3}$$

$$\begin{aligned}\therefore \frac{1}{(1-x)(1+2x)} &= \frac{1}{3}(1-x)^{-1} + \frac{2}{3}(1+2x)^{-1} \\ &= \frac{1}{3}(1+x+x^2+\dots+x^r+\dots) + \frac{2}{3}(1-2x+4x^2+\dots+(-2)^r \cdot x^r+\dots) \\ &= 1 - x + 3x^2 + \dots + \left[\frac{1}{3} + \frac{2}{3}(-2)^r\right]x^r + \dots \\ &= 1 - x + 3x^2 + \dots + \frac{1}{3}[1 - (-2)^{r+1}]x^r + \dots\end{aligned}$$

The expansion of $(1-x)^{-1}$ is valid for $|x| < 1$ and the expansion $(1+2x)^{-1}$ is valid for $|2x| < 1$, i.e. $|x| < \frac{1}{2}$

Combining the two validities $|x| < \frac{1}{2}$ is common. So the expansion

$$\frac{1}{(1-x)(1+2x)} \text{ is valid for } |x| < \frac{1}{2}$$

5. Expand e^{2x} in ascending powers of x giving the first three terms and term in x^r .

Solution

Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots$, for all values of x

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots + \frac{(2x)^r}{r!} + \dots$$

$$e^{2x} = 1 + 2x + 2x^2 + \dots + \frac{2^r}{r!}x^r + \dots$$

6. Find the first three terms in the expansion of e^{x+x^2} in ascending powers of x

Solution

$$e^{x+x^2} = 1 + (x + x^2) + \frac{(x+x^2)^2}{2!} + \dots$$

$$= 1 + x + x^2 + \frac{1}{2}x^2 + \dots$$

$$= 1 + x + \frac{3}{2}x^2 + \dots$$

7. Expand e^{2-x} in ascending powers of x as far as term in x^3

Solution

$$e^{2-x} = e^2 \times e^{-x} = e^2 \left[1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots + \frac{x^r}{r!} + \dots \right]$$

$$= e^2 \left[1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots + \frac{x^r}{r!} + \dots \right]$$

8. Expand $\ln(1-x)$ in ascending powers of x . Hence state the validity of the expansion.

Solution

$$\text{Using } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{r+1} \frac{x^r}{r} + \dots$$

, where $-1 < x \leq 1$

Replacing x by $-x$

$$\Rightarrow \ln(1-x) = x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \dots + (-1)^{r+1} \frac{(-x)^r}{r} + \dots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^r}{r} - \dots$$

where $-1 < -x \leq 1$ or $1 > x \geq -1$

$\therefore$ Validity is $-1 \leq x < 1$

9. Expand $\ln(2+3x)$ in ascending powers of x as far as the term in x^3 . State the values of x for which the expansion is valid.

Solution

$$\ln(2+3x) = \ln 2 + \ln \left(1 + \frac{3}{2}x \right)$$

$$= \ln 2 + \left(\frac{3}{2}x \right) - \frac{1}{2} \left(\frac{3}{2}x \right)^2 + \frac{1}{3} \left(\frac{3}{2}x \right)^3 - \dots$$

$$= \ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{8}x^3 - \dots$$

The expansion for validity is $-1 < \frac{3}{2}x \leq 1$ or $-\frac{2}{3} < x \leq \frac{2}{3}$

10. Expand $\ln \left(\frac{1+x}{\sqrt{1-2x}} \right)$ in ascending powers of x as far as the term in x^3 , stating the values of x for which the expansion is valid.

Solution

$$\begin{aligned}\ln\left(\frac{1+x}{\sqrt{1-2x}}\right) &= \ln(1+x) - \frac{1}{2}\ln(1-2x) \\ &= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots\right\} - \frac{1}{2}\left\{(-2x) - \frac{1}{2}(-2x)^2 + \frac{1}{3}(-2x)^3 - \dots\right\} \\ &= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots\right\} + \left\{x + x^2 + \frac{4}{3}x^3 - \dots\right\} \\ &= 2x + \frac{1}{2}x^2 + \frac{5}{3}x^3 + \dots\end{aligned}$$

The expansion of $\ln(1+x)$ is valid for $-1 < x \leq 1$, and the expansion of $\ln(1-2x)$ is valid for $-1 < -2x \leq 1$ or $-\frac{1}{2} \leq x < \frac{1}{2}$

- 11.** Assuming that x is so small that x^3 and higher power terms may be neglected, obtain an equivalent approximation to the function $\frac{1+e^x}{1+\ln(1+x)}$

Solution

$$1 + e^x = 1 + \left(1 + x + \frac{x^2}{2!} + \dots\right) \simeq 2 + x + \frac{1}{2}x^2$$

$$1 + \ln(1+x) = 1 + \left(x - \frac{x^2}{2} + \dots\right) \simeq 1 + x - \frac{1}{2}x^2$$

Using Binomial expansion

$$(1+y)^{-1} = 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \dots = 1 - y + y^2 - \dots$$

Substituting $y = x - \frac{1}{2}x^2$

$$\begin{aligned}\Rightarrow \frac{1}{1+\ln(1+x)} &\simeq \left[1 + \left(x - \frac{1}{2}x^2\right)\right]^{-1} \\ &= 1 - \left(x - \frac{1}{2}x^2\right) + \left(x - \frac{1}{2}x^2\right)^2 - \dots \\ &= 1 - x + \frac{1}{2}x^2 + \dots\end{aligned}$$

$$\begin{aligned}\therefore \frac{1+e^x}{1+\ln(1+x)} &\simeq \left(2 + x + \frac{1}{2}x^2\right) \cdot \left(-x + \frac{1}{2}x^2\right) \\ &= 2 - 2x + 3x^2 + x - x^2 + \frac{1}{2}x^2 \\ &= 2 - x + \frac{5}{2}x^2\end{aligned}$$

- 12.** Expand $e^{\sin x}$ in ascending powers of x , neglecting terms in x^4 and higher power terms of x .

Solution

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = x - \frac{1}{6}x^3 + \dots$$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \dots$$

Substituting $y = x - \frac{1}{6}x^3$

$$\begin{aligned}e^{\sin x} &= 1 + \left(x - \frac{1}{6}x^3\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{6}\left(x - \frac{1}{6}x^3\right)^3 + \dots \\ &= 1 + x - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ &= 1 + x + \frac{1}{2}x^2\end{aligned}$$

- 13.** Expand $\sec x$ in ascending powers of x , as far term in x^4

Solution

$$\begin{aligned}\sec x &= (\cos x)^{-1} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)^{-1} \\ &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)^{-1}\end{aligned}$$

Using Binomial expansion

$$(1 + y)^{-1} = 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \dots = 1 - y + y^2 - \dots$$

Thus substituting $y = -\frac{x^2}{2} + \frac{x^4}{24}$

$$\begin{aligned} \sec x &= 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 - \dots \\ &= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \dots \\ &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots \end{aligned}$$

- 14.** Find the expansion in ascending powers of x for $\ln(1 - x - 6x^2)$, giving the terms up to and including that in x^4 , and state the values of x for which the expansion is valid.

Solution

$$\ln(1 - x - 6x^2) = \ln[(1 - 3x)(1 + 2x)] = \ln(1 - 3x) + \ln(1 + 2x)$$

$$\text{Using } \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\begin{aligned} \Rightarrow \ln(1 - x - 6x^2) &= \left(-3x - \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3} - \frac{(-3x)^4}{4} + \dots\right) + \\ &\quad + \left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots\right) \end{aligned}$$

$$\therefore \ln(1 - x - 6x^2) = -x - \frac{13}{2}x^2 - \frac{19}{3}x^3 - \frac{97}{4}x^4 - \dots$$

The expansion for $\ln(1 - 3x)$ is valid for $-1 < -3x \leq 1$. i.e. $-\frac{1}{3} \leq x < \frac{1}{3}$

The expansion for $\ln(1 + 2x)$ is valid for $-1 < 2x \leq 1$. i.e. $-\frac{1}{2} \leq x < \frac{1}{2}$

As both expansions have been used, the expansion is valid for $-\frac{1}{3} \leq x < \frac{1}{3}$

- 15.** Use Maclaurin's theorem to find the first three terms non-zero terms in the expansion of $\sin x$. hence find $\sin(0.1)$ rad correct to 7 d.p

Solution

From the notes above a student must show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \text{ is the required expansion.}$$

$$\begin{aligned} \text{Thus } \sin(0.1) \text{ rad} &\approx 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} \\ &= 0.09983341 \end{aligned}$$

$$\therefore \sin(0.1) \text{ rad} = 0.09983341 \text{ Correct to 7 d.p}$$

Task

- 1.** Obtain the series of the expansion of $\frac{\ln(1+x)}{2+x}$ up to and including the term in x^4 and state the values of x for which the expansion is valid.

$$\text{Ans: } \frac{1}{2}x - \frac{1}{2}x^2 + \frac{5}{12}x^3 - \frac{1}{3}x^4; -1 < x \leq 1$$

- 2.** Obtain the expansion in ascending powers of x of the function $f(x) = \ln\left\{\frac{(1-3x)^2}{(1-2x)}\right\}$. Give the values of the coefficients of the powers of x up to and including x^3 , and give an expression for the coefficient of x^n . For what range of values of x is the expansion valid?

$$\text{Ans: } 1 - x + \frac{5}{2}x^2 - \frac{29}{6}x^3, |x| < \frac{1}{2}$$

INEQUALITIES IN ONE VARIABLE

To Solve and Graph Inequalities in One Variable

Properties of Inequalities

1. If $a < b$, then $a + c < b + c$. Here the symbol does not *reverse* when you add/subtract a real number on both sides of the inequality.
2. If $a < b$ and $c > 0$, then $ac < bc$ or $\frac{a}{c} < \frac{b}{c}$. Here the inequality symbol does not *reverse*; when you multiply/divide an inequality by a positive number.
3. If $a < b$ and $c < 0$, then $ac > bc$ or $\frac{a}{c} > \frac{b}{c}$. Here the inequality symbol *reverses*; when you multiply/divide an inequality by a negative number.

Examples

Solve the inequality

a) $3x - 4 \leq 10 + x$

b) $\frac{8-2x}{5} > 6$

Solution

a) $3x - 4 \leq 10 + x \Rightarrow 2x \leq 14 \quad \therefore x \leq 7$

b) $\frac{8-2x}{5} > 6 \Rightarrow 8 - 2x > 30$

$\Rightarrow -2x > 22 \quad \therefore x < -11$ (Symbol reverses)

Graphing Linear Inequalities

Inequalities containing symbols $\leq$ or $\geq$ represent end points of inequalities to be included in the graph. Usually it is represented as (●), solid dot and if $<$ or $>$ are used, the end points are not included in the graph. Usually it is denoted as (○)

Example

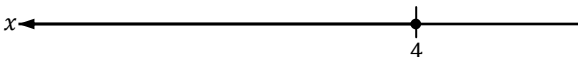
1. Represent the inequalities on a graph

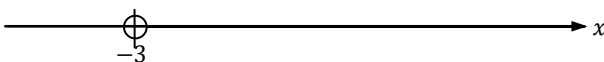
a) $x \leq 4$

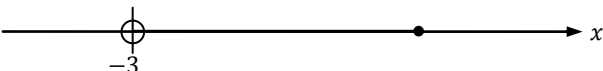
b) $x > -3$

c) $-3 < x \leq 4$

Solution

a) For $x \leq 4$ 

b) For $x > -3$ 

c) For $-3 < x \leq 4$ 

2. Solve the inequality and graph its solution

a) $\frac{4}{3}\left(x - \frac{1}{2}\right) + \frac{1}{2}x \geq \frac{2}{3}\left(2x - \frac{5}{2}\right)$

b) $\frac{x+2}{4} - \frac{2-x}{3} < 4 - \frac{4x-5}{6}$

Solution

a) $\frac{4}{3}\left(x - \frac{1}{2}\right) + \frac{1}{2}x \geq \frac{2}{3}\left(2x - \frac{5}{2}\right) \Rightarrow \frac{4x}{3} - \frac{2}{3} + \frac{1}{2}x \geq \frac{4x}{3} - \frac{5}{3}$

Multiply through by 6 (L.C.M of 2 and 3)

$$\frac{24x}{3} - \frac{12}{3} + \frac{6x}{3} \geq \frac{24x}{3} - \frac{30}{3}$$

$$8x - 4 + 2x \geq 8x - 10 \Rightarrow 2x \geq -6 \quad \therefore x \geq -3$$

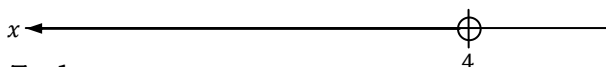
b) $\frac{x+2}{4} - \frac{2-x}{3} < 4 - \frac{4x-5}{6}$

Multiply through by 12 (L.C.M of 3, 4 and 6)

$$\frac{12(x+2)}{4} - \frac{12(2-x)}{3} < 4(12) - \frac{12(4x-5)}{6}, \text{ re arranging}$$

$$3x + 6 - 8 + 4x + 8x - 10 < 48$$

$$15x - 12 < 48 \quad \therefore x < 4$$



Task

Solve and graph the inequality

a) $\frac{15-6x}{3} > 5$

Ans: $x < 0$

b) $8x + 3(x + 1) > 5x - (9x - 6)$

Ans: $x > \frac{1}{5}$

POLYNOMIAL INEQUALITIES IN ONE VARIABLE

To Solve and Graph Polynomial Inequalities in One variable

Let $P(x)$ be any polynomial, then $P(x) < 0$ and $P(x) > 0$ are called *Polynomial Inequalities*

Two ways of solving polynomial inequalities are:

- Use a sign graph of $P(x)$
- Analyze the graph of $P(x)$

Note:

- If $P(x) > 0$, the graph $y = P(x)$ is said to be *Above the x - axis*
- If $P(x) < 0$, the graph $y = P(x)$ is said to be *Below the x - axis*

If $P(x)$ is factorisable, then the easiest method to be used is sign graph method.

Example

1. Solve the inequality and hence sketch $P(x)$

a) $x^3 - 2x^2 - 3x < 0$; $P(x) < 0$

b) $(x + 1)(x - 1)(x - 2) > 0$; $P(x) > 0$

c) $x^3 + 5x^2 + 3x \leq 9$; $P(x) \leq 0$

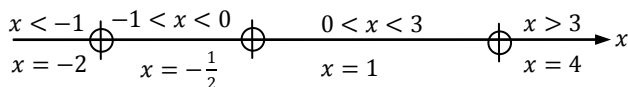
Solution

a) $x^3 - 2x^2 - 3x < 0$; $P(x) = x^3 - 2x^2 - 3x$

Finding the zeroes of $P(x) = x^3 - 2x^2 - 3x$

$$P(x) = x^3 - 2x^2 - 3x = 0 \quad \therefore x = 0, -1, 3$$

Plotting the zeroes on a number line



Use open dots since $P(x)$ is *less than* zero.

$P(x)$ in each interval: $P(x) = x(x + 1)(x - 3)$

$$P(-2) = -2(-2+1)(-2-3) = -$$

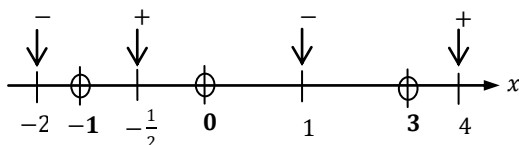
$$P\left(-\frac{1}{2}\right) = -\frac{1}{2}\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}-3\right) = +$$

$$P(1) = 1(1+1)(1-3) = -$$

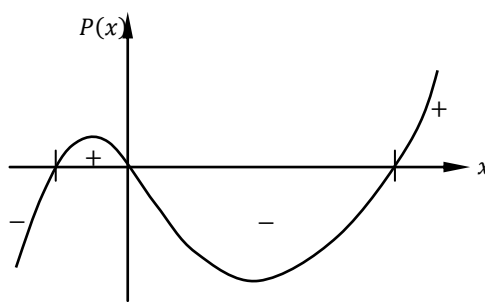
$$P(4) = 4(4+1)(4-3) = +$$

∴ The solution of $P(x) < 0$ is

$$x < -1 \text{ or } 0 < x < 3$$



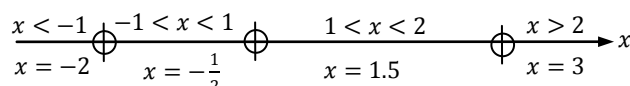
Now sketching $P(x)$



b) $(x+1)(x-1)(x-2) > 0$; $P(x) = (x+1)(x-1)(x-2)$

Finding the zeroes of $P(x) = (x+1)(x-1)(x-2)$ ∴ $x = \pm 1, 2$

Plotting the zeroes on a number line



Use open dots since $P(x)$ is *greater than* zero.

Testing $P(x)$ for each interval

$$P(x) = (x+1)(x-1)(x-2)$$

$$P(-2) = (-2+1)(-2-1)(-2-2) = -$$

$$P(-0.5) = (-0.5+1)(-0.5-1)(-0.5-2) = +$$

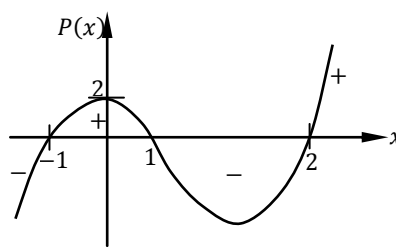
$$P(1.5) = (1.5+1)(1.5-1)(1.5-2) = -$$

$$P(2) = (2+1)(2-1)(2-2) = +$$

Since $P(x) > 0$, then the solution is:

$$-1 < x < 1 \text{ and } x > 2$$

Now sketching $P(x)$



c) $x^3 + 5x^2 + 3x \leq 9$; $P(x) = x^3 + 5x^2 + 3x - 9$

Let the zeroes of $P(x)$ be $x = \frac{p}{q}$ where p divides -9 and q divides 1

Possible values of p : $\pm 1, \pm 3, \pm 9$

For $P(1) = 1 + 5 + 3 - 9 = 0$ $(x+1)$ is a factor

For $P(-3) = (-3)^3 + 5(-3)^2 + 3(-3) - 9 = 0$ $(x+3)$ is a factor

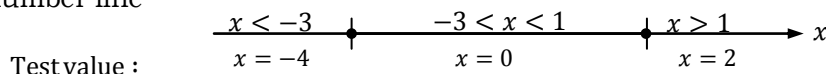
∴ $(x-1)(x+3) = x^2 + 2x - 3$ is a factor of $P(x)$

$$\begin{array}{r} x^2 + 2x - 3 \overline{) x^3 + 5x^2 + 3x - 9} \\ \underline{x^3 + 2x^2 - 3x} \\ 3x^2 + 6x^2 - 9 \\ \underline{3x^2 + 6x^2 - 9} \\ 0 \end{array}$$

∴ $P(x) = (x+3)(x+3)(x-1)$

∴ Zeroes of $P(x)$ are: $x = -3, 1$

On number line



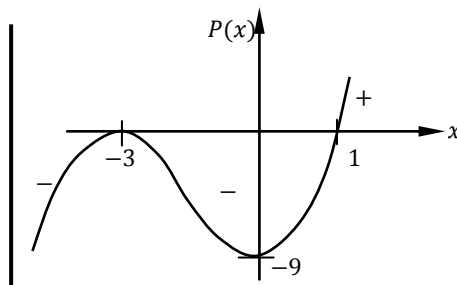
Solid circles are used.

Testing $P(x)$ for each interval

$$P(-4) = (-4 + 3)(-4 + 3)(-4 - 1) = -$$

$$P(0) = (0 + 3)(0 + 3)(0 - 1) = -$$

$$P(2) = (2 + 3)(2 + 3)(2 - 1) = +$$



2. Solve:

a) $(x^2 - 1)(x - 4)^2 \geq 0$

b) $x^4 - 3x^2 - 10 > 0$

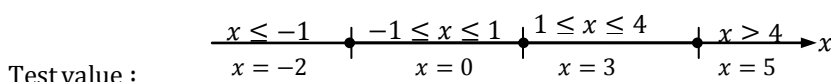
Solution

a) Let $P(x) = (x^2 - 1)(x - 4)^2 = (x + 1)(x - 1)(x - 4)^2$; $P(x) \geq 0$

Finding zeroes of $P(x) = (x^2 - 1)(x - 4)^2$

$$(x + 1)(x - 1)(x - 4)^2 = 0 \quad \therefore x = \pm 1, 4$$

Plotting the zeroes on a number line



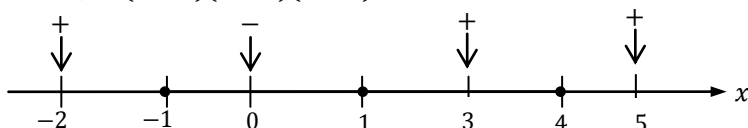
Testing $P(x)$ for each interval

$$P(-2) = (-2 + 1)(-2 - 1)(-2 - 4)^2 = +$$

$$P(0) = (0 + 1)(0 - 1)(0 - 4)^2 = -$$

$$P(3) = (3 + 1)(3 - 1)(3 - 4)^2 = +$$

$$P(5) = (5 + 1)(5 - 1)(5 - 4)^2 = +$$



$$\therefore P(x) \geq 0 \text{ has a solution } x \leq -1, x \geq 1$$

The above solution can be simplified in to a tabular solution as below.

Inequality	$x \leq -1$	$-1 \leq x \leq 1$	$1 \leq x \leq 4$	$x \geq 4$
Test value	-2	0	3	5
$x + 1$	-	+	+	+
$x - 1$	-	-	+	+
$(x - 4)^2$	+	+	+	+
$(x + 1)(x - 1)(x - 4)^2$	+	-	+	+

$$\therefore P(x) \geq 0 \text{ has a solution } x \leq -1, x \geq 1$$

b) Left as an exercise.

Ans: $x < \sqrt{5}, x > \sqrt{5}$

[HINT: Let $x^2 = y \Rightarrow y^2 - 3y - 10 > 0$]

Task

Solve the inequality

a) $4x^4 - 4x^3 - 3x^2 + 4x - 1 > 0$

Ans: $x < -1, x > 1$

b) $(x - 1)(x - 2)^2(x - 3)^2 < 0$

Ans: $x < 4, x \neq 2, x \neq 3$

c) $n^3 - 7n + 6 < 0$

Ans: $n < -3, 1 < n < 2$

SOLVING INEQUALITIES INVOLVING FUNCTION OF THE FORM $\frac{P(x)}{Q(x)}$ WHERE $P(x)$ AND $Q(x)$ ARE POLYNOMIALS/ PRODUCTS OF POLYNOMIALS

Example

Solve:

a) $\frac{(x+2)(x-5)^2}{x-4} \leq 0$

b) $\frac{(x-3)(x-4)}{(x-5)(x-6)^2} < 0$

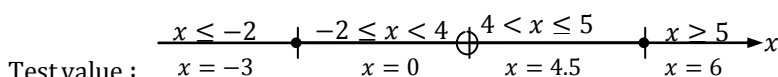
c) $\frac{x-1}{x-2} > \frac{x-2}{x+3}$

Solution

a) $\frac{(x+2)(x-5)^2}{x-4} \leq 0$, let $P(x) = \frac{(x+2)(x-5)^2}{x-4}$

Plotting the zeroes of all linear factors occurring in Numerator and Denominator

Zeroes of $P(x) = \frac{(x+2)(x-5)^2}{x-4}$ are: $x = -2, 4, 5$



Use solid dots/circles for -2 and 5 since $P(x)$ can equal to zero; and open dot/circle for 4 , to remind yourself to omit it from the solution since the denominator of $P(x)$ can never be equal to zero.

Do a sign analysis of $P(x) = \frac{(x+2)(x-5)^2}{x-4}$

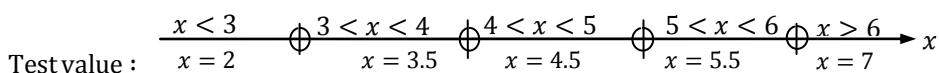
Inequality	$x \leq -2$	$-2 \leq x < 4$	$4 < x \leq 5$	$x \geq 5$
$x + 2$	—	+	+	+
$(x - 5)^2$	+	+	+	+
$x - 4$	—	—	+	+
$P(x) = \frac{(x+2)(x-5)^2}{x-4}$	+	—	+	+

$\therefore$ The solution of $P(x) \leq 0$ is: $-2 \leq x < 4$

b) $\frac{(x-3)(x-4)}{(x-5)(x-6)^2} < 0$, let $P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$

Plotting the zeroes of all linear factors occurring in Numerator and Denominator

Zeroes of $P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$ are: $x = 3, 4, 5, 6$



Use open dots/circles for 3 and 4 since $P(x)$ is less than zero; and open dot/circle for 5 and 6 since the denominator of $P(x)$ can never be equal to zero.

Do a sign analysis of $P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$

Inequality	$x < 3$	$3 < x < 4$	$4 < x < 5$	$5 < x < 6$	$x > 6$
$x - 3$	—	+	+	+	+
$x - 4$	—	+	+	+	+
$x - 5$	—	—	+	+	+
$(x - 6)^2$	+	—	+	+	+
$P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$	—	+	—	+	+

∴ The solution of $P(x) < 0$ is: $x < 3$ and $4 < x < 5$

c) $\frac{x-1}{x-2} > \frac{x-2}{x+3}$
 $\frac{x-1}{x-2} - \frac{x-2}{x+3} > 0$
 $\frac{(x-1)(x+3) - (x-2)(x-2)}{(x-2)(x+3)} > 0$ or $\frac{x^2+2x-3 - (x^2-4x+4)}{(x-2)(x+3)} > 0$
 $\frac{6x-7}{(x-2)(x+3)} > 0$, let $P(x) = \frac{6x-7}{(x-2)(x+3)}$

Finding the zeroes of $P(x) = \frac{6x-7}{(x-2)(x+3)}$ both for the numerator and denominator, zeroes are: $x = \frac{7}{6}, 2, -3$

Test value: $\frac{x < -3}{x = -4} \quad \bigoplus \quad \frac{-3 < x < \frac{7}{6}}{x = 0} \quad \bigoplus \quad \frac{\frac{7}{6} < x < 2}{x = 1.5} \quad \bigoplus \quad \frac{x > 2}{x = 3}$

Inequality	$x < -3$	$-3 < x < \frac{7}{6}$	$\frac{7}{6} < x < 2$	$x > 2$
$6x - 7$	—	—	+	+
$x - 2$	—	—	—	+
$x + 3$	—	+	+	+
$P(x) = \frac{6x-7}{(x-2)(x+3)}$	—	+	—	+

The solution of $P(x) > 0$ is: $x > 2$, $-3 < x < \frac{7}{6}$

STATING THE RANGE OF VALUES OF A QUADRATIC RATIONAL POLYNOMIAL FUNCTION

Example

If x is a real number, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ does not lie strictly between 2 and 6.

Solution

$$\text{Let } y = \frac{x^2+2x-11}{2(x-3)} \Rightarrow x^2 + 2x - 11 = 2xy - 6y$$

$$\Rightarrow x^2 + x(2 - 2y) + 6y - 11 = 0$$

For real values of x , $B^2 - 4AC \geq 0$

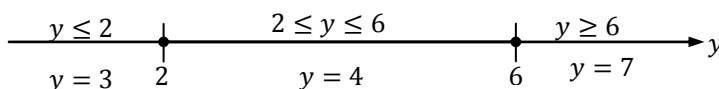
$$(2 - 2y)^2 - 4(6y - 11) \geq 0$$

$$4 - 8y + 4y^2 - 24y + 44 \geq 0 \text{ or } 4y^2 - 32y + 48 \geq 0$$

$\therefore y^2 - 8y + 12 \geq 0$ or $(y - 6)(y - 2) \geq 0$, let $P(y) = (y - 6)(y - 2)$

Finding the zeroes of $P(y)$

$\Rightarrow y = 2, 6$



Inequality	$y \leq 2$	$2 \leq y \leq 6$	$y \geq 6$
$y - 2$	-	+	+
$y - 6$	-	-	+
$P(y) = (y - 6)(y - 2)$	+	-	+

Since $P(y) \geq 0$, the interval $2 \leq y \leq 6$ does not solve the inequality.

Thus the expression $\frac{x^2 + 2x - 11}{2(x - 3)}$ does not strictly lie between 2 and 6.

Task

1. If x is a real number, prove that the expression $\frac{x}{x^2 - 5x + 9}$ lies between $-\frac{1}{11}$ and 1
2. If x is real, prove that $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ cannot lie strictly between 5 and 9
3. Show that for real x , $-\frac{1}{4} \leq \frac{x}{x^2 + 4} \leq \frac{1}{4}$

SOLVING INEQUALITIES INVOLVING MODULUS

Example

Find the set of values of x for which:

a) $ 2x - 3 > 5$	e) $\left \frac{2x - 4}{x + 1} \right < 4$
b) $ 4x + 2 < x + 8$	f) $\left \frac{x}{x - 3} \right < x$
c) $ x + 1 > x - 3 $	g) $\left \frac{2x + 5}{x^2 - 4} \right \geq \frac{1}{5}$
d) $3 x + 5 > x + 3 $	h) $ 2x - 3 > 4 x $

Solution

- a) $|2x - 3| > 5$

Note

$$|2x - 3| = \sqrt{(2x - 3)^2}$$

$$\Rightarrow \sqrt{(2x - 3)^2} > 5$$

Squaring both sides

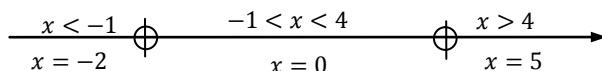
$$\Rightarrow (2x - 3)^2 > 25 \text{ or } (2x - 3)^2 - 25 > 0$$

$$\Rightarrow (2x - 3 + 5)(2x - 3 - 5) > 0$$

$$\Rightarrow (2x + 2)(2x - 8) > 0 \text{ or } (x + 1)(x - 4) > 0$$

$$\text{Let } P(x) = (x + 1)(x - 4)$$

$$\Rightarrow (x+1)(x-4) = 0 \quad \therefore x = -1, x = 4$$



Inequality	$x < -1$	$-1 < x < 4$	$x > 4$
Test value	-2	0	5
$x + 1$	-	+	+
$x - 4$	-	-	+
$(x + 1)(x - 4)$	+	-	+

$$\therefore x < -1, x > 4$$

b) Left as an exercise. **Ans:** $-2 < x < 2$

c) Left as an exercise. **Ans:** $x > 1$

d) $3|x+5| > |x+3|$

Squaring both sides to eliminate the modulus symbols

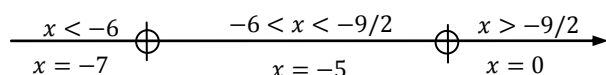
$$\Rightarrow 9(x+5)^2 > (x+3)^2 \quad \text{or} \quad 9(x+5)^2 - (x+3)^2 > 0$$

$$\Rightarrow (3x+15+x+3)(3x+15-x-3) > 0$$

$$\Rightarrow (4x+18)(2x+12) > 0 \quad \text{or} \quad (2x+9)(x+6) > 0$$

$$\text{Let } P(x) = (2x+9)(x+6)$$

$$\Rightarrow (2x+9)(x+6) = 0 \quad \therefore x = -\frac{9}{2}, x = -6$$



Inequality	$x < -6$	$-6 < x < -9/2$	$x > -9/2$
Test value	-7	-5	0
$2x + 9$	-	-	+
$x + 6$	-	+	+
$(x + 1)(x - 4)$	+	-	+

$$\therefore x < -6, x > -\frac{9}{2}$$

e) Left as an exercise. **Ans:** $x < -4, x > 0$

f) Left as an exercise. **Ans:** $0 < x < 2; x > 4$

g) Left as an exercise. **Ans:** $-7 \leq x \leq -3, -2.35 \leq x < -2, 2 < x \leq 12.35$

h) Left as an exercise. **Ans:** $-\frac{3}{2} < x < \frac{1}{2}$

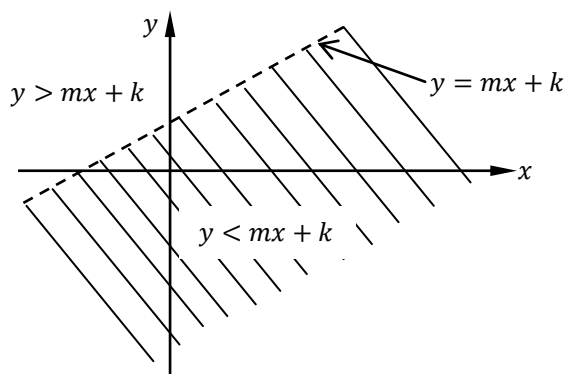
INEQUALITIES IN TWO VARIABLES

Polynomial Inequalities in Two variables and to Graph the solution set of a system of inequalities.

The solution to inequalities in two variables is an ordered pair of numbers. Thus graphs of inequalities in two variables consist of points in the plane.

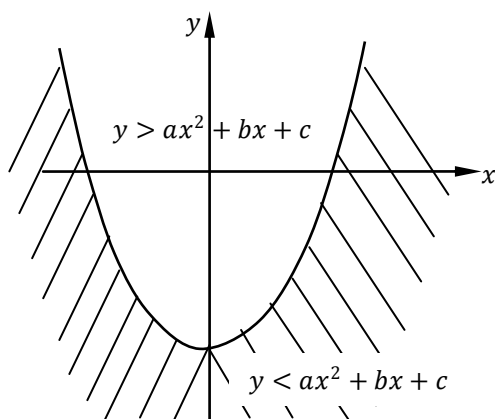
Graphing $y > mx + k$ and $y < mx + k$

Start by graphing the line $y = mx + k$ (Dotted if symbols $<$ or $>$ are used, and solid if symbols $\leq$ or $\geq$ are used)



Any point (x, y) below the line $y = mx + k$ satisfies the inequality $y < mx + k$ and above the line $y = mx + k$ satisfies the inequality $y > mx + k$

Graphing $y > ax^2 + bx + c$ and $y < ax^2 + bx + c$



Any point (x, y) below (Shaded) the curve $y = ax^2 + bx + c$ satisfies the inequality $y < ax^2 + bx + c$ and above (Not shaded) the curve $y = ax^2 + bx + c$ satisfies the inequality $y > ax^2 + bx + c$

Example

1. Sketch the graph;

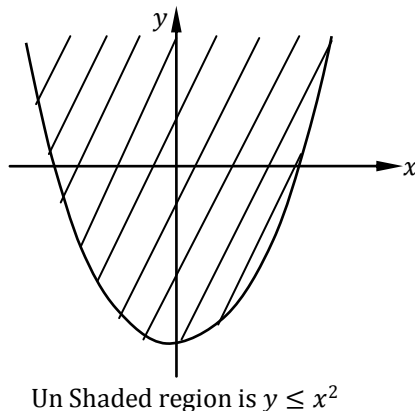
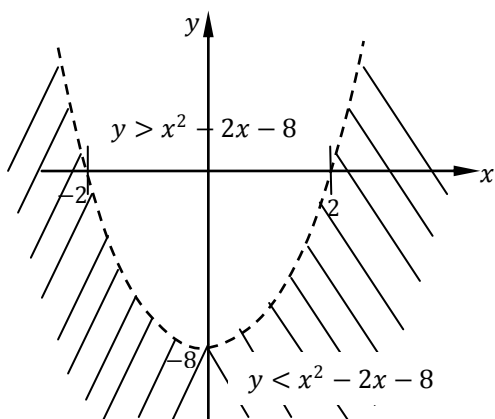
a) $y > x^2 - 2x - 8$

b) $y \leq x^2$

Solution

a) Sketching the curve $y = x^2 - 2x - 8$
(Dotted curve boundary)

b) Sketching the curve $y \leq x^2$
(Solid boundary)



2. Graph the solution set of the inequality

a) $y \leq 4 - x^2$ and $y > x + 2$

b) $y \leq x^2 + x - 2$ and $y \leq x^2 + x - 2$

Solution

Inequality

Test point

Result

$y \leq 4 - x^2$

$(0, 0)$

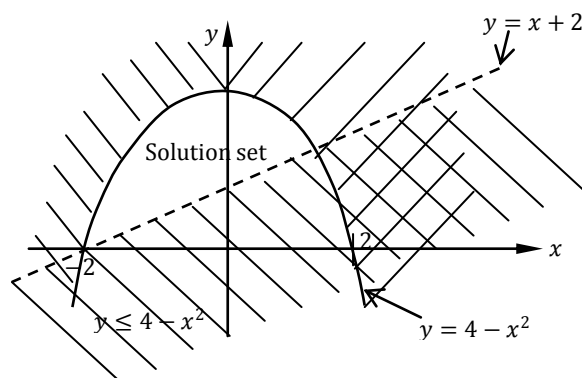
$0 \leq 4$, True (Required)

$y > x + 2$

$(0, 0)$

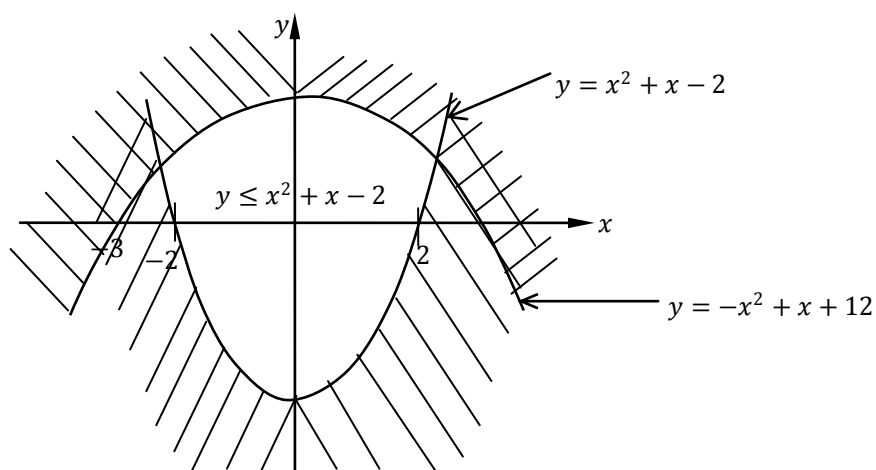
$0 < 4$, False (Not required)

The unshaded region is the solution set.



$$y = x^2 + x - 2$$

b)



Inequality

$$y \leq x^2 + x - 2$$

$$y \leq -x^2 + x + 12$$

Test point

$$(0, 0)$$

$$(0, 0)$$

Result

$$0 \leq -2, \text{ False}$$

$$0 \leq 12, \text{ True}$$